Isabelle/HOL — Higher-Order Logic

November 22, 2007

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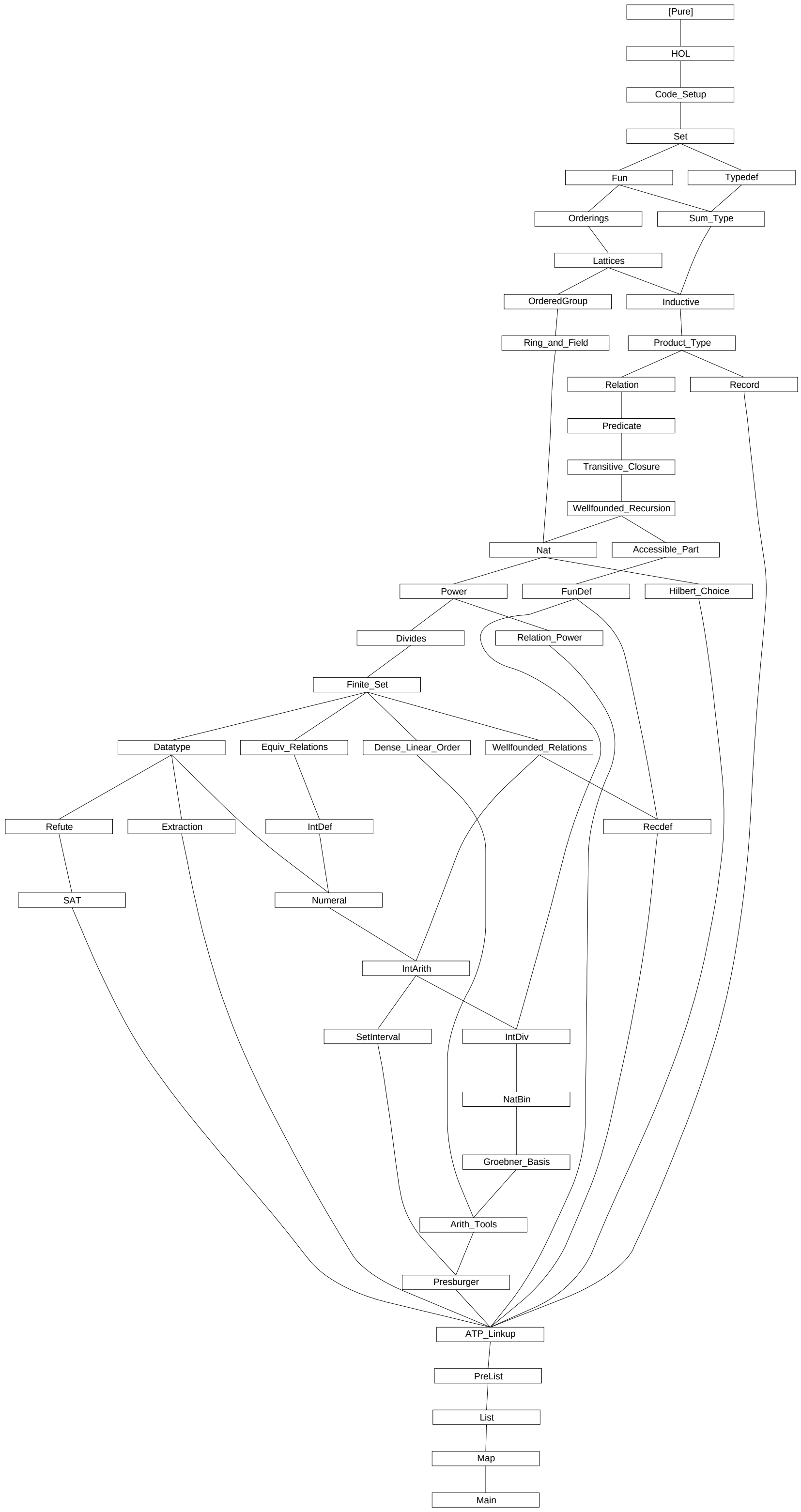
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1 HOL: The basis of Higher-Order Logic

theory *HOL*
imports *CPure*
uses
  (*hologic.ML*)
  ~~/*src*/*Tools*/*IsaPlanner*/*zipper.ML*
  ~~/*src*/*Tools*/*IsaPlanner*/*isand.ML*
  ~~/*src*/*Tools*/*IsaPlanner*/*rw-tools.ML*
  ~~/*src*/*Tools*/*IsaPlanner*/*rw-inst.ML*
  ~~/*src*/*Provers*/*project-rule.ML*
  ~~/*src*/*Provers*/*hypsubst.ML*
  ~~/*src*/*Provers*/*splitter.ML*
  ~~/*src*/*Provers*/*classical.ML*
  ~~/*src*/*Provers*/*blast.ML*
  ~~/*src*/*Provers*/*clasimp.ML*
  ~~/*src*/*Provers*/*eqsubst.ML*
  ~~/*src*/*Provers*/*quantifier1.ML*
  (*simpdata.ML*)
  ~~/*src*/*Tools*/*induct.ML*
  ~~/*src*/*Tools*/*code*/*code-name.ML*
  ~~/*src*/*Tools*/*code*/*code-funcgr.ML*
  ~~/*src*/*Tools*/*code*/*code-thingol.ML*
  ~~/*src*/*Tools*/*code*/*code-target.ML*
  ~~/*src*/*Tools*/*code*/*code-package.ML*
  ~~/*src*/*Tools*/*nbe.ML*
begin

1.1 Primitive logic

1.1.1 Core syntax

classes *type*
defaultsort *type*

global

typedecl *bool*

arities
  bool :: *type*
  fun :: (*type*, *type*) *type*

  itself :: (*type*) *type*

judgment
  Trueprop :: *bool* => *prop* ((-) *5*)

consts
  Not :: *bool* => *bool* (~ - [*40*] *40*)

True :: bool
False :: bool
arbitrary :: 'a

The :: ('a => bool) => 'a
All :: ('a => bool) => bool (**binder** ALL 10)
Ex :: ('a => bool) => bool (**binder** EX 10)
Ex1 :: ('a => bool) => bool (**binder** EX! 10)
Let :: ['a, 'a => 'b] => 'b

op = :: ['a, 'a] => bool (**infixl** = 50)
op & :: [bool, bool] => bool (**infixr** & 35)
op | :: [bool, bool] => bool (**infixr** | 30)
op --> :: [bool, bool] => bool (**infixr** --> 25)

local

consts
If :: [bool, 'a, 'a] => 'a ((if (-)/ then (-)/ else (-)) 10)

1.1.2 Additional concrete syntax

notation (**output**)
op = (**infix** = 50)

abbreviation
not-equal :: ['a, 'a] => bool (**infixl** ~= 50) **where**
x ~= y == ~ (x = y)

notation (**output**)
not-equal (**infix** ~= 50)

notation (xsymbols)
Not (¬ - [40] 40) **and**
op & (**infixr** ∧ 35) **and**
op | (**infixr** ∨ 30) **and**
op --> (**infixr** ⟶ 25) **and**
not-equal (**infix** ≠ 50)

notation (HTML **output**)
Not (¬ - [40] 40) **and**
op & (**infixr** ∧ 35) **and**
op | (**infixr** ∨ 30) **and**
not-equal (**infix** ≠ 50)

abbreviation (iff)
iff :: [bool, bool] => bool (**infixr** <-> 25) **where**
A <-> B == A = B

notation (*xsymbols*)
iff (**infixr** ⟷ *25*)

nonterminals
letbinds letbind
case-syn cases-syn

syntax
```
-The        :: [pttrn, bool] => 'a              ((3THE -./ -) [0, 10] 10)

-bind       :: [pttrn, 'a] => letbind           ((2- =/ -) 10)
            :: letbind => letbinds              (-)
-binds      :: [letbind, letbinds] => letbinds  (-;/ -)
-Let        :: [letbinds, 'a] => 'a             ((let (-)/ in (-)) 10)

-case-syntax:: ['a, cases-syn] => 'b            ((case - of/ -) 10)
-case1      :: ['a, 'b] => case-syn             ((2- =>/ -) 10)
            :: case-syn => cases-syn            (-)
-case2      :: [case-syn, cases-syn] => cases-syn  (-/ | -)
```

translations
```
THE x. P              == The (%x. P)
-Let (-binds b bs) e  == -Let b (-Let bs e)
let x = a in e        == Let a (%x. e)
```

⟨*ML*⟩

syntax (*xsymbols*)
```
-case1      :: ['a, 'b] => case-syn             ((2- ⇒/ -) 10)
```

notation (*xsymbols*)
All (**binder** ∀ *10*) **and**
Ex (**binder** ∃ *10*) **and**
Ex1 (**binder** ∃! *10*)

notation (*HTML* **output**)
All (**binder** ∀ *10*) **and**
Ex (**binder** ∃ *10*) **and**
Ex1 (**binder** ∃! *10*)

notation (*HOL*)
All (**binder** ! *10*) **and**
Ex (**binder** ? *10*) **and**
Ex1 (**binder** ?! *10*)

1.1.3 Axioms and basic definitions

axioms

eq-reflection: $(x{=}y) \text{ ==> } (x{==}y)$

refl: $t = (t{::}'a)$

ext: $(!!x{::}'a.\ (f\ x\ {::}'b) = g\ x) \text{ ==> } (\%x.\ f\ x) = (\%x.\ g\ x)$
— Extensionality is built into the meta-logic, and this rule expresses a related property. It is an eta-expanded version of the traditional rule, and similar to the ABS rule of HOL

the-eq-trivial: $(\mathit{THE}\ x.\ x = a) = (a{::}'a)$

impI: $(P \text{ ==> } Q) \text{ ==> } P{-}{-}{>}Q$
mp: $[|\ P{-}{-}{>}Q;\ \ P\ |] \text{ ==> } Q$

defs
True-def: $\mathit{True} \ == ((\%x{::}bool.\ x) = (\%x.\ x))$
All-def: $\mathit{All}(P) \ == (P = (\%x.\ \mathit{True}))$
Ex-def: $\mathit{Ex}(P) \ == !Q.\ (!x.\ P\ x \text{ --> } Q) \text{ --> } Q$
False-def: $\mathit{False} \ == (!P.\ P)$
not-def: $\tilde{}\ P \ == P{-}{-}{>}\mathit{False}$
and-def: $P\ \&\ Q \ == !R.\ (P{-}{-}{>}Q{-}{-}{>}R) \text{ --> } R$
or-def: $P\ |\ Q \ == !R.\ (P{-}{-}{>}R) \text{ --> } (Q{-}{-}{>}R) \text{ --> } R$
Ex1-def: $\mathit{Ex1}(P) \ == ?\ x.\ P(x)\ \&\ (!\ y.\ P(y) \text{ --> } y{=}x)$

axioms
iff: $(P{-}{-}{>}Q) \text{ --> } (Q{-}{-}{>}P) \text{ --> } (P{=}Q)$
True-or-False: $(P{=}\mathit{True})\ |\ (P{=}\mathit{False})$

defs
Let-def: $\mathit{Let}\ s\ f == f(s)$
if-def: $\mathit{If}\ P\ x\ y == \mathit{THE}\ z{::}'a.\ (P{=}\mathit{True} \text{ --> } z{=}x)\ \&\ (P{=}\mathit{False} \text{ --> } z{=}y)$

finalconsts
op =
op -->
The
arbitrary

axiomatization
undefined :: *'a*

axiomatization where
undefined-fun: *undefined* x = *undefined*

1.1.4 Generic classes and algebraic operations

class *default* = *type* +

fixes *default* :: $'a$

class *zero* = *type* +
fixes *zero* :: $'a$ (*0*)

class *one* = *type* +
fixes *one* :: $'a$ (*1*)

hide (**open**) *const zero one*

class *plus* = *type* +
fixes *plus* :: $'a \Rightarrow 'a \Rightarrow 'a$ (**infixl** + *65*)

class *minus* = *type* +
fixes *uminus* :: $'a \Rightarrow 'a$ (− - [*81*] *80*)
and *minus* :: $'a \Rightarrow 'a \Rightarrow 'a$ (**infixl** − *65*)

class *times* = *type* +
fixes *times* :: $'a \Rightarrow 'a \Rightarrow 'a$ (**infixl** ∗ *70*)

class *inverse* = *type* +
fixes *inverse* :: $'a \Rightarrow 'a$
and *divide* :: $'a \Rightarrow 'a \Rightarrow 'a$ (**infixl** '/ *70*)

class *abs* = *type* +
fixes *abs* :: $'a \Rightarrow 'a$
begin

notation (*xsymbols*)
abs (|-|)

notation (*HTML* **output**)
abs (|-|)

end

class *sgn* = *type* +
fixes *sgn* :: $'a \Rightarrow 'a$

class *ord* = *type* +
fixes *less-eq* :: $'a \Rightarrow 'a \Rightarrow bool$
and *less* :: $'a \Rightarrow 'a \Rightarrow bool$
begin

notation
less-eq (*op* <=) **and**
less-eq ((-/ <= -) [*51*, *51*] *50*) **and**
less (*op* <) **and**
less ((-/ < -) [*51*, *51*] *50*)

notation (*xsymbols*)
less-eq (*op* $\leq$) **and**
less-eq ((-/ $\leq$ -) [*51*, *51*] *50*)

notation (*HTML* **output**)
less-eq (*op* $\leq$) **and**
less-eq ((-/ $\leq$ -) [*51*, *51*] *50*)

abbreviation (*input*)
greater-eq (**infix** >= *50*) **where**
x >= *y* $\equiv$ *y* <= *x*

notation (*input*)
greater-eq (**infix** $\geq$ *50*)

abbreviation (*input*)
greater (**infix** > *50*) **where**
x > *y* $\equiv$ *y* < *x*

definition
Least :: ($'a \Rightarrow bool$) $\Rightarrow$ $'a$ (**binder** *LEAST* *10*) **where**
Least P == (*THE x*. *P x* $\wedge$ ($\forall$ *y*. *P y* $\longrightarrow$ *less-eq x y*))

end

syntax
-index1 :: *index* ($_1$)
translations
(*index*) $_1$ => (*index*) $\diamond$

⟨*ML*⟩

1.2 Fundamental rules

1.2.1 Equality

Thanks to Stephan Merz

lemma *subst*:
assumes *eq*: *s* = *t* **and** *p*: *P s*
shows *P t*
⟨*proof*⟩

lemma *sym*: *s* = *t* ==> *t* = *s*
⟨*proof*⟩

lemma *ssubst*: *t* = *s* ==> *P s* ==> *P t*
⟨*proof*⟩

lemma *trans*: $[| \ r{=}s; \ s{=}t \ |] ==> r{=}t$
⟨*proof*⟩

lemma *meta-eq-to-obj-eq*:
assumes *meq*: $A == B$
shows $A = B$
⟨*proof*⟩

Useful with *erule* for proving equalities from known equalities.

lemma *box-equals*: $[| \ a{=}b; \ \ a{=}c; \ \ b{=}d \ |] ==> c{=}d$
⟨*proof*⟩

For calculational reasoning:

lemma *forw-subst*: $a = b ==> P \ b ==> P \ a$
⟨*proof*⟩

lemma *back-subst*: $P \ a ==> a = b ==> P \ b$
⟨*proof*⟩

1.2.2 Congruence rules for application

lemma *fun-cong*: $(f{::}'a{=}{>}'b) = g ==> f(x){=}g(x)$
⟨*proof*⟩

lemma *arg-cong*: $x{=}y ==> f(x){=}f(y)$
⟨*proof*⟩

lemma *arg-cong2*: $[\![\ a = b; \ c = d \ ]\!] \Longrightarrow f \ a \ c = f \ b \ d$
⟨*proof*⟩

lemma *cong*: $[| \ f = g; \ (x{::}'a) = y \ |] ==> f(x) = g(y)$
⟨*proof*⟩

1.2.3 Equality of booleans – iff

lemma *iffI*: **assumes** $P ==> Q$ **and** $Q ==> P$ **shows** $P{=}Q$
⟨*proof*⟩

lemma *iffD2*: $[| \ P{=}Q; \ Q \ |] ==> P$
⟨*proof*⟩

lemma *rev-iffD2*: $[| \ Q; \ P{=}Q \ |] ==> P$
⟨*proof*⟩

lemma *iffD1*: $Q = P \Longrightarrow Q \Longrightarrow P$
⟨*proof*⟩

lemma *rev-iffD1*: $Q \Longrightarrow Q = P \Longrightarrow P$

⟨*proof*⟩

lemma *iffE*:
assumes *major*: $P=Q$
and *minor*: [| P −−> Q; Q −−> P |] ==> R
shows R
⟨*proof*⟩

1.2.4 True

lemma *TrueI*: *True*
⟨*proof*⟩

lemma *eqTrueI*: P ==> P = *True*
⟨*proof*⟩

lemma *eqTrueE*: P = *True* ==> P
⟨*proof*⟩

1.2.5 Universal quantifier

lemma *allI*: **assumes** !!x::$'a$. $P(x)$ **shows** *ALL* x. $P(x)$
⟨*proof*⟩

lemma *spec*: *ALL* x::$'a$. $P(x)$ ==> $P(x)$
⟨*proof*⟩

lemma *allE*:
assumes *major*: *ALL* x. $P(x)$
and *minor*: $P(x)$ ==> R
shows R
⟨*proof*⟩

lemma *all-dupE*:
assumes *major*: *ALL* x. $P(x)$
and *minor*: [| $P(x)$; *ALL* x. $P(x)$ |] ==> R
shows R
⟨*proof*⟩

1.2.6 False

Depends upon *spec*; it is impossible to do propositional logic before quantifiers!

lemma *FalseE*: *False* ==> P
⟨*proof*⟩

lemma *False-neq-True*: *False* = *True* ==> P
⟨*proof*⟩

1.2.7 Negation

lemma *notI*:
  assumes P ==> *False*
  shows ~P
  ⟨*proof*⟩

lemma *False-not-True*: *False* ~= *True*
  ⟨*proof*⟩

lemma *True-not-False*: *True* ~= *False*
  ⟨*proof*⟩

lemma *notE*: [| ~P; P |] ==> R
  ⟨*proof*⟩

lemma *notI2*: $(P \Longrightarrow \neg\ Pa) \Longrightarrow (P \Longrightarrow Pa) \Longrightarrow \neg\ P$
  ⟨*proof*⟩

