

The Coq Proof Assistant

The standard library

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The Coq Development Team

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Contents

This document is a short description of the COQ standard library. This library comes with the system as a complement of the core library (the **Init** library ; see the Reference Manual for a description of this library). It provides a set of modules directly available through the **Require** command.

The standard library is composed of the following subdirectories:

Logic Classical logic and dependent equality

Bool Booleans (basic functions and results)

Arith Basic Peano arithmetic

ZArith Basic integer arithmetic

Reals Classical Real Numbers and Analysis

Lists Monomorphic and polymorphic lists (basic functions and results), Streams (infinite sequences defined with co-inductive types)

Sets Sets (classical, constructive, finite, infinite, power set, etc.)

Relations Relations (definitions and basic results).

Sorting Sorted list (basic definitions and heapsort correctness).

Wellfounded Well-founded relations (basic results).

Program Tactics to deal with dependently-typed programs and their proofs.

Classes Standard type class instances on relations and Coq part of the setoid rewriting tactic.

Each of these subdirectories contains a set of modules, whose specifications (GALLINA files) have been roughly, and automatically, pasted in the following pages. There is also a version of this document in HTML format on the WWW, which you can access from the COQ home page at <http://coq.inria.fr/library>.

Chapter 1

Library `Coq.Init.Datatypes`

```
Require Import Notations.
```

```
Require Import Logic.
```

unit is a singleton datatype with sole inhabitant *tt*

```
Inductive unit : Set :=
```

```
  tt : unit.
```

bool is the datatype of the boolean values *true* and *false*

```
Inductive bool : Set :=
```

```
  | true : bool
```

```
  | false : bool.
```

```
Add Printing If bool.
```

```
Delimit Scope bool_scope with bool.
```

Basic boolean operators

```
Definition andb (b1 b2:bool) : bool := if b1 then b2 else false.
```

```
Definition orb (b1 b2:bool) : bool := if b1 then true else b2.
```

```
Definition implb (b1 b2:bool) : bool := if b1 then b2 else true.
```

```
Definition xorb (b1 b2:bool) : bool :=
```

```
  match b1, b2 with
```

```
    | true, true ⇒ false
```

```
    | true, false ⇒ true
```

```
    | false, true ⇒ true
```

```
    | false, false ⇒ false
```

```
  end.
```

```
Definition negb (b:bool) := if b then false else true.
```

```
Infix "||" := orb : bool_scope.
```

```
Infix "&&" := andb : bool_scope.
```

1.1 Properties of *andb*

Lemma *andb_prop* : $\forall a\ b:\text{bool}, \text{andb } a\ b = \text{true} \rightarrow a = \text{true} \wedge b = \text{true}$.

Hint Resolve *andb_prop*: *bool*.

Lemma *andb_true_intro* :

$\forall b1\ b2:\text{bool}, b1 = \text{true} \wedge b2 = \text{true} \rightarrow \text{andb } b1\ b2 = \text{true}$.

Hint Resolve *andb_true_intro*: *bool*.

Interpretation of booleans as propositions

Inductive *eq_true* : *bool* \rightarrow *Prop* := *is_eq_true* : *eq_true* *true*.

Hint Constructors *eq_true* : *eq_true*.

Another way of interpreting booleans as propositions

Definition *is_true* *b* := *b* = *true*.

is_true can be activated as a coercion by (Local) Coercion *is_true* : *bool* \rightarrow *Prop*.

Additional rewriting lemmas about *eq_true*

Lemma *eq_true_ind_r* :

$\forall (P : \text{bool} \rightarrow \text{Prop}) (b : \text{bool}), P\ b \rightarrow \text{eq_true } b \rightarrow P\ \text{true}$.

Lemma *eq_true_rec_r* :

$\forall (P : \text{bool} \rightarrow \text{Set}) (b : \text{bool}), P\ b \rightarrow \text{eq_true } b \rightarrow P\ \text{true}$.

Lemma *eq_true_rect_r* :

$\forall (P : \text{bool} \rightarrow \text{Type}) (b : \text{bool}), P\ b \rightarrow \text{eq_true } b \rightarrow P\ \text{true}$.

nat is the datatype of natural numbers built from *O* and successor *S*; note that the constructor name is the letter O. Numbers in *nat* can be denoted using a decimal notation; e.g. *3%nat* abbreviates *S (S (S O))*

Inductive *nat* : *Set* :=

| *O* : *nat*
| *S* : *nat* \rightarrow *nat*.

Delimit Scope *nat_scope* with *nat*.

Empty_set has no inhabitant

Inductive *Empty_set* : *Set* :=.

identity A a is the family of datatypes on *A* whose sole non-empty member is the singleton datatype *identity A a a* whose sole inhabitant is denoted *refl_identity A a*

Inductive *identity* (*A*:*Type*) (*a*:*A*) : *A* \rightarrow *Type* :=

identity_refl : *identity a a*.

Hint Resolve *identity_refl*: *core*.

Implicit Arguments *identity_ind* [*A*].

Implicit Arguments *identity_rec* [*A*].

Implicit Arguments *identity_rect* [*A*].

option A is the extension of *A* with an extra element *None*

Inductive *option* (*A*:*Type*) : *Type* :=

```
| Some : A → option A
| None : option A.
```

Implicit Arguments None [A].

```
Definition option_map (A B:Type) (f:A→B) o :=
  match o with
  | Some a ⇒ Some (f a)
  | None ⇒ None
end.
```

$\text{sum } A \ B$, written $A + B$, is the disjoint sum of A and B

```
Inductive sum (A B:Type) : Type :=
| inl : A → sum A B
| inr : B → sum A B.
```

Notation " $x + y$ " := ($\text{sum } x \ y$) : *type_scope*.

$\text{prod } A \ B$, written $A \times B$, is the product of A and B ; the pair $\text{pair } A \ B \ a \ b$ of a and b is abbreviated (a, b)

```
Inductive prod (A B:Type) : Type :=
  pair : A → B → prod A B.
```

Add Printing Let *prod*.

Notation " $x * y$ " := ($\text{prod } x \ y$) : *type_scope*.

Notation " (x, y, \dots, z) " := ($\text{pair } \dots (\text{pair } x \ y) \dots z$) : *core_scope*.

Section projections.

Variables $A \ B : \text{Type}$.

```
Definition fst (p:A × B) := match p with
| (x, y) ⇒ x
end.
```

```
Definition snd (p:A × B) := match p with
| (x, y) ⇒ y
end.
```

End projections.

Hint Resolve pair inl inr: *core*.

Lemma surjective_pairing :

```
  ∀ (A B:Type) (p:A × B), p = pair (fst p) (snd p).
```

Lemma injective_projections :

```
  ∀ (A B:Type) (p1 p2:A × B),
    fst p1 = fst p2 → snd p1 = snd p2 → p1 = p2.
```

```
Definition prod_uncurry (A B C:Type) (f:prod A B → C)
  (x:A) (y:B) : C := f (pair x y).
```

```
Definition prod_curry (A B C:Type) (f:A → B → C)
  (p:prod A B) : C := match p with
| pair x y ⇒ f x y
end.
```

Comparison

Inductive `comparison` : `Set` :=

| `Eq` : `comparison`
 | `Lt` : `comparison`
 | `Gt` : `comparison`.

Definition `CompOpp` (r :`comparison`) :=

`match` r `with`
 | `Eq` \Rightarrow `Eq`
 | `Lt` \Rightarrow `Gt`
 | `Gt` \Rightarrow `Lt`
`end`.

Lemma `CompOpp_involutive` : $\forall c, \text{CompOpp } (\text{CompOpp } c) = c$.

Lemma `CompOpp_inj` : $\forall c\ c', \text{CompOpp } c = \text{CompOpp } c' \rightarrow c = c'$.

Lemma `CompOpp_iff` : $\forall c\ c', \text{CompOpp } c = c' \leftrightarrow c = \text{CompOpp } c'$.

The *CompSpec* inductive will be used to relate a *compare* function (returning a comparison answer) and some equality and order predicates. Interest: *CompSpec* behave nicely with `case` and `destruct`.

Inductive `CompSpec` { A } ($eq\ lt : A \rightarrow A \rightarrow \text{Prop}$)($x\ y:A$) : `comparison` \rightarrow `Prop` :=

| `CompEq` : $eq\ x\ y \rightarrow \text{CompSpec } eq\ lt\ x\ y\ \text{Eq}$
 | `CompLt` : $lt\ x\ y \rightarrow \text{CompSpec } eq\ lt\ x\ y\ \text{Lt}$
 | `CompGt` : $lt\ y\ x \rightarrow \text{CompSpec } eq\ lt\ x\ y\ \text{Gt}$.

Hint *Constructors* `CompSpec`.

For having clean interfaces after extraction, *CompSpec* is declared in `Prop`. For some situations, it is nonetheless useful to have a version in `Type`. Interestingly, these two versions are equivalent.

Inductive `CompSpecT` { A } ($eq\ lt : A \rightarrow A \rightarrow \text{Prop}$)($x\ y:A$) : `comparison` \rightarrow `Type` :=

| `CompEqT` : $eq\ x\ y \rightarrow \text{CompSpecT } eq\ lt\ x\ y\ \text{Eq}$
 | `CompLtT` : $lt\ x\ y \rightarrow \text{CompSpecT } eq\ lt\ x\ y\ \text{Lt}$
 | `CompGtT` : $lt\ y\ x \rightarrow \text{CompSpecT } eq\ lt\ x\ y\ \text{Gt}$.

Hint *Constructors* `CompSpecT`.

Lemma `CompSpec2Type` : $\forall A\ (eq\ lt:A \rightarrow A \rightarrow \text{Prop})\ x\ y\ c,$
 $\text{CompSpec } eq\ lt\ x\ y\ c \rightarrow \text{CompSpecT } eq\ lt\ x\ y\ c$.

Identity

Definition `ID` := $\forall A:\text{Type}, A \rightarrow A$.

Definition `id` : `ID` := `fun` $A\ x \Rightarrow x$.

Polymorphic lists and some operations

Inductive `list` ($A : \text{Type}$) : `Type` :=

| `nil` : `list` A
 | `cons` : $A \rightarrow \text{list } A \rightarrow \text{list } A$.

Implicit Arguments `nil` [A].

Infix ":: :: " := `cons` (at *level* 60, *right associativity*) : *list_scope*.

Delimit Scope *list_scope* **with** *list*.

Local Open Scope *list_scope*.

Definition *length* (*A* : Type) : list *A* → nat :=
 fix *length* *l* :=
 match *l* with
 | nil ⇒ 0
 | _ :: *l'* ⇒ S (length *l'*)
end.

Concatenation of two lists

Definition *app* (*A* : Type) : list *A* → list *A* → list *A* :=
 fix *app* *l* *m* :=
 match *l* with
 | nil ⇒ *m*
 | *a* :: *l1* ⇒ *a* :: app *l1* *m*
end.

Infix "++" := app (right associativity, at level 60) : list_scope.

Chapter 2

Library Coq.Init.Logic_Type

This module defines type constructors for types in **Type** (*Datatypes.v* and *Logic.v* defined them for types in **Set**)

```
Require Import Datatypes.
```

```
Require Export Logic.
```

Negation of a type in **Type**

```
Definition notT (A:Type) := A → False.
```

Properties of *identity*

```
Section identity_is_a_congruence.
```

```
Variables A B : Type.
```

```
Variable f : A → B.
```

```
Variables x y z : A.
```

```
Lemma identity_sym : identity x y → identity y x.
```

```
Lemma identity_trans : identity x y → identity y z → identity x z.
```

```
Lemma identity_congr : identity x y → identity (f x) (f y).
```

```
Lemma not_identity_sym : notT (identity x y) → notT (identity y x).
```

```
End identity_is_a_congruence.
```

```
Definition identity_ind_r :
```

```
  ∀ (A:Type) (a:A) (P:A → Prop), P a → ∀ y:A, identity y a → P y.
```

```
Defined.
```

```
Definition identity_rec_r :
```

```
  ∀ (A:Type) (a:A) (P:A → Set), P a → ∀ y:A, identity y a → P y.
```

```
Defined.
```

```
Definition identity_rect_r :
```

```
  ∀ (A:Type) (a:A) (P:A → Type), P a → ∀ y:A, identity y a → P y.
```

```
Defined.
```

```
Hint Immediate identity_sym not_identity_sym: core v62.
```

Notation refl_id := identity_refl (*only parsing*).
Notation sym_id := identity_sym (*only parsing*).
Notation trans_id := identity_trans (*only parsing*).
Notation sym_not_id := not_identity_sym (*only parsing*).

Chapter 3

Library Coq.Init.Logic

Require Import Notations.

3.1 Propositional connectives

True is the always true proposition `Inductive True : Prop :=
| : True.`

False is the always false proposition `Inductive False : Prop :=.`

not A, written $\neg A$, is the negation of *A* `Definition not (A:Prop) := A → False.`

Notation `"~ x" := (not x) : type_scope.`

`Hint Unfold not: core.`

and A B, written $A \wedge B$, is the conjunction of *A* and *B*

conj p q is a proof of $A \wedge B$ as soon as *p* is a proof of *A* and *q* a proof of *B*

proj1 and *proj2* are first and second projections of a conjunction

`Inductive and (A B:Prop) : Prop :=
conj : A → B → A ∧ B`

where `"A /\ B" := (and A B) : type_scope.`

`Section Conjunction.`

`Variables A B : Prop.`

`Theorem proj1 : A ∧ B → A.`

`Theorem proj2 : A ∧ B → B.`

`End Conjunction.`

or A B, written $A \vee B$, is the disjunction of *A* and *B*

`Inductive or (A B:Prop) : Prop :=
| or_introl : A → A ∨ B
| or_intror : B → A ∨ B`

where "A \vee B" := (or A B) : type_scope.

iff A B, written $A \leftrightarrow B$, expresses the equivalence of A and B

Definition iff (A B:Prop) := (A \rightarrow B) \wedge (B \rightarrow A).

Notation "A \leftrightarrow B" := (iff A B) : type_scope.

Section Equivalence.

Theorem iff_refl : $\forall A:\text{Prop}, A \leftrightarrow A$.

Theorem iff_trans : $\forall A B C:\text{Prop}, (A \leftrightarrow B) \rightarrow (B \leftrightarrow C) \rightarrow (A \leftrightarrow C)$.

Theorem iff_sym : $\forall A B:\text{Prop}, (A \leftrightarrow B) \rightarrow (B \leftrightarrow A)$.

End Equivalence.

Hint Unfold iff: extcore.

Some equivalences

Theorem neg_false : $\forall A : \text{Prop}, \neg A \leftrightarrow (A \leftrightarrow \text{False})$.

Theorem and_cancel_l : $\forall A B C : \text{Prop},$
 $(B \rightarrow A) \rightarrow (C \rightarrow A) \rightarrow ((A \wedge B \leftrightarrow A \wedge C) \leftrightarrow (B \leftrightarrow C))$.

Theorem and_cancel_r : $\forall A B C : \text{Prop},$
 $(B \rightarrow A) \rightarrow (C \rightarrow A) \rightarrow ((B \wedge A \leftrightarrow C \wedge A) \leftrightarrow (B \leftrightarrow C))$.

Theorem and_comm : $\forall A B : \text{Prop}, A \wedge B \leftrightarrow B \wedge A$.

Theorem and_assoc : $\forall A B C : \text{Prop}, (A \wedge B) \wedge C \leftrightarrow A \wedge B \wedge C$.

Theorem or_cancel_l : $\forall A B C : \text{Prop},$
 $(B \rightarrow \neg A) \rightarrow (C \rightarrow \neg A) \rightarrow ((A \vee B \leftrightarrow A \vee C) \leftrightarrow (B \leftrightarrow C))$.

Theorem or_cancel_r : $\forall A B C : \text{Prop},$
 $(B \rightarrow \neg A) \rightarrow (C \rightarrow \neg A) \rightarrow ((B \vee A \leftrightarrow C \vee A) \leftrightarrow (B \leftrightarrow C))$.

Theorem or_comm : $\forall A B : \text{Prop}, (A \vee B) \leftrightarrow (B \vee A)$.

Theorem or_assoc : $\forall A B C : \text{Prop}, (A \vee B) \vee C \leftrightarrow A \vee B \vee C$.

Backward direction of the equivalences above does not need assumptions

Theorem and_iff_compat_l : $\forall A B C : \text{Prop},$
 $(B \leftrightarrow C) \rightarrow (A \wedge B \leftrightarrow A \wedge C)$.

Theorem and_iff_compat_r : $\forall A B C : \text{Prop},$
 $(B \leftrightarrow C) \rightarrow (B \wedge A \leftrightarrow C \wedge A)$.

Theorem or_iff_compat_l : $\forall A B C : \text{Prop},$
 $(B \leftrightarrow C) \rightarrow (A \vee B \leftrightarrow A \vee C)$.

Theorem or_iff_compat_r : $\forall A B C : \text{Prop},$
 $(B \leftrightarrow C) \rightarrow (B \vee A \leftrightarrow C \vee A)$.

Lemma iff_and : $\forall A B : \text{Prop}, (A \leftrightarrow B) \rightarrow (A \rightarrow B) \wedge (B \rightarrow A)$.

Lemma iff_to_and : $\forall A B : \text{Prop}, (A \leftrightarrow B) \leftrightarrow (A \rightarrow B) \wedge (B \rightarrow A)$.

(IF_then_else P Q R), written IF P then Q else R denotes either P and Q, or $\neg P$ and Q

Definition IF_then_else (P Q R:Prop) := P \wedge Q \vee $\neg P \wedge R$.

Notation "'IF' c1 'then' c2 'else' c3" := (IF_then_else c1 c2 c3)
 (at level 200, right associativity) : type_scope.

3.2 First-order quantifiers

$ex\ P$, or simply $\exists x, P\ x$, or also $\exists x:A, P\ x$, expresses the existence of an x of some type A in **Set** which satisfies the predicate P . This is existential quantification.

$ex2\ P\ Q$, or simply $exists2\ x, P\ x \ \&\ Q\ x$, or also $exists2\ x:A, P\ x \ \&\ Q\ x$, expresses the existence of an x of type A which satisfies both predicates P and Q .

Universal quantification is primitively written $\forall x:A, Q$. By symmetry with existential quantification, the construction $all\ P$ is provided too.

Remark: $\exists x, Q$ denotes $ex\ (\text{fun } x \Rightarrow Q)$ so that $\exists x, P\ x$ is in fact equivalent to $ex\ (\text{fun } x \Rightarrow P\ x)$ which may be not convertible to $ex\ P$ if P is not itself an abstraction

Inductive $ex\ (A:\text{Type})\ (P:A \rightarrow \text{Prop}) : \text{Prop} :=$
 $ex_intro : \forall x:A, P\ x \rightarrow ex\ (A:=A)\ P.$

Inductive $ex2\ (A:\text{Type})\ (P\ Q:A \rightarrow \text{Prop}) : \text{Prop} :=$
 $ex_intro2 : \forall x:A, P\ x \rightarrow Q\ x \rightarrow ex2\ (A:=A)\ P\ Q.$

Definition $all\ (A:\text{Type})\ (P:A \rightarrow \text{Prop}) := \forall x:A, P\ x.$

Notation "'exists' x , p" := (ex (fun x => p))
 (at level 200, x ident, right associativity) : type_scope.

Notation "'exists' x : t , p" := (ex (fun x:t => p))
 (at level 200, x ident, right associativity,
 format "'[' 'exists' '/' ' x : t , '/' ' p ']'")
 : type_scope.

Notation "'exists2' x , p & q" := (ex2 (fun x => p) (fun x => q))
 (at level 200, x ident, p at level 200, right associativity) : type_scope.

Notation "'exists2' x : t , p & q" := (ex2 (fun x:t => p) (fun x:t => q))
 (at level 200, x ident, t at level 200, p at level 200, right associativity,
 format "'[' 'exists2' '/' ' x : t , '/' ' [' p & '/' q ']' ']'")
 : type_scope.

Derived rules for universal quantification

Section universal_quantification.

Variable $A : \text{Type}.$

Variable $P : A \rightarrow \text{Prop}.$

Theorem $inst : \forall x:A, all\ (\text{fun } x \Rightarrow P\ x) \rightarrow P\ x.$

Theorem $gen : \forall (B:\text{Prop})\ (f:\forall y:A, B \rightarrow P\ y), B \rightarrow all\ P.$

End universal_quantification.

3.3 Equality

$eq\ x\ y$, or simply $x=y$ expresses the equality of x and y . Both x and y must belong to the same type A . The definition is inductive and states the reflexivity of the equality. The others properties (symmetry, transitivity, replacement of equals by equals) are proved below. The type of x and y can be made explicit using the notation $x = y :> A$. This is Leibniz equality as it expresses that x and y are equal iff every property on A which is true of x is also true of y

```
Inductive eq (A:Type) (x:A) : A → Prop :=
  eq_refl : x = x :> A
```

where "x = y :> A" := (@eq A x y) : type_scope.

Notation "x = y" := (x = y :>_) : type_scope.

Notation "x <> y :> T" := (¬ x = y :> T) : type_scope.

Notation "x <> y" := (x ≠ y :>_) : type_scope.

```
Implicit Arguments eq [A] ].
```

```
Implicit Arguments eq_ind [A].
```

```
Implicit Arguments eq_rec [A].
```

```
Implicit Arguments eq_rect [A].
```

```
Hint Resolve ! conj or_introl or_intror eq_refl: core.
```

```
Hint Resolve ex_intro ex_intro2: core.
```

Section Logic-lemmas.

```
Theorem absurd : ∀ A C:Prop, A → ¬ A → C.
```

Section equality.

```
Variables A B : Type.
```

```
Variable f : A → B.
```

```
Variables x y z : A.
```

```
Theorem eq_sym : x = y → y = x.
```

```
Opaque eq_sym.
```

```
Theorem eq_trans : x = y → y = z → x = z.
```

```
Opaque eq_trans.
```

```
Theorem f_equal : x = y → f x = f y.
```

```
Opaque f_equal.
```

```
Theorem not_eq_sym : x ≠ y → y ≠ x.
```

End equality.

```
Definition eq_ind_r :
```

```
  ∀ (A:Type) (x:A) (P:A → Prop), P x → ∀ y:A, y = x → P y.
```

```
Defined.
```

```
Definition eq_rec_r :
```

```
  ∀ (A:Type) (x:A) (P:A → Set), P x → ∀ y:A, y = x → P y.
```

```
Defined.
```

```
Definition eq_rect_r :
```

```
  ∀ (A:Type) (x:A) (P:A → Type), P x → ∀ y:A, y = x → P y.
```

```
Defined.
```

End Logic_lemmas.

Theorem f_equal2 :

$\forall (A1\ A2\ B:\text{Type}) (f:A1 \rightarrow A2 \rightarrow B) (x1\ y1:A1)$
 $(x2\ y2:A2), x1 = y1 \rightarrow x2 = y2 \rightarrow f\ x1\ x2 = f\ y1\ y2.$

Theorem f_equal3 :

$\forall (A1\ A2\ A3\ B:\text{Type}) (f:A1 \rightarrow A2 \rightarrow A3 \rightarrow B) (x1\ y1:A1)$
 $(x2\ y2:A2) (x3\ y3:A3),$
 $x1 = y1 \rightarrow x2 = y2 \rightarrow x3 = y3 \rightarrow f\ x1\ x2\ x3 = f\ y1\ y2\ y3.$

Theorem f_equal4 :

$\forall (A1\ A2\ A3\ A4\ B:\text{Type}) (f:A1 \rightarrow A2 \rightarrow A3 \rightarrow A4 \rightarrow B)$
 $(x1\ y1:A1) (x2\ y2:A2) (x3\ y3:A3) (x4\ y4:A4),$
 $x1 = y1 \rightarrow x2 = y2 \rightarrow x3 = y3 \rightarrow x4 = y4 \rightarrow f\ x1\ x2\ x3\ x4 = f\ y1\ y2\ y3\ y4.$

Theorem f_equal5 :

$\forall (A1\ A2\ A3\ A4\ A5\ B:\text{Type}) (f:A1 \rightarrow A2 \rightarrow A3 \rightarrow A4 \rightarrow A5 \rightarrow B)$
 $(x1\ y1:A1) (x2\ y2:A2) (x3\ y3:A3) (x4\ y4:A4) (x5\ y5:A5),$
 $x1 = y1 \rightarrow$
 $x2 = y2 \rightarrow$
 $x3 = y3 \rightarrow x4 = y4 \rightarrow x5 = y5 \rightarrow f\ x1\ x2\ x3\ x4\ x5 = f\ y1\ y2\ y3\ y4\ y5.$

Notation sym_eq := eq_sym (only parsing).

Notation trans_eq := eq_trans (only parsing).

Notation sym_not_eq := not_eq_sym (only parsing).

Notation refl_equal := eq_refl (only parsing).

Notation sym_equal := eq_sym (only parsing).

Notation trans_equal := eq_trans (only parsing).

Notation sym_not_equal := not_eq_sym (only parsing).

Hint Immediate eq_sym not_eq_sym: core.

Basic definitions about relations and properties

Definition subrelation (A B : Type) (R R' : A → B → Prop) :=

$\forall x\ y, R\ x\ y \rightarrow R'\ x\ y.$

Definition unique (A : Type) (P : A → Prop) (x:A) :=

$P\ x \wedge \forall (x':A), P\ x' \rightarrow x=x'.$

Definition uniqueness (A:Type) (P:A→Prop) := $\forall x\ y, P\ x \rightarrow P\ y \rightarrow x = y.$

Unique existence

Notation "'exists' ! x , P" := (ex (unique (fun x ⇒ P)))

(at level 200, x ident, right associativity,
format "['exists' ! ' / ' x , ' / ' P ']") : type_scope.

Notation "'exists' ! x : A , P" :=

(ex (unique (fun x:A ⇒ P)))
(at level 200, x ident, right associativity,
format "['exists' ! ' / ' x : A , ' / ' P ']") : type_scope.

Lemma unique_existence : $\forall (A:\text{Type}) (P:A \rightarrow \text{Prop}),$

$((\exists x, P\ x) \wedge \text{uniqueness}\ P) \leftrightarrow (\exists! x, P\ x).$

3.4 Being inhabited

The predicate *inhabited* can be used in different contexts. If A is thought as a type, *inhabited* A states that A is inhabited. If A is thought as a computationally relevant proposition, then *inhabited* A weakens A so as to hide its computational meaning. The so-weakened proof remains computationally relevant but only in a propositional context.

Inductive *inhabited* (A :Type) : Prop := *inhabits* : $A \rightarrow$ *inhabited* A .

Hint *Resolve* *inhabits*: *core*.

Lemma *exists_inhabited* : $\forall (A$:Type) (P : $A \rightarrow$ Prop),
($\exists x, P\ x$) \rightarrow *inhabited* A .

Declaration of *stepl* and *stepr* for *eq* and *iff*

Lemma *eq_stepl* : $\forall (A$: Type) ($x\ y\ z$: A), $x = y \rightarrow x = z \rightarrow z = y$.

Declare Left Step *eq_stepl*.

Declare Right Step *eq_trans*.

Lemma *iff_stepl* : $\forall A\ B\ C$: Prop, ($A \leftrightarrow B$) \rightarrow ($A \leftrightarrow C$) \rightarrow ($C \leftrightarrow B$).

Declare Left Step *iff_stepl*.

Declare Right Step *iff_trans*.

Chapter 4

Library **Coq.Init.Notations**

These are the notations whose level and associativity are imposed by Coq

Notations for propositional connectives

Reserved Notation " $x \leftrightarrow y$ " (at level 95, no associativity).

Reserved Notation " $x \wedge y$ " (at level 80, right associativity).

Reserved Notation " $x \vee y$ " (at level 85, right associativity).

Reserved Notation " $\sim x$ " (at level 75, right associativity).

Notations for equality and inequalities

Reserved Notation " $x = y :> T$ "

(at level 70, y at next level, no associativity).

Reserved Notation " $x = y$ " (at level 70, no associativity).

Reserved Notation " $x = y = z$ "

(at level 70, no associativity, y at next level).

Reserved Notation " $x <> y :> T$ "

(at level 70, y at next level, no associativity).

Reserved Notation " $x <> y$ " (at level 70, no associativity).

Reserved Notation " $x \leq y$ " (at level 70, no associativity).

Reserved Notation " $x < y$ " (at level 70, no associativity).

Reserved Notation " $x \geq y$ " (at level 70, no associativity).

Reserved Notation " $x > y$ " (at level 70, no associativity).

Reserved Notation " $x \leq y \leq z$ " (at level 70, y at next level).

Reserved Notation " $x \leq y < z$ " (at level 70, y at next level).

Reserved Notation " $x < y < z$ " (at level 70, y at next level).

Reserved Notation " $x < y \leq z$ " (at level 70, y at next level).

Arithmetical notations (also used for type constructors)

Reserved Notation " $x + y$ " (at level 50, left associativity).

Reserved Notation " $x - y$ " (at level 50, left associativity).

Reserved Notation " $x * y$ " (at level 40, left associativity).

Reserved Notation " x / y " (at level 40, left associativity).

Reserved Notation " $- x$ " (at level 35, right associativity).

Reserved Notation $/ x$ (at level 35, right associativity).

Reserved Notation $x \wedge y$ (at level 30, right associativity).

Notations for booleans

Reserved Notation $x \parallel y$ (at level 50, left associativity).

Reserved Notation $x \&\& y$ (at level 40, left associativity).

Notations for pairs

Reserved Notation (x, y, \dots, z) (at level 0).

Notation $\{x\}$ is reserved and has a special status as component of other notations such as $\{A\} + \{B\}$ and $A + \{B\}$ (which are at the same level than $x + y$); $\{x\}$ is at level 0 to factor with $\{x : A \mid P\}$

Reserved Notation $\{x\}$ (at level 0, x at level 99).

Notations for sigma-types or subsets

Reserved Notation $\{x \mid P\}$ (at level 0, x at level 99).

Reserved Notation $\{x \mid P \& Q\}$ (at level 0, x at level 99).

Reserved Notation $\{x : A \mid P\}$ (at level 0, x at level 99).

Reserved Notation $\{x : A \mid P \& Q\}$ (at level 0, x at level 99).

Reserved Notation $\{x : A \& P\}$ (at level 0, x at level 99).

Reserved Notation $\{x : A \& P \& Q\}$ (at level 0, x at level 99).

Delimit Scope *type_scope* with *type*.

Delimit Scope *core_scope* with *core*.

Open Scope *core_scope*.

Open Scope *type_scope*.

Chapter 5

Library `Coq.Init.Peano`

The type *nat* of Peano natural numbers (built from *O* and *S*) is defined in *Datatypes.v*

This module defines the following operations on natural numbers :

- predecessor *pred*
- addition *plus*
- multiplication *mult*
- less or equal order *le*
- less *lt*
- greater or equal *ge*
- greater *gt*

It states various lemmas and theorems about natural numbers, including Peano's axioms of arithmetic (in Coq, these are provable). Case analysis on *nat* and induction on $\text{nat} \times \text{nat}$ are provided too

```
Require Import Notations.
```

```
Require Import Datatypes.
```

```
Require Import Logic.
```

```
Open Scope nat_scope.
```

```
Definition eq_S := f_equal S.
```

```
Hint Resolve (f_equal S): v62.
```

```
Hint Resolve (f_equal (A:=nat)): core.
```

The predecessor function

```
Definition pred (n:nat) : nat := match n with
| O => n
| S u => u
end.
```

Hint Resolve (f_equal pred): v62.

Theorem pred_Sn : $\forall n:\text{nat}, n = \text{pred } (\text{S } n)$.

Injectivity of successor

Theorem eq_add_S : $\forall n m:\text{nat}, \text{S } n = \text{S } m \rightarrow n = m$.

Hint Immediate eq_add_S: core.

Theorem not_eq_S : $\forall n m:\text{nat}, n \neq m \rightarrow \text{S } n \neq \text{S } m$.

Hint Resolve not_eq_S: core.

Definition IsSucc (n:nat) : Prop :=

```
match n with
| O => False
| S p => True
end.
```

Zero is not the successor of a number

Theorem O_S : $\forall n:\text{nat}, 0 \neq \text{S } n$.

Hint Resolve O_S: core.

Theorem n_Sn : $\forall n:\text{nat}, n \neq \text{S } n$.

Hint Resolve n_Sn: core.

Addition

Fixpoint plus (n m:nat) : nat :=

```
match n with
| O => m
| S p => S (p + m)
end
```

where "n + m" := (plus n m) : nat_scope.

Hint Resolve (f_equal2 plus): v62.

Hint Resolve (f_equal2 (A1:=nat) (A2:=nat)): core.

Lemma plus_n_O : $\forall n:\text{nat}, n = n + 0$.

Hint Resolve plus_n_O: core.

Lemma plus_O_n : $\forall n:\text{nat}, 0 + n = n$.

Lemma plus_n_Sm : $\forall n m:\text{nat}, \text{S } (n + m) = n + \text{S } m$.

Hint Resolve plus_n_Sm: core.

Lemma plus_Sn_m : $\forall n m:\text{nat}, \text{S } n + m = \text{S } (n + m)$.

Standard associated names

Notation plus_0_r_reverse := plus_n_O (only parsing).

Notation plus_succ_r_reverse := plus_n_Sm (only parsing).

Multiplication

Fixpoint mult (n m:nat) : nat :=

```
match n with
```

```

| O ⇒ 0
| S p ⇒ m + p × m
end

```

where "n * m" := (mult n m) : nat_scope.

Hint Resolve (f_equal2 mult): core.

Lemma mult_n_O : ∀ n:nat, 0 = n × 0.

Hint Resolve mult_n_O: core.

Lemma mult_n_Sm : ∀ n m:nat, n × m + n = n × S m.

Hint Resolve mult_n_Sm: core.

Standard associated names

Notation mult_0_r_reverse := mult_n_O (only parsing).

Notation mult_succ_r_reverse := mult_n_Sm (only parsing).

Truncated subtraction: $m - n$ is 0 if $n \geq m$

```

Fixpoint minus (n m:nat) : nat :=
  match n, m with
  | O, _ ⇒ n
  | S k, O ⇒ n
  | S k, S l ⇒ k - l
  end

```

where "n - m" := (minus n m) : nat_scope.

Definition of the usual orders, the basic properties of le and lt can be found in files Le and Lt

```

Inductive le (n:nat) : nat → Prop :=
| le_n : n ≤ n
| le_S : ∀ m:nat, n ≤ m → n ≤ S m

```

where "n <= m" := (le n m) : nat_scope.

Hint Constructors le: core.

Definition lt (n m:nat) := S n ≤ m.

Hint Unfold lt: core.

Infix "<" := lt : nat_scope.

Definition ge (n m:nat) := m ≤ n.

Hint Unfold ge: core.

Infix "≥" := ge : nat_scope.

Definition gt (n m:nat) := m < n.

Hint Unfold gt: core.

Infix ">" := gt : nat_scope.

Notation "x <= y <= z" := (x ≤ y ∧ y ≤ z) : nat_scope.

Notation "x <= y < z" := (x ≤ y ∧ y < z) : nat_scope.

Notation " $x < y < z$ " $:= (x < y \wedge y < z) : \text{nat_scope}$.

Notation " $x < y \leq z$ " $:= (x < y \wedge y \leq z) : \text{nat_scope}$.

Case analysis

Theorem `nat_case` :

$\forall (n:\text{nat}) (P:\text{nat} \rightarrow \text{Prop}), P\ 0 \rightarrow (\forall m:\text{nat}, P\ (\text{S } m)) \rightarrow P\ n.$

Principle of double induction

Theorem `nat_double_ind` :

$\forall R:\text{nat} \rightarrow \text{nat} \rightarrow \text{Prop},$

$(\forall n:\text{nat}, R\ 0\ n) \rightarrow$

$(\forall n:\text{nat}, R\ (\text{S } n)\ 0) \rightarrow$

$(\forall n\ m:\text{nat}, R\ n\ m \rightarrow R\ (\text{S } n)\ (\text{S } m)) \rightarrow \forall n\ m:\text{nat}, R\ n\ m.$

Chapter 6

Library `Coq.Init.Prelude`

```
Require Export Notations.  
Require Export Logic.  
Require Export Datatypes.  
Require Export Specif.  
Require Export Peano.  
Require Export Coq.Init.Wf.  
Require Export Coq.Init.Tactics.
```

Chapter 7

Library **Coq.Init.Specif**

Basic specifications : sets that may contain logical information

Require Import Notations.

Require Import Datatypes.

Require Import Logic.

Subsets and Sigma-types

$(\text{sig } A \ P)$, or more suggestively $\{x:A \mid P \ x\}$, denotes the subset of elements of the type A which satisfy the predicate P . Similarly $(\text{sig2 } A \ P \ Q)$, or $\{x:A \mid P \ x \ \& \ Q \ x\}$, denotes the subset of elements of the type A which satisfy both P and Q .

Inductive **sig** ($A:\text{Type}$) ($P:A \rightarrow \text{Prop}$) : **Type** :=
 exist : $\forall x:A, P \ x \rightarrow \text{sig } P$.

Inductive **sig2** ($A:\text{Type}$) ($P \ Q:A \rightarrow \text{Prop}$) : **Type** :=
 exist2 : $\forall x:A, P \ x \rightarrow Q \ x \rightarrow \text{sig2 } P \ Q$.

$(\text{sigT } A \ P)$, or more suggestively $\{x:A \ \& \ (P \ x)\}$ is a Sigma-type. Similarly for $(\text{sigT2 } A \ P \ Q)$, also written $\{x:A \ \& \ (P \ x) \ \& \ (Q \ x)\}$.

Inductive **sigT** ($A:\text{Type}$) ($P:A \rightarrow \text{Type}$) : **Type** :=
 existT : $\forall x:A, P \ x \rightarrow \text{sigT } P$.

Inductive **sigT2** ($A:\text{Type}$) ($P \ Q:A \rightarrow \text{Type}$) : **Type** :=
 existT2 : $\forall x:A, P \ x \rightarrow Q \ x \rightarrow \text{sigT2 } P \ Q$.

Notation " $\{ x \mid P \}$ " := (**sig** (**fun** $x \Rightarrow P$)) : *type_scope*.

Notation " $\{ x \mid P \ \& \ Q \}$ " := (**sig2** (**fun** $x \Rightarrow P$) (**fun** $x \Rightarrow Q$)) : *type_scope*.

Notation " $\{ x : A \mid P \}$ " := (**sig** (**fun** $x:A \Rightarrow P$)) : *type_scope*.

Notation " $\{ x : A \mid P \ \& \ Q \}$ " := (**sig2** (**fun** $x:A \Rightarrow P$) (**fun** $x:A \Rightarrow Q$)) :
 type_scope.

Notation " $\{ x : A \ \& \ P \}$ " := (**sigT** (**fun** $x:A \Rightarrow P$)) : *type_scope*.

Notation " $\{ x : A \ \& \ P \ \& \ Q \}$ " := (**sigT2** (**fun** $x:A \Rightarrow P$) (**fun** $x:A \Rightarrow Q$)) :
 type_scope.

Add Printing Let *sig*.

Add Printing Let *sig2*.

Add *Printing* Let $sigT$.
Add *Printing* Let $sigT2$.

Projections of *sig*

An element y of a subset $\{x:A \ \& \ (P \ x)\}$ is the pair of an a of type A and of a proof h that a satisfies P . Then $(proj1_sig \ y)$ is the witness a and $(proj2_sig \ y)$ is the proof of $(P \ a)$

Section Subset_projections.

Variable A : Type.

Variable $P : A \rightarrow \text{Prop}$.

```

Definition proj1-sig (e:sig P) := match e with
| exist a b => a
end.

```

```

Definition proj2_sig (e:sig P) :=
  match e return P (proj1_sig e) with
  | exist a b => b
  end.

```

End Subset_projections.

Projections of $sigT$

An element x of a sigma-type $\{y:A \ \& \ P \ y\}$ is a dependent pair made of an a of type A and an h of type $P \ a$. Then, $(projT1 \ x)$ is the first projection and $(projT2 \ x)$ is the second projection, the type of which depends on the $projT1$.

Section Projections.

Variable A : Type.

Variable $P : A \rightarrow \text{Type}$.

```

Definition projT1 (x:sigT P) : A := match x with
    | existT a _ => a
end.

```

```

Definition projT2 (x: sigT P) : P (projT1 x) :=
  match x return P (projT1 x) with
  | existT _ h => h
  end.

```

End Projections.

$\text{sig}T$ of a predicate is equivalent to sig

$$\text{Lemma sig_of_sigT} : \forall (A:\text{Type}) (P:A \rightarrow \text{Prop}), \text{sigT } P \rightarrow \text{sig } P.$$
$$\text{Lemma sigT_of_sig : } \forall (A:\text{Type}) (P:A \rightarrow \text{Prop}), \text{sig } P \rightarrow \text{sigT } P.$$

Coercion `sigT_of_sig` : *sig* \rightarrow *sigT*.

Coercion `sig_of_sigT` : $\text{sig}T \rightarrow \text{sig}$.

sumbool is a boolean type equipped with the justification of their value

$$\text{Inductive sumbool } (A \ B:\text{Prop}) : \text{Set} :=$$

- left : $A \rightarrow \{A\} + \{B\}$
- right : $B \rightarrow \{A\} + \{B\}$

where " $\{ A \} + \{ B \}$ " := (sumbool $A B$) : type_scope.

Add Printing If sumbool.

sumor is an option type equipped with the justification of why it may not be a regular value

Inductive sumor ($A:Type$) ($B:Prop$) : Type :=

| inleft : $A \rightarrow A + \{B\}$

| inright : $B \rightarrow A + \{B\}$

where " $A + \{ B \}$ " := (sumor $A B$) : type_scope.

Add Printing If sumor.

Various forms of the axiom of choice for specifications

Section Choice_lemmas.

Variables $S S' : Set$.

Variable $R : S \rightarrow S' \rightarrow Prop$.

Variable $R' : S \rightarrow S' \rightarrow Set$.

Variables $R1 R2 : S \rightarrow Prop$.

Lemma Choice :

$(\forall x:S, \{y:S' \mid R x y\}) \rightarrow \{f:S \rightarrow S' \mid \forall z:S, R z (f z)\}$.

Lemma Choice2 :

$(\forall x:S, \{y:S' \ \& \ R' x y\}) \rightarrow \{f:S \rightarrow S' \ \& \ \forall z:S, R' z (f z)\}$.

Lemma bool_choice :

$(\forall x:S, \{R1 x\} + \{R2 x\}) \rightarrow$
 $\{f:S \rightarrow bool \mid \forall x:S, f x = true \wedge R1 x \vee f x = false \wedge R2 x\}$.

End Choice_lemmas.

Section Dependent_choice_lemmas.

Variables $X : Set$.

Variable $R : X \rightarrow X \rightarrow Prop$.

Lemma dependent_choice :

$(\forall x:X, \{y \mid R x y\}) \rightarrow$
 $\forall x0, \{f : nat \rightarrow X \mid f 0 = x0 \wedge \forall n, R (f n) (f (S n))\}$.

End Dependent_choice_lemmas.

A result of type ($Exc A$) is either a normal value of type A or an *error* :

Inductive Exc [$A:Type$] : Type := value : $A \rightarrow (Exc A)$ | error : ($Exc A$).

It is implemented using the option type.

Definition Exc := option.

Definition value := Some.

Definition error := @None.

Implicit Arguments error [A].

Definition except := False_rec.

Implicit Arguments except [P].

Theorem absurd_set : $\forall (A:Prop) (C:Set), A \rightarrow \neg A \rightarrow C$.

Hint Resolve left right inleft inright: core v62.

Hint Resolve exist exist2 existT existT2: core.

Notation sigS := sigT (only parsing).

Notation existS := existT (only parsing).

Notation sigS_rect := sigT_rect (only parsing).

Notation sigS_rec := sigT_rec (only parsing).

Notation sigS_ind := sigT_ind (only parsing).

Notation projS1 := projT1 (only parsing).

Notation projS2 := projT2 (only parsing).

Notation sigS2 := sigT2 (only parsing).

Notation existS2 := existT2 (only parsing).

Notation sigS2_rect := sigT2_rect (only parsing).

Notation sigS2_rec := sigT2_rec (only parsing).

Notation sigS2_ind := sigT2_ind (only parsing).

Chapter 8

Library `Coq.Init.Tactics`

```
Require Import Notations.  
Require Import Logic.  
Require Import Specif.
```

8.1 Useful tactics

Ex falso quodlibet : a tactic for proving `False` instead of the current goal. This is just a nicer name for tactics such as `elimtype False` and other `cut False`.

```
Ltac exfalso := elimtype False.
```

A tactic for proof by contradiction. With contradict `H`,

- $H: \neg A \vdash B$ gives $\vdash A$
- $H: \neg A \vdash \neg B$ gives $H: B \vdash A$
- $H: A \vdash B$ gives $\vdash \neg A$
- $H: A \vdash \neg B$ gives $H: B \vdash \neg A$
- $H: \text{False}$ leads to a resolved subgoal.

Moreover, negations may be in unfolded forms, and `A` or `B` may live in `Type`

```
Ltac contradict H :=  
  let save tac H := let x:=fresh in intro x; tac H; rename x into H  
  in  
  let negpos H := case H; clear H  
  in  
  let negneg H := save negpos H  
  in  
  let pospos H :=  
    let A := type of H in (exfalso; revert H; try fold (¬A))  
  in
```

```

let posneg H := save pospos H
in
let neg H := match goal with
| ⊢ (¬_) ⇒ negneg H
| ⊢ (_→False) ⇒ negneg H
| ⊢ _ ⇒ negpos H
end in
let pos H := match goal with
| ⊢ (¬_) ⇒ posneg H
| ⊢ (_→False) ⇒ posneg H
| ⊢ _ ⇒ pospos H
end in
match type of H with
| (¬_) ⇒ neg H
| (_→False) ⇒ neg H
| _ ⇒ (elim H;fail) || pos H
end.

Ltac swap H :=
  idtac "swap is OBSOLETE: use contradict instead.";
  intro; apply H; clear H.

Ltac absurd_hyp H :=
  idtac "absurd_hyp is OBSOLETE: use contradict instead.";
  let T := type of H in
  absurd T.

Ltac false_hyp H G :=
  let T := type of H in absurd T; [ apply G | assumption ].

Ltac case_eq x := generalize (refl_equal x); pattern x at -1; case x.

Tactic Notation "destruct_with_eqn" constr(x) :=
  destruct x as []_eqn.
Tactic Notation "destruct_with_eqn" ident(n) :=
  try intros until n; destruct n as []_eqn.
Tactic Notation "destruct_with_eqn" ":" ident(H) constr(x) :=
  destruct x as []_eqn:H.
Tactic Notation "destruct_with_eqn" ":" ident(H) ident(n) :=
  try intros until n; destruct n as []_eqn:H.

  Break every hypothesis of a certain type

Ltac destruct_all t :=
  match goal with
  | x : t ⊢ _ ⇒ destruct x; destruct_all t
  | _ ⇒ idtac
  end.

Tactic Notation "rewrite_all" constr(eq) := repeat rewrite eq in ×.

```

Tactic Notation "rewrite_all" "<-" *constr*(*eq*) := repeat rewrite ← *eq* in ×.

Tactics for applying equivalences.

The following code provides tactics "apply -> t", "apply <- t", "apply -> t in H" and "apply <- t in H". Here t is a term whose type consists of nested dependent and nondependent products with an equivalence $A \leftrightarrow B$ as the conclusion. The tactics with "->" in their names apply $A \rightarrow B$ while those with "<-" in the name apply $B \rightarrow A$.

```
Ltac find_equiv H :=
let T := type of H in
lazymatch T with
| ?A → ?B ⇒
  let H1 := fresh in
  let H2 := fresh in
  cut A;
  [intro H1; pose proof (H H1) as H2; clear H H1;
   rename H2 into H; find_equiv H |
   clear H]
| ∀ x : ?t, _ ⇒
  let a := fresh "a" with
    H1 := fresh "H" in
    evar (a : t); pose proof (H a) as H1; unfold a in H1;
    clear a; clear H; rename H1 into H; find_equiv H
| ?A ↔ ?B ⇒ idtac
| _ ⇒ fail "The given statement does not seem to end with an equivalence."
end.
```

```
Ltac bapply lemma todo :=
let H := fresh in
  pose proof lemma as H;
  find_equiv H; [todo H; clear H | .. ].
```

```
Tactic Notation "apply" "->" constr(lemma) :=
bapply lemma ltac:(fun H ⇒ destruct H as [H _]; apply H).
```

```
Tactic Notation "apply" "<-" constr(lemma) :=
bapply lemma ltac:(fun H ⇒ destruct H as [_ H]; apply H).
```

```
Tactic Notation "apply" "->" constr(lemma) "in" hyp(J) :=
bapply lemma ltac:(fun H ⇒ destruct H as [H _]; apply H in J).
```

```
Tactic Notation "apply" "<-" constr(lemma) "in" hyp(J) :=
bapply lemma ltac:(fun H ⇒ destruct H as [_ H]; apply H in J).
```

An experimental tactic simpler than auto that is useful for ending proofs "in one step"

```
Ltac easy :=
let rec use_hyp H :=
  match type of H with
  | _ ∧ _ ⇒ exact H || destruct_hyp H
  | _ ⇒ try solve [inversion H]
  end
```

```

with do_intro := let H := fresh in intro H; use_hyp H
with destruct_hyp H := case H; clear H; do_intro; do_intro in
let rec use_hyps :=
  match goal with
  | H : _ ∧ _ ⊢ _ ⇒ exact H || (destruct_hyp H; use_hyps)
  | H : _ ⊢ _ ⇒ solve [inversion H]
  | _ ⇒ idtac
  end in
let rec do_atom :=
  solve [reflexivity | symmetry; trivial] ||
  contradiction ||
  (split; do_atom)
with do_ccl := trivial with eq_true; repeat do_intro; do_atom in
(use_hyps; do_ccl) || fail "Cannot solve this goal".

```

Tactic Notation "now" *tactic*(*t*) := *t*; *easy*.

A tactic to document or check what is proved at some point of a script

Ltac *now_show c* := *change c*.

Support for rewriting decidability statements

```

Lemma decide_left : ∀ (C:Prop) (decide:{C}+{¬C}),
  C → ∀ P:{C}+{¬C}→Prop, (∀ H:C, P (left _ H)) → P decide.
Lemma decide_right : ∀ (C:Prop) (decide:{C}+{¬C}),
  ¬C → ∀ P:{C}+{¬C}→Prop, (∀ H:¬C, P (right _ H)) → P decide.
Tactic Notation "decide" constr(lemma) "with" constr(H) :=
  let try_to_merge_hyps H :=
    try (clear H; intro H) ||
    (let H' := fresh H "bis" in intro H'; try clear H') ||
    (let H' := fresh in intro H'; try clear H') in
  match type of H with
  | ¬ ?C ⇒ apply (decide_right lemma H); try_to_merge_hyps H
  | ?C → False ⇒ apply (decide_right lemma H); try_to_merge_hyps H
  | _ ⇒ apply (decide_left lemma H); try_to_merge_hyps H
  end.

```

Clear an hypothesis and its dependencies

```

Tactic Notation "clear" "dependent" hyp(h) :=
  let rec depclear h :=
    clear h ||
    match goal with
    | H : context [ h ] ⊢ _ ⇒ depclear H; depclear h
    end ||
    fail "hypothesis to clear is used in the conclusion (maybe indirectly)"
  in depclear h.

```

Revert an hypothesis and its dependencies : this is actually generalize dependent...

Tactic Notation "revert" "dependent" *hyp*(*h*) :=
generalize dependent *h*.

Chapter 9

Library `Coq.Init.Wf`

9.1 This module proves the validity of

- well-founded recursion (also known as course of values)
- well-founded induction

from a well-founded ordering on a given set

```
Require Import Notations.
```

```
Require Import Logic.
```

```
Require Import Datatypes.
```

Well-founded induction principle on `Prop`

```
Section Well_founded.
```

```
Variable A : Type.
```

```
Variable R : A → A → Prop.
```

The accessibility predicate is defined to be non-informative (`Acc_rect` is automatically defined because `Acc` is a singleton type)

```
Inductive Acc (x: A) : Prop :=
```

```
  Acc_intro : (∀ y:A, R y x → Acc y) → Acc x.
```

```
Lemma Acc_inv : ∀ x:A, Acc x → ∀ y:A, R y x → Acc y.
```

```
Global Implicit Arguments Acc_inv [x y] [x].
```

A relation is well-founded if every element is accessible

```
Definition well_founded := ∀ a:A, Acc a.
```

Well-founded induction on `Set` and `Prop`

```
Hypothesis Rwf : well_founded.
```

```
Theorem well_founded_induction_type :
```

```
  ∀ P:A → Type,
```

```
  (∀ x:A, (∀ y:A, R y x → P y) → P x) → ∀ a:A, P a.
```

```
Theorem well_founded_induction :
```

$\forall P:A \rightarrow \mathbf{Set},$
 $(\forall x:A, (\forall y:A, R\ y\ x \rightarrow P\ y) \rightarrow P\ x) \rightarrow \forall a:A, P\ a.$

Theorem `well_founded_ind` :

$\forall P:A \rightarrow \mathbf{Prop},$
 $(\forall x:A, (\forall y:A, R\ y\ x \rightarrow P\ y) \rightarrow P\ x) \rightarrow \forall a:A, P\ a.$

Well-founded fixpoints

Section `FixPoint`.

Variable $P : A \rightarrow \mathbf{Type}.$

Variable $F : \forall x:A, (\forall y:A, R\ y\ x \rightarrow P\ y) \rightarrow P\ x.$

Fixpoint `Fix_F` $(x:A) (a:\mathbf{Acc}\ x) : P\ x :=$
 $F\ (\mathbf{fun}\ (y:A) (h:R\ y\ x) \Rightarrow \mathbf{Fix_F}\ (\mathbf{Acc_inv}\ a\ h)).$

Scheme `Acc_inv_dep` := **Induction** for `Acc` Sort **Prop**.

Lemma `Fix_F_eq` :

$\forall (x:A) (r:\mathbf{Acc}\ x),$
 $F\ (\mathbf{fun}\ (y:A) (p:R\ y\ x) \Rightarrow \mathbf{Fix_F}\ (x:=y) (\mathbf{Acc_inv}\ r\ p)) = \mathbf{Fix_F}\ (x:=x)\ r.$

Definition `Fix` $(x:A) := \mathbf{Fix_F}\ (Rwf\ x).$

Proof that *well_founded_induction* satisfies the fixpoint equation. It requires an extra property of the functional

Hypothesis

$F_ext :$
 $\forall (x:A) (f\ g:\forall y:A, R\ y\ x \rightarrow P\ y),$
 $(\forall (y:A) (p:R\ y\ x), f\ y\ p = g\ y\ p) \rightarrow F\ f = F\ g.$

Lemma `Fix_F_inv` : $\forall (x:A) (r\ s:\mathbf{Acc}\ x), \mathbf{Fix_F}\ r = \mathbf{Fix_F}\ s.$

Lemma `Fix_eq` : $\forall x:A, \mathbf{Fix}\ x = F\ (\mathbf{fun}\ (y:A) (p:R\ y\ x) \Rightarrow \mathbf{Fix}\ y).$

End `FixPoint`.

End `Well_founded`.

Well-founded fixpoints over pairs

Section `Well_founded_2`.

Variables $A\ B : \mathbf{Type}.$

Variable $R : A \times B \rightarrow A \times B \rightarrow \mathbf{Prop}.$

Variable $P : A \rightarrow B \rightarrow \mathbf{Type}.$

Section `FixPoint_2`.

Variable

$F :$
 $\forall (x:A) (x':B),$
 $(\forall (y:A) (y':B), R\ (y, y')\ (x, x') \rightarrow P\ y\ y') \rightarrow P\ x\ x'.$

Fixpoint `Fix_F_2` $(x:A) (x':B) (a:\mathbf{Acc}\ R\ (x, x')) : P\ x\ x' :=$
 F

$(\mathbf{fun}\ (y:A) (y':B) (h:R\ (y, y')\ (x, x')) \Rightarrow$

$\text{Fix_F_2 } (x:=y) (x':=y') (\text{Acc_inv } a (y, y') h)).$

End FixPoint_2.

Hypothesis Rwf : well-founded R .

Theorem well_founded_induction_type_2 :

$(\forall (x:A) (x':B),$
 $(\forall (y:A) (y':B), R (y, y') (x, x') \rightarrow P y y') \rightarrow P x x') \rightarrow$
 $\forall (a:A) (b:B), P a b.$

End Well_founded_2.

Notation $\text{Acc_iter} := \text{Fix_F } (\text{only parsing}).$ *Notation* $\text{Acc_iter_2} := \text{Fix_F_2 } (\text{only parsing}).$

Chapter 10

Library **Coq.Logic.Berardi**

This file formalizes Berardi's paradox which says that in the calculus of constructions, excluded middle (EM) and axiom of choice (AC) imply proof irrelevance (PI). Here, the axiom of choice is not necessary because of the use of inductive types.

```
@article{Barbanera-Berardi:JFP96,
  author    = {F. Barbanera and S. Berardi},
  title     = {Proof-irrelevance out of Excluded-middle and Choice
              in the Calculus of Constructions},
  journal   = {Journal of Functional Programming},
  year      = {1996},
  volume    = {6},
  number    = {3},
  pages     = {519-525}
}
```

Section **Berardis_paradox**.

Excluded middle **Hypothesis** $EM : \forall P:\text{Prop}, P \vee \neg P$.

Conditional on any proposition. **Definition** $\text{IFProp} (P B:\text{Prop}) (e1 e2:P) :=$
match $EM\ B$ **with**
| **or_introl** $_ \Rightarrow e1$
| **or_intror** $_ \Rightarrow e2$
end.

Axiom of choice applied to disjunction. Provable in Coq because of dependent elimination.

Lemma $\text{AC_IF} :$

$\forall (P B:\text{Prop}) (e1 e2:P) (Q:P \rightarrow \text{Prop}),$
 $(B \rightarrow Q\ e1) \rightarrow (\neg B \rightarrow Q\ e2) \rightarrow Q\ (\text{IFProp}\ B\ e1\ e2).$

We assume a type with two elements. They play the role of booleans. The main theorem under the current assumptions is that $T=F$ **Variable** $Bool : \text{Prop}$.