1.2.8 Implication

lemma *impE*:
  assumes P-->Q P Q ==> R
  shows R
⟨*proof*⟩

lemma *rev-mp*: [| P; P --> Q |] ==> Q
⟨*proof*⟩

lemma *contrapos-nn*:
  assumes *major*: ~Q
    and *minor*: P==>Q
  shows ~P
⟨*proof*⟩

lemma *contrapos-pn*:
  assumes *major*: Q
    and *minor*: P ==> ~Q
  shows ~P
⟨*proof*⟩

lemma *not-sym*: t ~= s ==> s ~= t
  ⟨*proof*⟩

lemma *eq-neq-eq-imp-neq*: [| x = a ; a ~= b; b = y |] ==> x ~= y
  ⟨*proof*⟩

lemma *rev-contrapos*:
  assumes *pq*: $P ==> Q$
    and *nq*: $\sim Q$
  shows $\sim P$
⟨*proof*⟩

1.2.9 Existential quantifier

lemma *exI*: $P\ x ==> EX\ x{::}'a.\ P\ x$
⟨*proof*⟩

lemma *exE*:
  assumes *major*: $EX\ x{::}'a.\ P(x)$
    and *minor*: $!!x.\ P(x) ==> Q$
  shows Q
⟨*proof*⟩

1.2.10 Conjunction

lemma *conjI*: $[|\ P;\ Q\ |] ==> P\&Q$
⟨*proof*⟩

lemma *conjunct1*: $[|\ P\ \&\ Q\ |] ==> P$
⟨*proof*⟩

lemma *conjunct2*: $[|\ P\ \&\ Q\ |] ==> Q$
⟨*proof*⟩

lemma *conjE*:
  assumes *major*: $P\&Q$
    and *minor*: $[|\ P;\ Q\ |] ==> R$
  shows R
⟨*proof*⟩

lemma *context-conjI*:
  assumes $P\ P ==> Q$ **shows** $P\ \&\ Q$
⟨*proof*⟩

1.2.11 Disjunction

lemma *disjI1*: $P ==> P|Q$
⟨*proof*⟩

lemma *disjI2*: $Q ==> P|Q$
⟨*proof*⟩

lemma *disjE*:
  assumes *major*: $P|Q$
    and *minorP*: $P ==> R$
    and *minorQ*: $Q ==> R$

shows *R*
⟨*proof*⟩

1.2.12 Classical logic

lemma *classical*:
assumes *prem*: ~*P* ==> *P*
shows *P*
⟨*proof*⟩

lemmas *ccontr* = *FalseE* [*THEN classical*, *standard*]

lemma *rev-notE*:
assumes *premp*: *P*
and *premnot*: ~*R* ==> ~*P*
shows *R*
⟨*proof*⟩

lemma *notnotD*: ~~*P* ==> *P*
⟨*proof*⟩

lemma *contrapos-pp*:
assumes *p1*: *Q*
and *p2*: ~*P* ==> ~*Q*
shows *P*
⟨*proof*⟩

1.2.13 Unique existence

lemma *ex1I*:
assumes *P a* !!*x*. *P*(*x*) ==> *x*=*a*
shows *EX*! *x*. *P*(*x*)
⟨*proof*⟩

Sometimes easier to use: the premises have no shared variables. Safe!

lemma *ex-ex1I*:
assumes *ex-prem*: *EX x*. *P*(*x*)
and *eq*: !!*x y*. [| *P*(*x*); *P*(*y*) |] ==> *x*=*y*
shows *EX*! *x*. *P*(*x*)
⟨*proof*⟩

lemma *ex1E*:
assumes *major*: *EX*! *x*. *P*(*x*)
and *minor*: !!*x*. [| *P*(*x*); *ALL y*. *P*(*y*) --> *y*=*x* |] ==> *R*
shows *R*
⟨*proof*⟩

lemma *ex1-implies-ex*: *EX*! *x*. *P x* ==> *EX x*. *P x*

⟨*proof*⟩

1.2.14 THE: definite description operator

lemma *the-equality*:
assumes *prema*: $P\ a$
and *premx*: $!!x.\ P\ x\ ==>\ x=a$
shows $(THE\ x.\ P\ x) = a$
⟨*proof*⟩

lemma *theI*:
assumes $P\ a$ **and** $!!x.\ P\ x\ ==>\ x=a$
shows $P\ (THE\ x.\ P\ x)$
⟨*proof*⟩

lemma *theI'*: $EX!\ x.\ P\ x\ ==>\ P\ (THE\ x.\ P\ x)$
⟨*proof*⟩

lemma *theI2*:
assumes $P\ a\ !!x.\ P\ x\ ==>\ x=a\ !!x.\ P\ x\ ==>\ Q\ x$
shows $Q\ (THE\ x.\ P\ x)$
⟨*proof*⟩

lemma *the1I2*: **assumes** $EX!\ x.\ P\ x\ \bigwedge x.\ P\ x \Longrightarrow Q\ x$ **shows** $Q\ (THE\ x.\ P\ x)$
⟨*proof*⟩

lemma *the1-equality* [*elim?*]: $[|\ EX!x.\ P\ x;\ P\ a\ |]\ ==>\ (THE\ x.\ P\ x) = a$
⟨*proof*⟩

lemma *the-sym-eq-trivial*: $(THE\ y.\ x=y) = x$
⟨*proof*⟩

1.2.15 Classical intro rules for disjunction and existential quantifiers

lemma *disjCI*:
assumes $^\sim Q\ ==>\ P$ **shows** $P|Q$
⟨*proof*⟩

lemma *excluded-middle*: $^\sim P\ |\ P$
⟨*proof*⟩

case distinction as a natural deduction rule. Note that $\neg\ P$ is the second case, not the first

lemma *case-split-thm*:
assumes *prem1*: $P\ ==>\ Q$
and *prem2*: $^\sim P\ ==>\ Q$
shows Q

⟨*proof*⟩
lemmas *case-split* = *case-split-thm* [*case-names True False*]

lemma *impCE*:
  assumes *major*: $P-->Q$
    and *minor*: $\~P ==> R$ $Q ==> R$
  shows R
⟨*proof*⟩

lemma *impCE'*:
  assumes *major*: $P-->Q$
    and *minor*: $Q ==> R$ $\~P ==> R$
  shows R
⟨*proof*⟩

lemma *iffCE*:
  assumes *major*: $P=Q$
    and *minor*: $[| P; Q |] ==> R$ $[| \~P; \~Q |] ==> R$
  shows R
⟨*proof*⟩

lemma *exCI*:
  assumes $ALL\ x.\ \~P(x) ==> P(a)$
  shows $EX\ x.\ P(x)$
⟨*proof*⟩

1.2.16 Intuitionistic Reasoning

lemma *impE'*:
  assumes *1*: $P --> Q$
   and *2*: $Q ==> R$
   and *3*: $P --> Q ==> P$
  shows R
⟨*proof*⟩

lemma *allE'*:
  assumes *1*: $ALL\ x.\ P\ x$
   and *2*: $P\ x ==> ALL\ x.\ P\ x ==> Q$
  shows Q
⟨*proof*⟩

lemma *notE'*:
  assumes *1*: $\~\ P$
   and *2*: $\~\ P ==> P$
  shows R
⟨*proof*⟩

lemma *TrueE*: *True ==> P ==> P* ⟨*proof*⟩
lemma *notFalseE*: ~ *False ==> P ==> P* ⟨*proof*⟩

lemmas [*Pure.elim*!] = *disjE iffE FalseE conjE exE TrueE notFalseE*
  and [*Pure.intro*!] = *iffI conjI impI TrueI notI allI refl*
  and [*Pure.elim 2*] = *allE notE′ impE′*
  and [*Pure.intro*] = *exI disjI2 disjI1*

lemmas [*trans*] = *trans*
  and [*sym*] = *sym not-sym*
  and [*Pure.elim?*] = *iffD1 iffD2 impE*

⟨*ML*⟩

1.2.17 Atomizing meta-level connectives

lemma *atomize-all* [*atomize*]: (!!*x*. *P x*) == *Trueprop* (*ALL x*. *P x*)
⟨*proof*⟩

lemma *atomize-imp* [*atomize*]: (*A* ==> *B*) == *Trueprop* (*A* --> *B*)
⟨*proof*⟩

lemma *atomize-not*: (*A* ==> *False*) == *Trueprop* (~*A*)
⟨*proof*⟩

lemma *atomize-eq* [*atomize*]: (*x* == *y*) == *Trueprop* (*x* = *y*)
⟨*proof*⟩

lemma *atomize-conj* [*atomize*]:
  includes *meta-conjunction-syntax*
  shows (*A* && *B*) == *Trueprop* (*A* & *B*)
⟨*proof*⟩

lemmas [*symmetric*, *rulify*] = *atomize-all atomize-imp*
  and [*symmetric*, *defn*] = *atomize-all atomize-imp atomize-eq*

1.3 Package setup

1.3.1 Classical Reasoner setup

lemma *thin-refl*:
  $\bigwedge X.\ [\![\ x{=}x;\ PROP\ W\ ]\!] \Longrightarrow PROP\ W$ ⟨*proof*⟩

⟨*ML*⟩

ResBlacklist holds theorems blacklisted to sledgehammer. These theorems typically produce clauses that are prolific (match too many equality or membership literals) and relate to seldom-used facts. Some duplicate other rules.

⟨*ML*⟩

declare *iffI* [*intro*!]
  and *notI* [*intro*!]
  and *impI* [*intro*!]
  and *disjCI* [*intro*!]
  and *conjI* [*intro*!]
  and *TrueI* [*intro*!]
  and *refl* [*intro*!]

declare *iffCE* [*elim*!]
  and *FalseE* [*elim*!]
  and *impCE* [*elim*!]
  and *disjE* [*elim*!]
  and *conjE* [*elim*!]
  and *conjE* [*elim*!]

declare *ex-ex1I* [*intro*!]
  and *allI* [*intro*!]
  and *the-equality* [*intro*]
  and *exI* [*intro*]

declare *exE* [*elim*!]
  allE [*elim*]

⟨*ML*⟩

lemma *contrapos-np*: $\sim Q ==> (\sim P ==> Q) ==> P$
  ⟨*proof*⟩

declare *ex-ex1I* [*rule del*, *intro*! *2*]
  and *ex1I* [*intro*]

lemmas [*intro?*] = *ext*
  and [*elim?*] = *ex1-implies-ex*

lemma *alt-ex1E* [*elim*!]:
  assumes *major*: $\exists!x.\ P\ x$
    and *prem*: $\bigwedge x.\ [\![\ P\ x;\ \forall\ y\ y'.\ P\ y \wedge P\ y' \longrightarrow y = y'\ ]\!] \Longrightarrow R$
  shows R
⟨*proof*⟩

⟨*ML*⟩

1.3.2 Simplifier

lemma *eta-contract-eq*: $(\%s.\ f\ s) = f$ ⟨*proof*⟩

lemma *simp-thms*:

shows *not-not*: $(\sim\ \sim\ P) = P$
and *Not-eq-iff*: $((\sim P) = (\sim Q)) = (P = Q)$
and
$(P\ \sim= Q) = (P = (\sim Q))$
$(P \mid \sim P) = True \quad (\sim P \mid P) = True$
$(x = x) = True$
and *not-True-eq-False*: $(\neg\ True) = False$
and *not-False-eq-True*: $(\neg\ False) = True$
and
$(\sim P)\ \sim= P \ \ P\ \sim= (\sim P)$
$(True{=}P) = P$
and *eq-True*: $(P = True) = P$
and $(False{=}P) = (\sim P)$
and *eq-False*: $(P = False) = (\neg\ P)$
and
$(True\ {-}{-}{>}\ P) = P \ \ (False\ {-}{-}{>}\ P) = True$
$(P\ {-}{-}{>}\ True) = True \ \ (P\ {-}{-}{>}\ P) = True$
$(P\ {-}{-}{>}\ False) = (\sim P) \ \ (P\ {-}{-}{>}\ \sim P) = (\sim P)$
$(P\ \&\ True) = P \ \ (True\ \&\ P) = P$
$(P\ \&\ False) = False \ \ (False\ \&\ P) = False$
$(P\ \&\ P) = P \ \ (P\ \&\ (P\ \&\ Q)) = (P\ \&\ Q)$
$(P\ \&\ \sim P) = False \quad (\sim P\ \&\ P) = False$
$(P \mid True) = True \ \ (True \mid P) = True$
$(P \mid False) = P \ \ (False \mid P) = P$
$(P \mid P) = P \ \ (P \mid (P \mid Q)) = (P \mid Q)$ **and**
$(ALL\ x.\ P) = P \ \ (EX\ x.\ P) = P \ \ EX\ x.\ x{=}t \ \ EX\ x.\ t{=}x$
— needed for the one-point-rule quantifier simplification procs
— essential for termination!! **and**
$!!P.\ (EX\ x.\ x{=}t\ \&\ P(x)) = P(t)$
$!!P.\ (EX\ x.\ t{=}x\ \&\ P(x)) = P(t)$
$!!P.\ (ALL\ x.\ x{=}t\ {-}{-}{>}\ P(x)) = P(t)$
$!!P.\ (ALL\ x.\ t{=}x\ {-}{-}{>}\ P(x)) = P(t)$
⟨*proof*⟩

lemma *disj-absorb*: $(A \mid A) = A$
⟨*proof*⟩

lemma *disj-left-absorb*: $(A \mid (A \mid B)) = (A \mid B)$
⟨*proof*⟩

lemma *conj-absorb*: $(A\ \&\ A) = A$
⟨*proof*⟩

lemma *conj-left-absorb*: $(A\ \&\ (A\ \&\ B)) = (A\ \&\ B)$
⟨*proof*⟩

lemma *eq-ac*:
shows *eq-commute*: $(a{=}b) = (b{=}a)$
and *eq-left-commute*: $(P{=}(Q{=}R)) = (Q{=}(P{=}R))$

  and *eq-assoc*: $((P{=}Q){=}R) = (P{=}(Q{=}R))$ ⟨*proof*⟩
lemma *neq-commute*: $(a\tilde{}{=}b) = (b\tilde{}{=}a)$ ⟨*proof*⟩

lemma *conj-comms*:
 shows *conj-commute*: $(P\&Q) = (Q\&P)$
  and *conj-left-commute*: $(P\&(Q\&R)) = (Q\&(P\&R))$ ⟨*proof*⟩
lemma *conj-assoc*: $((P\&Q)\&R) = (P\&(Q\&R))$ ⟨*proof*⟩

lemmas *conj-ac* = *conj-commute conj-left-commute conj-assoc*

lemma *disj-comms*:
 shows *disj-commute*: $(P|Q) = (Q|P)$
  and *disj-left-commute*: $(P|(Q|R)) = (Q|(P|R))$ ⟨*proof*⟩
lemma *disj-assoc*: $((P|Q)|R) = (P|(Q|R))$ ⟨*proof*⟩

lemmas *disj-ac* = *disj-commute disj-left-commute disj-assoc*

lemma *conj-disj-distribL*: $(P\&(Q|R)) = (P\&Q \mid P\&R)$ ⟨*proof*⟩
lemma *conj-disj-distribR*: $((P|Q)\&R) = (P\&R \mid Q\&R)$ ⟨*proof*⟩

lemma *disj-conj-distribL*: $(P|(Q\&R)) = ((P|Q)\ \&\ (P|R))$ ⟨*proof*⟩
lemma *disj-conj-distribR*: $((P\&Q)|R) = ((P|R)\ \&\ (Q|R))$ ⟨*proof*⟩

lemma *imp-conjR*: $(P \dashrightarrow (Q\&R)) = ((P{-}{-}{>}Q)\ \&\ (P{-}{-}{>}R))$ ⟨*proof*⟩
lemma *imp-conjL*: $((P\&Q) \dashrightarrow R) = (P \dashrightarrow (Q \dashrightarrow R))$ ⟨*proof*⟩
lemma *imp-disjL*: $((P|Q) \dashrightarrow R) = ((P{-}{-}{>}R)\&(Q{-}{-}{>}R))$ ⟨*proof*⟩

These two are specialized, but *imp-disj-not1* is useful in *Auth/Yahalom*.

lemma *imp-disj-not1*: $(P \dashrightarrow Q \mid R) = (\tilde{}Q \dashrightarrow P \dashrightarrow R)$ ⟨*proof*⟩
lemma *imp-disj-not2*: $(P \dashrightarrow Q \mid R) = (\tilde{}R \dashrightarrow P \dashrightarrow Q)$ ⟨*proof*⟩

lemma *imp-disj1*: $((P{-}{-}{>}Q)|R) = (P{-}{-}{>}\ Q|R)$ ⟨*proof*⟩
lemma *imp-disj2*: $(Q|(P{-}{-}{>}R)) = (P{-}{-}{>}\ Q|R)$ ⟨*proof*⟩

lemma *imp-cong*: $(P = P') \Longrightarrow (P' \Longrightarrow (Q = Q')) \Longrightarrow ((P \dashrightarrow Q) = (P' \dashrightarrow Q'))$
 ⟨*proof*⟩

lemma *de-Morgan-disj*: $(\tilde{}(P \mid Q)) = (\tilde{}P\ \&\ \tilde{}Q)$ ⟨*proof*⟩
lemma *de-Morgan-conj*: $(\tilde{}(P\ \&\ Q)) = (\tilde{}P \mid \tilde{}Q)$ ⟨*proof*⟩
lemma *not-imp*: $(\tilde{}(P \dashrightarrow Q)) = (P\ \&\ \tilde{}Q)$ ⟨*proof*⟩
lemma *not-iff*: $(P\tilde{}{=}Q) = (P = (\tilde{}Q))$ ⟨*proof*⟩
lemma *disj-not1*: $(\tilde{}P \mid Q) = (P \dashrightarrow Q)$ ⟨*proof*⟩
lemma *disj-not2*: $(P \mid \tilde{}Q) = (Q \dashrightarrow P)$ — changes orientation :-(
 ⟨*proof*⟩
lemma *imp-conv-disj*: $(P \dashrightarrow Q) = ((\tilde{}P) \mid Q)$ ⟨*proof*⟩

lemma *iff-conv-conj-imp*: $(P = Q) = ((P \dashrightarrow Q)\ \&\ (Q \dashrightarrow P))$ ⟨*proof*⟩

lemma *cases-simp*: $((P \; --> \; Q) \; \& \; (\verb|~| P \; --> \; Q)) = Q$
— Avoids duplication of subgoals after *split-if*, when the true and false
— cases boil down to the same thing.
⟨*proof*⟩

lemma *not-all*: $(\verb|~| \; (! \; x. \; P(x))) = (? \; x.\verb|~| P(x))$ ⟨*proof*⟩
lemma *imp-all*: $((! \; x. \; P \; x) \; --> \; Q) = (? \; x. \; P \; x \; --> \; Q)$ ⟨*proof*⟩
lemma *not-ex*: $(\verb|~| \; (? \; x. \; P(x))) = (! \; x.\verb|~| P(x))$ ⟨*proof*⟩
lemma *imp-ex*: $((? \; x. \; P \; x) \; --> \; Q) = (! \; x. \; P \; x \; --> \; Q)$ ⟨*proof*⟩
lemma *all-not-ex*: $(ALL \; x. \; P \; x) = (\verb|~| \; (EX \; x. \; \verb|~| \; P \; x \;))$ ⟨*proof*⟩

declare *All-def* [*noatp*]

lemma *ex-disj-distrib*: $(? \; x. \; P(x) \mid Q(x)) = ((? \; x. \; P(x)) \mid (? \; x. \; Q(x)))$ ⟨*proof*⟩
lemma *all-conj-distrib*: $(!x. \; P(x) \; \& \; Q(x)) = ((! \; x. \; P(x)) \; \& \; (! \; x. \; Q(x)))$ ⟨*proof*⟩