Variable $T : Bool$.

Variable $F : Bool$.

The powerset operator **Definition** $\text{pow} (P:\text{Prop}) := P \rightarrow Bool$.

A piece of theory about retracts **Section** Retracts.

Variables $A\ B : \text{Prop}$.

Record $\text{retract} : \text{Prop} :=$

$\{i : A \rightarrow B; j : B \rightarrow A; \text{inv} : \forall a:A, j\ (i\ a) = a\}$.

Record $\text{retract_cond} : \text{Prop} :=$

$\{i2 : A \rightarrow B; j2 : B \rightarrow A; \text{inv2} : \text{retract} \rightarrow \forall a:A, j2\ (i2\ a) = a\}$.

The dependent elimination above implies the axiom of choice: **Lemma** $\text{AC} : \forall r:\text{retract_cond}, \text{retract} \rightarrow \forall a:A, j2\ r\ (i2\ r\ a) = a$.

End Retracts.

This lemma is basically a commutation of implication and existential quantification: $(\exists x \mid A \rightarrow P(x)) \Leftrightarrow (A \rightarrow \exists x \mid P(x))$ which is provable in classical logic (\Rightarrow is already provable in intuitionistic logic).

Lemma $\text{L1} : \forall A\ B:\text{Prop}, \text{retract_cond}\ (\text{pow}\ A)\ (\text{pow}\ B)$.

The paradoxical set **Definition** $\text{U} := \forall P:\text{Prop}, \text{pow}\ P$.

Bijection between U and $(\text{pow}\ \text{U})$ **Definition** $f\ (u:\text{U}) : \text{pow}\ \text{U} := u\ \text{U}$.

Definition $g\ (h:\text{pow}\ \text{U}) : \text{U} :=$

$\text{fun}\ X \Rightarrow \text{let}\ lX := j2\ (\text{L1}\ X\ \text{U})\ \text{in}\ \text{let}\ rU := i2\ (\text{L1}\ \text{U}\ \text{U})\ \text{in}\ lX\ (rU\ h)$.

We deduce that the powerset of U is a retract of U . This lemma is stated in Berardi's article, but is not used afterwards. **Lemma** $\text{retract_pow_U_U} : \text{retract}\ (\text{pow}\ \text{U})\ \text{U}$.

Encoding of Russel's paradox

The boolean negation. **Definition** $\text{Not_b}\ (b:\text{Bool}) := \text{IFProp}\ (b = \text{True})\ \text{False}\ \text{True}$.

the set of elements not belonging to itself **Definition** $\text{R} : \text{U} := g\ (\text{fun}\ u:\text{U} \Rightarrow \text{Not_b}\ (u\ \text{U}\ u))$.

Lemma $\text{not_has_fixpoint} : \text{R}\ \text{R} = \text{Not_b}\ (\text{R}\ \text{R})$.

Theorem $\text{classical_proof_irrelevance} : \text{True} = \text{False}$.

End Berardis_paradox.

Chapter 11

Library `Coq.Logic.ChoiceFacts`

Some facts and definitions concerning choice and description in intuitionistic logic.

We investigate the relations between the following choice and description principles

- `AC_rel` = relational form of the (non extensional) axiom of choice (a "set-theoretic" axiom of choice)
- `AC_fun` = functional form of the (non extensional) axiom of choice (a "type-theoretic" axiom of choice)
- `DC_fun` = functional form of the dependent axiom of choice
- `ACw_fun` = functional form of the countable axiom of choice
- `AC!` = functional relation reification (known as axiom of unique choice in topos theory, sometimes called principle of definite description in the context of constructive type theory)
- `GAC_rel` = guarded relational form of the (non extensional) axiom of choice
- `GAC_fun` = guarded functional form of the (non extensional) axiom of choice
- `GAC!` = guarded functional relation reification
- `OAC_rel` = "omniscient" relational form of the (non extensional) axiom of choice
- `OAC_fun` = "omniscient" functional form of the (non extensional) axiom of choice (called `AC*` in Bell [*Bell*])
- `OAC!`
- `ID_iota` = intuitionistic definite description
- `ID_epsilon` = intuitionistic indefinite description
- `D_iota` = (weakly classical) definite description principle
- `D_epsilon` = (weakly classical) indefinite description principle

- PI = proof irrelevance
- IGP = independence of general premises (an unconstrained generalisation of the constructive principle of independence of premises)
- Drinker = drinker's paradox (small form) (called Ex in Bell [Bell])

We let also

- IPL₂ = 2nd-order impredicative minimal predicate logic (with ex. quant.)
- IPL² = 2nd-order functional minimal predicate logic (with ex. quant.)
- IPL₂² = 2nd-order impredicative, 2nd-order functional minimal pred. logic (with ex. quant.)

with no prerequisite on the non-emptiness of domains

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5. Equivalence of choices on dependent or non dependent functional types

6. Non contradiction of constructive descriptions wrt functional choices

7. Definite description transports classical logic to the computational world

8. Choice -> Dependent choice -> Countable choice

References:

[Bell] John L. Bell, Choice principles in intuitionistic set theory, unpublished.

[Bell93] John L. Bell, Hilbert's Epsilon Operator in Intuitionistic Type Theories, Mathematical Logic Quarterly, volume 39, 1993.

[Carlström05] Jesper Carlström, Interpreting descriptions in intentional type theory, Journal of Symbolic Logic 70(2):488-514, 2005.

11.1 Definitions

Choice, reification and description schemes

Section ChoiceSchemes.

Variables A B :Type.

Variable P:A->Prop.

Variable R:A->B->Prop.

11.1.1 Constructive choice and description

AC_{rel}

Definition RelationalChoice_{on} :=

$$\begin{aligned} & \forall R:A \rightarrow B \rightarrow \mathbf{Prop}, \\ & (\forall x:A, \exists y:B, R\ x\ y) \rightarrow \\ & (\exists R' : A \rightarrow B \rightarrow \mathbf{Prop}, \text{subrelation } R' R \wedge \forall x, \exists! y, R' x\ y). \end{aligned}$$

AC_{fun}

Definition FunctionalChoice_{on} :=

$$\begin{aligned} & \forall R:A \rightarrow B \rightarrow \mathbf{Prop}, \\ & (\forall x:A, \exists y:B, R\ x\ y) \rightarrow \\ & (\exists f : A \rightarrow B, \forall x:A, R\ x\ (f\ x)). \end{aligned}$$

DC_{fun}

Definition FunctionalDependentChoice_{on} :=

$$\begin{aligned} & \forall (R:A \rightarrow A \rightarrow \mathbf{Prop}), \\ & (\forall x, \exists y, R\ x\ y) \rightarrow \forall x0, \\ & (\exists f : \mathbf{nat} \rightarrow A, f\ 0 = x0 \wedge \forall n, R\ (f\ n)\ (f\ (\mathbf{S}\ n))). \end{aligned}$$

AC_{Cw}_{fun}

Definition FunctionalCountableChoice_{on} :=

$$\begin{aligned} & \forall (R:\mathbf{nat} \rightarrow A \rightarrow \mathbf{Prop}), \\ & (\forall n, \exists y, R\ n\ y) \rightarrow \\ & (\exists f : \mathbf{nat} \rightarrow A, \forall n, R\ n\ (f\ n)). \end{aligned}$$

AC! or Functional Relation Reification (known as Axiom of Unique Choice in topos theory; also called principle of definite description)

Definition FunctionalRelReification_{on} :=

$$\begin{aligned} & \forall R:A \rightarrow B \rightarrow \mathbf{Prop}, \\ & (\forall x:A, \exists! y:B, R\ x\ y) \rightarrow \\ & (\exists f : A \rightarrow B, \forall x:A, R\ x\ (f\ x)). \end{aligned}$$

ID_{epsilon} (constructive version of indefinite description; combined with proof-irrelevance, it may be connected to Carlström's type theory with a constructive indefinite description operator)

Definition ConstructiveIndefiniteDescription_{on} :=

$$\begin{aligned} & \forall P:A \rightarrow \mathbf{Prop}, \\ & (\exists x, P\ x) \rightarrow \{x:A \mid P\ x\}. \end{aligned}$$

ID_{iota} (constructive version of definite description; combined with proof-irrelevance, it may be connected to Carlström's and Stenlund's type theory with a constructive definite description operator)

Definition ConstructiveDefiniteDescription_{on} :=

$$\begin{aligned} & \forall P:A \rightarrow \mathbf{Prop}, \\ & (\exists! x, P\ x) \rightarrow \{x:A \mid P\ x\}. \end{aligned}$$

11.1.2 Weakly classical choice and description

GAC_rel

Definition GuardedRelationalChoice_on :=

$$\begin{aligned} &\forall P : A \rightarrow \text{Prop}, \forall R : A \rightarrow B \rightarrow \text{Prop}, \\ &(\forall x : A, P x \rightarrow \exists y : B, R x y) \rightarrow \\ &(\exists R' : A \rightarrow B \rightarrow \text{Prop}, \\ &\quad \text{subrelation } R' R \wedge \forall x, P x \rightarrow \exists! y, R' x y). \end{aligned}$$

GAC_fun

Definition GuardedFunctionalChoice_on :=

$$\begin{aligned} &\forall P : A \rightarrow \text{Prop}, \forall R : A \rightarrow B \rightarrow \text{Prop}, \\ &\text{inhabited } B \rightarrow \\ &(\forall x : A, P x \rightarrow \exists y : B, R x y) \rightarrow \\ &(\exists f : A \rightarrow B, \forall x, P x \rightarrow R x (f x)). \end{aligned}$$

GFR_fun

Definition GuardedFunctionalRelReification_on :=

$$\begin{aligned} &\forall P : A \rightarrow \text{Prop}, \forall R : A \rightarrow B \rightarrow \text{Prop}, \\ &\text{inhabited } B \rightarrow \\ &(\forall x : A, P x \rightarrow \exists! y : B, R x y) \rightarrow \\ &(\exists f : A \rightarrow B, \forall x : A, P x \rightarrow R x (f x)). \end{aligned}$$

OAC_rel

Definition OmniscientRelationalChoice_on :=

$$\begin{aligned} &\forall R : A \rightarrow B \rightarrow \text{Prop}, \\ &\exists R' : A \rightarrow B \rightarrow \text{Prop}, \\ &\quad \text{subrelation } R' R \wedge \forall x : A, (\exists y : B, R x y) \rightarrow \exists! y, R' x y. \end{aligned}$$

OAC_fun

Definition OmniscientFunctionalChoice_on :=

$$\begin{aligned} &\forall R : A \rightarrow B \rightarrow \text{Prop}, \\ &\text{inhabited } B \rightarrow \\ &\exists f : A \rightarrow B, \forall x : A, (\exists y : B, R x y) \rightarrow R x (f x). \end{aligned}$$

D_epsilon

Definition EpsilonStatement_on :=

$$\begin{aligned} &\forall P : A \rightarrow \text{Prop}, \\ &\text{inhabited } A \rightarrow \{ x : A \mid (\exists x, P x) \rightarrow P x \}. \end{aligned}$$

D_iota

Definition IotaStatement_on :=

$$\begin{aligned} &\forall P : A \rightarrow \text{Prop}, \\ &\text{inhabited } A \rightarrow \{ x : A \mid (\exists! x, P x) \rightarrow P x \}. \end{aligned}$$

End ChoiceSchemes.

Generalized schemes

Notation RelationalChoice :=

$(\forall A B, \text{RelationalChoice_on } A B).$
Notation FunctionalChoice :=
 $(\forall A B, \text{FunctionalChoice_on } A B).$
Definition FunctionalDependentChoice :=
 $(\forall A, \text{FunctionalDependentChoice_on } A).$
Definition FunctionalCountableChoice :=
 $(\forall A, \text{FunctionalCountableChoice_on } A).$
Notation FunctionalChoiceOnInhabitedSet :=
 $(\forall A B, \text{inhabited } B \rightarrow \text{FunctionalChoice_on } A B).$
Notation FunctionalRelReification :=
 $(\forall A B, \text{FunctionalRelReification_on } A B).$
Notation GuardedRelationalChoice :=
 $(\forall A B, \text{GuardedRelationalChoice_on } A B).$
Notation GuardedFunctionalChoice :=
 $(\forall A B, \text{GuardedFunctionalChoice_on } A B).$
Notation GuardedFunctionalRelReification :=
 $(\forall A B, \text{GuardedFunctionalRelReification_on } A B).$
Notation OmniscientRelationalChoice :=
 $(\forall A B, \text{OmniscientRelationalChoice_on } A B).$
Notation OmniscientFunctionalChoice :=
 $(\forall A B, \text{OmniscientFunctionalChoice_on } A B).$
Notation ConstructiveDefiniteDescription :=
 $(\forall A, \text{ConstructiveDefiniteDescription_on } A).$
Notation ConstructiveIndefiniteDescription :=
 $(\forall A, \text{ConstructiveIndefiniteDescription_on } A).$
Notation IotaStatement :=
 $(\forall A, \text{IotaStatement_on } A).$
Notation EpsilonStatement :=
 $(\forall A, \text{EpsilonStatement_on } A).$
 Subclassical schemes
Definition ProofIrrelevance :=
 $\forall (A:\text{Prop}) (a1\ a2:A), a1 = a2.$
Definition IndependenceOfGeneralPremises :=
 $\forall (A:\text{Type}) (P:A \rightarrow \text{Prop}) (Q:\text{Prop}),$
 $\text{inhabited } A \rightarrow$
 $(Q \rightarrow \exists x, P\ x) \rightarrow \exists x, Q \rightarrow P\ x.$
Definition SmallDrinker'sParadox :=
 $\forall (A:\text{Type}) (P:A \rightarrow \text{Prop}), \text{inhabited } A \rightarrow$
 $\exists x, (\exists x, P\ x) \rightarrow P\ x.$

11.2 AC_rel + AC! = AC_fun

We show that the functional formulation of the axiom of Choice (usual formulation in type theory) is equivalent to its relational formulation (only formulation of set theory) + functional relation reification (aka axiom of unique choice, or, principle of (parametric) definite descriptions)

This shows that the axiom of choice can be assumed (under its relational formulation) without known inconsistency with classical logic, though functional relation reification conflicts with classical logic

Lemma `description_rel_choice_imp_func_choice` :

$\forall A B : \text{Type},$
 $\text{FunctionalRelReification_on } A B \rightarrow \text{RelationalChoice_on } A B \rightarrow \text{FunctionalChoice_on } A B.$

Lemma `func_choice_imp_rel_choice` :

$\forall A B, \text{FunctionalChoice_on } A B \rightarrow \text{RelationalChoice_on } A B.$

Lemma `func_choice_imp_description` :

$\forall A B, \text{FunctionalChoice_on } A B \rightarrow \text{FunctionalRelReification_on } A B.$

Corollary `FunChoice_Equiv_RelChoice_and_ParamDefinDescr` :

$\forall A B, \text{FunctionalChoice_on } A B \leftrightarrow$
 $\text{RelationalChoice_on } A B \wedge \text{FunctionalRelReification_on } A B.$

11.3 Connection between the guarded, non guarded and omniscient choices

We show that the guarded formulations of the axiom of choice are equivalent to their "omniscient" variant and comes from the non guarded formulation in presence either of the independance of general premises or subset types (themselves derivable from subtypes thanks to proof- irrelevance)

11.3.1 AC_rel + PI -> GAC_rel and AC_rel + IGP -> GAC_rel and GAC_rel = OAC_rel

Lemma `rel_choice_and_proof_irrel_imp_guarded_rel_choice` :

$\text{RelationalChoice} \rightarrow \text{ProofIrrelevance} \rightarrow \text{GuardedRelationalChoice}.$

Lemma `rel_choice_indep_of_general_premises_imp_guarded_rel_choice` :

$\forall A B, \text{inhabited } B \rightarrow \text{RelationalChoice_on } A B \rightarrow$
 $\text{IndependenceOfGeneralPremises} \rightarrow \text{GuardedRelationalChoice_on } A B.$

Lemma `guarded_rel_choice_imp_rel_choice` :

$\forall A B, \text{GuardedRelationalChoice_on } A B \rightarrow \text{RelationalChoice_on } A B.$

Lemma `subset_types_imp_guarded_rel_choice_iff_rel_choice` :

$\text{ProofIrrelevance} \rightarrow (\text{GuardedRelationalChoice} \leftrightarrow \text{RelationalChoice}).$

$\text{OAC_rel} = \text{GAC_rel}$

Corollary `guarded_iff_omniscient_rel_choice` :

$\text{GuardedRelationalChoice} \leftrightarrow \text{OmniscientRelationalChoice}.$

11.3.2 AC_fun + IGP = GAC_fun = OAC_fun = AC_fun + Drinker

AC_fun + IGP = GAC_fun

Lemma guarded_fun_choice_imp_indep_of_general_premises :
GuardedFunctionalChoice \rightarrow IndependenceOfGeneralPremises.

Lemma guarded_fun_choice_imp_fun_choice :
GuardedFunctionalChoice \rightarrow FunctionalChoiceOnInhabitedSet.

Lemma fun_choice_and_indep_general_prem_imp_guarded_fun_choice :
FunctionalChoiceOnInhabitedSet \rightarrow IndependenceOfGeneralPremises
 \rightarrow GuardedFunctionalChoice.

Corollary fun_choice_and_indep_general_prem_iff_guarded_fun_choice :
FunctionalChoiceOnInhabitedSet \wedge IndependenceOfGeneralPremises
 \leftrightarrow GuardedFunctionalChoice.

AC_fun + Drinker = OAC_fun

This was already observed by Bell [Bell]

Lemma omniscient_fun_choice_imp_small_drinker :
OmniscientFunctionalChoice \rightarrow SmallDrinker'sParadox.

Lemma omniscient_fun_choice_imp_fun_choice :
OmniscientFunctionalChoice \rightarrow FunctionalChoiceOnInhabitedSet.

Lemma fun_choice_and_small_drinker_imp_omniscient_fun_choice :
FunctionalChoiceOnInhabitedSet \rightarrow SmallDrinker'sParadox
 \rightarrow OmniscientFunctionalChoice.

Corollary fun_choice_and_small_drinker_iff_omniscient_fun_choice :
FunctionalChoiceOnInhabitedSet \wedge SmallDrinker'sParadox
 \leftrightarrow OmniscientFunctionalChoice.

OAC_fun = GAC_fun

This is derivable from the intuitionistic equivalence between IGP and Drinker but we give a direct proof

Theorem guarded_iff_omniscient_fun_choice :
GuardedFunctionalChoice \leftrightarrow OmniscientFunctionalChoice.

11.3.3 D_iota \rightarrow ID_iota and D_epsilon \leftrightarrow ID_epsilon + Drinker

D_iota \rightarrow ID_iota

Lemma iota_imp_constructive_definite_description :
IotaStatement \rightarrow ConstructiveDefiniteDescription.

ID_epsilon + Drinker \leftrightarrow D_epsilon

Lemma epsilon_imp_constructive_indefinite_description :
EpsilonStatement \rightarrow ConstructiveIndefiniteDescription.

Lemma constructive_indefinite_description_and_small_drinker_imp_epsilon :

SmallDrinker'sParadox \rightarrow ConstructiveIndefiniteDescription \rightarrow
EpsilonStatement.

Lemma epsilon_imp_small_drinker :
EpsilonStatement \rightarrow SmallDrinker'sParadox.

Theorem constructive_indefinite_description_and_small_drinker_iff_epsilon :
(SmallDrinker'sParadox \times ConstructiveIndefiniteDescription \rightarrow
EpsilonStatement) \times
(EpsilonStatement \rightarrow
SmallDrinker'sParadox \times ConstructiveIndefiniteDescription).

11.4 Derivability of choice for decidable relations with well-ordered codomain

Countable codomains, such as *nat*, can be equipped with a well-order, which implies the existence of a least element on inhabited decidable subsets. As a consequence, the relational form of the axiom of choice is derivable on *nat* for decidable relations.

We show instead that functional relation reification and the functional form of the axiom of choice are equivalent on decidable relation with *nat* as codomain

Require Import Wf_nat.

Require Import Decidable.

Definition FunctionalChoice_on_rel (A B:Type) (R:A \rightarrow B \rightarrow Prop) :=
($\forall x:A, \exists y : B, R x y$) \rightarrow
 $\exists f : A \rightarrow B, (\forall x:A, R x (f x))$.

Lemma classical_denumerable_description_imp_fun_choice :

$\forall A:\text{Type},$
FunctionalRelReification_on A nat \rightarrow
 $\forall R:A \rightarrow \text{nat} \rightarrow \text{Prop},$
($\forall x y, \text{decidable } (R x y)$) \rightarrow FunctionalChoice_on_rel R.

11.5 Choice on dependent and non dependent function types are equivalent

11.5.1 Choice on dependent and non dependent function types are equivalent

Definition DependentFunctionalChoice_on (A:Type) (B:A \rightarrow Type) :=
 $\forall R:\forall x:A, B x \rightarrow \text{Prop},$
($\forall x:A, \exists y : B x, R x y$) \rightarrow
($\exists f : (\forall x:A, B x), \forall x:A, R x (f x)$).

Notation DependentFunctionalChoice :=
($\forall A (B:A \rightarrow \text{Type}), \text{DependentFunctionalChoice_on } B$).

The easy part

Theorem `dep_non_dep_functional_choice` :
`DependentFunctionalChoice` \rightarrow `FunctionalChoice`.

Deriving choice on product types requires some computation on singleton propositional types, so we need computational conjunction projections and dependent elimination of conjunction and equality

Scheme `and_indd` := `Induction for and Sort Prop`.

Scheme `eq_indd` := `Induction for eq Sort Prop`.

Definition `proj1_inf` ($A\ B:\text{Prop}$) ($p : A \wedge B$) :=
`let` (a, b) := p `in` a .

Theorem `non_dep_dep_functional_choice` :
`FunctionalChoice` \rightarrow `DependentFunctionalChoice`.

11.5.2 Reification of dependent and non dependent functional relation are equivalent

Definition `DependentFunctionalRelReification_on` ($A:\text{Type}$) ($B:A \rightarrow \text{Type}$) :=
 $\forall (R:\forall x:A, B\ x \rightarrow \text{Prop}),$
 $(\forall x:A, \exists! y : B\ x, R\ x\ y) \rightarrow$
 $(\exists f : (\forall x:A, B\ x), \forall x:A, R\ x\ (f\ x)).$

Notation `DependentFunctionalRelReification` :=
 $(\forall A\ (B:A \rightarrow \text{Type}), \text{DependentFunctionalRelReification_on}\ B).$

The easy part

Theorem `dep_non_dep_functional_rel_reification` :
`DependentFunctionalRelReification` \rightarrow `FunctionalRelReification`.

Deriving choice on product types requires some computation on singleton propositional types, so we need computational conjunction projections and dependent elimination of conjunction and equality

Theorem `non_dep_dep_functional_rel_reification` :
`FunctionalRelReification` \rightarrow `DependentFunctionalRelReification`.

Corollary `dep_iff_non_dep_functional_rel_reification` :
`FunctionalRelReification` \leftrightarrow `DependentFunctionalRelReification`.

11.6 Non contradiction of constructive descriptions wrt functional axioms of choice

11.6.1 Non contradiction of indefinite description

Lemma `relative_non_contradiction_of_indefinite_descr` :
 $\forall C:\text{Prop}, (\text{ConstructiveIndefiniteDescription} \rightarrow C)$
 $\rightarrow (\text{FunctionalChoice} \rightarrow C).$

Lemma `constructive_indefinite_descr_fun_choice` :
`ConstructiveIndefiniteDescription` \rightarrow `FunctionalChoice`.

11.6.2 Non contradiction of definite description

Lemma `relative_non_contradiction_of_definite_descr` :

$\forall C:\text{Prop}, (\text{ConstructiveDefiniteDescription} \rightarrow C)$
 $\rightarrow (\text{FunctionalRelReification} \rightarrow C).$

Lemma `constructive_definite_descr_fun_reification` :

$\text{ConstructiveDefiniteDescription} \rightarrow \text{FunctionalRelReification}.$

Remark, the following corollaries morally hold:

Definition `In_propositional_context` $(A:\text{Type}) := \text{forall } C:\text{Prop}, (A \rightarrow C) \rightarrow C.$

Corollary `constructive_definite_descr_in_prop_context_iff_fun_reification` : `In_propositional_context`
`ConstructiveIndefiniteDescription` \leftrightarrow `FunctionalChoice`.

Corollary `constructive_definite_descr_in_prop_context_iff_fun_reification` : `In_propositional_context`
`ConstructiveDefiniteDescription` \leftrightarrow `FunctionalRelReification`.

but expecting *FunctionalChoice* (resp. *FunctionalRelReification*) to be applied on the same Type universes on both sides of the first (resp. second) equivalence breaks the stratification of universes.

11.7 Excluded-middle + definite description \Rightarrow computational excluded-middle

The idea for the following proof comes from [*ChicliPottierSimpson02*]

Classical logic and axiom of unique choice (i.e. functional relation reification), as shown in [*ChicliPottierSimpson02*], implies the double-negation of excluded-middle in **Set** (which is incompatible with the impredicativity of **Set**).

We adapt the proof to show that constructive definite description transports excluded-middle from **Prop** to **Set**.

[*ChicliPottierSimpson02*] Laurent Chicli, Loïc Pottier, Carlos Simpson, Mathematical Quotients and Quotient Types in Coq, Proceedings of TYPES 2002, Lecture Notes in Computer Science 2646, Springer Verlag.

Require Import Setoid.

Theorem `constructive_definite_descr_excluded_middle` :

$\text{ConstructiveDefiniteDescription} \rightarrow$
 $(\forall P:\text{Prop}, P \vee \neg P) \rightarrow (\forall P:\text{Prop}, \{P\} + \{\neg P\}).$

Corollary `fun_reification_descr_computational_excluded_middle_in_prop_context` :

$\text{FunctionalRelReification} \rightarrow$
 $(\forall P:\text{Prop}, P \vee \neg P) \rightarrow$
 $\forall C:\text{Prop}, ((\forall P:\text{Prop}, \{P\} + \{\neg P\}) \rightarrow C) \rightarrow C.$

11.8 Choice \Rightarrow Dependent choice \Rightarrow Countable choice

Require Import Arith.

Theorem `functional_choice_imp_functional_dependent_choice` :

$\text{FunctionalChoice} \rightarrow \text{FunctionalDependentChoice}.$

Theorem `functional_dependent_choice_imp_functional_countable_choice` :
 `FunctionalDependentChoice` \rightarrow `FunctionalCountableChoice`.

Chapter 12

Library `Coq.Logic.ClassicalChoice`

This file provides classical logic and functional choice; this especially provides both indefinite descriptions and choice functions but this is weaker than providing epsilon operator and classical logic as the indefinite descriptions provided by the axiom of choice can be used only in a propositional context (especially, they cannot be used to build choice functions outside the scope of a theorem proof)

This file extends `ClassicalUniqueChoice.v` with full choice. As `ClassicalUniqueChoice.v`, it implies the double-negation of excluded-middle in `Set` and leads to a classical world populated with non computable functions. Especially it conflicts with the impredicativity of `Set`, knowing that *true*≠*false* in `Set`.

`Require Export ClassicalUniqueChoice.`

`Require Export RelationalChoice.`

`Require Import ChoiceFacts.`

`Definition subset (U:Type) (P Q:U→Prop) : Prop := $\forall x, P\ x \rightarrow Q\ x$.`

`Theorem singleton_choice :`

`$\forall (A : \text{Type}) (P : A \rightarrow \text{Prop}),$
 $(\exists x : A, P\ x) \rightarrow \exists P' : A \rightarrow \text{Prop}, \text{subset } P' P \wedge \exists! x, P'\ x.$`

`Theorem choice :`

`$\forall (A\ B : \text{Type}) (R : A \rightarrow B \rightarrow \text{Prop}),$
 $(\forall x : A, \exists y : B, R\ x\ y) \rightarrow$
 $\exists f : A \rightarrow B, (\forall x : A, R\ x\ (f\ x)).$`

Chapter 13

Library `Coq.Logic.ClassicalDescription`

This file provides classical logic and definite description, which is equivalent to providing classical logic and Church's iota operator

Classical logic and definite descriptions implies excluded-middle in `Set` and leads to a classical world populated with non computable functions. It conflicts with the impredicativity of `Set`

`Require Export Classical.`

`Require Import ChoiceFacts.`

Notation `Local inhabited A := A (only parsing).`

Axiom `constructive_definite_description :`

$\forall (A : \text{Type}) (P : A \rightarrow \text{Prop}), (\exists! x : A, P x) \rightarrow \{x : A \mid P x\}.$

The idea for the following proof comes from *ChicliPottierSimpson02*

Theorem `excluded_middle_informative : $\forall P : \text{Prop}, \{P\} + \{\neg P\}.$`

Theorem `classical_definite_description :`

$\forall (A : \text{Type}) (P : A \rightarrow \text{Prop}), \text{inhabited } A \rightarrow \{x : A \mid (\exists! x : A, P x) \rightarrow P x\}.$

Church's iota operator

Definition `iota (A : Type) (i : inhabited A) (P : A → Prop) : A`
`:= proj1_sig (classical_definite_description P i).`

Definition `iota_spec (A : Type) (i : inhabited A) (P : A → Prop) :`
`($\exists! x : A, P x$) → P (iota i P)`
`:= proj2_sig (classical_definite_description P i).`

Axiom of unique "choice" (functional reification of functional relations) *Theorem* `dependent_unique_choice`

:

$\forall (A : \text{Type}) (B : A \rightarrow \text{Type}) (R : \forall x : A, B x \rightarrow \text{Prop}),$
 $(\forall x : A, \exists! y : B x, R x y) \rightarrow$
 $(\exists f : (\forall x : A, B x), \forall x : A, R x (f x)).$

Theorem `unique_choice :`

$\forall (A B : \text{Type}) (R : A \rightarrow B \rightarrow \text{Prop}),$
 $(\forall x : A, \exists! y : B, R x y) \rightarrow$

$(\exists f : A \rightarrow B, \forall x:A, R\ x\ (f\ x)).$

Compatibility lemmas

Definition `dependent_description` := `dependent_unique_choice`.

Definition `description` := `unique_choice`.

Chapter 14

Library `Coq.Logic.ClassicalEpsilon`

This file provides classical logic and indefinite description under the form of Hilbert's epsilon operator

Hilbert's epsilon operator and classical logic implies excluded-middle in `Set` and leads to a classical world populated with non computable functions. It conflicts with the impredicativity of `Set`

```
Require Export Classical.
```

```
Require Import ChoiceFacts.
```

```
Axiom constructive_indefinite_description :
```

```
  ∀ (A : Type) (P : A → Prop),  
    (∃ x, P x) → { x : A | P x }.
```

```
Lemma constructive_definite_description :
```

```
  ∀ (A : Type) (P : A → Prop),  
    (∃! x, P x) → { x : A | P x }.
```

```
Theorem excluded_middle_informative : ∀ P : Prop, {P} + {¬ P}.
```

```
Theorem classical_indefinite_description :
```

```
  ∀ (A : Type) (P : A → Prop), inhabited A →  
    { x : A | (∃ x, P x) → P x }.
```

Hilbert's epsilon operator

```
Definition epsilon (A : Type) (i : inhabited A) (P : A → Prop) : A  
:= proj1_sig (classical_indefinite_description P i).
```

```
Definition epsilon_spec (A : Type) (i : inhabited A) (P : A → Prop) :  
  (∃ x, P x) → P (epsilon i P)  
:= proj2_sig (classical_indefinite_description P i).
```

Open question: is `classical_indefinite_description` constructively provable from *relational_choice* and *constructive_definite_description* (at least, using the fact that *functional_choice* is provable from *relational_choice* and *unique_choice*, we know that the double negation of *classical_indefinite_description* is provable (see *relative_non_contradiction_of_indefinite_desc*).

A proof that if P is inhabited, *epsilon a P* does not depend on the actual proof that the domain of P is inhabited (proof idea kindly provided by Pierre Castéran)

Lemma `epsilon_inh_irrelevance` :
 $\forall (A:\text{Type}) (i\ j : \text{inhabited } A) (P:A \rightarrow \text{Prop}),$
 $(\exists x, P\ x) \rightarrow \text{epsilon } i\ P = \text{epsilon } j\ P.$
 Opaque `epsilon`.

Weaker lemmas (compatibility lemmas)

Theorem `choice` :
 $\forall (A\ B : \text{Type}) (R : A \rightarrow B \rightarrow \text{Prop}),$
 $(\forall x : A, \exists y : B, R\ x\ y) \rightarrow$
 $(\exists f : A \rightarrow B, \forall x : A, R\ x\ (f\ x)).$

Chapter 15

Library `Coq.Logic.ClassicalFacts`

Some facts and definitions about classical logic

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3. Weak classical axioms
 - 3.1. Weak excluded middle
 - 3.2. Gödel-Dummett axiom and right distributivity of implication over disjunction
 - 3.3. Independence of general premises and drinker's paradox

15.1 Prop degeneracy = excluded-middle + prop extensionality

i.e. $(\forall A, A = \text{True} \vee A = \text{False}) \leftrightarrow (\forall A, A \vee \neg A) \wedge (\forall A B, (A \leftrightarrow B) \rightarrow A = B)$

prop_degeneracy (also referred to as propositional completeness) asserts (up to consistency) that there are only two distinct formulas **Definition** `prop_degeneracy` := $\forall A:\text{Prop}, A = \text{True} \vee A = \text{False}$.

prop_extensionality asserts that equivalent formulas are equal **Definition** `prop_extensionality` := $\forall A B:\text{Prop}, (A \leftrightarrow B) \rightarrow A = B$.

excluded_middle asserts that we can reason by case on the truth or falsity of any formula **Definition** `excluded_middle` := $\forall A:\text{Prop}, A \vee \neg A$.

We show $\text{prop_degeneracy} \leftrightarrow (\text{prop_extensionality} \wedge \text{excluded_middle})$

Lemma `prop-degen-ext` : `prop-degeneracy` \rightarrow `prop-extensionality`.

Lemma `prop-degen-em` : `prop-degeneracy` \rightarrow `excluded_middle`.

Lemma `prop_ext_em_degen` :

`prop-extensionality` \rightarrow `excluded_middle` \rightarrow `prop-degeneracy`.

A weakest form of propositional extensionality: extensionality for provable propositions only

Definition `provable_prop_extensionality` := $\forall A:\text{Prop}, A \rightarrow A = \text{True}$.

Lemma `provable_prop_ext` :
`prop_extensionality` \rightarrow `provable_prop_extensionality`.

15.2 Classical logic and proof-irrelevance

15.2.1 CC |- prop ext + A inhabited -> (A = A->A) -> A has fixpoint

We successively show that:

prop_extensionality implies equality of A and $A \rightarrow A$ for inhabited A , which implies the existence of a (trivial) retract from $A \rightarrow A$ to A (just take the identity), which implies the existence of a fixpoint operator in A (e.g. take the Y combinator of lambda-calculus)

Notation `Local inhabited A` := A (*only parsing*).

Lemma `prop_ext_A_eq_A_imp_A` :
`prop_extensionality` $\rightarrow \forall A:\text{Prop}, \text{inhabited } A \rightarrow (A \rightarrow A) = A$.

Record `retract (A B:Prop) : Prop` :=
 $\{f1 : A \rightarrow B; f2 : B \rightarrow A; f1_o_f2 : \forall x:B, f1 (f2 x) = x\}$.

Lemma `prop_ext_retract_A_A_imp_A` :
`prop_extensionality` $\rightarrow \forall A:\text{Prop}, \text{inhabited } A \rightarrow \text{retract } A (A \rightarrow A)$.

Record `has_fixpoint (A:Prop) : Prop` :=
 $\{F : (A \rightarrow A) \rightarrow A; \text{Fix} : \forall f:A \rightarrow A, F f = f (F f)\}$.

Lemma `ext_prop_fixpoint` :
`prop_extensionality` $\rightarrow \forall A:\text{Prop}, \text{inhabited } A \rightarrow \text{has_fixpoint } A$.

Remark: *prop_extensionality* can be replaced in lemma *ext_prop_fixpoint* by the weakest property *provable_prop_extensionality*.

15.2.2 CC |- prop_ext /\ dep elim on bool -> proof-irrelevance

proof_irrelevance asserts equality of all proofs of a given formula **Definition** `proof_irrelevance` :=
 $\forall (A:\text{Prop}) (a1 a2:A), a1 = a2$.

Assume that we have booleans with the property that there is at most 2 booleans (which is equivalent to dependent case analysis). Consider the fixpoint of the negation function: it is either true or false by dependent case analysis, but also the opposite by fixpoint. Hence proof-irrelevance.

We then map equality of boolean proofs to proof irrelevance in all propositions.

Section `Proof_irrelevance_gen`.

Variable `bool` : `Prop`.
Variable `true` : `bool`.
Variable `false` : `bool`.
Hypothesis `bool_elim` : $\forall C:\text{Prop}, C \rightarrow C \rightarrow \text{bool} \rightarrow C$.
Hypothesis

```

    bool_elim_redl :  $\forall (C:\mathbf{Prop}) (c1\ c2:C), c1 = \text{bool\_elim } C\ c1\ c2\ \text{true}.$ 
Hypothesis
    bool_elim_redr :  $\forall (C:\mathbf{Prop}) (c1\ c2:C), c2 = \text{bool\_elim } C\ c1\ c2\ \text{false}.$ 
Let bool_dep_induction :=
 $\forall P:\mathbf{bool} \rightarrow \mathbf{Prop}, P\ \text{true} \rightarrow P\ \text{false} \rightarrow \forall b:\mathbf{bool}, P\ b.$ 
Lemma aux : prop_extensionality  $\rightarrow$  bool_dep_induction  $\rightarrow$  true = false.
Lemma ext_prop_dep_proof_irrel_gen :
    prop_extensionality  $\rightarrow$  bool_dep_induction  $\rightarrow$  proof_irrelevance.
End Proof_irrelevance_gen.

```

In the pure Calculus of Constructions, we can define the boolean proposition $\text{bool} = (C:\mathbf{Prop})C \rightarrow C \rightarrow C$ but we cannot prove that it has at most 2 elements.

Section Proof_irrelevance_Prop_Ext_CC.

```

Definition BoolP :=  $\forall C:\mathbf{Prop}, C \rightarrow C \rightarrow C.$ 
Definition TrueP : BoolP := fun C c1 c2  $\Rightarrow$  c1.
Definition FalseP : BoolP := fun C c1 c2  $\Rightarrow$  c2.
Definition BoolP_elim C c1 c2 (b:BoolP) := b C c1 c2.
Definition BoolP_elim_redl (C:Prop) (c1 c2:C) :
    c1 = BoolP_elim C c1 c2 TrueP := refl_equal c1.
Definition BoolP_elim_redr (C:Prop) (c1 c2:C) :
    c2 = BoolP_elim C c1 c2 FalseP := refl_equal c2.
Definition BoolP_dep_induction :=
     $\forall P:\mathbf{BoolP} \rightarrow \mathbf{Prop}, P\ \text{TrueP} \rightarrow P\ \text{FalseP} \rightarrow \forall b:\mathbf{BoolP}, P\ b.$ 
Lemma ext_prop_dep_proof_irrel_cc :
    prop_extensionality  $\rightarrow$  BoolP_dep_induction  $\rightarrow$  proof_irrelevance.
End Proof_irrelevance_Prop_Ext_CC.

```

Remark: *prop_extensionality* can be replaced in lemma *ext_prop_dep_proof_irrel_gen* by the weakest property *provable_prop_extensionality*.

15.2.3 CIC |- prop. ext. \rightarrow proof-irrelevance

In the Calculus of Inductive Constructions, inductively defined booleans enjoy dependent case analysis, hence directly proof-irrelevance from propositional extensionality.

Section Proof_irrelevance_CIC.

```

Inductive boolP : Prop :=
| trueP : boolP
| falseP : boolP.
Definition boolP_elim_redl (C:Prop) (c1 c2:C) :
    c1 = boolP_ind C c1 c2 trueP := refl_equal c1.
Definition boolP_elim_redr (C:Prop) (c1 c2:C) :
    c2 = boolP_ind C c1 c2 falseP := refl_equal c2.
Scheme boolP_indd := Induction for boolP Sort Prop.
Lemma ext_prop_dep_proof_irrel_cic : prop_extensionality  $\rightarrow$  proof_irrelevance.

```

End Proof_irrelevance_CIC.

Can we state proof irrelevance from propositional degeneracy (i.e. propositional extensionality + excluded middle) without dependent case analysis ?

Berardi [Berardi90] built a model of CC interpreting inhabited types by the set of all untyped lambda-terms. This model satisfies propositional degeneracy without satisfying proof-irrelevance (nor dependent case analysis). This implies that the previous results cannot be refined.

[Berardi90] Stefano Berardi, "Type dependence and constructive mathematics", Ph. D. thesis, Dipartimento Matematica, Università di Torino, 1990.

15.2.4 CC |- excluded-middle + dep elim on bool -> proof-irrelevance

This is a proof in the pure Calculus of Construction that classical logic in Prop + dependent elimination of disjunction entails proof-irrelevance.

Reference:

[Coquand90] T. Coquand, "Metamathematical Investigations of a Calculus of Constructions", Proceedings of Logic in Computer Science (LICS'90), 1990.

Proof skeleton: classical logic + dependent elimination of disjunction + discrimination of proofs implies the existence of a retract from Prop into bool, hence inconsistency by encoding any paradox of system U- (e.g. Hurkens' paradox).

Require Import Hurkens.

Section Proof_irrelevance_EM_CC.

Variable or : Prop → Prop → Prop.

Variable or_introl : ∀ A B:Prop, A → or A B.

Variable or_intror : ∀ A B:Prop, B → or A B.

Hypothesis or_elim : ∀ A B C:Prop, (A → C) → (B → C) → or A B → C.

Hypothesis

or_elim_redl :

∀ (A B C:Prop) (f:A → C) (g:B → C) (a:A),
f a = or_elim A B C f g (or_introl A B a).

Hypothesis

or_elim_redr :

∀ (A B C:Prop) (f:A → C) (g:B → C) (b:B),
g b = or_elim A B C f g (or_intror A B b).

Hypothesis

or_dep_elim :

∀ (A B:Prop) (P:or A B → Prop),
(∀ a:A, P (or_introl A B a)) →
(∀ b:B, P (or_intror A B b)) → ∀ b:or A B, P b.

Hypothesis em : ∀ A:Prop, or A (¬ A).

Variable B : Prop.

Variables b1 b2 : B.

p2b and b2p form a retract if ¬b1=b2

Definition p2b A := or_elim A (¬ A) B (fun _ => b1) (fun _ => b2) (em A).

Definition $\text{b2p } b := b1 = b$.

Lemma $\text{p2p1} : \forall A:\text{Prop}, A \rightarrow \text{b2p } (\text{p2b } A)$.

Lemma $\text{p2p2} : b1 \neq b2 \rightarrow \forall A:\text{Prop}, \text{b2p } (\text{p2b } A) \rightarrow A$.

Using excluded-middle a second time, we get proof-irrelevance

Theorem $\text{proof_irrelevance_cc} : b1 = b2$.

End $\text{Proof_irrelevance_EM_CC}$.

Remark: Hurkens' paradox still holds with a retract from the *negative* fragment of **Prop** into *bool*, hence weak classical logic, i.e. $\forall A, \neg A \setminus / \sim \sim A$, is enough for deriving proof-irrelevance.

15.2.5 CIC |- excluded-middle -> proof-irrelevance

Since, dependent elimination is derivable in the Calculus of Inductive Constructions (CCI), we get proof-irrelevance from classical logic in the CCI.

Section $\text{Proof_irrelevance_CCI}$.

Hypothesis $\text{em} : \forall A:\text{Prop}, A \vee \neg A$.

Definition $\text{or_elim_redl } (A B C:\text{Prop}) (f:A \rightarrow C) (g:B \rightarrow C)$
 $(a:A) : f a = \text{or_ind } f g (\text{or_introl } B a) := \text{refl_equal } (f a)$.

Definition $\text{or_elim_redr } (A B C:\text{Prop}) (f:A \rightarrow C) (g:B \rightarrow C)$
 $(b:B) : g b = \text{or_ind } f g (\text{or_intror } A b) := \text{refl_equal } (g b)$.

Scheme $\text{or_indd} := \text{Induction for or Sort Prop}$.

Theorem $\text{proof_irrelevance_cci} : \forall (B:\text{Prop}) (b1 b2:B), b1 = b2$.

End $\text{Proof_irrelevance_CCI}$.

Remark: in the Set-impredicative CCI, Hurkens' paradox still holds with *bool* in **Set** and since $\neg \text{true} = \text{false}$ for *true* and *false* in *bool* from **Set**, we get the inconsistency of $\text{em} : \forall A:\text{Prop}, \{A\} + \{\sim A\}$ in the Set-impredicative CCI.

15.3 Weak classical axioms

We show the following increasing in the strength of axioms:

- weak excluded-middle
- right distributivity of implication over disjunction and Gödel-Dummett axiom
- independence of general premises and drinker's paradox
- excluded-middle

15.3.1 Weak excluded-middle

The weak classical logic based on $\sim\sim A \vee \neg A$ is referred to with name KC in {[ChagrovZakharyashev97](#)] [[ChagrovZakharyashev97](#)] Alexander Chagrov and Michael Zakharyashev, "Modal Logic", Clarendon Press, 1997.

Definition `weak_excluded_middle` :=
 $\forall A:\text{Prop}, \sim\sim A \vee \neg A.$

The interest in the equivalent variant `weak_generalized_excluded_middle` is that it holds even in logic without a primitive `False` connective (like Gödel-Dummett axiom)

Definition `weak_generalized_excluded_middle` :=
 $\forall A B:\text{Prop}, ((A \rightarrow B) \rightarrow B) \vee (A \rightarrow B).$

15.3.2 Gödel-Dummett axiom

$(A \rightarrow B) \vee (B \rightarrow A)$ is studied in [[Dummett59](#)] and is based on [[Gödel33](#)].

[[Dummett59](#)] Michael A. E. Dummett. "A Propositional Calculus with a Denumerable Matrix", In the Journal of Symbolic Logic, Vol 24 No. 2(1959), pp 97-103.

[[Gödel33](#)] Kurt Gödel. "Zum intuitionistischen Aussagenkalkül", *Ergeb. Math. Koll.* 4 (1933), pp. 34-38.

Definition `GodelDummett` := $\forall A B:\text{Prop}, (A \rightarrow B) \vee (B \rightarrow A).$

Lemma `excluded_middle_Godel_Dummett` : `excluded_middle` \rightarrow `GodelDummett`.

$(A \rightarrow B) \vee (B \rightarrow A)$ is equivalent to $(C \rightarrow A \vee B) \rightarrow (C \rightarrow A) \vee (C \rightarrow B)$ (proof from [[Dummett59](#)])

Definition `RightDistributivityImplicationOverDisjunction` :=
 $\forall A B C:\text{Prop}, (C \rightarrow A \vee B) \rightarrow (C \rightarrow A) \vee (C \rightarrow B).$

Lemma `Godel_Dummett_iff_right_distr_implication_over_disjunction` :
`GodelDummett` \leftrightarrow `RightDistributivityImplicationOverDisjunction`.

$(A \rightarrow B) \vee (B \rightarrow A)$ is stronger than the weak excluded middle

Lemma `Godel_Dummett_weak_excluded_middle` :
`GodelDummett` \rightarrow `weak_excluded_middle`.

15.3.3 Independence of general premises and drinker's paradox

Independence of general premises is the unconstrained, non constructive, version of the Independence of Premises as considered in [[Troelstra73](#)].

It is a generalization to predicate logic of the right distributivity of implication over disjunction (hence of Gödel-Dummett axiom) whose own constructive form (obtained by a restricting the third formula to be negative) is called Kreisel-Putnam principle [[KreiselPutnam57](#)].

[[KreiselPutnam57](#)], Georg Kreisel and Hilary Putnam. "Eine Unableitsbarkeitsbeweismethode für den intuitionistischen Aussagenkalkül". *Archiv für Mathematische Logik und Grundlagenforschung*, 3:74- 78, 1957.

[[Troelstra73](#)], Anne Troelstra, editor. *Metamathematical Investigation of Intuitionistic Arithmetic and Analysis*, volume 344 of *Lecture Notes in Mathematics*, Springer-Verlag, 1973.

Definition IndependenceOfGeneralPremises :=

$\forall (A:\text{Type}) (P:A \rightarrow \text{Prop}) (Q:\text{Prop}),$
 $\text{inhabited } A \rightarrow (Q \rightarrow \exists x, P x) \rightarrow \exists x, Q \rightarrow P x.$

Lemma

`independence_general_premises_right_distr_implication_over_disjunction :`
`IndependenceOfGeneralPremises \rightarrow RightDistributivityImplicationOverDisjunction.`

Lemma `independence_general_premises_Godel_Dummett :`

`IndependenceOfGeneralPremises \rightarrow GodelDummett.`

Independence of general premises is equivalent to the drinker's paradox

Definition DrinkerParadox :=

$\forall (A:\text{Type}) (P:A \rightarrow \text{Prop}),$
 $\text{inhabited } A \rightarrow \exists x, (\exists x, P x) \rightarrow P x.$

Lemma `independence_general_premises_drinker :`

`IndependenceOfGeneralPremises \leftrightarrow DrinkerParadox.`

Independence of general premises is weaker than (generalized) excluded middle

Remark: generalized excluded middle is preferred here to avoid relying on the "ex falso quodlibet" property (i.e. $\text{False} \rightarrow \forall A, A$)

Definition `generalized_excluded_middle :=`

$\forall A B:\text{Prop}, A \vee (A \rightarrow B).$

Lemma `excluded_middle_independence_general_premises :`

`generalized_excluded_middle \rightarrow DrinkerParadox.`

Chapter 16

Library `Coq.Logic.Classical_Pred_Set`

This file is obsolete, use `Classical_Pred_Type.v` via `Classical.v` instead

Classical Predicate Logic on Set

`Require Import Classical_Pred_Type.`

`Section Generic.`

`Variable U : Set.`

de Morgan laws for quantifiers

`Lemma not_all_ex_not :`

`∀ P:U → Prop, ¬ (∀ n:U, P n) → ∃ n : U, ¬ P n.`

`Lemma not_all_not_ex :`

`∀ P:U → Prop, ¬ (∀ n:U, ¬ P n) → ∃ n : U, P n.`

`Lemma not_ex_all_not :`

`∀ P:U → Prop, ¬ (∃ n : U, P n) → ∀ n:U, ¬ P n.`

`Lemma not_ex_not_all :`

`∀ P:U → Prop, ¬ (∃ n : U, ¬ P n) → ∀ n:U, P n.`

`Lemma ex_not_not_all :`

`∀ P:U → Prop, (∃ n : U, ¬ P n) → ¬ (∀ n:U, P n).`

`Lemma all_not_not_ex :`

`∀ P:U → Prop, (∀ n:U, ¬ P n) → ¬ (∃ n : U, P n).`

`End Generic.`

Chapter 17

Library `Coq.Logic.Classical_Pred_Type`

Classical Predicate Logic on Type

`Require Import Classical_Prop.`

`Section Generic.`

`Variable U : Type.`

de Morgan laws for quantifiers

`Lemma not_all_not_ex :`

`∀ P:U → Prop, ¬ (∀ n:U, ¬ P n) → ∃ n : U, P n.`

`Lemma not_all_ex_not :`

`∀ P:U → Prop, ¬ (∀ n:U, P n) → ∃ n : U, ¬ P n.`

`Lemma not_ex_all_not :`

`∀ P:U → Prop, ¬ (∃ n : U, P n) → ∀ n:U, ¬ P n.`

`Lemma not_ex_not_all :`

`∀ P:U → Prop, ¬ (∃ n : U, ¬ P n) → ∀ n:U, P n.`

`Lemma ex_not_not_all :`

`∀ P:U → Prop, (∃ n : U, ¬ P n) → ¬ (∀ n:U, P n).`

`Lemma all_not_not_ex :`

`∀ P:U → Prop, (∀ n:U, ¬ P n) → ¬ (∃ n : U, P n).`

`End Generic.`

Chapter 18

Library `Coq.Logic.Classical_Prop`

Classical Propositional Logic

`Require Import ClassicalFacts.`

`Hint Unfold not: core.`

`Axiom classic : $\forall P:\text{Prop}, P \vee \neg P$.`

`Lemma NNPP : $\forall p:\text{Prop}, \neg \neg p \rightarrow p$.`

Peirce's law states $\forall P Q:\text{Prop}, ((P \rightarrow Q) \rightarrow P) \rightarrow P$. Thanks to $\forall P, \text{False} \rightarrow P$, it is equivalent to the following form

`Lemma Peirce : $\forall P:\text{Prop}, ((P \rightarrow \text{False}) \rightarrow P) \rightarrow P$.`

`Lemma not_imply_elim : $\forall P Q:\text{Prop}, \neg (P \rightarrow Q) \rightarrow P$.`

`Lemma not_imply_elim2 : $\forall P Q:\text{Prop}, \neg (P \rightarrow Q) \rightarrow \neg Q$.`

`Lemma imply_to_or : $\forall P Q:\text{Prop}, (P \rightarrow Q) \rightarrow \neg P \vee Q$.`

`Lemma imply_to_and : $\forall P Q:\text{Prop}, \neg (P \rightarrow Q) \rightarrow P \wedge \neg Q$.`

`Lemma or_to_imply : $\forall P Q:\text{Prop}, \neg P \vee Q \rightarrow P \rightarrow Q$.`

`Lemma not_and_or : $\forall P Q:\text{Prop}, \neg (P \wedge Q) \rightarrow \neg P \vee \neg Q$.`

`Lemma or_not_and : $\forall P Q:\text{Prop}, \neg P \vee \neg Q \rightarrow \neg (P \wedge Q)$.`

`Lemma not_or_and : $\forall P Q:\text{Prop}, \neg (P \vee Q) \rightarrow \neg P \wedge \neg Q$.`

`Lemma and_not_or : $\forall P Q:\text{Prop}, \neg P \wedge \neg Q \rightarrow \neg (P \vee Q)$.`

`Lemma imply_and_or : $\forall P Q:\text{Prop}, (P \rightarrow Q) \rightarrow P \vee Q \rightarrow Q$.`

`Lemma imply_and_or2 : $\forall P Q R:\text{Prop}, (P \rightarrow Q) \rightarrow P \vee R \rightarrow Q \vee R$.`

`Lemma proof_irrelevance : $\forall (P:\text{Prop}) (p1 p2:P), p1 = p2$.`

`Ltac classical_right := match goal with
| _ : _ \vdash ?X1 \vee _ \Rightarrow (elim (classic X1); intro; [left; trivial | right])
end.`

`Ltac classical_left := match goal with
| _ : _ \vdash _ \vee ?X1 \Rightarrow (elim (classic X1); intro; [right; trivial | left])
end.`

```

Require Export EqdepFacts.
Module EQ_RECT_EQ.

Lemma eq_rect_eq :
   $\forall (U:\text{Type}) (p:U) (Q:U \rightarrow \text{Type}) (x:Q\ p) (h:p = p), x = \text{eq\_rect } p\ Q\ x\ p\ h.$ 
End EQ_RECT_EQ.

Module EQDEPTHEORY := EQDEPTHEORY(EQ_RECT_EQ).
Export EqdepTheory.

```

Chapter 19

Library **Coq.Logic.Classical_Type**

This file is obsolete, use Classical.v instead

Classical Logic for Type

Require Export Classical_Prop.

Require Export Classical_Pred_Type.

Chapter 20

Library

Coq.Logic.ClassicalUniqueChoice

This file provides classical logic and unique choice; this is weaker than providing iota operator and classical logic as the definite descriptions provided by the axiom of unique choice can be used only in a propositional context (especially, they cannot be used to build functions outside the scope of a theorem proof)

Classical logic and unique choice, as shown in [ChicliPottierSimpson02], implies the double-negation of excluded-middle in **Set**, hence it implies a strongly classical world. Especially it conflicts with the impredicativity of **Set**.

[ChicliPottierSimpson02] Laurent Chicli, Loïc Pottier, Carlos Simpson, Mathematical Quotients and Quotient Types in Coq, Proceedings of TYPES 2002, Lecture Notes in Computer Science 2646, Springer Verlag.

Require Export Classical.

Axiom

dependent_unique_choice :
 $\forall (A:\mathbf{Type}) (B:A \rightarrow \mathbf{Type}) (R:\forall x:A, B\ x \rightarrow \mathbf{Prop}),$
 $(\forall x : A, \exists! y : B\ x, R\ x\ y) \rightarrow$
 $(\exists f : (\forall x:A, B\ x), \forall x:A, R\ x\ (f\ x)).$

Unique choice reifies functional relations into functions

Theorem *unique_choice* :

$\forall (A\ B:\mathbf{Type}) (R:A \rightarrow B \rightarrow \mathbf{Prop}),$
 $(\forall x:A, \exists! y : B, R\ x\ y) \rightarrow$
 $(\exists f:A \rightarrow B, \forall x:A, R\ x\ (f\ x)).$

The following proof comes from [ChicliPottierSimpson02]

Require Import Setoid.

Theorem *classic_set_in_prop_context* :

$\forall C:\mathbf{Prop}, ((\forall P:\mathbf{Prop}, \{P\} + \{\neg P\}) \rightarrow C) \rightarrow C.$

Corollary *not_not_classic_set* :

$((\forall P:\mathbf{Prop}, \{P\} + \{\neg P\}) \rightarrow \mathbf{False}) \rightarrow \mathbf{False}.$

Notation `classic_set` := `not_not_classic_set` (*only parsing*).

Chapter 21

Library **Coq.Logic.Classical**

Classical Logic

Require Export Classical_Prop.

Require Export Classical_Pred_Type.

Chapter 22

Library `Coq.Logic.ConstructiveEpsilon`

This module proves the constructive description schema, which infers the sigma-existence (i.e., `Set`-existence) of a witness to a predicate from the regular existence (i.e., `Prop`-existence). One requires that the underlying set is countable and that the predicate is decidable.