The & congruence rule: not included by default! May slow rewrite proofs down by as much as 50%

lemma *conj-cong*:
$(P = P') ==> (P' ==> (Q = Q')) ==> ((P \; \& \; Q) = (P' \; \& \; Q'))$
⟨*proof*⟩

lemma *rev-conj-cong*:
$(Q = Q') ==> (Q' ==> (P = P')) ==> ((P \; \& \; Q) = (P' \; \& \; Q'))$
⟨*proof*⟩

The | congruence rule: not included by default!

lemma *disj-cong*:
$(P = P') ==> (\verb|~| P' ==> (Q = Q')) ==> ((P \mid Q) = (P' \mid Q'))$
⟨*proof*⟩

if-then-else rules

lemma *if-True*: $(\mathit{if} \; True \; \mathit{then} \; x \; \mathit{else} \; y) = x$
⟨*proof*⟩

lemma *if-False*: $(\mathit{if} \; False \; \mathit{then} \; x \; \mathit{else} \; y) = y$
⟨*proof*⟩

lemma *if-P*: $P ==> (\mathit{if} \; P \; \mathit{then} \; x \; \mathit{else} \; y) = x$
⟨*proof*⟩

lemma *if-not-P*: $\verb|~| P ==> (\mathit{if} \; P \; \mathit{then} \; x \; \mathit{else} \; y) = y$
⟨*proof*⟩

lemma *split-if*: $P \; (\mathit{if} \; Q \; \mathit{then} \; x \; \mathit{else} \; y) = ((Q \; --> \; P(x)) \; \& \; (\verb|~| Q \; --> \; P(y)))$
⟨*proof*⟩

lemma *split-if-asm*: P (*if* Q *then* x *else* y) = (~((Q & ~P x) | (~Q & ~P y)))
⟨*proof*⟩

lemmas *if-splits* [*noatp*] = *split-if split-if-asm*

lemma *if-cancel*: (*if* c *then* x *else* x) = x
⟨*proof*⟩

lemma *if-eq-cancel*: (*if* $x = y$ *then* y *else* x) = x
⟨*proof*⟩

lemma *if-bool-eq-conj*: (*if* P *then* Q *else* R) = ((P-->Q) & (~P-->R))
— This form is useful for expanding *if*s on the RIGHT of the ==> symbol.
⟨*proof*⟩

lemma *if-bool-eq-disj*: (*if* P *then* Q *else* R) = ((P&Q) | (~P&R))
— And this form is useful for expanding *if*s on the LEFT.
⟨*proof*⟩

lemma *Eq-TrueI*: P ==> P == *True* ⟨*proof*⟩
lemma *Eq-FalseI*: ~P ==> P == *False* ⟨*proof*⟩

let rules for simproc

lemma *Let-folded*: $f\ x \equiv g\ x \Longrightarrow$ *Let* $x\ f \equiv$ *Let* $x\ g$
⟨*proof*⟩

lemma *Let-unfold*: $f\ x \equiv g \Longrightarrow$ *Let* $x\ f \equiv g$
⟨*proof*⟩

The following copy of the implication operator is useful for fine-tuning congruence rules. It instructs the simplifier to simplify its premise.

constdefs
simp-implies :: [*prop*, *prop*] => *prop* (**infixr** =*simp*=> *1*)
simp-implies ≡ *op* ==>

lemma *simp-impliesI*:
assumes *PQ*: (*PROP P* ⟹ *PROP Q*)
shows *PROP P* =*simp*=> *PROP Q*
⟨*proof*⟩

lemma *simp-impliesE*:
assumes *PQ*: *PROP P* =*simp*=> *PROP Q*
and *P*: *PROP P*
and *QR*: *PROP Q* ⟹ *PROP R*
shows *PROP R*
⟨*proof*⟩

lemma *simp-implies-cong*:

assumes PP' :*PROP P == PROP P'*
and $P'QQ'$: *PROP P' ==> (PROP Q == PROP Q')*
shows *(PROP P =simp=> PROP Q) == (PROP P' =simp=> PROP Q')*
⟨*proof*⟩

lemma *uncurry*:
assumes $P \longrightarrow Q \longrightarrow R$
shows $P \wedge Q \longrightarrow R$
⟨*proof*⟩

lemma *iff-allI*:
assumes $\bigwedge x.\ P\ x = Q\ x$
shows $(\forall x.\ P\ x) = (\forall x.\ Q\ x)$
⟨*proof*⟩

lemma *iff-exI*:
assumes $\bigwedge x.\ P\ x = Q\ x$
shows $(\exists x.\ P\ x) = (\exists x.\ Q\ x)$
⟨*proof*⟩

lemma *all-comm*:
$(\forall x\ y.\ P\ x\ y) = (\forall y\ x.\ P\ x\ y)$
⟨*proof*⟩

lemma *ex-comm*:
$(\exists x\ y.\ P\ x\ y) = (\exists y\ x.\ P\ x\ y)$
⟨*proof*⟩

⟨*ML*⟩

Simproc for proving $(y = x) == False$ from premise $\sim(x = y)$:

⟨*ML*⟩

lemma *True-implies-equals*: $(True \Longrightarrow PROP\ P) \equiv PROP\ P$
⟨*proof*⟩

lemma *ex-simps*:
!!*P Q*. *(EX x. P x & Q) = ((EX x. P x) & Q)*
!!*P Q*. *(EX x. P & Q x) = (P & (EX x. Q x))*
!!*P Q*. *(EX x. P x | Q) = ((EX x. P x) | Q)*
!!*P Q*. *(EX x. P | Q x) = (P | (EX x. Q x))*
!!*P Q*. *(EX x. P x --> Q) = ((ALL x. P x) --> Q)*
!!*P Q*. *(EX x. P --> Q x) = (P --> (EX x. Q x))*
— Miniscoping: pushing in existential quantifiers.
⟨*proof*⟩

lemma *all-simps*:
!!*P Q*. *(ALL x. P x & Q) = ((ALL x. P x) & Q)*

$!!P\ Q.\ (ALL\ x.\ P\ \&\ Q\ x)\ \ = (P\ \&\ (ALL\ x.\ Q\ x))$
$!!P\ Q.\ (ALL\ x.\ P\ x\ |\ Q)\ \ = ((ALL\ x.\ P\ x)\ |\ Q)$
$!!P\ Q.\ (ALL\ x.\ P\ |\ Q\ x)\ \ = (P\ |\ (ALL\ x.\ Q\ x))$
$!!P\ Q.\ (ALL\ x.\ P\ x\ {-}{-}{>}\ Q) = ((EX\ x.\ P\ x)\ {-}{-}{>}\ Q)$
$!!P\ Q.\ (ALL\ x.\ P\ {-}{-}{>}\ Q\ x) = (P\ {-}{-}{>}\ (ALL\ x.\ Q\ x))$
— Miniscoping: pushing in universal quantifiers.
⟨*proof*⟩

lemmas [*simp*] =
triv-forall-equality
True-implies-equals
if-True
if-False
if-cancel
if-eq-cancel
imp-disjL

conj-assoc
disj-assoc
de-Morgan-conj
de-Morgan-disj
imp-disj1
imp-disj2
not-imp
disj-not1
not-all
not-ex
cases-simp
the-eq-trivial
the-sym-eq-trivial
ex-simps
all-simps
simp-thms

lemmas [*cong*] = *imp-cong simp-implies-cong*
lemmas [*split*] = *split-if*

⟨*ML*⟩

Simplifies x assuming c and y assuming c

lemma *if-cong*:
assumes $b = c$
and $c \Longrightarrow x = u$
and $\neg\ c \Longrightarrow y = v$
shows $(\mathit{if}\ b\ \mathit{then}\ x\ \mathit{else}\ y) = (\mathit{if}\ c\ \mathit{then}\ u\ \mathit{else}\ v)$
⟨*proof*⟩

Prevents simplification of x and y: faster and allows the execution of functional programs.

lemma *if-weak-cong* [*cong*]:
assumes $b = c$
shows (*if b then x else y*) = (*if c then x else y*)
⟨*proof*⟩

Prevents simplification of t: much faster

lemma *let-weak-cong*:
assumes $a = b$
shows (*let x = a in t x*) = (*let x = b in t x*)
⟨*proof*⟩

To tidy up the result of a simproc. Only the RHS will be simplified.

lemma *eq-cong2*:
assumes $u = u'$
shows $(t \equiv u) \equiv (t \equiv u')$
⟨*proof*⟩

lemma *if-distrib*:
f (*if c then x else y*) = (*if c then f x else f y*)
⟨*proof*⟩

This lemma restricts the effect of the rewrite rule u=v to the left-hand side of an equality. Used in {*Integ*,*Real*}/*simproc.ML*

lemma *restrict-to-left*:
assumes $x = y$
shows $(x = z) = (y = z)$
⟨*proof*⟩

1.3.3 Generic cases and induction

Rule projections:

⟨*ML*⟩

constdefs
induct-forall **where** *induct-forall P* == $\forall x.\ P\ x$
induct-implies **where** *induct-implies A B* == $A \longrightarrow B$
induct-equal **where** *induct-equal x y* == $x = y$
induct-conj **where** *induct-conj A B* == $A \wedge B$

lemma *induct-forall-eq*: (!!x. $P\ x$) == *Trueprop* (*induct-forall* ($\lambda x.\ P\ x$))
⟨*proof*⟩

lemma *induct-implies-eq*: (*A* ==> *B*) == *Trueprop* (*induct-implies A B*)
⟨*proof*⟩

lemma *induct-equal-eq*: (*x* == *y*) == *Trueprop* (*induct-equal x y*)
⟨*proof*⟩

lemma *induct-conj-eq*:
  includes *meta-conjunction-syntax*
  shows $(A \&\& B) == Trueprop\ (induct\text{-}conj\ A\ B)$
  ⟨*proof*⟩

lemmas *induct-atomize* = *induct-forall-eq induct-implies-eq induct-equal-eq induct-conj-eq*
lemmas *induct-rulify* [*symmetric*, *standard*] = *induct-atomize*
lemmas *induct-rulify-fallback* =
  induct-forall-def induct-implies-def induct-equal-def induct-conj-def

lemma *induct-forall-conj*: $induct\text{-}forall\ (\lambda x.\ induct\text{-}conj\ (A\ x)\ (B\ x)) =$
    $induct\text{-}conj\ (induct\text{-}forall\ A)\ (induct\text{-}forall\ B)$
  ⟨*proof*⟩

lemma *induct-implies-conj*: $induct\text{-}implies\ C\ (induct\text{-}conj\ A\ B) =$
    $induct\text{-}conj\ (induct\text{-}implies\ C\ A)\ (induct\text{-}implies\ C\ B)$
  ⟨*proof*⟩

lemma *induct-conj-curry*: $(induct\text{-}conj\ A\ B ==> PROP\ C) == (A ==> B ==>$
$PROP\ C)$
⟨*proof*⟩

lemmas *induct-conj* = *induct-forall-conj induct-implies-conj induct-conj-curry*

hide *const induct-forall induct-implies induct-equal induct-conj*

Method setup.

⟨*ML*⟩

1.4 Other simple lemmas and lemma duplicates

lemma *Let-0* [*simp*]: $Let\ 0\ f = f\ 0$
  ⟨*proof*⟩

lemma *Let-1* [*simp*]: $Let\ 1\ f = f\ 1$
  ⟨*proof*⟩

lemma *ex1-eq* [*iff*]: $EX!\ x.\ x = t\ EX!\ x.\ t = x$
  ⟨*proof*⟩

lemma *choice-eq*: $(ALL\ x.\ EX!\ y.\ P\ x\ y) = (EX!\ f.\ ALL\ x.\ P\ x\ (f\ x))$
  ⟨*proof*⟩

lemma *mk-left-commute*:
  fixes f (**infix** $\otimes$ *60*)
  assumes a: $\bigwedge x\ y\ z.\ (x \otimes y) \otimes z = x \otimes (y \otimes z)$ **and**
        c: $\bigwedge x\ y.\ x \otimes y = y \otimes x$
  shows $x \otimes (y \otimes z) = y \otimes (x \otimes z)$

⟨*proof*⟩

lemmas *eq-sym-conv* = *eq-commute*

lemma *nnf-simps*:
$(\neg(P \wedge Q)) = (\neg\ P \vee \neg\ Q)$ $(\neg\ (P \vee Q)) = (\neg\ P \wedge \neg Q)$ $(P \longrightarrow Q) = (\neg P \vee Q)$
$(P = Q) = ((P \wedge Q) \vee (\neg P \wedge \neg\ Q))$ $(\neg(P = Q)) = ((P \wedge \neg\ Q) \vee (\neg P \wedge Q))$
$(\neg\ \neg(P)) = P$
⟨*proof*⟩

1.5 Basic ML bindings

⟨*ML*⟩

1.6 Code generator basic setup – see further *Code-Setup.thy*

⟨*ML*⟩

class *eq* (**attach** *op* =) = *type*

code-datatype *True False*

lemma [*code func*]:
shows *False* ∧ *x* ⟷ *False*
and *True* ∧ *x* ⟷ *x*
and *x* ∧ *False* ⟷ *False*
and *x* ∧ *True* ⟷ *x* ⟨*proof*⟩

lemma [*code func*]:
shows *False* ∨ *x* ⟷ *x*
and *True* ∨ *x* ⟷ *True*
and *x* ∨ *False* ⟷ *x*
and *x* ∨ *True* ⟷ *True* ⟨*proof*⟩

lemma [*code func*]:
shows ¬ *True* ⟷ *False*
and ¬ *False* ⟷ *True* ⟨*proof*⟩

instance *bool* :: *eq* ⟨*proof*⟩

lemma [*code func*]:
shows *False* = *P* ⟷ ¬ *P*
and *True* = *P* ⟷ *P*
and *P* = *False* ⟷ ¬ *P*
and *P* = *True* ⟷ *P* ⟨*proof*⟩

code-datatype *Trueprop prop*

code-datatype $TYPE('a)$

lemma *Let-case-cert*:
  assumes *CASE* ≡ (λ*x*. *Let x f*)
  shows *CASE x* ≡ *f x*
  ⟨*proof*⟩

lemma *If-case-cert*:
  includes *meta-conjunction-syntax*
  assumes *CASE* ≡ (λ*b*. *If b f g*)
  shows (*CASE True* ≡ *f*) && (*CASE False* ≡ *g*)
  ⟨*proof*⟩

⟨*ML*⟩

1.7 Legacy tactics and ML bindings

⟨*ML*⟩

end

2 Code-Setup: Setup of code generators and derived tools

theory *Code-Setup*
imports *HOL*
uses ~~/*src*/*HOL*/*Tools*/*recfun-codegen.ML*
begin

2.1 SML code generator setup

⟨*ML*⟩

types-code
  bool (*bool*)
attach (*term-of*) ⟪
fun term-of-bool b = *if b then HOLogic.true-const else HOLogic.false-const*;
⟫
attach (*test*) ⟪
fun gen-bool i = *one-of* [*false*, *true*];
⟫
  prop (*bool*)
attach (*term-of*) ⟪
fun term-of-prop b =
  HOLogic.mk-Trueprop (*if b then HOLogic.true-const else HOLogic.false-const*);
⟫

consts-code
  Trueprop ((-))

True (*true*)
False (*false*)
Not (*Bool.not*)
op | ((- *orelse*/ -))
op & ((- *andalso*/ -))
If ((*if* -/ *then* -/ *else* -))

⟨*ML*⟩

quickcheck-params [*size* = *5*, *iterations* = *50*]

Evaluation

⟨*ML*⟩

2.2 Generic code generator setup

using built-in Haskell equality

code-class *eq*
(*Haskell Eq* **where** *op* = ≡ (==))

code-const *op* =
(*Haskell* **infixl** *4* ==)

type bool

lemmas [*code*] = *imp-conv-disj*

code-type *bool*
(*SML bool*)
(*OCaml bool*)
(*Haskell Bool*)

code-instance *bool* :: *eq*
(*Haskell* −)

code-const *op* = :: *bool* ⇒ *bool* ⇒ *bool*
(*Haskell* **infixl** *4* ==)

code-const *True* **and** *False* **and** *Not* **and** *op* & **and** *op* | **and** *If*
(*SML true* **and** *false* **and** *not*
and infixl *1 andalso* **and infixl** *0 orelse*
and !(*if* (-)/ *then* (-)/ *else* (-)))
(*OCaml true* **and** *false* **and** *not*
and infixl *4* && **and infixl** *2* ||
and !(*if* (-)/ *then* (-)/ *else* (-)))
(*Haskell True* **and** *False* **and** *not*
and infixl *3* && **and infixl** *2* ||
and !(*if* (-)/ *then* (-)/ *else* (-)))

code-reserved *SML*
bool true false not

code-reserved *OCaml*
bool not

code generation for undefined as exception

code-const *undefined*
(*SML raise/ Fail/ undefined*)
(*OCaml failwith/ undefined*)
(*Haskell error/ undefined*)

Let and If

lemmas [*code func*] = *Let-def if-True if-False*

2.3 Evaluation oracle

⟨*ML*⟩

2.4 Normalization by evaluation

⟨*ML*⟩

end

3 Set: Set theory for higher-order logic

theory *Set*
imports *Code-Setup*
begin

A set in HOL is simply a predicate.