Coq does not allow case analysis on sort `Prop` when the goal is in `Set`. Therefore, one cannot eliminate $\exists n, P\ n$ in order to show $\{n : \text{nat} \mid P\ n\}$. However, one can perform a recursion on an inductive predicate in sort `Prop` so that the returning type of the recursion is in `Set`. This trick is described in Coq'Art book, Sect. 14.2.3 and 15.4. In particular, this trick is used in the proof of *Fix_F* in the module `Coq.Init.Wf`. There, recursion is done on an inductive predicate *Acc* and the resulting type is in `Type`.

To find a witness of *P* constructively, we program the well-known linear search algorithm that tries *P* on all natural numbers starting from 0 and going up. Such an algorithm needs a suitable termination certificate. We offer two ways for providing this termination certificate: a direct one, based on an ad-hoc predicate called *before_witness*, and another one based on the predicate *Acc*. For the first one we provide explicit and short proof terms.

Based on ideas from Benjamin Werner and Jean-François Monin

Contributed by Yevgeniy Makarov and Jean-François Monin

Section `ConstructiveIndefiniteDescription.Direct`.

Variable *P* : `nat` → `Prop`.

Hypothesis *P_dec* : $\forall n, \{P\ n\} + \{\sim(P\ n)\}$.

The termination argument is *before_witness* *n*, which says that any number before any witness (not necessarily the *x* of $\exists x : A, P\ x$) makes the search eventually stops.

Inductive *before_witness* : `nat` → `Prop` :=

| **stop** : $\forall n, P\ n \rightarrow \text{before_witness}\ n$
| **next** : $\forall n, \text{before_witness}\ (\text{S } n) \rightarrow \text{before_witness}\ n$.

Fixpoint *O_witness* (*n* : `nat`) : `before_witness n → before_witness 0 :=`

match *n* **return** (`before_witness n → before_witness 0) with
| 0 ⇒ fun b ⇒ b
| S n ⇒ fun b ⇒ O_witness n (next n b)`

end.

Definition inv_before_witness :

```

 $\forall n, \text{before\_witness } n \rightarrow \neg(P\ n) \rightarrow \text{before\_witness } (S\ n) :=$ 
fun n b  $\Rightarrow$ 
  match b in before_witness n return  $\neg P\ n \rightarrow \text{before\_witness } (S\ n)$  with
  | stop n p  $\Rightarrow$  fun not_p  $\Rightarrow$  match (not_p p) with end
  | next n b  $\Rightarrow$  fun _  $\Rightarrow$  b
end.

```

Fixpoint linear_search m (b : before_witness m) : {n : nat | P n} :=

```

match P_dec m with
| left yes  $\Rightarrow$  exist (fun n  $\Rightarrow$  P n) m yes
| right no  $\Rightarrow$  linear_search (S m) (inv_before_witness m b no)
end.

```

Definition constructive_indefinite_description_nat :

```

( $\exists n, P\ n$ )  $\rightarrow$  {n:nat | P n} :=
fun e  $\Rightarrow$  linear_search O (let (n, p) := e in O_witness n (stop n p)).

```

End ConstructiveIndefiniteDescription_Direct.

Require Import Arith.

Section ConstructiveIndefiniteDescription_Acc.

Variable P : nat \rightarrow Prop.

Hypothesis P_decidable : $\forall n : \text{nat}, \{P\ n\} + \{\neg P\ n\}$.

The predicate *Acc* delineates elements that are accessible via a given relation *R*. An element is accessible if there are no infinite *R*-descending chains starting from it.

To use *Fix_F*, we define a relation *R* and prove that if $\exists n, P\ n$ then 0 is accessible with respect to *R*. Then, by induction on the definition of *Acc* *R* 0, we show $\{n : \text{nat} \mid P\ n\}$.

The relation *R* describes the connection between the two successive numbers we try. Namely, *y* is *R*-less than *x* if we try *y* after *x*, i.e., $y = S\ x$ and $P\ x$ is false. Then the absence of an infinite *R*-descending chain from 0 is equivalent to the termination of our searching algorithm.

Let $R\ (x\ y : \text{nat}) : \text{Prop} := x = S\ y \wedge \neg P\ y$.

Notation Local acc x := (Acc R x).

Lemma P_implies_acc : $\forall x : \text{nat}, P\ x \rightarrow \text{acc } x$.

Lemma P_eventually_implies_acc : $\forall (x : \text{nat}) (n : \text{nat}), P\ (n + x) \rightarrow \text{acc } x$.

Corollary P_eventually_implies_acc_ex : $(\exists n : \text{nat}, P\ n) \rightarrow \text{acc } 0$.

In the following statement, we use the trick with recursion on *Acc*. This is also where decidability of *P* is used.

Theorem acc_implies_P_eventually : $\text{acc } 0 \rightarrow \{n : \text{nat} \mid P\ n\}$.

Theorem constructive_indefinite_description_nat_Acc :

```

( $\exists n : \text{nat}, P\ n$ )  $\rightarrow$  {n : nat | P n}.

```

End ConstructiveIndefiniteDescription_Acc.

Section ConstructiveEpsilon_nat.

Variable $P : \text{nat} \rightarrow \text{Prop}$.

Hypothesis $P_decidable : \forall x : \text{nat}, \{P\ x\} + \{\neg P\ x\}$.

Definition constructive_epsilon_nat ($E : \exists n : \text{nat}, P\ n$) : nat
:= proj1_sig (constructive_indefinite_description_nat $P\ P_decidable\ E$).

Definition constructive_epsilon_spec ($E : (\exists n, P\ n)$) : P (constructive_epsilon_nat E)
:= proj2_sig (constructive_indefinite_description_nat $P\ P_decidable\ E$).

End ConstructiveEpsilon_nat.

Section ConstructiveEpsilon.

For the current purpose, we say that a set A is countable if there are functions $f : A \rightarrow \text{nat}$ and $g : \text{nat} \rightarrow A$ such that g is a left inverse of f .

Variable $A : \text{Set}$.

Variable $f : A \rightarrow \text{nat}$.

Variable $g : \text{nat} \rightarrow A$.

Hypothesis $gof_eq_id : \forall x : A, g\ (f\ x) = x$.

Variable $P : A \rightarrow \text{Prop}$.

Hypothesis $P_decidable : \forall x : A, \{P\ x\} + \{\neg P\ x\}$.

Definition P' ($x : \text{nat}$) : Prop := $P\ (g\ x)$.

Lemma $P'_decidable : \forall n : \text{nat}, \{P'\ n\} + \{\neg P'\ n\}$.

Lemma constructive_indefinite_description : $(\exists x : A, P\ x) \rightarrow \{x : A \mid P\ x\}$.

Lemma constructive_definite_description : $(\exists! x : A, P\ x) \rightarrow \{x : A \mid P\ x\}$.

Definition constructive_epsilon ($E : \exists x : A, P\ x$) : A
:= proj1_sig (constructive_indefinite_description E).

Definition constructive_epsilon_spec ($E : (\exists x, P\ x)$) : P (constructive_epsilon E)
:= proj2_sig (constructive_indefinite_description E).

End ConstructiveEpsilon.

Chapter 23

Library `Coq.Logic.Decidable`

Properties of decidable propositions

Definition `decidable` ($P:\text{Prop}$) := $P \vee \neg P$.

Theorem `dec_not_not` : $\forall P:\text{Prop}, \text{decidable } P \rightarrow (\neg P \rightarrow \text{False}) \rightarrow P$.

Theorem `dec_True` : `decidable True`.

Theorem `dec_False` : `decidable False`.

Theorem `dec_or` :

$\forall A B:\text{Prop}, \text{decidable } A \rightarrow \text{decidable } B \rightarrow \text{decidable } (A \vee B)$.

Theorem `dec_and` :

$\forall A B:\text{Prop}, \text{decidable } A \rightarrow \text{decidable } B \rightarrow \text{decidable } (A \wedge B)$.

Theorem `dec_not` : $\forall A:\text{Prop}, \text{decidable } A \rightarrow \text{decidable } (\neg A)$.

Theorem `dec_imp` :

$\forall A B:\text{Prop}, \text{decidable } A \rightarrow \text{decidable } B \rightarrow \text{decidable } (A \rightarrow B)$.

Theorem `dec_iff` :

$\forall A B:\text{Prop}, \text{decidable } A \rightarrow \text{decidable } B \rightarrow \text{decidable } (A \leftrightarrow B)$.

Theorem `not_not` : $\forall P:\text{Prop}, \text{decidable } P \rightarrow \neg \neg P \rightarrow P$.

Theorem `not_or` : $\forall A B:\text{Prop}, \neg (A \vee B) \rightarrow \neg A \wedge \neg B$.

Theorem `not_and` : $\forall A B:\text{Prop}, \text{decidable } A \rightarrow \neg (A \wedge B) \rightarrow \neg A \vee \neg B$.

Theorem `not_imp` : $\forall A B:\text{Prop}, \text{decidable } A \rightarrow \neg (A \rightarrow B) \rightarrow A \wedge \neg B$.

Theorem `imp_simp` : $\forall A B:\text{Prop}, \text{decidable } A \rightarrow (A \rightarrow B) \rightarrow \neg A \vee B$.

Theorem `not_iff` :

$\forall A B:\text{Prop}, \text{decidable } A \rightarrow \text{decidable } B \rightarrow$
 $\neg (A \leftrightarrow B) \rightarrow (A \wedge \neg B) \vee (\neg A \wedge B)$.

Results formulated with `iff`, used in `FSetDecide`. Negation are expanded since it is unclear whether setoid rewrite will always perform conversion.

We begin with lemmas that, when read from left to right, can be understood as ways to eliminate uses of *not*.

Theorem `not_true_iff` : $(\text{True} \rightarrow \text{False}) \leftrightarrow \text{False}$.

```

Theorem not_false_iff : (False → False) ↔ True.
Theorem not_not_iff : ∀ A:Prop, decidable A →
  ((A → False) → False) ↔ A).
Theorem contrapositive : ∀ A B:Prop, decidable A →
  ((A → False) → (B → False)) ↔ (B → A).
Lemma or_not_l_iff_1 : ∀ A B: Prop, decidable A →
  ((A → False) ∨ B ↔ (A → B)).
Lemma or_not_l_iff_2 : ∀ A B: Prop, decidable B →
  ((A → False) ∨ B ↔ (A → B)).
Lemma or_not_r_iff_1 : ∀ A B: Prop, decidable A →
  (A ∨ (B → False) ↔ (B → A)).
Lemma or_not_r_iff_2 : ∀ A B: Prop, decidable B →
  (A ∨ (B → False) ↔ (B → A)).
Lemma imp_not_l : ∀ A B: Prop, decidable A →
  ((A → False) → B) ↔ (A ∨ B).

```

Moving Negations Around: We have four lemmas that, when read from left to right, describe how to push negations toward the leaves of a proposition and, when read from right to left, describe how to pull negations toward the top of a proposition.

```

Theorem not_or_iff : ∀ A B:Prop,
  (A ∨ B → False) ↔ (A → False) ∧ (B → False).
Lemma not_and_iff : ∀ A B:Prop,
  (A ∧ B → False) ↔ (A → B → False).
Lemma not_imp_iff : ∀ A B:Prop, decidable A →
  ((A → B) → False) ↔ A ∧ (B → False).
Lemma not_imp_rev_iff : ∀ A B : Prop, decidable A →
  ((A → B) → False) ↔ (B → False) ∧ A).

```

With the following hint database, we can leverage `auto` to check decidability of propositions.

```

Hint Resolve dec_True dec_False dec_or dec_and dec_imp dec_not dec_iff
: decidable_prop.

```

`solve_decidable using lib` will solve goals about the decidability of a proposition, assisted by an auxiliary database of lemmas. The database is intended to contain lemmas stating the decidability of base propositions, (e.g., the decidability of equality on a particular inductive type).

```

Tactic Notation "solve_decidable" "using" ident(db) :=
  match goal with
  | ⊢ decidable _ =>
    solve [ auto 100 with decidable_prop db ]
  end.
Tactic Notation "solve_decidable" :=
  solve_decidable using core.

```

Chapter 24

Library `Coq.Logic.Description`

This file provides a constructive form of definite description; it allows to build functions from the proof of their existence in any context; this is weaker than Church's iota operator

```
Require Import ChoiceFacts.
```

```
Axiom constructive_definite_description :
```

```
   $\forall (A : \mathbf{Type}) (P : A \rightarrow \mathbf{Prop}),$   
   $(\exists! x, P x) \rightarrow \{ x : A \mid P x \}.$ 
```

Chapter 25

Library **Coq.Logic.Diaconescu**

Diaconescu showed that the Axiom of Choice entails Excluded-Middle in topoi [Diaconescu75](#). Lacas and Werner adapted the proof to show that the axiom of choice in equivalence classes entails Excluded-Middle in Type Theory [LacasWerner99](#).

Three variants of Diaconescu’s result in type theory are shown below.

A. A proof that the relational form of the Axiom of Choice + Extensionality for Predicates entails Excluded-Middle (by Hugo Herbelin)

B. A proof that the relational form of the Axiom of Choice + Proof Irrelevance entails Excluded-Middle for Equality Statements (by Benjamin Werner)

C. A proof that extensional Hilbert epsilon’s description operator entails excluded-middle (taken from Bell [Bell93](#))

See also [Carlström](#) for a discussion of the connection between the Extensional Axiom of Choice and Excluded-Middle

[Diaconescu75](#) Radu Diaconescu, Axiom of Choice and Complementation, in Proceedings of AMS, vol 51, pp 176-178, 1975.

[LacasWerner99](#) Samuel Lacas, Benjamin Werner, Which Choices imply the excluded middle?, preprint, 1999.

[Bell93](#) John L. Bell, Hilbert’s epsilon operator and classical logic, Journal of Philosophical Logic, 22: 1-18, 1993

[Carlström04](#) Jesper Carlström, EM + Ext + AC_int <-> AC_ext, Mathematical Logic Quarterly, vol 50(3), pp 236-240, 2004.

25.1 Pred. Ext. + Rel. Axiom of Choice -> Excluded-Middle

Section `PredExt_RelChoice_imp_EM`.

The axiom of extensionality for predicates

Definition `PredicateExtensionality` :=

$\forall P Q : \text{bool} \rightarrow \text{Prop}, (\forall b : \text{bool}, P\ b \leftrightarrow Q\ b) \rightarrow P = Q.$

From predicate extensionality we get propositional extensionality hence proof-irrelevance

Require Import `ClassicalFacts`.

Variable `pred_extensionality` : `PredicateExtensionality`.

Lemma prop_ext : $\forall A B:\text{Prop}, (A \leftrightarrow B) \rightarrow A = B$.

Lemma proof_irrel : $\forall (A:\text{Prop}) (a1\ a2:A), a1 = a2$.

From proof-irrelevance and relational choice, we get guarded relational choice

Require Import ChoiceFacts.

Variable rel_choice : RelationalChoice.

Lemma guarded_rel_choice : GuardedRelationalChoice.

The form of choice we need: there is a functional relation which chooses an element in any non empty subset of bool

Require Import Bool.

Lemma AC_bool_subset_to_bool :

$\exists R : (\text{bool} \rightarrow \text{Prop}) \rightarrow \text{bool} \rightarrow \text{Prop},$
 $(\forall P:\text{bool} \rightarrow \text{Prop},$
 $(\exists b : \text{bool}, P\ b) \rightarrow$
 $\exists b : \text{bool}, P\ b \wedge R\ P\ b \wedge (\forall b':\text{bool}, R\ P\ b' \rightarrow b = b')).$

The proof of the excluded middle Remark: P could have been in Set or Type

Theorem pred_ext_and_rel_choice_imp_EM : $\forall P:\text{Prop}, P \vee \neg P$.

first we exhibit the choice functional relation R the actual "decision": is (R class_of_true) = true or false? the actual "decision": is (R class_of_false) = true or false? case where P is false: (R class_of_true)=true \wedge (R class_of_false)=false cases where P is true

End PredExt_RelChoice_imp_EM.

25.2 B. Proof-Irrel. + Rel. Axiom of Choice \rightarrow Excl.-Middle for Equality

This is an adaptation of Diaconescu's paradox exploiting that proof-irrelevance is some form of extensionality

Section ProofIrrel_RelChoice_imp_EqEM.

Variable rel_choice : RelationalChoice.

Variable proof_irrelevance : $\forall P:\text{Prop}, \forall x\ y:P, x=y$.

Let $a1$ and $a2$ be two elements in some type A

Variable A :Type.

Variables a1 a2 : A.

We build the subset A' of A made of $a1$ and $a2$

Definition A' := sigT (fun x \Rightarrow $x=a1 \vee x=a2$).

Definition a1':A'.

Defined.

Definition a2':A'.

Defined.

By proof-irrelevance, projection is a retraction

Lemma `projT1_injective` : $a1=a2 \rightarrow a1'=a2'$.

But from the actual proofs of being in A' , we can assert in the proof-irrelevant world the existence of relevant boolean witnesses

Lemma `decide` : $\forall x:A', \exists y:\text{bool},$
 $(\text{projT1 } x = a1 \wedge y = \text{true}) \vee (\text{projT1 } x = a2 \wedge y = \text{false}).$

Thanks to the axiom of choice, the boolean witnesses move from the propositional world to the relevant world

Theorem `proof_irrel_rel_choice_imp_eq_dec` : $a1=a2 \vee \neg a1=a2.$

An alternative more concise proof can be done by directly using the guarded relational choice

Lemma `proof_irrel_rel_choice_imp_eq_dec'` : $a1=a2 \vee \neg a1=a2.$

End `ProofIrrel_RelChoice_imp_EqEM`.

25.3 Extensional Hilbert's epsilon description operator \rightarrow Excluded-Middle

Proof sketch from Bell *Bell93* (with thanks to P. Castéran)

Notation Local `inhabited` $A := A$ (*only parsing*).

Section `ExtensionalEpsilon_imp_EM`.

Variable `epsilon` : $\forall A : \text{Type}, \text{inhabited } A \rightarrow (A \rightarrow \text{Prop}) \rightarrow A.$

Hypothesis `epsilon_spec` :
 $\forall (A:\text{Type}) (i:\text{inhabited } A) (P:A \rightarrow \text{Prop}),$
 $(\exists x, P x) \rightarrow P (\text{epsilon } A i P).$

Hypothesis `epsilon_extensionality` :
 $\forall (A:\text{Type}) (i:\text{inhabited } A) (P Q:A \rightarrow \text{Prop}),$
 $(\forall a, P a \leftrightarrow Q a) \rightarrow \text{epsilon } A i P = \text{epsilon } A i Q.$

Notation Local `eps` := (`epsilon bool true`) (*only parsing*).

Theorem `extensional_epsilon_imp_EM` : $\forall P:\text{Prop}, P \vee \neg P.$

End `ExtensionalEpsilon_imp_EM`.

Chapter 26

Library Coq.Logic.Epsilon

This file provides indefinite description under the form of Hilbert's epsilon operator; it does not assume classical logic.

Require Import ChoiceFacts.

Hilbert's epsilon: operator and specification in one statement

Axiom *epsilon_statement* :

$\forall (A : \text{Type}) (P : A \rightarrow \text{Prop}), \text{inhabited } A \rightarrow$
 $\{ x : A \mid (\exists x, P x) \rightarrow P x \}.$

Lemma *constructive_indefinite_description* :

$\forall (A : \text{Type}) (P : A \rightarrow \text{Prop}),$
 $(\exists x, P x) \rightarrow \{ x : A \mid P x \}.$

Lemma *small_drinkers'_paradox* :

$\forall (A : \text{Type}) (P : A \rightarrow \text{Prop}), \text{inhabited } A \rightarrow$
 $\exists x, (\exists x, P x) \rightarrow P x.$

Theorem *iota_statement* :

$\forall (A : \text{Type}) (P : A \rightarrow \text{Prop}), \text{inhabited } A \rightarrow$
 $\{ x : A \mid (\exists! x : A, P x) \rightarrow P x \}.$

Lemma *constructive_definite_description* :

$\forall (A : \text{Type}) (P : A \rightarrow \text{Prop}),$
 $(\exists! x, P x) \rightarrow \{ x : A \mid P x \}.$

Hilbert's epsilon operator and its specification

Definition *epsilon* ($A : \text{Type}$) ($i : \text{inhabited } A$) ($P : A \rightarrow \text{Prop}$) : A
:= *proj1_sig* (*epsilon_statement* $P i$).

Definition *epsilon_spec* ($A : \text{Type}$) ($i : \text{inhabited } A$) ($P : A \rightarrow \text{Prop}$) :
 $(\exists x, P x) \rightarrow P (\text{epsilon } i P)$
:= *proj2_sig* (*epsilon_statement* $P i$).

Church's iota operator and its specification

Definition *iota* ($A : \text{Type}$) ($i : \text{inhabited } A$) ($P : A \rightarrow \text{Prop}$) : A
:= *proj1_sig* (*iota_statement* $P i$).

```

Definition iota_spec ( $A : \text{Type}$ ) ( $i : \text{inhabited } A$ ) ( $P : A \rightarrow \text{Prop}$ ) :
  ( $\exists ! x : A, P\ x$ )  $\rightarrow P\ (\text{iota } i\ P)$ 
:= proj2_sig (iota_statement  $P\ i$ ).

```

Chapter 27

Library `Coq.Logic.Eqdep_dec`

We prove that there is only one proof of $x=x$, i.e. `refl_equal x`. This holds if the equality upon the set of x is decidable. A corollary of this theorem is the equality of the right projections of two equal dependent pairs.

Author: Thomas Kleymann |<tms@dcs.ed.ac.uk>| in Lego adapted to Coq by B. Barras

Credit: Proofs up to `K_dec` follow an outline by Michael Hedberg

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1. Streicher's K and injectivity of dependent pair hold on decidable types

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27.1 Streicher's K and injectivity of dependent pair hold on decidable types

Section `EqdepDec`.

`Variable A : Type.`

`Let comp (x y y':A) (eq1:x = y) (eq2:x = y') : y = y' :=
 eq_ind _ (fun a => a = y') eq2 _ eq1.`

`Remark trans_sym_eq : ∀ (x y:A) (u:x = y), comp u u = refl_equal y.`

`Variable eq_dec : ∀ x y:A, x = y ∨ x ≠ y.`

`Variable x : A.`

`Let nu (y:A) (u:x = y) : x = y :=
 match eq_dec x y with
 | or_introl eqxy => eqxy
 | or_intror neqxy => False_ind _ (neqxy u)
end.`

`Let nu_constant : ∀ (y:A) (u v:x = y), nu u = nu v.`

Qed.

Let $nu_inv (y:A) (v:x = y) : x = y := comp (nu (refl_equal x)) v$.

Remark $nu_left_inv : \forall (y:A) (u:x = y), nu_inv (nu u) = u$.

Theorem $eq_proofs_unicity : \forall (y:A) (p1 p2:x = y), p1 = p2$.

Theorem $K_dec :$

$\forall P:x = x \rightarrow Prop, P (refl_equal x) \rightarrow \forall p:x = x, P p$.

The corollary

Let $proj (P:A \rightarrow Prop) (exP:ex P) (def:P x) : P x :=$

```
match exP with
| ex_intro x' prf =>
  match eq_dec x' x with
  | or_introl eqprf => eq_ind x' P prf x eqprf
  | _ => def
end
end.
```

Theorem $inj_right_pair :$

$\forall (P:A \rightarrow Prop) (y y':P x),$
 $ex_intro P x y = ex_intro P x y' \rightarrow y = y'.$

End EqdepDec.

Require Import EqdepFacts.

We deduce axiom K for (decidable) types $Theorem K_dec_type :$

$\forall A:Type,$
 $(\forall x y:A, \{x = y\} + \{x \neq y\}) \rightarrow$
 $\forall (x:A) (P:x = x \rightarrow Prop), P (refl_equal x) \rightarrow \forall p:x = x, P p.$

Theorem $K_dec_set :$

$\forall A:Set,$
 $(\forall x y:A, \{x = y\} + \{x \neq y\}) \rightarrow$
 $\forall (x:A) (P:x = x \rightarrow Prop), P (refl_equal x) \rightarrow \forall p:x = x, P p.$

We deduce the eq_rect_eq axiom for (decidable) types $Theorem eq_rect_eq_dec :$

$\forall A:Type,$
 $(\forall x y:A, \{x = y\} + \{x \neq y\}) \rightarrow$
 $\forall (p:A) (Q:A \rightarrow Type) (x:Q p) (h:p = p), x = eq_rect p Q x p h.$

We deduce the injectivity of dependent equality for decidable types $Theorem eq_dep_eq_dec :$

$\forall A:Type,$
 $(\forall x y:A, \{x = y\} + \{x \neq y\}) \rightarrow$
 $\forall (P:A \rightarrow Type) (p:A) (x y:P p), eq_dep A P p x p y \rightarrow x = y.$

Theorem $UIP_dec :$

$\forall (A:Type),$
 $(\forall x y:A, \{x = y\} + \{x \neq y\}) \rightarrow$
 $\forall (x y:A) (p1 p2:x = y), p1 = p2.$

27.1.1 Definition of the functor that builds properties of dependent equalities on decidable sets in Type

The signature of decidable sets in `Type` `Module Type DECIDABLETYPE`.

```

Parameter U:Type.
Axiom eq_dec : ∀ x y:U, {x = y} + {x ≠ y}.
End DECIDABLETYPE.

The module DecidableEqDep collects equality properties for decidable set in Type
Module DECIDABLEEQDEP (M:DECIDABLETYPE).

Import M.

Invariance by Substitution of Reflexive Equality Proofs
Lemma eq_rect_eq :
  ∀ (p:U) (Q:U → Type) (x:Q p) (h:p = p), x = eq_rect p Q x p h.
Injectivity of Dependent Equality    Theorem eq_dep_eq :
  ∀ (P:U→Type) (p:U) (x y:P p), eq_dep U P p x p y → x = y.
Uniqueness of Identity Proofs (UIP)   Lemma UIP : ∀ (x y:U) (p1 p2:x = y), p1 = p2.
Uniqueness of Reflexive Identity Proofs Lemma UIP_refl : ∀ (x:U) (p:x = x), p = refl_equal x.
Streicher's axiom K    Lemma Streicher_K :
  ∀ (x:U) (P:x = x → Prop), P (refl_equal x) → ∀ p:x = x, P p.
Injectivity of equality on dependent pairs in Type    Lemma inj_pairT2 :
  ∀ (P:U → Type) (p:U) (x y:P p),
    existT P p x = existT P p y → x = y.
Proof-irrelevance on subsets of decidable sets    Lemma inj_pairP2 :
  ∀ (P:U → Prop) (x:U) (p q:P x),
    ex_intro P x p = ex_intro P x q → p = q.
End DECIDABLEEQDEP.

```

27.1.2 Definition of the functor that builds properties of dependent equalities on decidable sets in Set

The signature of decidable sets in `Set`

```

Module Type DECIDABLESET.

Parameter U:Type.
Axiom eq_dec : ∀ x y:U, {x = y} + {x ≠ y}.
End DECIDABLESET.

The module DecidableEqDepSet collects equality properties for decidable set in Set
Module DECIDABLEEQDEPSET (M:DECIDABLESET).

Import M.
Module N:=DECIDABLEEQDEP(M).

Invariance by Substitution of Reflexive Equality Proofs
Lemma eq_rect_eq :

```

$\forall (p:U) (Q:U \rightarrow \text{Type}) (x:Q\ p) (h:p = p), x = \text{eq_rect } p\ Q\ x\ p\ h.$
 Injectivity of Dependent Equality **Theorem** `eq_dep_eq` :
 $\forall (P:U \rightarrow \text{Type}) (p:U) (x\ y:P\ p), \text{eq_dep } U\ P\ p\ x\ p\ y \rightarrow x = y.$
 Uniqueness of Identity Proofs (UIP) **Lemma** `UIP` : $\forall (x\ y:U) (p1\ p2:x = y), p1 = p2.$
 Uniqueness of Reflexive Identity Proofs **Lemma** `UIP_refl` : $\forall (x:U) (p:x = x), p = \text{refl_equal } x.$
 Streicher's axiom K **Lemma** `Streicher_K` :
 $\forall (x:U) (P:x = x \rightarrow \text{Prop}), P (\text{refl_equal } x) \rightarrow \forall p:x = x, P\ p.$
 Proof-irrelevance on subsets of decidable sets **Lemma** `inj_pairP2` :
 $\forall (P:U \rightarrow \text{Prop}) (x:U) (p\ q:P\ x),$
 $\quad \text{ex_intro } P\ x\ p = \text{ex_intro } P\ x\ q \rightarrow p = q.$
 Injectivity of equality on dependent pairs in `Type` **Lemma** `inj_pair2` :
 $\forall (P:U \rightarrow \text{Type}) (p:U) (x\ y:P\ p),$
 $\quad \text{existS } P\ p\ x = \text{existS } P\ p\ y \rightarrow x = y.$
 Injectivity of equality on dependent pairs with second component in `Type` **Notation** `inj_pairT2`
`:= inj_pair2.`
End `DECIDABLEEQDEPSET.`
 From decidability to `inj_pair2` **Lemma** `inj_pair2_eq_dec` : $\forall A:\text{Type}, (\forall x\ y:A, \{x=y\} + \{x \neq y\}) \rightarrow$
 $(\forall (P:A \rightarrow \text{Type}) (p:A) (x\ y:P\ p), \text{existT } P\ p\ x = \text{existT } P\ p\ y \rightarrow x = y).$

Chapter 28

Library `Coq.Logic.EqdepFacts`

This file defines dependent equality and shows its equivalence with equality on dependent pairs (inhabiting sigma-types). It derives the consequence of axiomatizing the invariance by substitution of reflexive equality proofs and shows the equivalence between the 4 following statements

- Invariance by Substitution of Reflexive Equality Proofs.
- Injectivity of Dependent Equality
- Uniqueness of Identity Proofs
- Uniqueness of Reflexive Identity Proofs
- Streicher's Axiom K

These statements are independent of the calculus of constructions 2.

References:

1 T. Streicher, Semantical Investigations into Intensional Type Theory, Habilitationsschrift, LMU München, 1993. 2 M. Hofmann, T. Streicher, The groupoid interpretation of type theory, Proceedings of the meeting Twenty-five years of constructive type theory, Venice, Oxford University Press, 1998

Table of contents:

1. Definition of dependent equality and equivalence with equality
2. `Eq_rect_eq` \leftrightarrow `Eq_dep_eq` \leftrightarrow `UIP` \leftrightarrow `UIP_refl` \leftrightarrow `K`
3. Definition of the functor that builds properties of dependent equalities assuming axiom `eq_rect_eq`

28.1 Definition of dependent equality and equivalence with equality of dependent pairs

Section `Dependent_Equality`.

Variable `U` : Type.

Variable `P` : `U` \rightarrow Type.

Dependent equality

```

Inductive eq_dep (p:U) (x:P p) : ∀ q:U, P q → Prop :=
  eq_dep_intro : eq_dep p x p x.
Hint Constructors eq_dep: core.

Lemma eq_dep_refl : ∀ (p:U) (x:P p), eq_dep p x p x.
Lemma eq_dep_sym :
  ∀ (p q:U) (x:P p) (y:P q), eq_dep p x q y → eq_dep q y p x.
Hint Immediate eq_dep_sym: core.

Lemma eq_dep_trans :
  ∀ (p q r:U) (x:P p) (y:P q) (z:P r),
    eq_dep p x q y → eq_dep q y r z → eq_dep p x r z.
Scheme eq_indd := Induction for eq Sort Prop.

```

Equivalent definition of dependent equality expressed as a non dependent inductive type

```

Inductive eq_dep1 (p:U) (x:P p) (q:U) (y:P q) : Prop :=
  eq_dep1_intro : ∀ h:q = p, x = eq_rect q P y p h → eq_dep1 p x q y.

Lemma eq_dep1_dep :
  ∀ (p:U) (x:P p) (q:U) (y:P q), eq_dep1 p x q y → eq_dep p x q y.

Lemma eq_dep_dep1 :
  ∀ (p q:U) (x:P p) (y:P q), eq_dep p x q y → eq_dep1 p x q y.

```

End Dependent_Equality.

```

Implicit Arguments eq_dep [U P].
Implicit Arguments eq_dep1 [U P].

```

Dependent equality is equivalent to equality on dependent pairs

```

Lemma eq_sigT_eq_dep :
  ∀ (U:Type) (P:U → Type) (p q:U) (x:P p) (y:P q),
    existT P p x = existT P q y → eq_dep p x q y.

Notation eq_sigS_eq_dep := eq_sigT_eq_dep (only parsing).

```

```

Lemma equiv_eqex_eqdep :
  ∀ (U:Type) (P:U → Type) (p q:U) (x:P p) (y:P q),
    existT P p x = existT P q y ↔ eq_dep p x q y.

```

```

Lemma eq_dep_eq_sigT :
  ∀ (U:Type) (P:U → Type) (p q:U) (x:P p) (y:P q),
    eq_dep p x q y → existT P p x = existT P q y.

```

Exported hints

```

Hint Resolve eq_dep_intro: core.
Hint Immediate eq_dep_sym: core.

```

28.2 Eq_rect_eq <-> Eq_dep_eq <-> UIP <-> UIP_refl <-> K

Section Equivalences.

Variable $U:\text{Type}$.

Invariance by Substitution of Reflexive Equality Proofs

Definition $\text{Eq_rect_eq} :=$

$\forall (p:U) (Q:U \rightarrow \text{Type}) (x:Q\ p) (h:p = p), x = \text{eq_rect}\ p\ Q\ x\ p\ h.$

Injectivity of Dependent Equality

Definition $\text{Eq_dep_eq} :=$

$\forall (P:U \rightarrow \text{Type}) (p:U) (x\ y:P\ p), \text{eq_dep}\ p\ x\ p\ y \rightarrow x = y.$

Uniqueness of Identity Proofs (UIP)

Definition $\text{UIP_} :=$

$\forall (x\ y:U) (p1\ p2:x = y), p1 = p2.$

Uniqueness of Reflexive Identity Proofs

Definition $\text{UIP_refl_} :=$

$\forall (x:U) (p:x = x), p = \text{refl_equal}\ x.$

Streicher's axiom K

Definition $\text{Streicher_K_} :=$

$\forall (x:U) (P:x = x \rightarrow \text{Prop}), P (\text{refl_equal}\ x) \rightarrow \forall p:x = x, P\ p.$

Injectivity of Dependent Equality is a consequence of Invariance by Substitution of Reflexive Equality Proof

Lemma $\text{eq_rect_eq_eq_dep1_eq} :$

$\text{Eq_rect_eq} \rightarrow \forall (P:U \rightarrow \text{Type}) (p:U) (x\ y:P\ p), \text{eq_dep1}\ p\ x\ p\ y \rightarrow x = y.$

Lemma $\text{eq_rect_eq_eq_dep_eq} : \text{Eq_rect_eq} \rightarrow \text{Eq_dep_eq}.$

Uniqueness of Identity Proofs (UIP) is a consequence of Injectivity of Dependent Equality

Lemma $\text{eq_dep_eq_UIP} : \text{Eq_dep_eq} \rightarrow \text{UIP_}.$

Uniqueness of Reflexive Identity Proofs is a direct instance of UIP

Lemma $\text{UIP_UIP_refl} : \text{UIP_} \rightarrow \text{UIP_refl_}.$

Streicher's axiom K is a direct consequence of Uniqueness of Reflexive Identity Proofs

Lemma $\text{UIP_refl_Streicher_K} : \text{UIP_refl_} \rightarrow \text{Streicher_K_}.$

We finally recover from K the Invariance by Substitution of Reflexive Equality Proofs

Lemma $\text{Streicher_K_eq_rect_eq} : \text{Streicher_K_} \rightarrow \text{Eq_rect_eq}.$

Remark: It is reasonable to think that eq_rect_eq is strictly stronger than eq_rec_eq (which is eq_rect_eq restricted on Set):

Definition $\text{Eq_rec_eq} := \forall (P:U \rightarrow \text{Set}) (p:U) (x:P\ p) (h:p = p), x = \text{eq_rec}\ p\ P\ x\ p\ h.$

Typically, eq_rect_eq allows to prove UIP and Streicher's K what does not seem possible with eq_rec_eq . In particular, the proof of UIP requires to use eq_rect_eq on $\text{fun } y \rightarrow x=y$ which is in Type but not in Set .

End Equivalences.

Section Corollaries.

Variable $U:\text{Type}$.

UIP implies the injectivity of equality on dependent pairs in Type

Definition Inj_dep_pair :=

$\forall (P:U \rightarrow \text{Type}) (p:U) (x\ y:P\ p), \text{existT } P\ p\ x = \text{existT } P\ p\ y \rightarrow x = y.$

Lemma eq_dep_eq_inj_pair2 : Eq_dep_eq $U \rightarrow$ Inj_dep_pair.

End Corollaries.

Notation Inj_dep_pairS := Inj_dep_pair.

Notation Inj_dep_pairT := Inj_dep_pair.

Notation eq_dep_eq_inj_pairT2 := eq_dep_eq_inj_pair2.

28.3 Definition of the functor that builds properties of dependent equalities assuming axiom eq_rect_eq

Module Type EQDEPELIMINATION.

Axiom eq_rect_eq :

$\forall (U:\text{Type}) (p:U) (Q:U \rightarrow \text{Type}) (x:Q\ p) (h:p = p),$
 $x = \text{eq_rect } p\ Q\ x\ p\ h.$

End EQDEPELIMINATION.

Module EQDEPTHEORY ($M:\text{EQDEPELIMINATION}$).

Section Axioms.

Variable $U:\text{Type}.$

Invariance by Substitution of Reflexive Equality Proofs

Lemma eq_rect_eq :

$\forall (p:U) (Q:U \rightarrow \text{Type}) (x:Q\ p) (h:p = p), x = \text{eq_rect } p\ Q\ x\ p\ h.$

Lemma eq_rec_eq :

$\forall (p:U) (Q:U \rightarrow \text{Set}) (x:Q\ p) (h:p = p), x = \text{eq_rec } p\ Q\ x\ p\ h.$

Injectivity of Dependent Equality **Lemma** eq_dep_eq : $\forall (P:U \rightarrow \text{Type}) (p:U) (x\ y:P\ p), \text{eq_dep } p\ x\ p\ y \rightarrow x = y.$

Uniqueness of Identity Proofs (UIP) is a consequence of Injectivity of Dependent Equality

Lemma UIP : $\forall (x\ y:U) (p1\ p2:x = y), p1 = p2.$

Uniqueness of Reflexive Identity Proofs is a direct instance of UIP **Lemma** UIP_refl : $\forall (x:U) (p:x = x), p = \text{refl_equal } x.$

Streicher's axiom K is a direct consequence of Uniqueness of Reflexive Identity Proofs **Lemma** Streicher_K :

$\forall (x:U) (P:x = x \rightarrow \text{Prop}), P (\text{refl_equal } x) \rightarrow \forall p:x = x, P\ p.$

End Axioms.

UIP implies the injectivity of equality on dependent pairs in Type

Lemma inj_pair2 :

$\forall (U:\text{Type}) (P:U \rightarrow \text{Type}) (p:U) (x\ y:P\ p),$
 $\text{existT } P\ p\ x = \text{existT } P\ p\ y \rightarrow x = y.$

Notation inj_pairT2 := inj_pair2.

End EQDEPTHEORY.

```
Implicit Arguments eq_dep [].  
Implicit Arguments eq_depl [].
```

Chapter 29

Library **Coq.Logic.Eqdep**

This file axiomatizes the invariance by substitution of reflexive equality proofs [Streicher93] and exports its consequences, such as the injectivity of the projection of the dependent pair.

[Streicher93] T. Streicher, Semantical Investigations into Intensional Type Theory, Habilitationsschrift, LMU München, 1993.

```
Require Export EqdepFacts.
```

```
Module EQ_RECT_EQ.
```

```
Axiom eq_rect_eq :
```

```
  ∀ (U:Type) (p:U) (Q:U → Type) (x:Q p) (h:p = p), x = eq_rect p Q x p h.
```

```
End EQ_RECT_EQ.
```

```
Module EQDEPTHEORY := EQDEPTHEORY(EQ_RECT_EQ).
```

```
Export EqdepTheory.
```

Exported hints

```
Hint Resolve eq_dep_eq: eqdep v62.
```

```
Hint Resolve inj_pair2 inj_pairT2: eqdep.
```

Chapter 30

Library

Coq.Logic.FunctionalExtensionality

This module states the axiom of (dependent) functional extensionality and (dependent) eta-expansion. It introduces a tactic `extensionality` to apply the axiom of extensionality to an equality goal.

The converse of functional extensionality.

Lemma `equal_f` : $\forall \{A\ B : \text{Type}\} \{f\ g : A \rightarrow B\},$
 $f = g \rightarrow \forall x, f\ x = g\ x.$

Statements of functional extensionality for simple and dependent functions.

Axiom `functional_extensionality_dep` : $\forall \{A\} \{B : A \rightarrow \text{Type}\},$
 $\forall (f\ g : \forall x : A, B\ x),$
 $(\forall x, f\ x = g\ x) \rightarrow f = g.$

Lemma `functional_extensionality` $\{A\ B\} (f\ g : A \rightarrow B) :$
 $(\forall x, f\ x = g\ x) \rightarrow f = g.$

Apply `functional_extensionality`, introducing variable `x`.

Tactic Notation "extensionality" *ident*(`x`) :=
`match goal with`
 $[\vdash ?X = ?Y] \Rightarrow$
 `(apply (@functional_extensionality - - X Y) ||`
 `apply (@functional_extensionality_dep - - X Y)) ; intro x`
`end.`

Eta expansion follows from extensionality.

Lemma `eta_expansion_dep` $\{A\} \{B : A \rightarrow \text{Type}\} (f : \forall x : A, B\ x) :$
 $f = \text{fun } x \Rightarrow f\ x.$

Lemma `eta_expansion` $\{A\ B\} (f : A \rightarrow B) : f = \text{fun } x \Rightarrow f\ x.$

Chapter 31

Library Coq.Logic.Hurkens

This is Hurkens paradox *Hurkens* in system U-, adapted by Herman Geuvers *Geuvers* to show the inconsistency in the pure calculus of constructions of a retract from Prop into a small type.

References:

- *Hurkens* A. J. Hurkens, "A simplification of Girard's paradox", Proceedings of the 2nd international conference Typed Lambda-Calculi and Applications (TLCA'95), 1995.
- *Geuvers* "Inconsistency of Classical Logic in Type Theory", 2001 (see <http://www.cs.kun.nl/~herman/note.ps>).

Section Paradox.

Variable *bool* : Prop.

Variable *p2b* : Prop → bool.

Variable *b2p* : bool → Prop.

Hypothesis *p2p1* : $\forall A:\text{Prop}, b2p (p2b A) \rightarrow A$.

Hypothesis *p2p2* : $\forall A:\text{Prop}, A \rightarrow b2p (p2b A)$.

Variable *B* : Prop.

Definition *V* := $\forall A:\text{Prop}, ((A \rightarrow bool) \rightarrow A \rightarrow bool) \rightarrow A \rightarrow bool$.

Definition *U* := *V* → bool.

Definition *sb* (*z*:*V*) : *V* := fun *A r a* => *r* (*z* *A r*) *a*.

Definition *le* (*i*:*U* → bool) (*x*:*U*) : bool :=

x (fun *A r a* => *i* (fun *v* => sb *v* *A r*) *a*)).

Definition *induct* (*i*:*U* → bool) : Prop :=

$\forall x:U, b2p (le\ i\ x) \rightarrow b2p (i\ x)$.

Definition *WF* : *U* := fun *z* => *p2b* (induct (*z* *U* *le*)).

Definition *I* (*x*:*U*) : Prop :=

$(\forall i:U \rightarrow bool, b2p (le\ i\ x) \rightarrow b2p (i\ (\text{fun } v \Rightarrow sb\ v\ U\ le\ x))) \rightarrow B$.

Lemma *Omega* : $\forall i:U \rightarrow bool, induct\ i \rightarrow b2p (i\ WF)$.

Lemma *lemmal* : induct (fun *u* => *p2b* (*I* *u*)).

Lemma *lemma2* : $(\forall i:U \rightarrow bool, induct\ i \rightarrow b2p (i\ WF)) \rightarrow B$.

Theorem *paradox* : *B*.

End Paradox.

Chapter 32

Library **Coq.Logic.IndefiniteDescription**

This file provides a constructive form of indefinite description that allows to build choice functions; this is weaker than Hilbert's epsilon operator (which implies weakly classical properties) but stronger than the axiom of choice (which cannot be used outside the context of a theorem proof).

Require Import ChoiceFacts.

Axiom *constructive_indefinite_description* :

$\forall (A : \mathbf{Type}) (P : A \rightarrow \mathbf{Prop}),$
 $(\exists x, P x) \rightarrow \{ x : A \mid P x \}.$

Lemma *constructive_definite_description* :

$\forall (A : \mathbf{Type}) (P : A \rightarrow \mathbf{Prop}),$
 $(\exists! x, P x) \rightarrow \{ x : A \mid P x \}.$

Lemma *functional_choice* :

$\forall (A B : \mathbf{Type}) (R : A \rightarrow B \rightarrow \mathbf{Prop}),$
 $(\forall x : A, \exists y : B, R x y) \rightarrow$
 $(\exists f : A \rightarrow B, \forall x : A, R x (f x)).$

Chapter 33

Library **Coq.Logic.JMeq**

John Major's Equality as proposed by Conor McBride

Reference:

McBride Elimination with a Motive, Proceedings of TYPES 2000, LNCS 2277, pp 197-216, 2002.

Inductive JMeq (A:Type) (x:A) : $\forall B:\text{Type}, B \rightarrow \text{Prop} :=$
JMeq_refl : JMeq x x.

Hint Resolve JMeq_refl.

Lemma JMeq_sym : $\forall (A B:\text{Type}) (x:A) (y:B), \text{JMeq } x \ y \rightarrow \text{JMeq } y \ x.$

Hint Immediate JMeq_sym.

Lemma JMeq_trans :

$\forall (A B C:\text{Type}) (x:A) (y:B) (z:C), \text{JMeq } x \ y \rightarrow \text{JMeq } y \ z \rightarrow \text{JMeq } x \ z.$

Axiom JMeq_eq : $\forall (A:\text{Type}) (x \ y:A), \text{JMeq } x \ y \rightarrow x = y.$

Lemma JMeq_ind : $\forall (A:\text{Type}) (x:A) (P:A \rightarrow \text{Prop}),$
 $P \ x \rightarrow \forall y, \text{JMeq } x \ y \rightarrow P \ y.$

Lemma JMeq_rec : $\forall (A:\text{Type}) (x:A) (P:A \rightarrow \text{Set}),$
 $P \ x \rightarrow \forall y, \text{JMeq } x \ y \rightarrow P \ y.$

Lemma JMeq_rect : $\forall (A:\text{Type}) (x:A) (P:A \rightarrow \text{Type}),$
 $P \ x \rightarrow \forall y, \text{JMeq } x \ y \rightarrow P \ y.$

Lemma JMeq_ind_r : $\forall (A:\text{Type}) (x:A) (P:A \rightarrow \text{Prop}),$
 $P \ x \rightarrow \forall y, \text{JMeq } y \ x \rightarrow P \ y.$

Lemma JMeq_rec_r : $\forall (A:\text{Type}) (x:A) (P:A \rightarrow \text{Set}),$
 $P \ x \rightarrow \forall y, \text{JMeq } y \ x \rightarrow P \ y.$

Lemma JMeq_rect_r : $\forall (A:\text{Type}) (x:A) (P:A \rightarrow \text{Type}),$
 $P \ x \rightarrow \forall y, \text{JMeq } y \ x \rightarrow P \ y.$

Lemma JMeq_congr :

$\forall (A:\text{Type}) (x:A) (B:\text{Type}) (f:A \rightarrow B) (y:A), \text{JMeq } x \ y \rightarrow f \ x = f \ y.$

$JMeq$ is equivalent to $eq_dep \text{ Type } (\text{fun } X \Rightarrow X)$

Require Import Eqdep.

Lemma $JMeq_eq_dep_id$:

$\forall (A B:\text{Type}) (x:A) (y:B), JMeq\ x\ y \rightarrow eq_dep\ \text{Type}\ (\text{fun } X \Rightarrow X)\ A\ x\ B\ y.$

Lemma $eq_dep_id_JMeq$:

$\forall (A B:\text{Type}) (x:A) (y:B), eq_dep\ \text{Type}\ (\text{fun } X \Rightarrow X)\ A\ x\ B\ y \rightarrow JMeq\ x\ y.$

$eq_dep\ U\ P\ p\ x\ q\ y$ is strictly finer than $JMeq\ (P\ p)\ x\ (P\ q)\ y$

Lemma eq_dep_JMeq :

$\forall U\ P\ p\ x\ q\ y, eq_dep\ U\ P\ p\ x\ q\ y \rightarrow JMeq\ x\ y.$

Lemma $eq_dep_strictly_stronger_JMeq$:

$\exists U, \exists P, \exists p, \exists q, \exists x, \exists y,$

$JMeq\ x\ y \wedge \neg eq_dep\ U\ P\ p\ x\ q\ y.$

However, when the dependencies are equal, $JMeq\ (P\ p)\ x\ (P\ q)\ y$ is as strong as $eq_dep\ U\ P\ p\ x\ q\ y$ (this uses $JMeq_eq$)

Lemma $JMeq_eq_dep$:

$\forall U\ (P:U \rightarrow \text{Prop})\ p\ q\ (x:P\ p)\ (y:P\ q),$

$p = q \rightarrow JMeq\ x\ y \rightarrow eq_dep\ U\ P\ p\ x\ q\ y.$

Notation $\text{sym_JMeq} := JMeq_sym$ (*only parsing*).

Notation $\text{trans_JMeq} := JMeq_trans$ (*only parsing*).

Chapter 34

Library **Coq.Logic.ProofIrrelevanceFacts**

This defines the functor that build consequences of proof-irrelevance

```
Require Export EqdepFacts.
```

```
Module Type PROOFIRRELEVANCE.
```

```
  Axiom proof_irrelevance :  $\forall (P:\text{Prop}) (p1\ p2:P), p1 = p2$ .
```

```
End PROOFIRRELEVANCE.
```

```
Module PROOFIRRELEVANCETHEORY (M:PROOFIRRELEVANCE).
```

Proof-irrelevance implies uniqueness of reflexivity proofs

```
Module EQ_RECT_EQ.
```

```
  Lemma eq_rect_eq :
```

```
     $\forall (U:\text{Type}) (p:U) (Q:U \rightarrow \text{Type}) (x:Q\ p) (h:p = p),$   
     $x = \text{eq\_rect}\ p\ Q\ x\ p\ h.$ 
```

```
End EQ_RECT_EQ.
```

Export the theory of injective dependent elimination

```
Module EQDEPTHEORY := EQDEPTHEORY(EQ_RECT_EQ).
```

```
Export EqdepTheory.
```

```
Scheme eq_indd := Induction for eq Sort Prop.
```

We derive the irrelevance of the membership property for subsets

```
Lemma subset_eq_compat :
```

```
   $\forall (U:\text{Set}) (P:U \rightarrow \text{Prop}) (x\ y:U) (p:P\ x) (q:P\ y),$   
   $x = y \rightarrow \text{exist}\ P\ x\ p = \text{exist}\ P\ y\ q.$ 
```

```
Lemma subsetT_eq_compat :
```

```
   $\forall (U:\text{Type}) (P:U \rightarrow \text{Prop}) (x\ y:U) (p:P\ x) (q:P\ y),$   
   $x = y \rightarrow \text{existT}\ P\ x\ p = \text{existT}\ P\ y\ q.$ 
```

```
End PROOFIRRELEVANCETHEORY.
```

Chapter 35

Library Coq.Logic.ProofIrrelevance

This file axiomatizes proof-irrelevance and derives some consequences

```
Require Import ProofIrrelevanceFacts.
```

```
Axiom proof_irrelevance :  $\forall$  (P:Prop) (p1 p2:P), p1 = p2.
```

```
Module PI. Definition proof_irrelevance := proof_irrelevance. End PI.
```

```
Module PROOFIRRELEVANCETHEORY := PROOFIRRELEVANCETHEORY(PI).
```

```
Export ProofIrrelevanceTheory.
```

Chapter 36

Library **Coq.Logic.RelationalChoice**

This file axiomatizes the relational form of the axiom of choice

```
Axiom relational_choice :  
  ∀ (A B : Type) (R : A → B → Prop),  
    (∀ x : A, ∃ y : B, R x y) →  
      ∃ R' : A → B → Prop,  
        subrelation R' R ∧ ∀ x : A, ∃! y : B, R' x y.
```

Chapter 37

Library `Coq.Logic.SetIsType`

37.1 The Set universe seen as a synonym for Type

After loading this file, Set becomes just another name for Type. This allows to easily perform a Set-to-Type migration, or at least test whether a development relies or not on specific features of Set: simply insert some Require Export of this file at starting points of the development and try to recompile...

Notation `"'Set'" := Type` (*only parsing*).

Chapter 38

Library `Coq.Arith.Arith_base`

```
Require Export Le.  
Require Export Lt.  
Require Export Plus.  
Require Export Gt.  
Require Export Minus.  
Require Export Mult.  
Require Export Between.  
Require Export Peano_dec.  
Require Export Compare_dec.  
Require Export Factorial.  
Require Export EqNat.  
Require Export Wf_nat.
```

Chapter 39

Library **Coq.Arith.Arith**

```
Require Export Arith_base.  
Require Export ArithRing.
```

Chapter 40

Library **Coq.Arith.Between**

```
Require Import Le.
Require Import Lt.

Open Local Scope nat_scope.

Implicit Types k l p q r : nat.

Section Between.
  Variables P Q : nat → Prop.

  Inductive between k : nat → Prop :=
    | bet_emp : between k k
    | bet_S : ∀ l, between k l → P l → between k (S l).

  Hint Constructors between: arith v62.

  Lemma bet_eq : ∀ k l, l = k → between k l.

  Hint Resolve bet_eq: arith v62.

  Lemma between_le : ∀ k l, between k l → k ≤ l.
  Hint Immediate between_le: arith v62.

  Lemma between_Sk_l : ∀ k l, between k l → S k ≤ l → between (S k) l.
  Hint Resolve between_Sk_l: arith v62.

  Lemma between_restr :
    ∀ k l (m:nat), k ≤ l → l ≤ m → between k m → between l m.

  Inductive exists_between k : nat → Prop :=
    | exists_S : ∀ l, exists_between k l → exists_between k (S l)
    | exists_le : ∀ l, k ≤ l → Q l → exists_between k (S l).

  Hint Constructors exists_between: arith v62.

  Lemma exists_le_S : ∀ k l, exists_between k l → S k ≤ l.

  Lemma exists_lt : ∀ k l, exists_between k l → k < l.
  Hint Immediate exists_le_S exists_lt: arith v62.

  Lemma exists_S_le : ∀ k l, exists_between k (S l) → k ≤ l.
  Hint Immediate exists_S_le: arith v62.
```

Definition `in_int` $p\ q\ r := p \leq r \wedge r < q$.
Lemma `in_int_intro` : $\forall p\ q\ r, p \leq r \rightarrow r < q \rightarrow \text{in_int } p\ q\ r$.
Hint Resolve `in_int_intro`: *arith v62*.
Lemma `in_int_lt` : $\forall p\ q\ r, \text{in_int } p\ q\ r \rightarrow p < q$.
Lemma `in_int_p_Sq` :
 $\forall p\ q\ r, \text{in_int } p\ (\text{S } q)\ r \rightarrow \text{in_int } p\ q\ r \vee r = q$:>**nat**.
Lemma `in_int_S` : $\forall p\ q\ r, \text{in_int } p\ q\ r \rightarrow \text{in_int } p\ (\text{S } q)\ r$.
Hint Resolve `in_int_S`: *arith v62*.
Lemma `in_int_Sp_q` : $\forall p\ q\ r, \text{in_int } (\text{S } p)\ q\ r \rightarrow \text{in_int } p\ q\ r$.
Hint Immediate `in_int_Sp_q`: *arith v62*.
Lemma `between_in_int` :
 $\forall k\ l, \text{between } k\ l \rightarrow \forall r, \text{in_int } k\ l\ r \rightarrow P\ r$.
Lemma `in_int_between` :
 $\forall k\ l, k \leq l \rightarrow (\forall r, \text{in_int } k\ l\ r \rightarrow P\ r) \rightarrow \text{between } k\ l$.
Lemma `exists_in_int` :
 $\forall k\ l, \text{exists_between } k\ l \rightarrow \text{exists2 } m : \text{nat}, \text{in_int } k\ l\ m \ \& \ Q\ m$.
Lemma `in_int_exists` : $\forall k\ l\ r, \text{in_int } k\ l\ r \rightarrow Q\ r \rightarrow \text{exists_between } k\ l$.
Lemma `between_or_exists` :
 $\forall k\ l,$
 $k \leq l \rightarrow$
 $(\forall n:\text{nat}, \text{in_int } k\ l\ n \rightarrow P\ n \vee Q\ n) \rightarrow$
 $\text{between } k\ l \vee \text{exists_between } k\ l$.
Lemma `between_not_exists` :
 $\forall k\ l,$
 $\text{between } k\ l \rightarrow$
 $(\forall n:\text{nat}, \text{in_int } k\ l\ n \rightarrow P\ n \rightarrow \neg Q\ n) \rightarrow \neg \text{exists_between } k\ l$.
Inductive `P_nth` (*init*:**nat**) : **nat** \rightarrow **nat** \rightarrow **Prop** :=
| `nth_O` : `P_nth` *init* *init* 0
| `nth_S` :
 $\forall k\ l\ (n:\text{nat}),$
 $\text{P_nth } \text{init } k\ n \rightarrow \text{between } (\text{S } k)\ l \rightarrow Q\ l \rightarrow \text{P_nth } \text{init } l\ (\text{S } n)$.
Lemma `nth_le` : $\forall (init:\text{nat})\ l\ (n:\text{nat}), \text{P_nth } \text{init } l\ n \rightarrow \text{init} \leq l$.
Definition `eventually` (*n*:**nat**) := `exists2` $k : \text{nat}, k \leq n \ \& \ Q\ k$.
Lemma `event_O` : `eventually` 0 $\rightarrow Q\ 0$.
End `Between`.
Hint Resolve `nth_O` `bet_S` `bet_emp` `bet_eq` `between_Sk_l` `exists_S` `exists_le`
`in_int_S` `in_int_intro`: *arith v62*.
Hint Immediate `in_int_Sp_q` `exists_le_S` `exists_S_le`: *arith v62*.