3.1 Basic syntax

global

typedecl *'a set*
arities *set* :: (*type*) *type*

consts
{} :: *'a set* ({})
UNIV :: *'a set*
insert :: *'a => 'a set => 'a set*
Collect :: (*'a => bool*) => *'a set* — comprehension
op Int :: *'a set => 'a set => 'a set* (**infixl** *Int 70*)
op Un :: *'a set => 'a set => 'a set* (**infixl** *Un 65*)

UNION :: *'a set => ('a => 'b set) => 'b set* — general union
INTER :: *'a set => ('a => 'b set) => 'b set* — general intersection
Union :: *'a set set => 'a set* — union of a set
Inter :: *'a set set => 'a set* — intersection of a set
Pow :: *'a set => 'a set set* — powerset
Ball :: *'a set => ('a => bool) => bool* — bounded universal quantifiers

Bex :: *'a set => ('a => bool) => bool* — bounded existential quantifiers
Bex1 :: *'a set => ('a => bool) => bool* — bounded unique existential quantifiers
image :: *('a => 'b) => 'a set => 'b set* (**infixr** ' *90*)
op : :: *'a => 'a set => bool* — membership

notation
op : (*op* :) **and**
op : ((-/ : -) [*50*, *51*] *50*)

local

3.2 Additional concrete syntax

abbreviation
range :: (*'a => 'b*) => *'b set* **where** — of function
range f == *f* ' *UNIV*

abbreviation
not-mem x A == ~ (*x* : *A*) — non-membership

notation
not-mem (*op* ~:) **and**
not-mem ((-/ ~: -) [*50*, *51*] *50*)

notation (*xsymbols*)
op Int (**infixl** ∩ *70*) **and**
op Un (**infixl** ∪ *65*) **and**
op : (*op* ∈) **and**
op : ((-/ ∈ -) [*50*, *51*] *50*) **and**
not-mem (*op* ∉) **and**
not-mem ((-/ ∉ -) [*50*, *51*] *50*) **and**
Union (⋃ - [*90*] *90*) **and**
Inter (⋂ - [*90*] *90*)

notation (*HTML* **output**)
op Int (**infixl** ∩ *70*) **and**
op Un (**infixl** ∪ *65*) **and**
op : (*op* ∈) **and**
op : ((-/ ∈ -) [*50*, *51*] *50*) **and**
not-mem (*op* ∉) **and**

not-mem ((-/ ∉ -) [50, 51] 50)

syntax
@*Finset* :: *args* => '*a set* ({(-)})
@*Coll* :: *pttrn* => *bool* => '*a set* ((1{-./ -}))
@*SetCompr* :: '*a* => *idts* => *bool* => '*a set* ((1{- |/-./ -}))
@*Collect* :: *idt* => '*a set* => *bool* => '*a set* ((1{- :/ -./ -}))
@*INTER1* :: *pttrns* => '*b set* => '*b set* ((3INT -./ -) [0, 10] 10)
@*UNION1* :: *pttrns* => '*b set* => '*b set* ((3UN -./ -) [0, 10] 10)
@*INTER* :: *pttrn* => '*a set* => '*b set* => '*b set* ((3INT -:-./ -) [0, 10] 10)
@*UNION* :: *pttrn* => '*a set* => '*b set* => '*b set* ((3UN -:-./ -) [0, 10] 10)
-*Ball* :: *pttrn* => '*a set* => *bool* => *bool* ((3ALL -:-./ -) [0, 0, 10] 10)
-*Bex* :: *pttrn* => '*a set* => *bool* => *bool* ((3EX -:-./ -) [0, 0, 10] 10)
-*Bex1* :: *pttrn* => '*a set* => *bool* => *bool* ((3EX! -:-./ -) [0, 0, 10] 10)
-*Bleast* :: *id* => '*a set* => *bool* => '*a* ((3LEAST -:-./ -) [0, 0, 10] 10)

syntax (*HOL*)
-*Ball* :: *pttrn* => '*a set* => *bool* => *bool* ((3! -:-./ -) [0, 0, 10] 10)
-*Bex* :: *pttrn* => '*a set* => *bool* => *bool* ((3? -:-./ -) [0, 0, 10] 10)
-*Bex1* :: *pttrn* => '*a set* => *bool* => *bool* ((3?! -:-./ -) [0, 0, 10] 10)

translations
{*x*, *xs*} == *insert x* {*xs*}
{*x*} == *insert x* {}
{*x*. *P*} == *Collect* (%*x*. *P*)
{*x*:*A*. *P*} => {*x*. *x*:*A* & *P*}
UN x y. *B* == *UN x*. *UN y*. *B*
UN x. *B* == *UNION UNIV* (%*x*. *B*)
UN x. *B* == *UN x*:*UNIV*. *B*
INT x y. *B* == *INT x*. *INT y*. *B*
INT x. *B* == *INTER UNIV* (%*x*. *B*)
INT x. *B* == *INT x*:*UNIV*. *B*
UN x:*A*. *B* == *UNION A* (%*x*. *B*)
INT x:*A*. *B* == *INTER A* (%*x*. *B*)
ALL x:*A*. *P* == *Ball A* (%*x*. *P*)
EX x:*A*. *P* == *Bex A* (%*x*. *P*)
EX! *x*:*A*. *P* == *Bex1 A* (%*x*. *P*)
LEAST x:*A*. *P* => *LEAST x*. *x*:*A* & *P*

syntax (*xsymbols*)
-*Ball* :: *pttrn* => '*a set* => *bool* => *bool* ((3∀ -∈-./ -) [0, 0, 10] 10)
-*Bex* :: *pttrn* => '*a set* => *bool* => *bool* ((3∃ -∈-./ -) [0, 0, 10] 10)
-*Bex1* :: *pttrn* => '*a set* => *bool* => *bool* ((3∃!-∈-./ -) [0, 0, 10] 10)
-*Bleast* :: *id* => '*a set* => *bool* => '*a* ((3LEAST-∈-./ -) [0, 0, 10] 10)

syntax (*HTML* **output**)
-*Ball* :: *pttrn* => '*a set* => *bool* => *bool* ((3∀ -∈-./ -) [0, 0, 10] 10)

$-Bex$ $::$ $pttrn$ $=>$ $'a\ set$ $=>$ $bool$ $=>$ $bool$ $((3\exists\ \text{-}\in\text{-}./\ \text{-})\ [0,\ 0,\ 10]\ 10)$
$-Bex1$ $::$ $pttrn$ $=>$ $'a\ set$ $=>$ $bool$ $=>$ $bool$ $((3\exists!\text{-}\in\text{-}./\ \text{-})\ [0,\ 0,\ 10]\ 10)$

syntax (*xsymbols*)
@*Collect* $::$ idt $=>$ $'a\ set$ $=>$ $bool$ $=>$ $'a\ set$ $((1\{\text{-}\ \in/\ \text{-}./\ \text{-}\}))$
@*UNION1* $::$ $pttrns$ $=>$ $'b\ set$ $=>$ $'b\ set$ $((3\bigcup\text{-}./\ \text{-})\ [0,\ 10]\ 10)$
@*INTER1* $::$ $pttrns$ $=>$ $'b\ set$ $=>$ $'b\ set$ $((3\bigcap\text{-}./\ \text{-})\ [0,\ 10]\ 10)$
@*UNION* $::$ $pttrn$ $=>$ $'a\ set$ $=>$ $'b\ set$ $=>$ $'b\ set$ $((3\bigcup\text{-}\in\text{-}./\ \text{-})\ [0,\ 10]\ 10)$
@*INTER* $::$ $pttrn$ $=>$ $'a\ set$ $=>$ $'b\ set$ $=>$ $'b\ set$ $((3\bigcap\text{-}\in\text{-}./\ \text{-})\ [0,\ 10]\ 10)$

syntax (*latex* **output**)
@*UNION1* $::$ $pttrns$ $=>$ $'b\ set$ $=>$ $'b\ set$ $((3\bigcup(00\text{-})/\ \text{-})\ [0,\ 10]\ 10)$
@*INTER1* $::$ $pttrns$ $=>$ $'b\ set$ $=>$ $'b\ set$ $((3\bigcap(00\text{-})/\ \text{-})\ [0,\ 10]\ 10)$
@*UNION* $::$ $pttrn$ $=>$ $'a\ set$ $=>$ $'b\ set$ $=>$ $'b\ set$ $((3\bigcup(00_{\text{-}\in\text{-}})/\ \text{-})\ [0,\ 10]$
$10)$
@*INTER* $::$ $pttrn$ $=>$ $'a\ set$ $=>$ $'b\ set$ $=>$ $'b\ set$ $((3\bigcap(00_{\text{-}\in\text{-}})/\ \text{-})\ [0,\ 10]$
$10)$

Note the difference between ordinary xsymbol syntax of indexed unions and intersections (e.g. $\bigcup a_1{\in}A_1.\ B$) and their LaTeX rendition: $\bigcup_{a_1\in A_1} B$. The former does not make the index expression a subscript of the union/intersection symbol because this leads to problems with nested subscripts in Proof General.

instance *set* :: (*type*) *ord*
subset-def: $A \le B \equiv \forall x{\in}A.\ x \in B$
psubset-def: $A < B \equiv A \le B \wedge A \neq B$ ⟨*proof*⟩
lemmas [*code func del*] = *subset-def psubset-def*

abbreviation
subset :: $'a\ set \Rightarrow 'a\ set \Rightarrow bool$ **where**
subset ≡ *less*

abbreviation
subset-eq :: $'a\ set \Rightarrow 'a\ set \Rightarrow bool$ **where**
subset-eq ≡ *less-eq*

notation (**output**)
subset (*op* <) **and**
subset $((\text{-}/ < \text{-})\ [50,\ 51]\ 50)$ **and**
subset-eq (*op* <=) **and**
subset-eq $((\text{-}/ <= \text{-})\ [50,\ 51]\ 50)$

notation (*xsymbols*)
subset (*op* ⊂) **and**
subset $((\text{-}/ \subset \text{-})\ [50,\ 51]\ 50)$ **and**
subset-eq (*op* ⊆) **and**
subset-eq $((\text{-}/ \subseteq \text{-})\ [50,\ 51]\ 50)$

notation (*HTML* **output**)

subset (*op* ⊂) **and**
subset ((-/ ⊂ -) [*50*, *51*] *50*) **and**
subset-eq (*op* ⊆) **and**
subset-eq ((-/ ⊆ -) [*50*, *51*] *50*)

abbreviation (*input*)
supset :: *'a set* ⇒ *'a set* ⇒ *bool* **where**
supset ≡ *greater*

abbreviation (*input*)
supset-eq :: *'a set* ⇒ *'a set* ⇒ *bool* **where**
supset-eq ≡ *greater-eq*

notation (*xsymbols*)
supset (*op* ⊃) **and**
supset ((-/ ⊃ -) [*50*, *51*] *50*) **and**
supset-eq (*op* ⊇) **and**
supset-eq ((-/ ⊇ -) [*50*, *51*] *50*)

3.2.1 Bounded quantifiers

syntax (**output**)
-setlessAll :: [*idt*, *'a*, *bool*] => *bool* ((*3ALL* -<-./ -) [*0*, *0*, *10*] *10*)
-setlessEx :: [*idt*, *'a*, *bool*] => *bool* ((*3EX* -<-./ -) [*0*, *0*, *10*] *10*)
-setleAll :: [*idt*, *'a*, *bool*] => *bool* ((*3ALL* -<=-./ -) [*0*, *0*, *10*] *10*)
-setleEx :: [*idt*, *'a*, *bool*] => *bool* ((*3EX* -<=-./ -) [*0*, *0*, *10*] *10*)
-setleEx1 :: [*idt*, *'a*, *bool*] => *bool* ((*3EX*! -<=-./ -) [*0*, *0*, *10*] *10*)

syntax (*xsymbols*)
-setlessAll :: [*idt*, *'a*, *bool*] => *bool* ((*3*∀ -⊂-./ -) [*0*, *0*, *10*] *10*)
-setlessEx :: [*idt*, *'a*, *bool*] => *bool* ((*3*∃ -⊂-./ -) [*0*, *0*, *10*] *10*)
-setleAll :: [*idt*, *'a*, *bool*] => *bool* ((*3*∀ -⊆-./ -) [*0*, *0*, *10*] *10*)
-setleEx :: [*idt*, *'a*, *bool*] => *bool* ((*3*∃ -⊆-./ -) [*0*, *0*, *10*] *10*)
-setleEx1 :: [*idt*, *'a*, *bool*] => *bool* ((*3*∃!-⊆-./ -) [*0*, *0*, *10*] *10*)

syntax (*HOL* **output**)
-setlessAll :: [*idt*, *'a*, *bool*] => *bool* ((*3*! -<-./ -) [*0*, *0*, *10*] *10*)
-setlessEx :: [*idt*, *'a*, *bool*] => *bool* ((*3?* -<-./ -) [*0*, *0*, *10*] *10*)
-setleAll :: [*idt*, *'a*, *bool*] => *bool* ((*3*! -<=-./ -) [*0*, *0*, *10*] *10*)
-setleEx :: [*idt*, *'a*, *bool*] => *bool* ((*3?* -<=-./ -) [*0*, *0*, *10*] *10*)
-setleEx1 :: [*idt*, *'a*, *bool*] => *bool* ((*3?*! -<=-./ -) [*0*, *0*, *10*] *10*)

syntax (*HTML* **output**)
-setlessAll :: [*idt*, *'a*, *bool*] => *bool* ((*3*∀ -⊂-./ -) [*0*, *0*, *10*] *10*)
-setlessEx :: [*idt*, *'a*, *bool*] => *bool* ((*3*∃ -⊂-./ -) [*0*, *0*, *10*] *10*)
-setleAll :: [*idt*, *'a*, *bool*] => *bool* ((*3*∀ -⊆-./ -) [*0*, *0*, *10*] *10*)
-setleEx :: [*idt*, *'a*, *bool*] => *bool* ((*3*∃ -⊆-./ -) [*0*, *0*, *10*] *10*)
-setleEx1 :: [*idt*, *'a*, *bool*] => *bool* ((*3*∃!-⊆-./ -) [*0*, *0*, *10*] *10*)

translations
$\forall A{\subset}B.\ P$ => $ALL\ A.\ A \subset B$ --> P
$\exists A{\subset}B.\ P$ => $EX\ A.\ A \subset B\ \&\ P$
$\forall A{\subseteq}B.\ P$ => $ALL\ A.\ A \subseteq B$ --> P
$\exists A{\subseteq}B.\ P$ => $EX\ A.\ A \subseteq B\ \&\ P$
$\exists!A{\subseteq}B.\ P$ => $EX!\ A.\ A \subseteq B\ \&\ P$

⟨*ML*⟩

Translate between $\{e \mid x1...xn.\ P\}$ and $\{u.\ EX\ x1..xn.\ u = e\ \&\ P\}$; $\{y.\ EX\ x1..xn.\ y = e\ \&\ P\}$ is only translated if $[0..n]$ *subset bvs*(e).

⟨*ML*⟩

3.3 Rules and definitions

Isomorphisms between predicates and sets.

axioms
mem-Collect-eq: $(a : \{x.\ P(x)\}) = P(a)$
Collect-mem-eq: $\{x.\ x{:}A\} = A$
finalconsts
Collect
op :

defs
Ball-def: $Ball\ A\ P$ == $ALL\ x.\ x{:}A$ --> $P(x)$
Bex-def: $Bex\ A\ P$ == $EX\ x.\ x{:}A\ \&\ P(x)$
Bex1-def: $Bex1\ A\ P$ == $EX!\ x.\ x{:}A\ \&\ P(x)$

instance *set* :: (*type*) *minus*
Compl-def: $-\ A$ == $\{x.\ \tilde{}x{:}A\}$
set-diff-def: $A - B$ == $\{x.\ x{:}A\ \&\ \tilde{}x{:}B\}$ ⟨*proof*⟩

lemmas [*code func del*] = *Compl-def set-diff-def*

defs
Un-def: $A\ Un\ B$ == $\{x.\ x{:}A \mid x{:}B\}$
Int-def: $A\ Int\ B$ == $\{x.\ x{:}A\ \&\ x{:}B\}$
INTER-def: $INTER\ A\ B$ == $\{y.\ ALL\ x{:}A.\ y{:}\ B(x)\}$
UNION-def: $UNION\ A\ B$ == $\{y.\ EX\ x{:}A.\ y{:}\ B(x)\}$
Inter-def: $Inter\ S$ == $(INT\ x{:}S.\ x)$
Union-def: $Union\ S$ == $(UN\ x{:}S.\ x)$
Pow-def: $Pow\ A$ == $\{B.\ B <= A\}$
empty-def: $\{\}$ == $\{x.\ False\}$
UNIV-def: $UNIV$ == $\{x.\ True\}$
insert-def: $insert\ a\ B$ == $\{x.\ x{=}a\}\ Un\ B$
image-def: $f`A$ == $\{y.\ EX\ x{:}A.\ y = f(x)\}$

3.4 Lemmas and proof tool setup

3.4.1 Relating predicates and sets

declare *mem-Collect-eq* [*iff*] *Collect-mem-eq* [*simp*]

lemma *CollectI*: $P(a)$ ==> a : {x. $P(x)$}
⟨*proof*⟩

lemma *CollectD*: a : {x. $P(x)$} ==> $P(a)$
⟨*proof*⟩

lemma *Collect-cong*: (!!x. P x = Q x) ==> {x. $P(x)$} = {x. $Q(x)$}
⟨*proof*⟩

lemmas *CollectE* = *CollectD* [*elim-format*]

3.4.2 Bounded quantifiers

lemma *ballI* [*intro*!]: (!!x. x:A ==> P x) ==> *ALL* x:A. P x
⟨*proof*⟩

lemmas *strip* = *impI allI ballI*

lemma *bspec* [*dest?*]: *ALL* x:A. P x ==> x:A ==> P x
⟨*proof*⟩

lemma *ballE* [*elim*]: *ALL* x:A. P x ==> (P x ==> Q) ==> (x ~: A ==> Q) ==> Q
⟨*proof*⟩

⟨*ML*⟩

This tactic takes assumptions $\forall x \in A.\ P\ x$ and $a \in A$; creates assumption P a.