Chapter 41

Library **Coq.Arith.Bool_nat**

```
Require Export Compare_dec.  
Require Export Peano_dec.  
Require Import Sumbool.  
Open Local Scope nat_scope.  
Implicit Types m n x y : nat.
```

The decidability of equality and order relations over type *nat* give some boolean functions with the adequate specification.

```
Definition notzerop n := sumbool_not _ _ (zerop n).  
Definition lt_ge_dec :  $\forall x y, \{x < y\} + \{x \geq y\}$  :=  
  fun n m  $\Rightarrow$  sumbool_not _ _ (le_lt_dec m n).  
Definition nat_lt_ge_bool x y := bool_of_sumbool (lt_ge_dec x y).  
Definition nat_ge_lt_bool x y :=  
  bool_of_sumbool (sumbool_not _ _ (lt_ge_dec x y)).  
Definition nat_le_gt_bool x y := bool_of_sumbool (le_gt_dec x y).  
Definition nat_gt_le_bool x y :=  
  bool_of_sumbool (sumbool_not _ _ (le_gt_dec x y)).  
Definition nat_eq_bool x y := bool_of_sumbool (eq_nat_dec x y).  
Definition nat_noteq_bool x y :=  
  bool_of_sumbool (sumbool_not _ _ (eq_nat_dec x y)).  
Definition zerop_bool x := bool_of_sumbool (zerop x).  
Definition notzerop_bool x := bool_of_sumbool (notzerop x).
```

Chapter 42

Library `Coq.Arith.Compare_dec`

```
Require Import Le.
Require Import Lt.
Require Import Gt.
Require Import Decidable.

Open Local Scope nat_scope.

Implicit Types m n x y : nat.

Definition zerop n : {n = 0} + {0 < n}.

Definition lt_eq_lt_dec : ∀ n m, {n < m} + {n = m} + {m < n}.

Definition gt_eq_gt_dec : ∀ n m, {m > n} + {n = m} + {n > m}.

Definition le_lt_dec : ∀ n m, {n ≤ m} + {m < n}.

Definition le_le_S_dec n m : {n ≤ m} + {S m ≤ n}.

Definition le_ge_dec n m : {n ≤ m} + {n ≥ m}.

Definition le_gt_dec n m : {n ≤ m} + {n > m}.

Definition le_lt_eq_dec n m : n ≤ m → {n < m} + {n = m}.

Theorem le_dec : ∀ n m, {n ≤ m} + {¬ n ≤ m}.

Theorem lt_dec : ∀ n m, {n < m} + {¬ n < m}.

Theorem gt_dec : ∀ n m, {n > m} + {¬ n > m}.

Theorem ge_dec : ∀ n m, {n ≥ m} + {¬ n ≥ m}.
```

Proofs of decidability

```
Theorem dec_le : ∀ n m, decidable (n ≤ m).
Theorem dec_lt : ∀ n m, decidable (n < m).
Theorem dec_gt : ∀ n m, decidable (n > m).
Theorem dec_ge : ∀ n m, decidable (n ≥ m).
Theorem not_eq : ∀ n m, n ≠ m → n < m ∨ m < n.
Theorem not_le : ∀ n m, ¬ n ≤ m → n > m.
```

Theorem `not_gt` : $\forall n\ m, \neg n > m \rightarrow n \leq m$.

Theorem `not_ge` : $\forall n\ m, \neg n \geq m \rightarrow n < m$.

Theorem `not_lt` : $\forall n\ m, \neg n < m \rightarrow n \geq m$.

A ternary comparison function in the spirit of *Zcompare*.

Fixpoint `nat_compare` $n\ m$:=

```

  match  $n, m$  with
  | O, O  $\Rightarrow$  Eq
  | O, S _  $\Rightarrow$  Lt
  | S _, O  $\Rightarrow$  Gt
  | S  $n'$ , S  $m'$   $\Rightarrow$  nat_compare  $n'$   $m'$ 
  end.

```

Lemma `nat_compare_S` : $\forall n\ m, \text{nat_compare } (S\ n) (S\ m) = \text{nat_compare } n\ m$.

Lemma `nat_compare_eq_iff` : $\forall n\ m, \text{nat_compare } n\ m = \text{Eq} \leftrightarrow n = m$.

Lemma `nat_compare_eq` : $\forall n\ m, \text{nat_compare } n\ m = \text{Eq} \rightarrow n = m$.

Lemma `nat_compare_lt` : $\forall n\ m, n < m \leftrightarrow \text{nat_compare } n\ m = \text{Lt}$.

Lemma `nat_compare_gt` : $\forall n\ m, n > m \leftrightarrow \text{nat_compare } n\ m = \text{Gt}$.

Lemma `nat_compare_le` : $\forall n\ m, n \leq m \leftrightarrow \text{nat_compare } n\ m \neq \text{Gt}$.

Lemma `nat_compare_ge` : $\forall n\ m, n \geq m \leftrightarrow \text{nat_compare } n\ m \neq \text{Lt}$.

Lemma `nat_compare_spec` : $\forall x\ y, \text{CompSpec eq lt } x\ y (\text{nat_compare } x\ y)$.

Some projections of the above equivalences.

Lemma `nat_compare_Lt_Lt` : $\forall n\ m, \text{nat_compare } n\ m = \text{Lt} \rightarrow n < m$.

Lemma `nat_compare_Gt_gt` : $\forall n\ m, \text{nat_compare } n\ m = \text{Gt} \rightarrow n > m$.

A previous definition of *nat_compare* in terms of *lt_eq_lt_dec*. The new version avoids the creation of proof parts.

Definition `nat_compare_alt` ($n\ m$:nat) :=

```

  match lt_eq_lt_dec  $n\ m$  with
  | inleft (left _)  $\Rightarrow$  Lt
  | inleft (right _)  $\Rightarrow$  Eq
  | inright _  $\Rightarrow$  Gt
  end.

```

Lemma `nat_compare_equiv`: $\forall n\ m,$
 $\text{nat_compare } n\ m = \text{nat_compare_alt } n\ m$.

A boolean version of *le* over *nat*.

Fixpoint `leb` (m :nat) : nat \rightarrow bool :=

```

  match  $m$  with
  | O  $\Rightarrow$  fun _:nat  $\Rightarrow$  true
  | S  $m'$   $\Rightarrow$ 
    fun  $n$ :nat  $\Rightarrow$  match  $n$  with
      | O  $\Rightarrow$  false

```

```

      | S n' => leb m' n'
    end

  end.

Lemma leb_correct :  $\forall m\ n, m \leq n \rightarrow \text{leb } m\ n = \text{true}$ .
Lemma leb_complete :  $\forall m\ n, \text{leb } m\ n = \text{true} \rightarrow m \leq n$ .
Lemma leb_iff :  $\forall m\ n, \text{leb } m\ n = \text{true} \leftrightarrow m \leq n$ .
Lemma leb_correct_conv :  $\forall m\ n, m < n \rightarrow \text{leb } n\ m = \text{false}$ .
Lemma leb_complete_conv :  $\forall m\ n, \text{leb } n\ m = \text{false} \rightarrow m < n$ .
Lemma leb_iff_conv :  $\forall m\ n, \text{leb } n\ m = \text{false} \leftrightarrow m < n$ .
Lemma leb_compare :  $\forall n\ m, \text{leb } n\ m = \text{true} \leftrightarrow \text{nat\_compare } n\ m \neq \text{Gt}$ .

```

Chapter 43

Library **Coq.Arith.Compare**

Equality is decidable on *nat*

Open Local Scope *nat_scope*.

Notation *not_eq_sym* := *sym_not_eq*.

Implicit Types *m n p q* : *nat*.

Require Import *Arith_base*.

Require Import *Peano_dec*.

Require Import *Compare_dec*.

Definition *le_or_le_S* := *le_le_S_dec*.

Definition *Pcompare* := *gt_eq_gt_dec*.

Lemma *le_dec* : $\forall n\ m, \{n \leq m\} + \{m \leq n\}$.

Definition *lt_or_eq* *n m* := $\{m > n\} + \{n = m\}$.

Lemma *le_decide* : $\forall n\ m, n \leq m \rightarrow \text{lt_or_eq } n\ m$.

Lemma *le_le_S_eq* : $\forall n\ m, n \leq m \rightarrow \text{S } n \leq m \vee n = m$.

Lemma *discrete_nat* :

$\forall n\ m, n < m \rightarrow \text{S } n = m \vee (\exists r : \text{nat}, m = \text{S } (\text{S } (n + r)))$.

Require Export *Wf_nat*.

Require Export *Min Max*.

Chapter 44

Library Coq.Arith.Div2

```
Require Import Lt.
Require Import Plus.
Require Import Compare_dec.
Require Import Even.

Open Local Scope nat_scope.

Implicit Type n : nat.
```

Here we define $n/2$ and prove some of its properties

```
Fixpoint div2 n : nat :=
  match n with
  | 0 => 0
  | S 0 => 0
  | S (S n') => S (div2 n')
  end.
```

Since *div2* is recursively defined on 0, 1 and $(S (S n))$, it is useful to prove the corresponding induction principle

```
Lemma ind_0_1_SS :
  ∀ P:nat → Prop,
    P 0 → P 1 → (∀ n, P n → P (S (S n))) → ∀ n, P n.

0 < n ⇒ n/2 < n
```

```
Lemma lt_div2 : ∀ n, 0 < n → div2 n < n.
```

```
Hint Resolve lt_div2: arith.
```

Properties related to the parity

```
Lemma even_div2 : ∀ n, even n → div2 n = div2 (S n)
with odd_div2 : ∀ n, odd n → S (div2 n) = div2 (S n).

Lemma div2_even : ∀ n, div2 n = div2 (S n) → even n
with div2_odd : ∀ n, S (div2 n) = div2 (S n) → odd n.

Hint Resolve even_div2 div2_even odd_div2 div2_odd: arith.

Lemma even_odd_div2 :
```

$\forall n,$
 $(\text{even } n \leftrightarrow \text{div2 } n = \text{div2 } (\text{S } n)) \wedge (\text{odd } n \leftrightarrow \text{S } (\text{div2 } n) = \text{div2 } (\text{S } n)).$

Properties related to the double ($2n$)

Definition $\text{double } n := n + n.$

Hint Unfold double : *arith*.

Lemma $\text{double_S} : \forall n, \text{double } (\text{S } n) = \text{S } (\text{double } n).$

Lemma $\text{double_plus} : \forall n (m:\text{nat}), \text{double } (n + m) = \text{double } n + \text{double } m.$

Hint Resolve double_S : *arith*.

Lemma $\text{even_odd_double} :$

$\forall n,$
 $(\text{even } n \leftrightarrow n = \text{double } (\text{div2 } n)) \wedge (\text{odd } n \leftrightarrow n = \text{S } (\text{double } (\text{div2 } n))).$

Specializations

Lemma $\text{even_double} : \forall n, \text{even } n \rightarrow n = \text{double } (\text{div2 } n).$

Lemma $\text{double_even} : \forall n, n = \text{double } (\text{div2 } n) \rightarrow \text{even } n.$

Lemma $\text{odd_double} : \forall n, \text{odd } n \rightarrow n = \text{S } (\text{double } (\text{div2 } n)).$

Lemma $\text{double_odd} : \forall n, n = \text{S } (\text{double } (\text{div2 } n)) \rightarrow \text{odd } n.$

Hint Resolve even_double double_even odd_double double_odd : *arith*.

Application:

- if n is even then there is a p such that $n = 2p$
- if n is odd then there is a p such that $n = 2p+1$

(Immediate: it is $n/2$) **Lemma** $\text{even_2n} : \forall n, \text{even } n \rightarrow \{p : \text{nat} \mid n = \text{double } p\}.$

Lemma $\text{odd_S2n} : \forall n, \text{odd } n \rightarrow \{p : \text{nat} \mid n = \text{S } (\text{double } p)\}.$

Doubling before dividing by two brings back to the initial number.

Lemma $\text{div2_double} : \forall n:\text{nat}, \text{div2 } (2 \times n) = n.$

Lemma $\text{div2_double_plus_one} : \forall n:\text{nat}, \text{div2 } (\text{S } (2 \times n)) = n.$

Chapter 45

Library `Coq.Arith.EqNat`

Equality on natural numbers

`Open Local Scope nat_scope.`

`Implicit Types m n x y : nat.`

45.1 Propositional equality

```
Fixpoint eq_nat n m : Prop :=  
  match n, m with  
  | O, O => True  
  | O, S _ => False  
  | S _, O => False  
  | S n1, S m1 => eq_nat n1 m1  
end.
```

`Theorem eq_nat_refl : $\forall n$, eq_nat n n.`

`Hint Resolve eq_nat_refl: arith v62.`

eq restricted to *nat* and *eq_nat* are equivalent

`Lemma eq_eq_nat : $\forall n m$, $n = m \rightarrow \text{eq_nat } n m$.`

`Hint Immediate eq_eq_nat: arith v62.`

`Lemma eq_nat_eq : $\forall n m$, $\text{eq_nat } n m \rightarrow n = m$.`

`Hint Immediate eq_nat_eq: arith v62.`

`Theorem eq_nat_is_eq : $\forall n m$, $\text{eq_nat } n m \leftrightarrow n = m$.`

`Theorem eq_nat_elim :`

`$\forall n (P:\text{nat} \rightarrow \text{Prop})$, $P n \rightarrow \forall m$, $\text{eq_nat } n m \rightarrow P m$.`

`Theorem eq_nat_decide : $\forall n m$, $\{\text{eq_nat } n m\} + \{\neg \text{eq_nat } n m\}$.`

45.2 Boolean equality on *nat*

```
Fixpoint beq_nat n m : bool :=
```

```

match n, m with
| O, O ⇒ true
| O, S _ ⇒ false
| S _, O ⇒ false
| S n1, S m1 ⇒ beq_nat n1 m1
end.

Lemma beq_nat_refl : ∀ n, true = beq_nat n n.

Definition beq_nat_eq : ∀ x y, true = beq_nat x y → x = y.

Lemma beq_nat_true : ∀ x y, beq_nat x y = true → x=y.

Lemma beq_nat_false : ∀ x y, beq_nat x y = false → x≠y.

Lemma beq_nat_true_iff : ∀ x y, beq_nat x y = true ↔ x=y.

Lemma beq_nat_false_iff : ∀ x y, beq_nat x y = false ↔ x≠y.

```

Chapter 46

Library **Coq.Arith.Euclid**

```
Require Import Mult.
Require Import Compare_dec.
Require Import Wf_nat.

Open Local Scope nat_scope.

Implicit Types a b n q r : nat.

Inductive diveucl a b : Set :=
  divex :  $\forall q\ r, b > r \rightarrow a = q \times b + r \rightarrow \text{diveucl } a\ b$ .

Lemma eucl_dev :  $\forall n, n > 0 \rightarrow \forall m:\text{nat}, \text{diveucl } m\ n$ .

Lemma quotient :
   $\forall n,$ 
   $n > 0 \rightarrow$ 
   $\forall m:\text{nat}, \{q : \text{nat} \mid \exists r : \text{nat}, m = q \times n + r \wedge n > r\}$ .

Lemma modulo :
   $\forall n,$ 
   $n > 0 \rightarrow$ 
   $\forall m:\text{nat}, \{r : \text{nat} \mid \exists q : \text{nat}, m = q \times n + r \wedge n > r\}$ .
```

Chapter 47

Library **Coq.Arith.Even**

Here we define the predicates *even* and *odd* by mutual induction and we prove the decidability and the exclusion of those predicates. The main results about parity are proved in the module Div2.

Open Local Scope *nat_scope*.

Implicit Types *m n* : **nat**.

47.1 Definition of *even* and *odd*, and basic facts

```
Inductive even : nat → Prop :=  
  | even_O : even 0  
  | even_S : ∀ n, odd n → even (S n)  
with odd : nat → Prop :=  
  odd_S : ∀ n, even n → odd (S n).  
  
Hint Constructors even: arith.  
Hint Constructors odd: arith.  
  
Lemma even_or_odd : ∀ n, even n ∨ odd n.  
Lemma even_odd_dec : ∀ n, {even n} + {odd n}.  
Lemma not_even_and_odd : ∀ n, even n → odd n → False.
```

47.2 Facts about *even* & *odd* wrt. *plus*

```
Lemma even_plus_split : ∀ n m,  
  (even (n + m) → even n ∧ even m ∨ odd n ∧ odd m)  
with odd_plus_split : ∀ n m,  
  odd (n + m) → odd n ∧ even m ∨ even n ∧ odd m.  
  
Lemma even_even_plus : ∀ n m, even n → even m → even (n + m)  
with odd_plus_l : ∀ n m, odd n → even m → odd (n + m).  
  
Lemma odd_plus_r : ∀ n m, even n → odd m → odd (n + m)  
with odd_even_plus : ∀ n m, odd n → odd m → even (n + m).
```

Lemma even_plus_aux : $\forall n m,$
 $(\text{odd } (n + m) \leftrightarrow \text{odd } n \wedge \text{even } m \vee \text{even } n \wedge \text{odd } m) \wedge$
 $(\text{even } (n + m) \leftrightarrow \text{even } n \wedge \text{even } m \vee \text{odd } n \wedge \text{odd } m).$
 Lemma even_plus_even_inv_r : $\forall n m, \text{even } (n + m) \rightarrow \text{even } n \rightarrow \text{even } m.$
 Lemma even_plus_even_inv_l : $\forall n m, \text{even } (n + m) \rightarrow \text{even } m \rightarrow \text{even } n.$
 Lemma even_plus_odd_inv_r : $\forall n m, \text{even } (n + m) \rightarrow \text{odd } n \rightarrow \text{odd } m.$
 Lemma even_plus_odd_inv_l : $\forall n m, \text{even } (n + m) \rightarrow \text{odd } m \rightarrow \text{odd } n.$
 Hint Resolve even_even_plus odd_even_plus: *arith*.
 Lemma odd_plus_even_inv_l : $\forall n m, \text{odd } (n + m) \rightarrow \text{odd } m \rightarrow \text{even } n.$
 Lemma odd_plus_even_inv_r : $\forall n m, \text{odd } (n + m) \rightarrow \text{odd } n \rightarrow \text{even } m.$
 Lemma odd_plus_odd_inv_l : $\forall n m, \text{odd } (n + m) \rightarrow \text{even } m \rightarrow \text{odd } n.$
 Lemma odd_plus_odd_inv_r : $\forall n m, \text{odd } (n + m) \rightarrow \text{even } n \rightarrow \text{odd } m.$
 Hint Resolve odd_plus_l odd_plus_r: *arith*.

47.3 Facts about *even* and *odd* wrt. *mult*

Lemma even_mult_aux :
 $\forall n m,$
 $(\text{odd } (n \times m) \leftrightarrow \text{odd } n \wedge \text{odd } m) \wedge (\text{even } (n \times m) \leftrightarrow \text{even } n \vee \text{even } m).$
 Lemma even_mult_l : $\forall n m, \text{even } n \rightarrow \text{even } (n \times m).$
 Lemma even_mult_r : $\forall n m, \text{even } m \rightarrow \text{even } (n \times m).$
 Hint Resolve even_mult_l even_mult_r: *arith*.
 Lemma even_mult_inv_r : $\forall n m, \text{even } (n \times m) \rightarrow \text{odd } n \rightarrow \text{even } m.$
 Lemma even_mult_inv_l : $\forall n m, \text{even } (n \times m) \rightarrow \text{odd } m \rightarrow \text{even } n.$
 Lemma odd_mult : $\forall n m, \text{odd } n \rightarrow \text{odd } m \rightarrow \text{odd } (n \times m).$
 Hint Resolve even_mult_l even_mult_r odd_mult: *arith*.
 Lemma odd_mult_inv_l : $\forall n m, \text{odd } (n \times m) \rightarrow \text{odd } n.$
 Lemma odd_mult_inv_r : $\forall n m, \text{odd } (n \times m) \rightarrow \text{odd } m.$

Chapter 48

Library `Coq.Arith.Factorial`

```
Require Import Plus.
Require Import Mult.
Require Import Lt.
Open Local Scope nat_scope.

Factorial

Boxed Fixpoint fact (n:nat) : nat :=
  match n with
  | 0 ⇒ 1
  | S n ⇒ S n × fact n
  end.

Lemma lt_O_fact : ∀ n:nat, 0 < fact n.
Lemma fact_neq_0 : ∀ n:nat, fact n ≠ 0.
Lemma fact_le : ∀ n m:nat, n ≤ m → fact n ≤ fact m.
```

Chapter 49

Library **Coq.Arith.Gt**

Theorems about *gt* in *nat*. *gt* is defined in *Init/Peano.v* as:

Definition *gt* (*n m*:*nat*) := *m* < *n*.

```
Require Import Le.  
Require Import Lt.  
Require Import Plus.  
Open Local Scope nat_scope.  
Implicit Types m n p : nat.
```

49.1 Order and successor

```
Theorem gt_Sn_O :  $\forall n, S\ n > 0$ .  
Hint Resolve gt_Sn_O: arith v62.  
  
Theorem gt_Sn_n :  $\forall n, S\ n > n$ .  
Hint Resolve gt_Sn_n: arith v62.  
  
Theorem gt_n_S :  $\forall n\ m, n > m \rightarrow S\ n > S\ m$ .  
Hint Resolve gt_n_S: arith v62.  
  
Lemma gt_S_n :  $\forall n\ m, S\ m > S\ n \rightarrow m > n$ .  
Hint Immediate gt_S_n: arith v62.  
  
Theorem gt_S :  $\forall n\ m, S\ n > m \rightarrow n > m \vee m = n$ .  
  
Lemma gt_pred :  $\forall n\ m, m > S\ n \rightarrow \text{pred}\ m > n$ .  
Hint Immediate gt_pred: arith v62.
```

49.2 Irreflexivity

```
Lemma gt_irrefl :  $\forall n, \neg n > n$ .  
Hint Resolve gt_irrefl: arith v62.
```

49.3 Asymmetry

Lemma `gt_asym` : $\forall n\ m, n > m \rightarrow \neg m > n$.

Hint `Resolve gt_asym`: *arith v62*.

49.4 Relating strict and large orders

Lemma `le_not_gt` : $\forall n\ m, n \leq m \rightarrow \neg n > m$.

Hint `Resolve le_not_gt`: *arith v62*.

Lemma `gt_not_le` : $\forall n\ m, n > m \rightarrow \neg n \leq m$.

Hint `Resolve gt_not_le`: *arith v62*.

Theorem `le_S_gt` : $\forall n\ m, \mathbf{S}\ n \leq m \rightarrow m > n$.

Hint `Immediate le_S_gt`: *arith v62*.

Lemma `gt_S_le` : $\forall n\ m, \mathbf{S}\ m > n \rightarrow n \leq m$.

Hint `Immediate gt_S_le`: *arith v62*.

Lemma `gt_le_S` : $\forall n\ m, m > n \rightarrow \mathbf{S}\ n \leq m$.

Hint `Resolve gt_le_S`: *arith v62*.

Lemma `le_gt_S` : $\forall n\ m, n \leq m \rightarrow \mathbf{S}\ m > n$.

Hint `Resolve le_gt_S`: *arith v62*.

49.5 Transitivity

Theorem `le_gt_trans` : $\forall n\ m\ p, m \leq n \rightarrow m > p \rightarrow n > p$.

Theorem `gt_le_trans` : $\forall n\ m\ p, n > m \rightarrow p \leq m \rightarrow n > p$.

Lemma `gt_trans` : $\forall n\ m\ p, n > m \rightarrow m > p \rightarrow n > p$.

Theorem `gt_trans_S` : $\forall n\ m\ p, \mathbf{S}\ n > m \rightarrow m > p \rightarrow n > p$.

Hint `Resolve gt_trans_S le_gt_trans gt_le_trans`: *arith v62*.

49.6 Comparison to 0

Theorem `gt_0_eq` : $\forall n, n > 0 \vee 0 = n$.

49.7 Simplification and compatibility

Lemma `plus_gt_reg_l` : $\forall n\ m\ p, p + n > p + m \rightarrow n > m$.

Lemma `plus_gt_compat_l` : $\forall n\ m\ p, n > m \rightarrow p + n > p + m$.

Hint `Resolve plus_gt_compat_l`: *arith v62*.

Chapter 50

Library **Coq.Arith.Le**

Order on natural numbers. *le* is defined in *Init/Peano.v* as:

```
Inductive le (n:nat) : nat -> Prop :=  
  | le_n : n <= n  
  | le_S : forall m:nat, n <= m -> n <= S m
```

where "n <= m" := (le n m) : nat_scope.

Open Local Scope *nat_scope*.

Implicit Types *m n p* : nat.

50.1 *le* is a pre-order

Reflexivity **Theorem** *le_refl* : $\forall n, n \leq n$.

Transitivity **Theorem** *le_trans* : $\forall n m p, n \leq m \rightarrow m \leq p \rightarrow n \leq p$.

Hint Resolve *le_trans*: *arith v62*.

50.2 Properties of *le* w.r.t. successor, predecessor and 0

Comparison to 0

Theorem *le_0_n* : $\forall n, 0 \leq n$.

Theorem *le_Sn_0* : $\forall n, \neg S n \leq 0$.

Hint Resolve *le_0_n le_Sn_0*: *arith v62*.

Theorem *le_n_0_eq* : $\forall n, n \leq 0 \rightarrow 0 = n$.

Hint Immediate *le_n_0_eq*: *arith v62*.

le and successor

Theorem *le_n_S* : $\forall n m, n \leq m \rightarrow S n \leq S m$.

Theorem *le_n_Sn* : $\forall n, n \leq S n$.

Hint Resolve le_n_S le_n_Sn : *arith v62*.

Theorem le_Sn_le : $\forall n\ m, \mathbf{S}\ n \leq m \rightarrow n \leq m$.

Hint Immediate le_Sn_le: *arith v62*.

Theorem le_S_n : $\forall n\ m, \mathbf{S}\ n \leq \mathbf{S}\ m \rightarrow n \leq m$.

Hint Immediate le_S_n: *arith v62*.

Theorem le_Sn_n : $\forall n, \neg \mathbf{S}\ n \leq n$.

Hint Resolve le_Sn_n: *arith v62*.

le and predecessor

Theorem le_pred_n : $\forall n, \text{pred } n \leq n$.

Hint Resolve le_pred_n: *arith v62*.

Theorem le_pred : $\forall n\ m, n \leq m \rightarrow \text{pred } n \leq \text{pred } m$.

50.3 *le* is a order on *nat*

Antisymmetry

Theorem le_antisym : $\forall n\ m, n \leq m \rightarrow m \leq n \rightarrow n = m$.

Hint Immediate le_antisym: *arith v62*.

50.4 A different elimination principle for the order on natural numbers

Lemma le_elim_rel :

$\forall P:\text{nat} \rightarrow \text{nat} \rightarrow \mathbf{Prop},$

$(\forall p, P\ 0\ p) \rightarrow$

$(\forall p\ (q:\text{nat}), p \leq q \rightarrow P\ p\ q \rightarrow P\ (\mathbf{S}\ p)\ (\mathbf{S}\ q)) \rightarrow$

$\forall n\ m, n \leq m \rightarrow P\ n\ m.$

Chapter 51

Library **Coq.Arith.Lt**

Theorems about *lt* in *nat*. *lt* is defined in library *Init/Peano.v* as:

```
Definition lt (n m:nat) := S n <= m.  
Infix "<" := lt : nat_scope.
```

```
Require Import Le.  
Open Local Scope nat_scope.  
Implicit Types m n p : nat.
```

51.1 Irreflexivity

```
Theorem lt_irrefl :  $\forall n, \neg n < n$ .  
Hint Resolve lt_irrefl: arith v62.
```

51.2 Relationship between *le* and *lt*

```
Theorem lt_le_S :  $\forall n m, n < m \rightarrow S n \leq m$ .  
Hint Immediate lt_le_S: arith v62.  
  
Theorem lt_n_Sm_le :  $\forall n m, n < S m \rightarrow n \leq m$ .  
Hint Immediate lt_n_Sm_le: arith v62.  
  
Theorem le_lt_n_Sm :  $\forall n m, n \leq m \rightarrow n < S m$ .  
Hint Immediate le_lt_n_Sm: arith v62.  
  
Theorem le_not_lt :  $\forall n m, n \leq m \rightarrow \neg m < n$ .  
Theorem lt_not_le :  $\forall n m, n < m \rightarrow \neg m \leq n$ .  
Hint Immediate le_not_lt lt_not_le: arith v62.
```

51.3 Asymmetry

```
Theorem lt_asym :  $\forall n m, n < m \rightarrow \neg m < n$ .
```

51.4 Order and successor

Theorem `lt_n_Sn` : $\forall n, n < S\ n$.