⟨*ML*⟩

Gives better instantiation for bound:

⟨*ML*⟩

lemma *bexI* [*intro*]: P x ==> x:A ==> *EX* x:A. P x
— Normally the best argument order: P x constrains the choice of $x \in A$.
⟨*proof*⟩

lemma *rev-bexI* [*intro?*]: x:A ==> P x ==> *EX* x:A. P x
— The best argument order when there is only one $x \in A$.
⟨*proof*⟩

lemma *bexCI*: (*ALL* x:A. ~P x ==> P a) ==> a:A ==> *EX* x:A. P x
⟨*proof*⟩

lemma *bexE* [*elim*!]: *EX x*:*A*. *P x* ==> (!!*x*. *x*:*A* ==> *P x* ==> *Q*) ==> *Q*
⟨*proof*⟩

lemma *ball-triv* [*simp*]: (*ALL x*:*A*. *P*) = ((*EX x*. *x*:*A*) --> *P*)
— Trival rewrite rule.
⟨*proof*⟩

lemma *bex-triv* [*simp*]: (*EX x*:*A*. *P*) = ((*EX x*. *x*:*A*) & *P*)
— Dual form for existentials.
⟨*proof*⟩

lemma *bex-triv-one-point1* [*simp*]: (*EX x*:*A*. *x* = *a*) = (*a*:*A*)
⟨*proof*⟩

lemma *bex-triv-one-point2* [*simp*]: (*EX x*:*A*. *a* = *x*) = (*a*:*A*)
⟨*proof*⟩

lemma *bex-one-point1* [*simp*]: (*EX x*:*A*. *x* = *a* & *P x*) = (*a*:*A* & *P a*)
⟨*proof*⟩

lemma *bex-one-point2* [*simp*]: (*EX x*:*A*. *a* = *x* & *P x*) = (*a*:*A* & *P a*)
⟨*proof*⟩

lemma *ball-one-point1* [*simp*]: (*ALL x*:*A*. *x* = *a* --> *P x*) = (*a*:*A* --> *P a*)
⟨*proof*⟩

lemma *ball-one-point2* [*simp*]: (*ALL x*:*A*. *a* = *x* --> *P x*) = (*a*:*A* --> *P a*)
⟨*proof*⟩

⟨*ML*⟩

3.4.3 Congruence rules

lemma *ball-cong*:
A = *B* ==> (!!*x*. *x*:*B* ==> *P x* = *Q x*) ==>
(*ALL x*:*A*. *P x*) = (*ALL x*:*B*. *Q x*)
⟨*proof*⟩

lemma *strong-ball-cong* [*cong*]:
A = *B* ==> (!!*x*. *x*:*B* =*simp*=> *P x* = *Q x*) ==>
(*ALL x*:*A*. *P x*) = (*ALL x*:*B*. *Q x*)
⟨*proof*⟩

lemma *bex-cong*:
A = *B* ==> (!!*x*. *x*:*B* ==> *P x* = *Q x*) ==>
(*EX x*:*A*. *P x*) = (*EX x*:*B*. *Q x*)
⟨*proof*⟩

lemma *strong-bex-cong* [*cong*]:
$A = B ==> (!!x.\ x{:}B =simp=> P\ x = Q\ x) ==>$
$(EX\ x{:}A.\ P\ x) = (EX\ x{:}B.\ Q\ x)$
⟨*proof*⟩

3.4.4 Subsets

lemma *subsetI* [*atp*,*intro*!]: $(!!x.\ x{:}A ==> x{:}B) ==> A \subseteq B$
⟨*proof*⟩

Map the type *'a set => anything* to just *'a*; for overloading constants whose first argument has type *'a set*.

lemma *subsetD* [*elim*]: $A \subseteq B ==> c \in A ==> c \in B$
— Rule in Modus Ponens style.
⟨*proof*⟩

declare *subsetD* [*intro?*] — FIXME

lemma *rev-subsetD*: $c \in A ==> A \subseteq B ==> c \in B$
— The same, with reversed premises for use with *erule* – cf *rev-mp*.
⟨*proof*⟩

declare *rev-subsetD* [*intro?*] — FIXME

Converts $A \subseteq B$ to $x \in A \Longrightarrow x \in B$.

⟨*ML*⟩

lemma *subsetCE* [*elim*]: $A \subseteq\ B ==> (c \notin A ==> P) ==> (c \in B ==> P) ==> P$
— Classical elimination rule.
⟨*proof*⟩

Takes assumptions $A \subseteq B$; $c \in A$ and creates the assumption $c \in B$.

⟨*ML*⟩

lemma *contra-subsetD*: $A \subseteq B ==> c \notin B ==> c \notin A$
⟨*proof*⟩

lemma *subset-refl* [*simp*,*atp*]: $A \subseteq A$
⟨*proof*⟩

lemma *subset-trans*: $A \subseteq B ==> B \subseteq C ==> A \subseteq C$
⟨*proof*⟩

3.4.5 Equality

lemma *set-ext*: **assumes** *prem*: $(!!x.\ (x{:}A) = (x{:}B))$ **shows** $A = B$

⟨*proof*⟩

lemma *expand-set-eq*: $(A = B) = (ALL\ x.\ (x{:}A) = (x{:}B))$
⟨*proof*⟩

lemma *subset-antisym* [*intro*!]: $A \subseteq B ==> B \subseteq A ==> A = B$
— Anti-symmetry of the subset relation.
⟨*proof*⟩

lemmas *equalityI* [*intro*!] = *subset-antisym*

Equality rules from ZF set theory – are they appropriate here?

lemma *equalityD1*: $A = B ==> A \subseteq B$
⟨*proof*⟩

lemma *equalityD2*: $A = B ==> B \subseteq A$
⟨*proof*⟩

Be careful when adding this to the claset as *subset-empty* is in the simpset: $A = \{\}$ goes to $\{\} \subseteq A$ and $A \subseteq \{\}$ and then back to $A = \{\}$!

lemma *equalityE*: $A = B ==> (A \subseteq B ==> B \subseteq A ==> P) ==> P$
⟨*proof*⟩

lemma *equalityCE* [*elim*]:
$A = B ==> (c \in A ==> c \in B ==> P) ==> (c \notin A ==> c \notin B ==> P) ==> P$
⟨*proof*⟩

lemma *eqset-imp-iff*: $A = B ==> (x : A) = (x : B)$
⟨*proof*⟩

lemma *eqelem-imp-iff*: $x = y ==> (x : A) = (y : A)$
⟨*proof*⟩

3.4.6 The universal set – UNIV

lemma *UNIV-I* [*simp*]: $x : UNIV$
⟨*proof*⟩

declare *UNIV-I* [*intro*] — unsafe makes it less likely to cause problems

lemma *UNIV-witness* [*intro?*]: $EX\ x.\ x : UNIV$
⟨*proof*⟩

lemma *subset-UNIV* [*simp*]: $A \subseteq UNIV$
⟨*proof*⟩

Eta-contracting these two rules (to remove P) causes them to be ignored because of their interaction with congruence rules.

lemma *ball-UNIV* [*simp*]: $Ball\ UNIV\ P = All\ P$
⟨*proof*⟩

lemma *bex-UNIV* [*simp*]: $Bex\ UNIV\ P = Ex\ P$
⟨*proof*⟩

3.4.7 The empty set

lemma *empty-iff* [*simp*]: $(c : \{\}) = False$
⟨*proof*⟩

lemma *emptyE* [*elim!*]: $a : \{\} ==> P$
⟨*proof*⟩

lemma *empty-subsetI* [*iff*]: $\{\} \subseteq A$
— One effect is to delete the ASSUMPTION $\{\} \subseteq A$
⟨*proof*⟩

lemma *equals0I*: $(!!y.\ y \in A ==> False) ==> A = \{\}$
⟨*proof*⟩

lemma *equals0D*: $A = \{\} ==> a \notin A$
— Use for reasoning about disjointness: $A \cap B = \{\}$
⟨*proof*⟩

lemma *ball-empty* [*simp*]: $Ball\ \{\}\ P = True$
⟨*proof*⟩

lemma *bex-empty* [*simp*]: $Bex\ \{\}\ P = False$
⟨*proof*⟩

lemma *UNIV-not-empty* [*iff*]: $UNIV\ \tilde{}{=}\ \{\}$
⟨*proof*⟩

3.4.8 The Powerset operator – Pow

lemma *Pow-iff* [*iff*]: $(A \in Pow\ B) = (A \subseteq B)$
⟨*proof*⟩

lemma *PowI*: $A \subseteq B ==> A \in Pow\ B$
⟨*proof*⟩

lemma *PowD*: $A \in Pow\ B ==> A \subseteq B$
⟨*proof*⟩

lemma *Pow-bottom*: $\{\} \in Pow\ B$
⟨*proof*⟩

lemma *Pow-top*: $A \in Pow\ A$
⟨*proof*⟩

3.4.9 Set complement

lemma *Compl-iff* [*simp*]: $(c \in -A) = (c \notin A)$
⟨*proof*⟩

lemma *ComplI* [*intro*!]: $(c \in A ==> False) ==> c \in -A$
⟨*proof*⟩

This form, with negated conclusion, works well with the Classical prover. Negated assumptions behave like formulae on the right side of the notional turnstile ...

lemma *ComplD* [*dest*!]: $c : -A ==> c^{\sim}{:}A$
⟨*proof*⟩

lemmas *ComplE* = *ComplD* [*elim-format*]

3.4.10 Binary union – Un

lemma *Un-iff* [*simp*]: $(c : A\ Un\ B) = (c{:}A \mid c{:}B)$
⟨*proof*⟩

lemma *UnI1* [*elim?*]: $c{:}A ==> c : A\ Un\ B$
⟨*proof*⟩

lemma *UnI2* [*elim?*]: $c{:}B ==> c : A\ Un\ B$
⟨*proof*⟩

Classical introduction rule: no commitment to A vs B.

lemma *UnCI* [*intro*!]: $(c^{\sim}{:}B ==> c{:}A) ==> c : A\ Un\ B$
⟨*proof*⟩

lemma *UnE* [*elim*!]: $c : A\ Un\ B ==> (c{:}A ==> P) ==> (c{:}B ==> P) ==> P$
⟨*proof*⟩

3.4.11 Binary intersection – Int

lemma *Int-iff* [*simp*]: $(c : A\ Int\ B) = (c{:}A\ \&\ c{:}B)$
⟨*proof*⟩

lemma *IntI* [*intro*!]: $c{:}A ==> c{:}B ==> c : A\ Int\ B$
⟨*proof*⟩

lemma *IntD1*: $c : A\ Int\ B ==> c{:}A$

⟨*proof*⟩

lemma *IntD2*: c : A *Int* B ==> c:B
⟨*proof*⟩

lemma *IntE* [*elim*!]: c : A *Int* B ==> (c:A ==> c:B ==> P) ==> P
⟨*proof*⟩

3.4.12 Set difference

lemma *Diff-iff* [*simp*]: (c : A − B) = (c:A & c~:B)
⟨*proof*⟩

lemma *DiffI* [*intro*!]: c : A ==> c ~: B ==> c : A − B
⟨*proof*⟩

lemma *DiffD1*: c : A − B ==> c : A
⟨*proof*⟩

lemma *DiffD2*: c : A − B ==> c : B ==> P
⟨*proof*⟩

lemma *DiffE* [*elim*!]: c : A − B ==> (c:A ==> c~:B ==> P) ==> P
⟨*proof*⟩

3.4.13 Augmenting a set − insert

lemma *insert-iff* [*simp*]: (a : *insert* b A) = (a = b | a:A)
⟨*proof*⟩

lemma *insertI1*: a : *insert* a B
⟨*proof*⟩

lemma *insertI2*: a : B ==> a : *insert* b B
⟨*proof*⟩

lemma *insertE* [*elim*!]: a : *insert* b A ==> (a = b ==> P) ==> (a:A ==> P) ==> P
⟨*proof*⟩

lemma *insertCI* [*intro*!]: (a~:B ==> a = b) ==> a: *insert* b B
— Classical introduction rule.
⟨*proof*⟩

lemma *subset-insert-iff*: ($A \subseteq$ *insert* x B) = (*if* x:A *then* $A - \{x\} \subseteq B$ *else* $A \subseteq B$)
⟨*proof*⟩

lemma *set-insert*:
assumes $x \in A$

obtains B **where** $A = insert\ x\ B$ **and** $x \notin B$
⟨*proof*⟩

lemma *insert-ident*: x ~: A ==> x ~: B ==> ($insert\ x\ A = insert\ x\ B$) = ($A = B$)
⟨*proof*⟩

3.4.14 Singletons, using insert

lemma *singletonI* [*intro*!,*noatp*]: a : $\{a\}$
— Redundant? But unlike *insertCI*, it proves the subgoal immediately!
⟨*proof*⟩

lemma *singletonD* [*dest*!,*noatp*]: b : $\{a\}$ ==> $b = a$
⟨*proof*⟩

lemmas *singletonE* = *singletonD* [*elim-format*]

lemma *singleton-iff*: (b : $\{a\}$) = ($b = a$)
⟨*proof*⟩

lemma *singleton-inject* [*dest*!]: $\{a\} = \{b\}$ ==> $a = b$
⟨*proof*⟩

lemma *singleton-insert-inj-eq* [*iff*,*noatp*]:
($\{b\} = insert\ a\ A$) = ($a = b$ & $A \subseteq \{b\}$)
⟨*proof*⟩

lemma *singleton-insert-inj-eq'* [*iff*,*noatp*]:
($insert\ a\ A = \{b\}$) = ($a = b$ & $A \subseteq \{b\}$)
⟨*proof*⟩

lemma *subset-singletonD*: $A \subseteq \{x\}$ ==> $A = \{\}$ | $A = \{x\}$
⟨*proof*⟩

lemma *singleton-conv* [*simp*]: $\{x.\ x = a\} = \{a\}$
⟨*proof*⟩

lemma *singleton-conv2* [*simp*]: $\{x.\ a = x\} = \{a\}$
⟨*proof*⟩

lemma *diff-single-insert*: $A - \{x\} \subseteq B$ ==> $x \in A$ ==> $A \subseteq insert\ x\ B$
⟨*proof*⟩

lemma *doubleton-eq-iff*: ($\{a,b\} = \{c,d\}$) = ($a=c$ & $b=d$ | $a=d$ & $b=c$)
⟨*proof*⟩

3.4.15 Unions of families

UN x:A. B x is $\bigcup B$ ' *A*.

declare *UNION-def* [*noatp*]

lemma *UN-iff* [*simp*]: (*b*: (*UN x*:*A*. *B x*)) = (*EX x*:*A*. *b*: *B x*)
⟨*proof*⟩

lemma *UN-I* [*intro*]: *a*:*A* ==> *b*: *B a* ==> *b*: (*UN x*:*A*. *B x*)
— The order of the premises presupposes that *A* is rigid; *b* may be flexible.
⟨*proof*⟩

lemma *UN-E* [*elim*!]: *b* : (*UN x*:*A*. *B x*) ==> (!!*x*. *x*:*A* ==> *b*: *B x* ==> *R*) ==> *R*
⟨*proof*⟩

lemma *UN-cong* [*cong*]:
A = *B* ==> (!!*x*. *x*:*B* ==> *C x* = *D x*) ==> (*UN x*:*A*. *C x*) = (*UN x*:*B*. *D x*)
⟨*proof*⟩

3.4.16 Intersections of families

INT x:A. B x is $\bigcap B$ ' *A*.

lemma *INT-iff* [*simp*]: (*b*: (*INT x*:*A*. *B x*)) = (*ALL x*:*A*. *b*: *B x*)
⟨*proof*⟩

lemma *INT-I* [*intro*!]: (!!*x*. *x*:*A* ==> *b*: *B x*) ==> *b* : (*INT x*:*A*. *B x*)
⟨*proof*⟩

lemma *INT-D* [*elim*]: *b* : (*INT x*:*A*. *B x*) ==> *a*:*A* ==> *b*: *B a*
⟨*proof*⟩

lemma *INT-E* [*elim*]: *b* : (*INT x*:*A*. *B x*) ==> (*b*: *B a* ==> *R*) ==> (*a*~:*A* ==> *R*) ==> *R*
— "Classical" elimination – by the Excluded Middle on $a \in A$.
⟨*proof*⟩

lemma *INT-cong* [*cong*]:
A = *B* ==> (!!*x*. *x*:*B* ==> *C x* = *D x*) ==> (*INT x*:*A*. *C x*) = (*INT x*:*B*. *D x*)
⟨*proof*⟩

3.4.17 Union

lemma *Union-iff* [*simp*,*noatp*]: (*A* : *Union C*) = (*EX X*:*C*. *A*:*X*)
⟨*proof*⟩

lemma *UnionI* [*intro*]: *X*:*C* ==> *A*:*X* ==> *A* : *Union C*

— The order of the premises presupposes that C is rigid; A may be flexible.
⟨*proof*⟩

lemma *UnionE* [*elim*!]: A : *Union* C ==> (!!X. A:X ==> X:C ==> R) ==> R
⟨*proof*⟩

3.4.18 Inter

lemma *Inter-iff* [*simp*,*noatp*]: (A : *Inter* C) = (*ALL* X:C. A:X)
⟨*proof*⟩

lemma *InterI* [*intro*!]: (!!X. X:C ==> A:X) ==> A : *Inter* C
⟨*proof*⟩

A "destruct" rule – every X in C contains A as an element, but $A \in X$ can hold when $X \in C$ does not! This rule is analogous to *spec*.

lemma *InterD* [*elim*]: A : *Inter* C ==> X:C ==> A:X
⟨*proof*⟩

lemma *InterE* [*elim*]: A : *Inter* C ==> (X~:C ==> R) ==> (A:X ==> R) ==> R
— "Classical" elimination rule – does not require proving $X \in C$.
⟨*proof*⟩

Image of a set under a function. Frequently b does not have the syntactic form of $f\ x$.

declare *image-def* [*noatp*]

lemma *image-eqI* [*simp*, *intro*]: b = f x ==> x:A ==> b : f‘A
⟨*proof*⟩

lemma *imageI*: x : A ==> f x : f ‘ A
⟨*proof*⟩

lemma *rev-image-eqI*: x:A ==> b = f x ==> b : f‘A
— This version's more effective when we already have the required x.
⟨*proof*⟩

lemma *imageE* [*elim*!]:
b : (%x. f x)‘A ==> (!!x. b = f x ==> x:A ==> P) ==> P
— The eta-expansion gives variable-name preservation.
⟨*proof*⟩

lemma *image-Un*: f‘(A *Un* B) = f‘A *Un* f‘B
⟨*proof*⟩

lemma *image-iff*: (z : f‘A) = (*EX* x:A. z = f x)

⟨*proof*⟩

lemma *image-subset-iff*: $(f`A \subseteq B) = (\forall x \in A.\ f\ x \in B)$
— This rewrite rule would confuse users if made default.
⟨*proof*⟩

lemma *subset-image-iff*: $(B \subseteq f`A) = (EX\ AA.\ AA \subseteq A\ \&\ B = f`AA)$
⟨*proof*⟩

lemma *image-subsetI*: $(!!x.\ x \in A ==> f\ x \in B) ==> f`A \subseteq B$
— Replaces the three steps *subsetI*, *imageE*, *hypsubst*, but breaks too many existing proofs.
⟨*proof*⟩

Range of a function – just a translation for image!

lemma *range-eqI*: $b = f\ x ==> b \in range\ f$
⟨*proof*⟩

lemma *rangeI*: $f\ x \in range\ f$
⟨*proof*⟩

lemma *rangeE* [*elim?*]: $b \in range\ (\lambda x.\ f\ x) ==> (!!x.\ b = f\ x ==> P) ==> P$
⟨*proof*⟩

3.4.19 Set reasoning tools

Rewrite rules for boolean case-splitting: faster than *split-if* [*split*].

lemma *split-if-eq1*: $((if\ Q\ then\ x\ else\ y) = b) = ((Q\ --> x = b)\ \&\ (\sim\ Q\ --> y = b))$
⟨*proof*⟩

lemma *split-if-eq2*: $(a = (if\ Q\ then\ x\ else\ y)) = ((Q\ --> a = x)\ \&\ (\sim\ Q\ --> a = y))$
⟨*proof*⟩

Split ifs on either side of the membership relation. Not for [*simp*] – can cause goals to blow up!

lemma *split-if-mem1*: $((if\ Q\ then\ x\ else\ y) : b) = ((Q\ --> x : b)\ \&\ (\sim\ Q\ --> y : b))$
⟨*proof*⟩

lemma *split-if-mem2*: $(a : (if\ Q\ then\ x\ else\ y)) = ((Q\ --> a : x)\ \&\ (\sim\ Q\ --> a : y))$
⟨*proof*⟩

lemmas *split-ifs* = *if-bool-eq-conj split-if-eq1 split-if-eq2 split-if-mem1 split-if-mem2*

lemmas *mem-simps* =

insert-iff empty-iff Un-iff Int-iff Compl-iff Diff-iff
mem-Collect-eq UN-iff Union-iff INT-iff Inter-iff
— Each of these has ALREADY been added [*simp*] above.

⟨*ML*⟩

3.4.20 The "proper subset" relation

lemma *psubsetI* [*intro*!,*noatp*]: $A \subseteq B ==> A \neq B ==> A \subset B$
⟨*proof*⟩

lemma *psubsetE* [*elim*!,*noatp*]:
$[|A \subset B;\ [|A \subseteq B;\ {}^\sim\ (B{\subseteq}A)|] ==> R|] ==> R$
⟨*proof*⟩

lemma *psubset-insert-iff*:
$(A \subset \mathit{insert}\ x\ B) = (\mathit{if}\ x \in B\ \mathit{then}\ A \subset B\ \mathit{else}\ \mathit{if}\ x \in A\ \mathit{then}\ A - \{x\} \subset B\ \mathit{else}\ A \subseteq B)$
⟨*proof*⟩

lemma *psubset-eq*: $(A \subset B) = (A \subseteq B\ \&\ A \neq B)$
⟨*proof*⟩

lemma *psubset-imp-subset*: $A \subset B ==> A \subseteq B$
⟨*proof*⟩

lemma *psubset-trans*: $[|\ A \subset B;\ B \subset C\ |] ==> A \subset C$
⟨*proof*⟩

lemma *psubsetD*: $[|\ A \subset B;\ c \in A\ |] ==> c \in B$
⟨*proof*⟩

lemma *psubset-subset-trans*: $A \subset B ==> B \subseteq C ==> A \subset C$
⟨*proof*⟩

lemma *subset-psubset-trans*: $A \subseteq B ==> B \subset C ==> A \subset C$
⟨*proof*⟩

lemma *psubset-imp-ex-mem*: $A \subset B ==> \exists\, b.\ b \in (B - A)$
⟨*proof*⟩

lemma *atomize-ball*:
$(!!x.\ x \in A ==> P\ x) == \mathit{Trueprop}\ (\forall\, x{\in}A.\ P\ x)$
⟨*proof*⟩

lemmas [*symmetric*, *rulify*] = *atomize-ball*
and [*symmetric*, *defn*] = *atomize-ball*

3.5 Further set-theory lemmas

3.5.1 Derived rules involving subsets.

insert.

lemma *subset-insertI*: $B \subseteq insert\ a\ B$
⟨*proof*⟩

lemma *subset-insertI2*: $A \subseteq B \Longrightarrow A \subseteq insert\ b\ B$
⟨*proof*⟩

lemma *subset-insert*: $x \notin A$ ==> $(A \subseteq insert\ x\ B) = (A \subseteq B)$
⟨*proof*⟩

Big Union – least upper bound of a set.

lemma *Union-upper*: $B \in A$ ==> $B \subseteq Union\ A$
⟨*proof*⟩

lemma *Union-least*: (!!X. $X \in A$ ==> $X \subseteq C$) ==> $Union\ A \subseteq C$
⟨*proof*⟩

General union.

lemma *UN-upper*: $a \in A$ ==> $B\ a \subseteq (\bigcup x{\in}A.\ B\ x)$
⟨*proof*⟩

lemma *UN-least*: (!!x. $x \in A$ ==> $B\ x \subseteq C$) ==> $(\bigcup x{\in}A.\ B\ x) \subseteq C$
⟨*proof*⟩

Big Intersection – greatest lower bound of a set.

lemma *Inter-lower*: $B \in A$ ==> $Inter\ A \subseteq B$
⟨*proof*⟩

lemma *Inter-subset*:
[| !!X. $X \in A$ ==> $X \subseteq B$; A ~= {} |] ==> $\bigcap A \subseteq B$
⟨*proof*⟩

lemma *Inter-greatest*: (!!X. $X \in A$ ==> $C \subseteq X$) ==> $C \subseteq Inter\ A$
⟨*proof*⟩

lemma *INT-lower*: $a \in A$ ==> $(\bigcap x{\in}A.\ B\ x) \subseteq B\ a$
⟨*proof*⟩

lemma *INT-greatest*: (!!x. $x \in A$ ==> $C \subseteq B\ x$) ==> $C \subseteq (\bigcap x{\in}A.\ B\ x)$
⟨*proof*⟩

Finite Union – the least upper bound of two sets.

lemma *Un-upper1*: $A \subseteq A \cup B$

⟨*proof*⟩

lemma *Un-upper2*: $B \subseteq A \cup B$
⟨*proof*⟩

lemma *Un-least*: $A \subseteq C ==> B \subseteq C ==> A \cup B \subseteq C$
⟨*proof*⟩

Finite Intersection – the greatest lower bound of two sets.

lemma *Int-lower1*: $A \cap B \subseteq A$
⟨*proof*⟩

lemma *Int-lower2*: $A \cap B \subseteq B$
⟨*proof*⟩

lemma *Int-greatest*: $C \subseteq A ==> C \subseteq B ==> C \subseteq A \cap B$
⟨*proof*⟩

Set difference.

lemma *Diff-subset*: $A - B \subseteq A$
⟨*proof*⟩

lemma *Diff-subset-conv*: $(A - B \subseteq C) = (A \subseteq B \cup C)$
⟨*proof*⟩

3.5.2 Equalities involving union, intersection, inclusion, etc.

$\{\}$.

lemma *Collect-const* [*simp*]: $\{s.\ P\} = (\textit{if } P \textit{ then } UNIV \textit{ else } \{\})$
— supersedes *Collect-False-empty*
⟨*proof*⟩

lemma *subset-empty* [*simp*]: $(A \subseteq \{\}) = (A = \{\})$
⟨*proof*⟩

lemma *not-psubset-empty* [*iff*]: $\neg\ (A < \{\})$
⟨*proof*⟩

lemma *Collect-empty-eq* [*simp*]: $(Collect\ P = \{\}) = (\forall x.\ \neg\ P\ x)$
⟨*proof*⟩

lemma *empty-Collect-eq* [*simp*]: $(\{\} = Collect\ P) = (\forall x.\ \neg\ P\ x)$
⟨*proof*⟩

lemma *Collect-neg-eq*: $\{x.\ \neg\ P\ x\} = -\ \{x.\ P\ x\}$
⟨*proof*⟩

lemma *Collect-disj-eq*: $\{x.\ P\ x \mid Q\ x\} = \{x.\ P\ x\} \cup \{x.\ Q\ x\}$
⟨*proof*⟩

lemma *Collect-imp-eq*: $\{x.\ P\ x \dashrightarrow Q\ x\} = -\{x.\ P\ x\} \cup \{x.\ Q\ x\}$
⟨*proof*⟩

lemma *Collect-conj-eq*: $\{x.\ P\ x\ \&\ Q\ x\} = \{x.\ P\ x\} \cap \{x.\ Q\ x\}$
⟨*proof*⟩

lemma *Collect-all-eq*: $\{x.\ \forall\, y.\ P\ x\ y\} = (\bigcap y.\ \{x.\ P\ x\ y\})$
⟨*proof*⟩

lemma *Collect-ball-eq*: $\{x.\ \forall\, y{\in}A.\ P\ x\ y\} = (\bigcap y{\in}A.\ \{x.\ P\ x\ y\})$
⟨*proof*⟩

lemma *Collect-ex-eq* [*noatp*]: $\{x.\ \exists\, y.\ P\ x\ y\} = (\bigcup y.\ \{x.\ P\ x\ y\})$
⟨*proof*⟩

lemma *Collect-bex-eq* [*noatp*]: $\{x.\ \exists\, y{\in}A.\ P\ x\ y\} = (\bigcup y{\in}A.\ \{x.\ P\ x\ y\})$
⟨*proof*⟩

insert.

lemma *insert-is-Un*: $insert\ a\ A = \{a\}\ Un\ A$
— NOT SUITABLE FOR REWRITING since $\{a\} == insert\ a\ \{\}$
⟨*proof*⟩

lemma *insert-not-empty* [*simp*]: $insert\ a\ A \neq \{\}$
⟨*proof*⟩

lemmas *empty-not-insert* = *insert-not-empty* [*symmetric*, *standard*]
declare *empty-not-insert* [*simp*]

lemma *insert-absorb*: $a \in A \Longrightarrow insert\ a\ A = A$
— [*simp*] causes recursive calls when there are nested inserts
— with *quadratic* running time
⟨*proof*⟩

lemma *insert-absorb2* [*simp*]: $insert\ x\ (insert\ x\ A) = insert\ x\ A$
⟨*proof*⟩

lemma *insert-commute*: $insert\ x\ (insert\ y\ A) = insert\ y\ (insert\ x\ A)$
⟨*proof*⟩

lemma *insert-subset* [*simp*]: $(insert\ x\ A \subseteq B) = (x \in B\ \&\ A \subseteq B)$
⟨*proof*⟩

lemma *mk-disjoint-insert*: $a \in A \Longrightarrow \exists\, B.\ A = insert\ a\ B\ \&\ a \notin B$
— use new B rather than $A - \{a\}$ to avoid infinite unfolding
⟨*proof*⟩

lemma *insert-Collect*: $insert\ a\ (Collect\ P) = \{u.\ u \neq a\ {-}{-}{>}\ P\ u\}$
⟨*proof*⟩

lemma *UN-insert-distrib*: $u \in A ==> (\bigcup x{\in}A.\ insert\ a\ (B\ x)) = insert\ a\ (\bigcup x{\in}A.\ B\ x)$
⟨*proof*⟩

lemma *insert-inter-insert*[*simp*]: $insert\ a\ A \cap insert\ a\ B = insert\ a\ (A \cap B)$
⟨*proof*⟩

lemma *insert-disjoint* [*simp*,*noatp*]:
$(insert\ a\ A \cap B = \{\}) = (a \notin B \wedge A \cap B = \{\})$
$(\{\} = insert\ a\ A \cap B) = (a \notin B \wedge \{\} = A \cap B)$
⟨*proof*⟩

lemma *disjoint-insert* [*simp*,*noatp*]:
$(B \cap insert\ a\ A = \{\}) = (a \notin B \wedge B \cap A = \{\})$
$(\{\} = A \cap insert\ b\ B) = (b \notin A \wedge \{\} = A \cap B)$
⟨*proof*⟩

image.

lemma *image-empty* [*simp*]: $f`\{\} = \{\}$
⟨*proof*⟩

lemma *image-insert* [*simp*]: $f\ `\ insert\ a\ B = insert\ (f\ a)\ (f`B)$
⟨*proof*⟩

lemma *image-constant*: $x \in A ==> (\lambda x.\ c)\ `\ A = \{c\}$
⟨*proof*⟩

lemma *image-constant-conv*: $(\%x.\ c)\ `\ A = (if\ A = \{\}\ then\ \{\}\ else\ \{c\})$
⟨*proof*⟩

lemma *image-image*: $f\ `\ (g\ `\ A) = (\lambda x.\ f\ (g\ x))\ `\ A$
⟨*proof*⟩

lemma *insert-image* [*simp*]: $x \in A ==> insert\ (f\ x)\ (f`A) = f`A$
⟨*proof*⟩

lemma *image-is-empty* [*iff*]: $(f`A = \{\}) = (A = \{\})$
⟨*proof*⟩

lemma *image-Collect* [*noatp*]: $f\ `\ \{x.\ P\ x\} = \{f\ x \mid x.\ P\ x\}$
— NOT suitable as a default simprule: the RHS isn't simpler than the LHS, with its implicit quantifier and conjunction. Also image enjoys better equational properties than does the RHS.
⟨*proof*⟩

lemma *if-image-distrib* [*simp*]:
$(\lambda x.\ \mathit{if}\ P\ x\ \mathit{then}\ f\ x\ \mathit{else}\ g\ x)$ ' S
$= (f$ ' $(S \cap \{x.\ P\ x\})) \cup (g$ ' $(S \cap \{x.\ \neg\ P\ x\}))$
⟨*proof*⟩

lemma *image-cong*: $M = N$ ==> $(!!x.\ x \in N$ ==> $f\ x = g\ x)$ ==> $f`M = g`N$
⟨*proof*⟩

range.

lemma *full-SetCompr-eq* [*noatp*]: $\{u.\ \exists\, x.\ u = f\ x\} = \mathit{range}\ f$
⟨*proof*⟩

lemma *range-composition* [*simp*]: $\mathit{range}\ (\lambda x.\ f\ \ (g\ x)) = f`\mathit{range}\ g$
⟨*proof*⟩

Int

lemma *Int-absorb* [*simp*]: $A \cap A = A$
⟨*proof*⟩

lemma *Int-left-absorb*: $A \cap (A \cap B) = A \cap B$
⟨*proof*⟩

lemma *Int-commute*: $A \cap B = B \cap A$
⟨*proof*⟩

lemma *Int-left-commute*: $A \cap (B \cap C) = B \cap (A \cap C)$
⟨*proof*⟩

lemma *Int-assoc*: $(A \cap B) \cap C = A \cap (B \cap C)$
⟨*proof*⟩

lemmas *Int-ac* = *Int-assoc Int-left-absorb Int-commute Int-left-commute*
— Intersection is an AC-operator

lemma *Int-absorb1*: $B \subseteq A$ ==> $A \cap B = B$
⟨*proof*⟩

lemma *Int-absorb2*: $A \subseteq B$ ==> $A \cap B = A$
⟨*proof*⟩

lemma *Int-empty-left* [*simp*]: $\{\} \cap B = \{\}$
⟨*proof*⟩

lemma *Int-empty-right* [*simp*]: $A \cap \{\} = \{\}$
⟨*proof*⟩

lemma *disjoint-eq-subset-Compl*: $(A \cap B = \{\}) = (A \subseteq -B)$

⟨*proof*⟩

lemma *disjoint-iff-not-equal*: $(A \cap B = \{\}) = (\forall x \in A.\ \forall y \in B.\ x \neq y)$
⟨*proof*⟩

lemma *Int-UNIV-left* [*simp*]: $UNIV \cap B = B$
⟨*proof*⟩

lemma *Int-UNIV-right* [*simp*]: $A \cap UNIV = A$
⟨*proof*⟩

lemma *Int-eq-Inter*: $A \cap B = \bigcap\{A, B\}$
⟨*proof*⟩

lemma *Int-Un-distrib*: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
⟨*proof*⟩

lemma *Int-Un-distrib2*: $(B \cup C) \cap A = (B \cap A) \cup (C \cap A)$
⟨*proof*⟩

lemma *Int-UNIV* [*simp,noatp*]: $(A \cap B = UNIV) = (A = UNIV\ \&\ B = UNIV)$
⟨*proof*⟩

lemma *Int-subset-iff* [*simp*]: $(C \subseteq A \cap B) = (C \subseteq A\ \&\ C \subseteq B)$
⟨*proof*⟩

lemma *Int-Collect*: $(x \in A \cap \{x.\ P\ x\}) = (x \in A\ \&\ P\ x)$
⟨*proof*⟩

Un.