Hint `Resolve lt_n_Sn`: *arith v62*.

Theorem `lt_S` : $\forall n\ m, n < m \rightarrow n < S\ m$.

Hint `Resolve lt_S`: *arith v62*.

Theorem `lt_n_S` : $\forall n\ m, n < m \rightarrow S\ n < S\ m$.

Hint `Resolve lt_n_S`: *arith v62*.

Theorem `lt_S_n` : $\forall n\ m, S\ n < S\ m \rightarrow n < m$.

Hint `Immediate lt_S_n`: *arith v62*.

Theorem `lt_0_Sn` : $\forall n, 0 < S\ n$.

Hint `Resolve lt_0_Sn`: *arith v62*.

Theorem `lt_n_0` : $\forall n, \neg n < 0$.

Hint `Resolve lt_n_0`: *arith v62*.

51.5 Predecessor

Lemma `S_pred` : $\forall n\ m, m < n \rightarrow n = S\ (\text{pred } n)$.

Lemma `lt_pred` : $\forall n\ m, S\ n < m \rightarrow n < \text{pred } m$.

Hint `Immediate lt_pred`: *arith v62*.

Lemma `lt_pred_n_n` : $\forall n, 0 < n \rightarrow \text{pred } n < n$.

Hint `Resolve lt_pred_n_n`: *arith v62*.

51.6 Transitivity properties

Theorem `lt_trans` : $\forall n\ m\ p, n < m \rightarrow m < p \rightarrow n < p$.

Theorem `lt_le_trans` : $\forall n\ m\ p, n < m \rightarrow m \leq p \rightarrow n < p$.

Theorem `le_lt_trans` : $\forall n\ m\ p, n \leq m \rightarrow m < p \rightarrow n < p$.

Hint `Resolve lt_trans lt_le_trans le_lt_trans`: *arith v62*.

51.7 Large = strict or equal

Theorem `le_lt_or_eq` : $\forall n\ m, n \leq m \rightarrow n < m \vee n = m$.

Theorem `le_lt_or_eq_iff` : $\forall n\ m, n \leq m \leftrightarrow n < m \vee n = m$.

Theorem `lt_le_weak` : $\forall n\ m, n < m \rightarrow n \leq m$.

Hint `Immediate lt_le_weak`: *arith v62*.

51.8 Dichotomy

Theorem `le_or_lt` : $\forall n\ m, n \leq m \vee m < n$.

Theorem `nat_total_order` : $\forall n\ m, n \neq m \rightarrow n < m \vee m < n$.

51.9 Comparison to 0

Theorem `neq_0_lt` : $\forall n, 0 \neq n \rightarrow 0 < n$.

Hint Immediate `neq_0_lt`: *arith v62*.

Theorem `lt_0_neq` : $\forall n, 0 < n \rightarrow 0 \neq n$.

Hint Immediate `lt_0_neq`: *arith v62*.

Chapter 52

Library Coq.Arith.Max

THIS FILE IS DEPRECATED. Use *MinMax* instead.

Require Export MinMax.

Local Open Scope nat_scope.

Implicit Types m n p : nat.

Notation max := MinMax.max (*only parsing*).

Definition max_0_l := max_0_l.

Definition max_0_r := max_0_r.

Definition succ_max_distr := succ_max_distr.

Definition plus_max_distr_l := plus_max_distr_l.

Definition plus_max_distr_r := plus_max_distr_r.

Definition max_case_strong := max_case_strong.

Definition max_spec := max_spec.

Definition max_dec := max_dec.

Definition max_case := max_case.

Definition max_idempotent := max_id.

Definition max_assoc := max_assoc.

Definition max_comm := max_comm.

Definition max_l := max_l.

Definition max_r := max_r.

Definition le_max_l := le_max_l.

Definition le_max_r := le_max_r.

Definition max_lub_l := max_lub_l.

Definition max_lub_r := max_lub_r.

Definition max_lub := max_lub.

Chapter 53

Library **Coq.Arith.Minus**

minus (difference between two natural numbers) is defined in *Init/Peano.v* as:

```
Fixpoint minus (n m:nat) : nat :=
  match n, m with
  | 0, _ => n
  | S k, 0 => S k
  | S k, S l => k - l
  end
where "n - m" := (minus n m) : nat_scope.
```

```
Require Import Lt.
Require Import Le.
Open Local Scope nat_scope.
Implicit Types m n p : nat.
```

53.1 0 is right neutral

Lemma `minus_n_O` : $\forall n, n = n - 0$.
Hint Resolve `minus_n_O`: *arith v62*.

53.2 Permutation with successor

Lemma `minus_Sn_m` : $\forall n\ m, m \leq n \rightarrow S\ (n - m) = S\ n - m$.
Hint Resolve `minus_Sn_m`: *arith v62*.
Theorem `pred_of_minus` : $\forall n, \text{pred } n = n - 1$.

53.3 Diagonal

Lemma `minus_diag` : $\forall n, n - n = 0$.

Lemma minus_diag_reverse : $\forall n, 0 = n - n$.
 Hint Resolve minus_diag_reverse: *arith v62*.
 Notation minus_n_n := minus_diag_reverse.

53.4 Simplification

Lemma minus_plus_simpl_l_reverse : $\forall n m p, n - m = p + n - (p + m)$.
 Hint Resolve minus_plus_simpl_l_reverse: *arith v62*.

53.5 Relation with plus

Lemma plus_minus : $\forall n m p, n = m + p \rightarrow p = n - m$.
 Hint Immediate plus_minus: *arith v62*.
 Lemma minus_plus : $\forall n m, n + m - n = m$.
 Hint Resolve minus_plus: *arith v62*.
 Lemma le_plus_minus : $\forall n m, n \leq m \rightarrow m = n + (m - n)$.
 Hint Resolve le_plus_minus: *arith v62*.
 Lemma le_plus_minus_r : $\forall n m, n \leq m \rightarrow n + (m - n) = m$.
 Hint Resolve le_plus_minus_r: *arith v62*.

53.6 Relation with order

Theorem minus_le_compat_r : $\forall n m p : \mathbf{nat}, n \leq m \rightarrow n - p \leq m - p$.
 Theorem minus_le_compat_l : $\forall n m p : \mathbf{nat}, n \leq m \rightarrow p - m \leq p - n$.
 Corollary le_minus : $\forall n m, n - m \leq n$.
 Lemma lt_minus : $\forall n m, m \leq n \rightarrow 0 < m \rightarrow n - m < n$.
 Hint Resolve lt_minus: *arith v62*.
 Lemma lt_O_minus_lt : $\forall n m, 0 < n - m \rightarrow m < n$.
 Hint Immediate lt_O_minus_lt: *arith v62*.
 Theorem not_le_minus_0 : $\forall n m, \neg m \leq n \rightarrow n - m = 0$.

Chapter 54

Library **Coq.Arith.Min**

THIS FILE IS DEPRECATED. Use *MinMax* instead.

Require Export MinMax.

Open Local Scope *nat_scope*.

Implicit Types *m n p* : **nat**.

Notation **min** := MinMax.min (*only parsing*).

Definition min_0_l := min_0_l.

Definition min_0_r := min_0_r.

Definition succ_min_distr := succ_min_distr.

Definition plus_min_distr_l := plus_min_distr_l.

Definition plus_min_distr_r := plus_min_distr_r.

Definition min_case_strong := min_case_strong.

Definition min_spec := min_spec.

Definition min_dec := min_dec.

Definition min_case := min_case.

Definition min_idempotent := min_id.

Definition min_assoc := min_assoc.

Definition min_comm := min_comm.

Definition min_l := min_l.

Definition min_r := min_r.

Definition le_min_l := le_min_l.

Definition le_min_r := le_min_r.

Definition min_glb_l := min_glb_l.

Definition min_glb_r := min_glb_r.

Definition min_glb := min_glb.

Chapter 55

Library **Coq.Arith.Mult**

```
Require Export Plus.  
Require Export Minus.  
Require Export Lt.  
Require Export Le.  
Open Local Scope nat_scope.  
Implicit Types m n p : nat.
```

Theorems about multiplication in *nat*. *mult* is defined in module *Init/Peano.v*.

55.1 *nat* is a semi-ring

55.1.1 Zero property

```
Lemma mult_0_r :  $\forall n, n \times 0 = 0$ .  
Lemma mult_0_l :  $\forall n, 0 \times n = 0$ .
```

55.1.2 1 is neutral

```
Lemma mult_1_l :  $\forall n, 1 \times n = n$ .  
Hint Resolve mult_1_l: arith v62.  
Lemma mult_1_r :  $\forall n, n \times 1 = n$ .  
Hint Resolve mult_1_r: arith v62.
```

55.1.3 Commutativity

```
Lemma mult_comm :  $\forall n m, n \times m = m \times n$ .  
Hint Resolve mult_comm: arith v62.
```

55.1.4 Distributivity

Lemma `mult_plus_distr_r` : $\forall n\ m\ p, (n + m) \times p = n \times p + m \times p$.

Hint `Resolve` `mult_plus_distr_r`: *arith v62*.

Lemma `mult_plus_distr_l` : $\forall n\ m\ p, n \times (m + p) = n \times m + n \times p$.

Lemma `mult_minus_distr_r` : $\forall n\ m\ p, (n - m) \times p = n \times p - m \times p$.

Hint `Resolve` `mult_minus_distr_r`: *arith v62*.

Lemma `mult_minus_distr_l` : $\forall n\ m\ p, n \times (m - p) = n \times m - n \times p$.

Hint `Resolve` `mult_minus_distr_l`: *arith v62*.

55.1.5 Associativity

Lemma `mult_assoc_reverse` : $\forall n\ m\ p, n \times m \times p = n \times (m \times p)$.

Hint `Resolve` `mult_assoc_reverse`: *arith v62*.

Lemma `mult_assoc` : $\forall n\ m\ p, n \times (m \times p) = n \times m \times p$.

Hint `Resolve` `mult_assoc`: *arith v62*.

55.1.6 Inversion lemmas

Lemma `mult_is_O` : $\forall n\ m, n \times m = 0 \rightarrow n = 0 \vee m = 0$.

Lemma `mult_is_one` : $\forall n\ m, n \times m = 1 \rightarrow n = 1 \wedge m = 1$.

55.1.7 Multiplication and successor

Lemma `mult_succ_l` : $\forall n\ m:\text{nat}, S\ n \times m = n \times m + m$.

Lemma `mult_succ_r` : $\forall n\ m:\text{nat}, n \times S\ m = n \times m + n$.

55.2 Compatibility with orders

Lemma `mult_O_le` : $\forall n\ m, m = 0 \vee n \leq m \times n$.

Hint `Resolve` `mult_O_le`: *arith v62*.

Lemma `mult_le_compat_l` : $\forall n\ m\ p, n \leq m \rightarrow p \times n \leq p \times m$.

Hint `Resolve` `mult_le_compat_l`: *arith*.

Lemma `mult_le_compat_r` : $\forall n\ m\ p, n \leq m \rightarrow n \times p \leq m \times p$.

Lemma `mult_le_compat` :

$\forall n\ m\ p\ (q:\text{nat}), n \leq m \rightarrow p \leq q \rightarrow n \times p \leq m \times q$.

Lemma `mult_S_lt_compat_l` : $\forall n\ m\ p, m < p \rightarrow S\ n \times m < S\ n \times p$.

Hint `Resolve` `mult_S_lt_compat_l`: *arith*.

Lemma `mult_lt_compat_r` : $\forall n\ m\ p, n < m \rightarrow 0 < p \rightarrow n \times p < m \times p$.

Lemma `mult_S_le_reg_l` : $\forall n\ m\ p, S\ n \times m \leq S\ n \times p \rightarrow m \leq p$.

55.3 $n \mapsto 2 \cdot n$ and $n \mapsto 2n+1$ have disjoint image

Theorem `odd_even_lem` : $\forall p\ q, 2 \times p + 1 \neq 2 \times q$.

55.4 Tail-recursive `mult`

`tail_mult` is an alternative definition for `mult` which is tail-recursive, whereas `mult` is not. This can be useful when extracting programs.

```
Fixpoint mult_acc (s:nat) m n : nat :=  
  match n with  
  | O  $\Rightarrow$  s  
  | S p  $\Rightarrow$  mult_acc (tail_plus m s) m p  
  end.
```

Lemma `mult_acc_aux` : $\forall n\ m\ p, m + n \times p = \text{mult_acc}\ m\ p\ n$.

Definition `tail_mult` $n\ m := \text{mult_acc}\ 0\ m\ n$.

Lemma `mult_tail_mult` : $\forall n\ m, n \times m = \text{tail_mult}\ n\ m$.

TailSimpl transforms any `tail_plus` and `tail_mult` into `plus` and `mult` and simplify

```
Ltac tail_simpl :=  
  repeat rewrite  $\leftarrow$  plus_tail_plus; repeat rewrite  $\leftarrow$  mult_tail_mult;  
  simpl in  $\vdash$   $\times$ .
```

Chapter 56

Library `Coq.Arith.Peano_dec`

```
Require Import Decidable.  
Open Local Scope nat_scope.  
Implicit Types m n x y : nat.  
Theorem O_or_S :  $\forall n, \{m : \text{nat} \mid S\ m = n\} + \{0 = n\}$ .  
Theorem eq_nat_dec :  $\forall n\ m, \{n = m\} + \{n \neq m\}$ .  
Hint Resolve O_or_S eq_nat_dec: arith.  
Theorem dec_eq_nat :  $\forall n\ m, \text{decidable } (n = m)$ .
```

Chapter 57

Library `Coq.Arith.Plus`

Properties of addition. `add` is defined in *Init/Peano.v* as:

```
Fixpoint plus (n m:nat) : nat :=
  match n with
  | 0 => m
  | S p => S (p + m)
  end
where "n + m" := (plus n m) : nat_scope.
```

```
Require Import Le.
Require Import Lt.
Open Local Scope nat_scope.
Implicit Types m n p q : nat.
```

57.1 Zero is neutral

Lemma `plus_0_l` : $\forall n, 0 + n = n$.

Lemma `plus_0_r` : $\forall n, n + 0 = n$.

57.2 Commutativity

Lemma `plus_comm` : $\forall n m, n + m = m + n$.

Hint Immediate `plus_comm`: *arith v62*.

57.3 Associativity

Lemma `plus_Snm_nSm` : $\forall n m, S n + m = n + S m$.

Lemma `plus_assoc` : $\forall n m p, n + (m + p) = n + m + p$.

Hint Resolve `plus_assoc`: *arith v62*.

Lemma plus_permute : $\forall n m p, n + (m + p) = m + (n + p)$.

Lemma plus_assoc_reverse : $\forall n m p, n + m + p = n + (m + p)$.

Hint Resolve plus_assoc_reverse: arith v62.

57.4 Simplification

Lemma plus_reg_l : $\forall n m p, p + n = p + m \rightarrow n = m$.

Lemma plus_le_reg_l : $\forall n m p, p + n \leq p + m \rightarrow n \leq m$.

Lemma plus_lt_reg_l : $\forall n m p, p + n < p + m \rightarrow n < m$.

57.5 Compatibility with order

Lemma plus_le_compat_l : $\forall n m p, n \leq m \rightarrow p + n \leq p + m$.

Hint Resolve plus_le_compat_l: arith v62.

Lemma plus_le_compat_r : $\forall n m p, n \leq m \rightarrow n + p \leq m + p$.

Hint Resolve plus_le_compat_r: arith v62.

Lemma le_plus_l : $\forall n m, n \leq n + m$.

Hint Resolve le_plus_l: arith v62.

Lemma le_plus_r : $\forall n m, m \leq n + m$.

Hint Resolve le_plus_r: arith v62.

Theorem le_plus_trans : $\forall n m p, n \leq m \rightarrow n \leq m + p$.

Hint Resolve le_plus_trans: arith v62.

Theorem lt_plus_trans : $\forall n m p, n < m \rightarrow n < m + p$.

Hint Immediate lt_plus_trans: arith v62.

Lemma plus_lt_compat_l : $\forall n m p, n < m \rightarrow p + n < p + m$.

Hint Resolve plus_lt_compat_l: arith v62.

Lemma plus_lt_compat_r : $\forall n m p, n < m \rightarrow n + p < m + p$.

Hint Resolve plus_lt_compat_r: arith v62.

Lemma plus_le_compat : $\forall n m p q, n \leq m \rightarrow p \leq q \rightarrow n + p \leq m + q$.

Lemma plus_le_lt_compat : $\forall n m p q, n \leq m \rightarrow p < q \rightarrow n + p < m + q$.

Lemma plus_lt_le_compat : $\forall n m p q, n < m \rightarrow p \leq q \rightarrow n + p < m + q$.

Lemma plus_lt_compat : $\forall n m p q, n < m \rightarrow p < q \rightarrow n + p < m + q$.

57.6 Inversion lemmas

Lemma plus_is_O : $\forall n m, n + m = 0 \rightarrow n = 0 \wedge m = 0$.

Definition plus_is_one :

$\forall m n, m + n = 1 \rightarrow \{m = 0 \wedge n = 1\} + \{m = 1 \wedge n = 0\}$.

57.7 Derived properties

Lemma `plus_permute_2_in_4` : $\forall n\ m\ p\ q, n + m + (p + q) = n + p + (m + q)$.

57.8 Tail-recursive plus

`tail_plus` is an alternative definition for `plus` which is tail-recursive, whereas `plus` is not. This can be useful when extracting programs.

```
Fixpoint tail_plus n m : nat :=  
  match n with  
  | O  $\Rightarrow$  m  
  | S n  $\Rightarrow$  tail_plus n (S m)  
  end.
```

Lemma `plus_tail_plus` : $\forall n\ m, n + m = \text{tail_plus } n\ m$.

57.9 Discrimination

Lemma `succ_plus_discr` : $\forall n\ m, n \neq S (\text{plus } m\ n)$.

Lemma `n_SS_n` : $\forall n, n \neq S (S\ n)$.

Lemma `n_SSS_n` : $\forall n, n \neq S (S (S\ n))$.

Lemma `n_SSSS_n` : $\forall n, n \neq S (S (S (S\ n)))$.

Chapter 58

Library **Coq.Arith.Wf_nat**

Well-founded relations and natural numbers

Require Import Lt.

Open Local Scope *nat_scope*.

Implicit Types *m n p* : **nat**.

Section Well_founded_Nat.

Variable *A* : **Type**.

Variable *f* : *A* → **nat**.

Definition ltof (*a b*:*A*) := *f a* < *f b*.

Definition gtof (*a b*:*A*) := *f b* > *f a*.

Theorem well_founded_ltof : well_founded ltof.

Theorem well_founded_gtof : well_founded gtof.

It is possible to directly prove the induction principle going back to primitive recursion on natural numbers (*induction_ltof1*) or to use the previous lemmas to extract a program with a fixpoint (*induction_ltof2*)

the ML-like program for *induction_ltof1* is :

let *induction_ltof1 f F a* =

let *rec indrec n k* =
 match *n* **with**
 | *O* → *error*
 | *S m* → *F k* (*indrec m*)

in *indrec (f a + 1) a*

the ML-like program for *induction_ltof2* is :

let *induction_ltof2 F a* = *indrec a*
 where *rec indrec a* = *F a indrec*;;

Theorem induction_ltof1 :

∀ *P*:*A* → **Set**,
 (**∀** *x*:*A*, (**∀** *y*:*A*, ltof *y x* → *P y*) → *P x*) → **∀** *a*:*A*, *P a*.

Theorem `induction_gtof1` :

$\forall P:A \rightarrow \mathbf{Set},$
 $(\forall x:A, (\forall y:A, \text{gtof } y \ x \rightarrow P \ y) \rightarrow P \ x) \rightarrow \forall a:A, P \ a.$

Theorem `induction_ltof2` :

$\forall P:A \rightarrow \mathbf{Set},$
 $(\forall x:A, (\forall y:A, \text{ltof } y \ x \rightarrow P \ y) \rightarrow P \ x) \rightarrow \forall a:A, P \ a.$

Theorem `induction_gtof2` :

$\forall P:A \rightarrow \mathbf{Set},$
 $(\forall x:A, (\forall y:A, \text{gtof } y \ x \rightarrow P \ y) \rightarrow P \ x) \rightarrow \forall a:A, P \ a.$

If a relation R is compatible with lt i.e. if $x \ R \ y \Rightarrow f(x) < f(y)$ then R is well-founded.

Variable $R : A \rightarrow A \rightarrow \mathbf{Prop}.$

Hypothesis $H_compat : \forall x \ y:A, R \ x \ y \rightarrow f \ x < f \ y.$

Theorem `well_founded_lt_compat` : `well_founded` $R.$

End `Well_founded_Nat.`

Lemma `lt_wf` : `well_founded` $lt.$

Lemma `lt_wf_rec1` :

$\forall n \ (P:\mathbf{nat} \rightarrow \mathbf{Set}), (\forall n, (\forall m, m < n \rightarrow P \ m) \rightarrow P \ n) \rightarrow P \ n.$

Lemma `lt_wf_rec` :

$\forall n \ (P:\mathbf{nat} \rightarrow \mathbf{Set}), (\forall n, (\forall m, m < n \rightarrow P \ m) \rightarrow P \ n) \rightarrow P \ n.$

Lemma `lt_wf_ind` :

$\forall n \ (P:\mathbf{nat} \rightarrow \mathbf{Prop}), (\forall n, (\forall m, m < n \rightarrow P \ m) \rightarrow P \ n) \rightarrow P \ n.$

Lemma `gt_wf_rec` :

$\forall n \ (P:\mathbf{nat} \rightarrow \mathbf{Set}), (\forall n, (\forall m, n > m \rightarrow P \ m) \rightarrow P \ n) \rightarrow P \ n.$

Lemma `gt_wf_ind` :

$\forall n \ (P:\mathbf{nat} \rightarrow \mathbf{Prop}), (\forall n, (\forall m, n > m \rightarrow P \ m) \rightarrow P \ n) \rightarrow P \ n.$

Lemma `lt_wf_double_rec` :

$\forall P:\mathbf{nat} \rightarrow \mathbf{nat} \rightarrow \mathbf{Set},$
 $(\forall n \ m,$
 $(\forall p \ q, p < n \rightarrow P \ p \ q) \rightarrow$
 $(\forall p, p < m \rightarrow P \ n \ p) \rightarrow P \ n \ m) \rightarrow \forall n \ m, P \ n \ m.$

Lemma `lt_wf_double_ind` :

$\forall P:\mathbf{nat} \rightarrow \mathbf{nat} \rightarrow \mathbf{Prop},$
 $(\forall n \ m,$
 $(\forall p \ (q:\mathbf{nat}), p < n \rightarrow P \ p \ q) \rightarrow$
 $(\forall p, p < m \rightarrow P \ n \ p) \rightarrow P \ n \ m) \rightarrow \forall n \ m, P \ n \ m.$

Hint `Resolve` `lt_wf`: *arith*.

Hint `Resolve` `well_founded_lt_compat`: *arith*.

Section `LT_WF_REL.`

Variable $A : \mathbf{Set}.$

Variable $R : A \rightarrow A \rightarrow \mathbf{Prop}.$

Variable $F : A \rightarrow \mathbf{nat} \rightarrow \mathbf{Prop}.$

```

Definition inv_lt_rel x y := exists2 n, F x n & (∀ m, F y m → n < m).

Hypothesis F_compat : ∀ x y:A, R x y → inv_lt_rel x y.
Remark acc_lt_rel : ∀ x:A, (∃ n, F x n) → Acc R x.

Theorem well_founded_inv_lt_rel_compat : well_founded R.

End LT_WF_REL.

Lemma well_founded_inv_rel_inv_lt_rel :
  ∀ (A:Set) (F:A → nat → Prop), well_founded (inv_lt_rel A F).

  A constructive proof that any non empty decidable subset of natural numbers has a least element

Require Import Le.
Require Import Compare_dec.
Require Import Decidable.

Definition has_unique_least_element (A:Type) (R:A→A→Prop) (P:A→Prop) :=
  ∃! x, P x ∧ ∀ x', P x' → R x x'.

Lemma dec_inh_nat_subset_has_unique_least_element :
  ∀ P:nat→Prop, (∀ n, P n ∨ ¬ P n) →
    (∃ n, P n) → has_unique_least_element le P.

  nth iteration of the function f

Fixpoint iter_nat (n:nat) (A:Type) (f:A → A) (x:A) : A :=
  match n with
  | 0 ⇒ x
  | S n' ⇒ f (iter_nat n' A f x)
  end.

Theorem iter_nat_plus :
  ∀ (n m:nat) (A:Type) (f:A → A) (x:A),
    iter_nat (n + m) A f x = iter_nat n A f (iter_nat m A f x).

```

Chapter 59

Library `Coq.Arith.NatOrderedType`

```
Require Import Lt Peano_dec Compare_dec EqNat
Equalities Orders OrdersTac.
```

59.1 DecidableType structure for Peano numbers

```
Module NAT_AS_UBE <: USUALBOOLEQ.
  Definition t := nat.
  Definition eq := @eq nat.
  Definition eqb := beq_nat.
  Definition eqb_eq := beq_nat_true_iff.
End NAT_AS_UBE.

Module NAT_AS_DT <: USUALDECIDABLETYPEFULL := MAKE_UDTF NAT_AS_UBE.
```

Note that the last module fulfills by subtyping many other interfaces, such as *DecidableType* or *EqualityType*.

59.2 OrderedType structure for Peano numbers

```
Module NAT_AS_OT <: ORDEREDTYPEFULL.
  Include NAT_AS_DT.
  Definition lt := lt.
  Definition le := le.
  Definition compare := nat_compare.
  Instance lt_strorder : StrictOrder lt.
  Instance lt_compat : Proper (Logic.eq==>Logic.eq==>iff) lt.
  Definition le_lteq := le_lt_or_eq_iff.
  Definition compare_spec := nat_compare_spec.
End NAT_AS_OT.
```

Note that *Nat_as_OT* can also be seen as a *UsualOrderedType* and a *OrderedType* (and also as a *DecidableType*).

59.3 An *order* tactic for Peano numbers

```
Module NATORDER := OTF_TO_ORDER_TAC NAT_AS_OT.  
Ltac nat_order := NatOrder.order.
```

Note that *nat_order* is domain-agnostic: it will not prove $1 \leq 2$ or $x \leq x+x$, but rather things like $x \leq y \rightarrow y \leq x \rightarrow x=y$.

```
Section Test.
```

```
Let test :  $\forall x y : \text{nat}, x \leq y \rightarrow y \leq x \rightarrow x=y$ .
```

```
End Test.
```

Chapter 60

Library **Coq.Arith.MinMax**

Require Import Orders NatOrderedType GenericMinMax.

60.1 Maximum and Minimum of two natural numbers

```
Fixpoint max n m : nat :=  
  match n, m with  
    | O, _  $\Rightarrow$  m  
    | S n', O  $\Rightarrow$  n  
    | S n', S m'  $\Rightarrow$  S (max n' m')  
  end.  
  
Fixpoint min n m : nat :=  
  match n, m with  
    | O, _  $\Rightarrow$  O  
    | S n', O  $\Rightarrow$  O  
    | S n', S m'  $\Rightarrow$  S (min n' m')  
  end.
```

These functions implement indeed a maximum and a minimum

Lemma max_l : $\forall x y, y \leq x \rightarrow \text{max } x y = x$.

Lemma max_r : $\forall x y, x \leq y \rightarrow \text{max } x y = y$.

Lemma min_l : $\forall x y, x \leq y \rightarrow \text{min } x y = x$.

Lemma min_r : $\forall x y, y \leq x \rightarrow \text{min } x y = y$.

Module NATHASMINMAX <: HASMINMAX NAT_AS_OT.

Definition max := max.

Definition min := min.

Definition max_l := max_l.

Definition max_r := max_r.

Definition min_l := min_l.

Definition min_r := min_r.

End NATHASMINMAX.

We obtain hence all the generic properties of *max* and *min*, see file *GenericMinMax* or use SearchAbout.

Module Export MMP := USUALMINMAXPROPERTIES NAT_AS_OT NATHASMINMAX.

60.2 Properties specific to the *nat* domain

Simplifications

Lemma max_0_l : $\forall n, \max 0 n = n$.

Lemma max_0_r : $\forall n, \max n 0 = n$.

Lemma min_0_l : $\forall n, \min 0 n = 0$.

Lemma min_0_r : $\forall n, \min n 0 = 0$.

Compatibilities (consequences of monotonicity)

Lemma succ_max_distr : $\forall n m, S (\max n m) = \max (S n) (S m)$.

Lemma succ_min_distr : $\forall n m, S (\min n m) = \min (S n) (S m)$.

Lemma plus_max_distr_l : $\forall n m p, \max (p + n) (p + m) = p + \max n m$.

Lemma plus_max_distr_r : $\forall n m p, \max (n + p) (m + p) = \max n m + p$.

Lemma plus_min_distr_l : $\forall n m p, \min (p + n) (p + m) = p + \min n m$.

Lemma plus_min_distr_r : $\forall n m p, \min (n + p) (m + p) = \min n m + p$.

Hint Resolve

max_l max_r le_max_l le_max_r
min_l min_r le_min_l le_min_r : *arith v62*.