lemma *Un-absorb* [*simp*]: $A \cup A = A$
⟨*proof*⟩

lemma *Un-left-absorb*: $A \cup (A \cup B) = A \cup B$
⟨*proof*⟩

lemma *Un-commute*: $A \cup B = B \cup A$
⟨*proof*⟩

lemma *Un-left-commute*: $A \cup (B \cup C) = B \cup (A \cup C)$
⟨*proof*⟩

lemma *Un-assoc*: $(A \cup B) \cup C = A \cup (B \cup C)$
⟨*proof*⟩

lemmas *Un-ac* = *Un-assoc Un-left-absorb Un-commute Un-left-commute*
— Union is an AC-operator

lemma *Un-absorb1*: $A \subseteq B \Longrightarrow A \cup B = B$

⟨*proof*⟩

lemma *Un-absorb2*: $B \subseteq A ==> A \cup B = A$
⟨*proof*⟩

lemma *Un-empty-left* [*simp*]: $\{\} \cup B = B$
⟨*proof*⟩

lemma *Un-empty-right* [*simp*]: $A \cup \{\} = A$
⟨*proof*⟩

lemma *Un-UNIV-left* [*simp*]: $\mathit{UNIV} \cup B = \mathit{UNIV}$
⟨*proof*⟩

lemma *Un-UNIV-right* [*simp*]: $A \cup \mathit{UNIV} = \mathit{UNIV}$
⟨*proof*⟩

lemma *Un-eq-Union*: $A \cup B = \bigcup\{A, B\}$
⟨*proof*⟩

lemma *Un-insert-left* [*simp*]: $(\mathit{insert}\ a\ B) \cup C = \mathit{insert}\ a\ (B \cup C)$
⟨*proof*⟩

lemma *Un-insert-right* [*simp*]: $A \cup (\mathit{insert}\ a\ B) = \mathit{insert}\ a\ (A \cup B)$
⟨*proof*⟩

lemma *Int-insert-left*:
$(\mathit{insert}\ a\ B)\ \mathit{Int}\ C = (\mathit{if}\ a \in C\ \mathit{then}\ \mathit{insert}\ a\ (B \cap C)\ \mathit{else}\ B \cap C)$
⟨*proof*⟩

lemma *Int-insert-right*:
$A \cap (\mathit{insert}\ a\ B) = (\mathit{if}\ a \in A\ \mathit{then}\ \mathit{insert}\ a\ (A \cap B)\ \mathit{else}\ A \cap B)$
⟨*proof*⟩

lemma *Un-Int-distrib*: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
⟨*proof*⟩

lemma *Un-Int-distrib2*: $(B \cap C) \cup A = (B \cup A) \cap (C \cup A)$
⟨*proof*⟩

lemma *Un-Int-crazy*:
$(A \cap B) \cup (B \cap C) \cup (C \cap A) = (A \cup B) \cap (B \cup C) \cap (C \cup A)$
⟨*proof*⟩

lemma *subset-Un-eq*: $(A \subseteq B) = (A \cup B = B)$
⟨*proof*⟩

lemma *Un-empty* [*iff*]: $(A \cup B = \{\}) = (A = \{\}\ \&\ B = \{\})$
⟨*proof*⟩

lemma *Un-subset-iff* [*simp*]: $(A \cup B \subseteq C) = (A \subseteq C \ \& \ B \subseteq C)$
⟨*proof*⟩

lemma *Un-Diff-Int*: $(A - B) \cup (A \cap B) = A$
⟨*proof*⟩

lemma *Diff-Int2*: $A \cap C - B \cap C = A \cap C - B$
⟨*proof*⟩

Set complement

lemma *Compl-disjoint* [*simp*]: $A \cap -A = \{\}$
⟨*proof*⟩

lemma *Compl-disjoint2* [*simp*]: $-A \cap A = \{\}$
⟨*proof*⟩

lemma *Compl-partition*: $A \cup -A = UNIV$
⟨*proof*⟩

lemma *Compl-partition2*: $-A \cup A = UNIV$
⟨*proof*⟩

lemma *double-complement* [*simp*]: $-(-A) = (A::'a\ set)$
⟨*proof*⟩

lemma *Compl-Un* [*simp*]: $-(A \cup B) = (-A) \cap (-B)$
⟨*proof*⟩

lemma *Compl-Int* [*simp*]: $-(A \cap B) = (-A) \cup (-B)$
⟨*proof*⟩

lemma *Compl-UN* [*simp*]: $-(\bigcup x{\in}A.\ B\ x) = (\bigcap x{\in}A.\ -B\ x)$
⟨*proof*⟩

lemma *Compl-INT* [*simp*]: $-(\bigcap x{\in}A.\ B\ x) = (\bigcup x{\in}A.\ -B\ x)$
⟨*proof*⟩

lemma *subset-Compl-self-eq*: $(A \subseteq -A) = (A = \{\})$
⟨*proof*⟩

lemma *Un-Int-assoc-eq*: $((A \cap B) \cup C = A \cap (B \cup C)) = (C \subseteq A)$
— Halmos, Naive Set Theory, page 16.
⟨*proof*⟩

lemma *Compl-UNIV-eq* [*simp*]: $-UNIV = \{\}$
⟨*proof*⟩

lemma *Compl-empty-eq* [*simp*]: $-\{\} = UNIV$

⟨*proof*⟩

lemma *Compl-subset-Compl-iff* [*iff*]: $(-A \subseteq -B) = (B \subseteq A)$
⟨*proof*⟩

lemma *Compl-eq-Compl-iff* [*iff*]: $(-A = -B) = (A = (B::'a\ set))$
⟨*proof*⟩

Union.

lemma *Union-empty* [*simp*]: $Union(\{\}) = \{\}$
⟨*proof*⟩

lemma *Union-UNIV* [*simp*]: $Union\ UNIV = UNIV$
⟨*proof*⟩

lemma *Union-insert* [*simp*]: $Union\ (insert\ a\ B) = a \cup \bigcup B$
⟨*proof*⟩

lemma *Union-Un-distrib* [*simp*]: $\bigcup(A\ \ Un\ B) = \bigcup A \cup \bigcup B$
⟨*proof*⟩

lemma *Union-Int-subset*: $\bigcup(A \cap B) \subseteq \bigcup A \cap \bigcup B$
⟨*proof*⟩

lemma *Union-empty-conv* [*simp,noatp*]: $(\bigcup A = \{\}) = (\forall x{\in}A.\ x = \{\})$
⟨*proof*⟩

lemma *empty-Union-conv* [*simp,noatp*]: $(\{\} = \bigcup A) = (\forall x{\in}A.\ x = \{\})$
⟨*proof*⟩

lemma *Union-disjoint*: $(\bigcup C \cap A = \{\}) = (\forall B{\in}C.\ B \cap A = \{\})$
⟨*proof*⟩

Inter.

lemma *Inter-empty* [*simp*]: $\bigcap\{\} = UNIV$
⟨*proof*⟩

lemma *Inter-UNIV* [*simp*]: $\bigcap UNIV = \{\}$
⟨*proof*⟩

lemma *Inter-insert* [*simp*]: $\bigcap(insert\ a\ B) = a \cap \bigcap B$
⟨*proof*⟩

lemma *Inter-Un-subset*: $\bigcap A \cup \bigcap B \subseteq \bigcap(A \cap B)$
⟨*proof*⟩

lemma *Inter-Un-distrib*: $\bigcap(A \cup B) = \bigcap A \cap \bigcap B$
⟨*proof*⟩

lemma *Inter-UNIV-conv* [*simp*,*noatp*]:
$(\bigcap A = UNIV) = (\forall x{\in}A.\ x = UNIV)$
$(UNIV = \bigcap A) = (\forall x{\in}A.\ x = UNIV)$
⟨*proof*⟩

UN and *INT*.

Basic identities:

lemma *UN-empty* [*simp*,*noatp*]: $(\bigcup x{\in}\{\}.\ B\ x) = \{\}$
⟨*proof*⟩

lemma *UN-empty2* [*simp*]: $(\bigcup x{\in}A.\ \{\}) = \{\}$
⟨*proof*⟩

lemma *UN-singleton* [*simp*]: $(\bigcup x{\in}A.\ \{x\}) = A$
⟨*proof*⟩

lemma *UN-absorb*: $k \in I ==> A\ k \cup (\bigcup i{\in}I.\ A\ i) = (\bigcup i{\in}I.\ A\ i)$
⟨*proof*⟩

lemma *INT-empty* [*simp*]: $(\bigcap x{\in}\{\}.\ B\ x) = UNIV$
⟨*proof*⟩

lemma *INT-absorb*: $k \in I ==> A\ k \cap (\bigcap i{\in}I.\ A\ i) = (\bigcap i{\in}I.\ A\ i)$
⟨*proof*⟩

lemma *UN-insert* [*simp*]: $(\bigcup x{\in}insert\ a\ A.\ B\ x) = B\ a \cup UNION\ A\ B$
⟨*proof*⟩

lemma *UN-Un*[*simp*]: $(\bigcup i \in A \cup B.\ M\ i) = (\bigcup i{\in}A.\ M\ i) \cup (\bigcup i{\in}B.\ M\ i)$
⟨*proof*⟩

lemma *UN-UN-flatten*: $(\bigcup x \in (\bigcup y{\in}A.\ B\ y).\ C\ x) = (\bigcup y{\in}A.\ \bigcup x{\in}B\ y.\ C\ x)$
⟨*proof*⟩

lemma *UN-subset-iff*: $((\bigcup i{\in}I.\ A\ i) \subseteq B) = (\forall i{\in}I.\ A\ i \subseteq B)$
⟨*proof*⟩

lemma *INT-subset-iff*: $(B \subseteq (\bigcap i{\in}I.\ A\ i)) = (\forall i{\in}I.\ B \subseteq A\ i)$
⟨*proof*⟩

lemma *INT-insert* [*simp*]: $(\bigcap x \in insert\ a\ A.\ B\ x) = B\ a \cap INTER\ A\ B$
⟨*proof*⟩

lemma *INT-Un*: $(\bigcap i \in A \cup B.\ M\ i) = (\bigcap i \in A.\ M\ i) \cap (\bigcap i{\in}B.\ M\ i)$
⟨*proof*⟩

lemma *INT-insert-distrib*:
$u \in A ==> (\bigcap x{\in}A.\ insert\ a\ (B\ x)) = insert\ a\ (\bigcap x{\in}A.\ B\ x)$

⟨*proof*⟩

lemma *Union-image-eq* [*simp*]: $\bigcup(B`A) = (\bigcup x \in A.\ B\ x)$
⟨*proof*⟩

lemma *image-Union*: $f\ `\ \bigcup S = (\bigcup x \in S.\ f\ `\ x)$
⟨*proof*⟩

lemma *Inter-image-eq* [*simp*]: $\bigcap(B`A) = (\bigcap x \in A.\ B\ x)$
⟨*proof*⟩

lemma *UN-constant* [*simp*]: $(\bigcup y \in A.\ c) = (\mathit{if}\ A = \{\}\ \mathit{then}\ \{\}\ \mathit{else}\ c)$
⟨*proof*⟩

lemma *INT-constant* [*simp*]: $(\bigcap y \in A.\ c) = (\mathit{if}\ A = \{\}\ \mathit{then}\ \mathit{UNIV}\ \mathit{else}\ c)$
⟨*proof*⟩

lemma *UN-eq*: $(\bigcup x \in A.\ B\ x) = \bigcup(\{Y.\ \exists x \in A.\ Y = B\ x\})$
⟨*proof*⟩

lemma *INT-eq*: $(\bigcap x \in A.\ B\ x) = \bigcap(\{Y.\ \exists x \in A.\ Y = B\ x\})$
— Look: it has an *existential* quantifier
⟨*proof*⟩

lemma *UNION-empty-conv*[*simp*]:
$(\{\} = (\mathit{UN}\ x{:}A.\ B\ x)) = (\forall x \in A.\ B\ x = \{\})$
$((\mathit{UN}\ x{:}A.\ B\ x) = \{\}) = (\forall x \in A.\ B\ x = \{\})$
⟨*proof*⟩

lemma *INTER-UNIV-conv*[*simp*]:
$(\mathit{UNIV} = (\mathit{INT}\ x{:}A.\ B\ x)) = (\forall x \in A.\ B\ x = \mathit{UNIV})$
$((\mathit{INT}\ x{:}A.\ B\ x) = \mathit{UNIV}) = (\forall x \in A.\ B\ x = \mathit{UNIV})$
⟨*proof*⟩

Distributive laws:

lemma *Int-Union*: $A \cap \bigcup B = (\bigcup C \in B.\ A \cap C)$
⟨*proof*⟩

lemma *Int-Union2*: $\bigcup B \cap A = (\bigcup C \in B.\ C \cap A)$
⟨*proof*⟩

lemma *Un-Union-image*: $(\bigcup x \in C.\ A\ x \cup B\ x) = \bigcup(A`C) \cup \bigcup(B`C)$
— Devlin, Fundamentals of Contemporary Set Theory, page 12, exercise 5:
— Union of a family of unions
⟨*proof*⟩

lemma *UN-Un-distrib*: $(\bigcup i \in I.\ A\ i \cup B\ i) = (\bigcup i \in I.\ A\ i) \cup (\bigcup i \in I.\ B\ i)$
— Equivalent version
⟨*proof*⟩

lemma *Un-Inter*: $A \cup \bigcap B = (\bigcap C{\in}B.\ A \cup C)$
⟨*proof*⟩

lemma *Int-Inter-image*: $(\bigcap x{\in}C.\ A\ x \cap B\ x) = \bigcap (A`C) \cap \bigcap (B`C)$
⟨*proof*⟩

lemma *INT-Int-distrib*: $(\bigcap i{\in}I.\ A\ i \cap B\ i) = (\bigcap i{\in}I.\ A\ i) \cap (\bigcap i{\in}I.\ B\ i)$
— Equivalent version
⟨*proof*⟩

lemma *Int-UN-distrib*: $B \cap (\bigcup i{\in}I.\ A\ i) = (\bigcup i{\in}I.\ B \cap A\ i)$
— Halmos, Naive Set Theory, page 35.
⟨*proof*⟩

lemma *Un-INT-distrib*: $B \cup (\bigcap i{\in}I.\ A\ i) = (\bigcap i{\in}I.\ B \cup A\ i)$
⟨*proof*⟩

lemma *Int-UN-distrib2*: $(\bigcup i{\in}I.\ A\ i) \cap (\bigcup j{\in}J.\ B\ j) = (\bigcup i{\in}I.\ \bigcup j{\in}J.\ A\ i \cap B\ j)$
⟨*proof*⟩

lemma *Un-INT-distrib2*: $(\bigcap i{\in}I.\ A\ i) \cup (\bigcap j{\in}J.\ B\ j) = (\bigcap i{\in}I.\ \bigcap j{\in}J.\ A\ i \cup B\ j)$
⟨*proof*⟩

Bounded quantifiers.

The following are not added to the default simpset because (a) they duplicate the body and (b) there are no similar rules for *Int*.

lemma *ball-Un*: $(\forall\, x \in A \cup B.\ P\ x) = ((\forall\, x{\in}A.\ P\ x)\ \&\ (\forall\, x{\in}B.\ P\ x))$
⟨*proof*⟩

lemma *bex-Un*: $(\exists\, x \in A \cup B.\ P\ x) = ((\exists\, x{\in}A.\ P\ x)\ |\ (\exists\, x{\in}B.\ P\ x))$
⟨*proof*⟩

lemma *ball-UN*: $(\forall\, z \in \mathit{UNION}\ A\ B.\ P\ z) = (\forall\, x{\in}A.\ \forall\, z \in B\ x.\ P\ z)$
⟨*proof*⟩

lemma *bex-UN*: $(\exists\, z \in \mathit{UNION}\ A\ B.\ P\ z) = (\exists\, x{\in}A.\ \exists\, z{\in}B\ x.\ P\ z)$
⟨*proof*⟩

Set difference.

lemma *Diff-eq*: $A - B = A \cap (-B)$
⟨*proof*⟩

lemma *Diff-eq-empty-iff* [*simp*,*noatp*]: $(A - B = \{\}) = (A \subseteq B)$
⟨*proof*⟩

lemma *Diff-cancel* [*simp*]: $A - A = \{\}$
$\langle proof \rangle$

lemma *Diff-idemp* [*simp*]: $(A - B) - B = A - (B::'a\ set)$
$\langle proof \rangle$

lemma *Diff-triv*: $A \cap B = \{\} ==> A - B = A$
$\langle proof \rangle$

lemma *empty-Diff* [*simp*]: $\{\} - A = \{\}$
$\langle proof \rangle$

lemma *Diff-empty* [*simp*]: $A - \{\} = A$
$\langle proof \rangle$

lemma *Diff-UNIV* [*simp*]: $A - UNIV = \{\}$
$\langle proof \rangle$

lemma *Diff-insert0* [*simp,noatp*]: $x \notin A ==> A - insert\ x\ B = A - B$
$\langle proof \rangle$

lemma *Diff-insert*: $A - insert\ a\ B = A - B - \{a\}$
— NOT SUITABLE FOR REWRITING since $\{a\} == insert\ a\ 0$
$\langle proof \rangle$

lemma *Diff-insert2*: $A - insert\ a\ B = A - \{a\} - B$
— NOT SUITABLE FOR REWRITING since $\{a\} == insert\ a\ 0$
$\langle proof \rangle$

lemma *insert-Diff-if*: $insert\ x\ A - B = (if\ x \in B\ then\ A - B\ else\ insert\ x\ (A - B))$
$\langle proof \rangle$

lemma *insert-Diff1* [*simp*]: $x \in B ==> insert\ x\ A - B = A - B$
$\langle proof \rangle$

lemma *insert-Diff-single*[*simp*]: $insert\ a\ (A - \{a\}) = insert\ a\ A$
$\langle proof \rangle$

lemma *insert-Diff*: $a \in A ==> insert\ a\ (A - \{a\}) = A$
$\langle proof \rangle$

lemma *Diff-insert-absorb*: $x \notin A ==> (insert\ x\ A) - \{x\} = A$
$\langle proof \rangle$

lemma *Diff-disjoint* [*simp*]: $A \cap (B - A) = \{\}$
$\langle proof \rangle$

lemma *Diff-partition*: $A \subseteq B ==> A \cup (B - A) = B$

⟨*proof*⟩

lemma *double-diff*: $A \subseteq B ==> B \subseteq C ==> B - (C - A) = A$
⟨*proof*⟩

lemma *Un-Diff-cancel* [*simp*]: $A \cup (B - A) = A \cup B$
⟨*proof*⟩

lemma *Un-Diff-cancel2* [*simp*]: $(B - A) \cup A = B \cup A$
⟨*proof*⟩

lemma *Diff-Un*: $A - (B \cup C) = (A - B) \cap (A - C)$
⟨*proof*⟩

lemma *Diff-Int*: $A - (B \cap C) = (A - B) \cup (A - C)$
⟨*proof*⟩

lemma *Un-Diff*: $(A \cup B) - C = (A - C) \cup (B - C)$
⟨*proof*⟩

lemma *Int-Diff*: $(A \cap B) - C = A \cap (B - C)$
⟨*proof*⟩

lemma *Diff-Int-distrib*: $C \cap (A - B) = (C \cap A) - (C \cap B)$
⟨*proof*⟩

lemma *Diff-Int-distrib2*: $(A - B) \cap C = (A \cap C) - (B \cap C)$
⟨*proof*⟩

lemma *Diff-Compl* [*simp*]: $A - (- B) = A \cap B$
⟨*proof*⟩

lemma *Compl-Diff-eq* [*simp*]: $- (A - B) = -A \cup B$
⟨*proof*⟩

Quantification over type *bool*.

lemma *bool-induct*: $P\ True \Longrightarrow P\ False \Longrightarrow P\ x$
⟨*proof*⟩

lemma *all-bool-eq*: $(\forall b.\ P\ b) \longleftrightarrow P\ True \wedge P\ False$
⟨*proof*⟩

lemma *bool-contrapos*: $P\ x \Longrightarrow \neg\ P\ False \Longrightarrow P\ True$
⟨*proof*⟩

lemma *ex-bool-eq*: $(\exists b.\ P\ b) \longleftrightarrow P\ True \vee P\ False$
⟨*proof*⟩

lemma *Un-eq-UN*: $A \cup B = (\bigcup b.\ if\ b\ then\ A\ else\ B)$

⟨*proof*⟩

lemma *UN-bool-eq*: $(\bigcup b::bool.\ A\ b) = (A\ True \cup A\ False)$
⟨*proof*⟩

lemma *INT-bool-eq*: $(\bigcap b::bool.\ A\ b) = (A\ True \cap A\ False)$
⟨*proof*⟩

Pow

lemma *Pow-empty* [*simp*]: $Pow\ \{\} = \{\{\}\}$
⟨*proof*⟩

lemma *Pow-insert*: $Pow\ (insert\ a\ A) = Pow\ A \cup (insert\ a\ `\ Pow\ A)$
⟨*proof*⟩

lemma *Pow-Compl*: $Pow\ (-\ A) = \{-B \mid B.\ A \in Pow\ B\}$
⟨*proof*⟩

lemma *Pow-UNIV* [*simp*]: $Pow\ UNIV = UNIV$
⟨*proof*⟩

lemma *Un-Pow-subset*: $Pow\ A \cup Pow\ B \subseteq Pow\ (A \cup B)$
⟨*proof*⟩

lemma *UN-Pow-subset*: $(\bigcup x{\in}A.\ Pow\ (B\ x)) \subseteq Pow\ (\bigcup x{\in}A.\ B\ x)$
⟨*proof*⟩

lemma *subset-Pow-Union*: $A \subseteq Pow\ (\bigcup A)$
⟨*proof*⟩

lemma *Union-Pow-eq* [*simp*]: $\bigcup (Pow\ A) = A$
⟨*proof*⟩

lemma *Pow-Int-eq* [*simp*]: $Pow\ (A \cap B) = Pow\ A \cap Pow\ B$
⟨*proof*⟩

lemma *Pow-INT-eq*: $Pow\ (\bigcap x{\in}A.\ B\ x) = (\bigcap x{\in}A.\ Pow\ (B\ x))$
⟨*proof*⟩

Miscellany.

lemma *set-eq-subset*: $(A = B) = (A \subseteq B\ \&\ B \subseteq A)$
⟨*proof*⟩

lemma *subset-iff*: $(A \subseteq B) = (\forall\, t.\ t \in A \ {-}{-}{>}\ t \in B)$
⟨*proof*⟩

lemma *subset-iff-psubset-eq*: $(A \subseteq B) = ((A \subset B) \mid (A = B))$
⟨*proof*⟩

lemma *all-not-in-conv* [*simp*]: $(\forall x.\ x \notin A) = (A = \{\})$
⟨*proof*⟩

lemma *ex-in-conv*: $(\exists x.\ x \in A) = (A \neq \{\})$
⟨*proof*⟩

lemma *distinct-lemma*: $f\ x \neq f\ y$ ==> $x \neq y$
⟨*proof*⟩

Miniscoping: pushing in quantifiers and big Unions and Intersections.

lemma *UN-simps* [*simp*]:
!!a B C. (UN x:C. insert a (B x)) = (if C={} then {} else insert a (UN x:C. B x))
!!A B C. (UN x:C. A x Un B) = ((if C={} then {} else (UN x:C. A x) Un B))
!!A B C. (UN x:C. A Un B x) = ((if C={} then {} else A Un (UN x:C. B x)))
!!A B C. (UN x:C. A x Int B) = ((UN x:C. A x) Int B)
!!A B C. (UN x:C. A Int B x) = (A Int (UN x:C. B x))
!!A B C. (UN x:C. A x − B) = ((UN x:C. A x) − B)
!!A B C. (UN x:C. A − B x) = (A − (INT x:C. B x))
!!A B. (UN x: Union A. B x) = (UN y:A. UN x:y. B x)
!!A B C. (UN z: UNION A B. C z) = (UN x:A. UN z: B(x). C z)
!!A B f. (UN x:f'A. B x) = (UN a:A. B (f a))
⟨*proof*⟩

lemma *INT-simps* [*simp*]:
!!A B C. (INT x:C. A x Int B) = (if C={} then UNIV else (INT x:C. A x) Int B)
!!A B C. (INT x:C. A Int B x) = (if C={} then UNIV else A Int (INT x:C. B x))
!!A B C. (INT x:C. A x − B) = (if C={} then UNIV else (INT x:C. A x) − B)
!!A B C. (INT x:C. A − B x) = (if C={} then UNIV else A − (UN x:C. B x))
!!a B C. (INT x:C. insert a (B x)) = insert a (INT x:C. B x)
!!A B C. (INT x:C. A x Un B) = ((INT x:C. A x) Un B)
!!A B C. (INT x:C. A Un B x) = (A Un (INT x:C. B x))
!!A B. (INT x: Union A. B x) = (INT y:A. INT x:y. B x)
!!A B C. (INT z: UNION A B. C z) = (INT x:A. INT z: B(x). C z)
!!A B f. (INT x:f'A. B x) = (INT a:A. B (f a))
⟨*proof*⟩

lemma *ball-simps* [*simp*,*noatp*]:
!!A P Q. (ALL x:A. P x | Q) = ((ALL x:A. P x) | Q)
!!A P Q. (ALL x:A. P | Q x) = (P | (ALL x:A. Q x))
!!A P Q. (ALL x:A. P --> Q x) = (P --> (ALL x:A. Q x))
!!A P Q. (ALL x:A. P x --> Q) = ((EX x:A. P x) --> Q)

!!P. (ALL x:{}. P x) = True
!!P. (ALL x:UNIV. P x) = (ALL x. P x)
!!a B P. (ALL x:insert a B. P x) = (P a & (ALL x:B. P x))
!!A P. (ALL x:Union A. P x) = (ALL y:A. ALL x:y. P x)
!!A B P. (ALL x: UNION A B. P x) = (ALL a:A. ALL x: B a. P x)
!!P Q. (ALL x:Collect Q. P x) = (ALL x. Q x --> P x)
!!A P f. (ALL x:f‘A. P x) = (ALL x:A. P (f x))
!!A P. (~(ALL x:A. P x)) = (EX x:A. ~P x)
⟨proof⟩

lemma *bex-simps* [*simp*,*noatp*]:
!!A P Q. (EX x:A. P x & Q) = ((EX x:A. P x) & Q)
!!A P Q. (EX x:A. P & Q x) = (P & (EX x:A. Q x))
!!P. (EX x:{}. P x) = False
!!P. (EX x:UNIV. P x) = (EX x. P x)
!!a B P. (EX x:insert a B. P x) = (P(a) | (EX x:B. P x))
!!A P. (EX x:Union A. P x) = (EX y:A. EX x:y. P x)
!!A B P. (EX x: UNION A B. P x) = (EX a:A. EX x:B a. P x)
!!P Q. (EX x:Collect Q. P x) = (EX x. Q x & P x)
!!A P f. (EX x:f‘A. P x) = (EX x:A. P (f x))
!!A P. (~(EX x:A. P x)) = (ALL x:A. ~P x)
⟨proof⟩

lemma *ball-conj-distrib*:
(ALL x:A. P x & Q x) = ((ALL x:A. P x) & (ALL x:A. Q x))
⟨proof⟩

lemma *bex-disj-distrib*:
(EX x:A. P x | Q x) = ((EX x:A. P x) | (EX x:A. Q x))
⟨proof⟩

Maxiscoping: pulling out big Unions and Intersections.

lemma *UN-extend-simps*:
!!a B C. insert a (UN x:C. B x) = (if C={} then {a} else (UN x:C. insert a (B x)))
!!A B C. (UN x:C. A x) Un B = (if C={} then B else (UN x:C. A x Un B))
!!A B C. A Un (UN x:C. B x) = (if C={} then A else (UN x:C. A Un B x))
!!A B C. ((UN x:C. A x) Int B) = (UN x:C. A x Int B)
!!A B C. (A Int (UN x:C. B x)) = (UN x:C. A Int B x)
!!A B C. ((UN x:C. A x) − B) = (UN x:C. A x − B)
!!A B C. (A − (INT x:C. B x)) = (UN x:C. A − B x)
!!A B. (UN y:A. UN x:y. B x) = (UN x: Union A. B x)
!!A B C. (UN x:A. UN z: B(x). C z) = (UN z: UNION A B. C z)
!!A B f. (UN a:A. B (f a)) = (UN x:f‘A. B x)
⟨proof⟩

lemma *INT-extend-simps*:
!!A B C. (INT x:C. A x) Int B = (if C={} then B else (INT x:C. A x Int B))
!!A B C. A Int (INT x:C. B x) = (if C={} then A else (INT x:C. A Int B x))

$!!A\ B\ C.\ (INT\ x{:}C.\ A\ x) - B\ \ = (if\ C{=}\{\}\ then\ UNIV{-}B\ else\ (INT\ x{:}C.\ A\ x - B))$
$!!A\ B\ C.\ A - (UN\ x{:}C.\ B\ x)\ \ = (if\ C{=}\{\}\ then\ A\ else\ (INT\ x{:}C.\ A - B\ x))$
$!!a\ B\ C.\ insert\ a\ (INT\ x{:}C.\ B\ x) = (INT\ x{:}C.\ insert\ a\ (B\ x))$
$!!A\ B\ C.\ ((INT\ x{:}C.\ A\ x)\ Un\ B)\ = (INT\ x{:}C.\ A\ x\ Un\ B)$
$!!A\ B\ C.\ A\ Un\ (INT\ x{:}C.\ B\ x)\ = (INT\ x{:}C.\ A\ Un\ B\ x)$
$!!A\ B.\ (INT\ y{:}A.\ INT\ x{:}y.\ B\ x) = (INT\ x{:}\ Union\ A.\ B\ x)$
$!!A\ B\ C.\ (INT\ x{:}A.\ INT\ z{:}\ B(x).\ C\ z) = (INT\ z{:}\ UNION\ A\ B.\ C\ z)$
$!!A\ B\ f.\ (INT\ a{:}A.\ B\ (f\ a))\ \ = (INT\ x{:}f`A.\ B\ x)$
⟨*proof*⟩

3.5.3 Monotonicity of various operations

lemma *image-mono*: $A \subseteq B ==> f`A \subseteq f`B$
⟨*proof*⟩

lemma *Pow-mono*: $A \subseteq B ==> Pow\ A \subseteq Pow\ B$
⟨*proof*⟩

lemma *Union-mono*: $A \subseteq B ==> \bigcup A \subseteq \bigcup B$
⟨*proof*⟩

lemma *Inter-anti-mono*: $B \subseteq A ==> \bigcap A \subseteq \bigcap B$
⟨*proof*⟩

lemma *UN-mono*:
$A \subseteq B ==> (!!x.\ x \in A ==> f\ x \subseteq g\ x) ==>$
$(\bigcup x{\in}A.\ f\ x) \subseteq (\bigcup x{\in}B.\ g\ x)$
⟨*proof*⟩

lemma *INT-anti-mono*:
$B \subseteq A ==> (!!x.\ x \in A ==> f\ x \subseteq g\ x) ==>$
$(\bigcap x{\in}A.\ f\ x) \subseteq (\bigcap x{\in}A.\ g\ x)$
— The last inclusion is POSITIVE!
⟨*proof*⟩

lemma *insert-mono*: $C \subseteq D ==> insert\ a\ C \subseteq insert\ a\ D$
⟨*proof*⟩

lemma *Un-mono*: $A \subseteq C ==> B \subseteq D ==> A \cup B \subseteq C \cup D$
⟨*proof*⟩

lemma *Int-mono*: $A \subseteq C ==> B \subseteq D ==> A \cap B \subseteq C \cap D$
⟨*proof*⟩

lemma *Diff-mono*: $A \subseteq C ==> D \subseteq B ==> A - B \subseteq C - D$
⟨*proof*⟩

lemma *Compl-anti-mono*: $A \subseteq B ==> -B \subseteq -A$

⟨*proof*⟩

Monotonicity of implications.

lemma *in-mono*: $A \subseteq B ==> x \in A --> x \in B$
⟨*proof*⟩

lemma *conj-mono*: $P1 --> Q1 ==> P2 --> Q2 ==> (P1 \ \& \ P2) --> (Q1 \ \& \ Q2)$
⟨*proof*⟩

lemma *disj-mono*: $P1 --> Q1 ==> P2 --> Q2 ==> (P1 \mid P2) --> (Q1 \mid Q2)$
⟨*proof*⟩

lemma *imp-mono*: $Q1 --> P1 ==> P2 --> Q2 ==> (P1 --> P2) --> (Q1 --> Q2)$
⟨*proof*⟩

lemma *imp-refl*: $P --> P$ ⟨*proof*⟩

lemma *ex-mono*: $(!!x.\ P\ x --> Q\ x) ==> (EX\ x.\ P\ x) --> (EX\ x.\ Q\ x)$
⟨*proof*⟩

lemma *all-mono*: $(!!x.\ P\ x --> Q\ x) ==> (ALL\ x.\ P\ x) --> (ALL\ x.\ Q\ x)$
⟨*proof*⟩

lemma *Collect-mono*: $(!!x.\ P\ x --> Q\ x) ==> Collect\ P \subseteq Collect\ Q$
⟨*proof*⟩

lemma *Int-Collect-mono*:
$A \subseteq B ==> (!!x.\ x \in A ==> P\ x --> Q\ x) ==> A \cap Collect\ P \subseteq B \cap Collect\ Q$
⟨*proof*⟩

lemmas *basic-monos* =
subset-refl imp-refl disj-mono conj-mono
ex-mono Collect-mono in-mono

lemma *eq-to-mono*: $a = b ==> c = d ==> b --> d ==> a --> c$
⟨*proof*⟩

lemma *eq-to-mono2*: $a = b ==> c = d ==> \sim b --> \sim d ==> \sim a --> \sim c$
⟨*proof*⟩

3.6 Inverse image of a function

constdefs
vimage :: $('a => 'b) => 'b\ set => 'a\ set$ (**infixr** $-`$ *90*)
$f -` B == \{x.\ f\ x : B\}$

3.6.1 Basic rules

lemma *vimage-eq* [*simp*]: $(a : f -` B) = (f\ a : B)$
⟨*proof*⟩

lemma *vimage-singleton-eq*: $(a : f -` \{b\}) = (f\ a = b)$
⟨*proof*⟩

lemma *vimageI* [*intro*]: $f\ a = b ==> b{:}B ==> a : f -` B$
⟨*proof*⟩

lemma *vimageI2*: $f\ a : A ==> a : f -` A$
⟨*proof*⟩

lemma *vimageE* [*elim*!]: $a{:}\ f -` B ==> (!!x.\ f\ a = x ==> x{:}B ==> P) ==> P$
⟨*proof*⟩

lemma *vimageD*: $a : f -` A ==> f\ a : A$
⟨*proof*⟩

3.6.2 Equations

lemma *vimage-empty* [*simp*]: $f -` \{\} = \{\}$
⟨*proof*⟩

lemma *vimage-Compl*: $f -` (-A) = -(f -` A)$
⟨*proof*⟩

lemma *vimage-Un* [*simp*]: $f -` (A\ \mathit{Un}\ B) = (f -` A)\ \mathit{Un}\ (f -` B)$
⟨*proof*⟩

lemma *vimage-Int* [*simp*]: $f -` (A\ \mathit{Int}\ B) = (f -` A)\ \mathit{Int}\ (f -` B)$
⟨*proof*⟩

lemma *vimage-Union*: $f -` (\mathit{Union}\ A) = (\mathit{UN}\ X{:}A.\ f -` X)$
⟨*proof*⟩

lemma *vimage-UN*: $f{-}`(\mathit{UN}\ x{:}A.\ B\ x) = (\mathit{UN}\ x{:}A.\ f -` B\ x)$
⟨*proof*⟩

lemma *vimage-INT*: $f{-}`(\mathit{INT}\ x{:}A.\ B\ x) = (\mathit{INT}\ x{:}A.\ f -` B\ x)$
⟨*proof*⟩

lemma *vimage-Collect-eq* [*simp*]: $f -` \mathit{Collect}\ P = \{y.\ P\ (f\ y)\}$
⟨*proof*⟩

lemma *vimage-Collect*: $(!!x.\ P\ (f\ x) = Q\ x) ==> f -` (\mathit{Collect}\ P) = \mathit{Collect}\ Q$
⟨*proof*⟩

lemma *vimage-insert*: $f-`(insert\ a\ B) = (f-`\{a\})\ Un\ (f-`B)$
— NOT suitable for rewriting because of the recurrence of $\{a\}$.
⟨*proof*⟩

lemma *vimage-Diff*: $f\ -`\ (A - B) = (f\ -`\ A) - (f\ -`\ B)$
⟨*proof*⟩

lemma *vimage-UNIV* [*simp*]: $f\ -`\ UNIV = UNIV$
⟨*proof*⟩

lemma *vimage-eq-UN*: $f-`B = (UN\ y: B.\ f-`\{y\})$
— NOT suitable for rewriting
⟨*proof*⟩

lemma *vimage-mono*: $A \subseteq B ==> f\ -`\ A \subseteq f\ -`\ B$
— monotonicity
⟨*proof*⟩

3.7 Getting the Contents of a Singleton Set

definition
contents :: $'a\ set \Rightarrow\ 'a$
where
[*code func del*]: $contents\ X = (THE\ x.\ X = \{x\})$

lemma *contents-eq* [*simp*]: $contents\ \{x\} = x$
⟨*proof*⟩

3.8 Transitivity rules for calculational reasoning

lemma *set-rev-mp*: $x{:}A ==> A \subseteq B ==> x{:}B$
⟨*proof*⟩

lemma *set-mp*: $A \subseteq B ==> x{:}A ==> x{:}B$
⟨*proof*⟩

3.9 Code generation for finite sets

code-datatype {} *insert*

3.9.1 Primitive predicates

definition
is-empty :: $'a\ set \Rightarrow bool$
where
[*code func del*]: $is\text{-}empty\ A \longleftrightarrow A = \{\}$
lemmas [*code inline*] = *is-empty-def* [*symmetric*]

lemma *is-empty-insert* [*code func*]:
$is\text{-}empty\ (insert\ a\ A) \longleftrightarrow False$

⟨*proof*⟩

lemma *is-empty-empty* [*code func*]:
is-empty $\{\} \longleftrightarrow True$
⟨*proof*⟩

lemma *Ball-insert* [*code func*]:
$Ball\ (insert\ a\ A)\ P \longleftrightarrow P\ a \wedge Ball\ A\ P$
⟨*proof*⟩

lemma *Ball-empty* [*code func*]:
$Ball\ \{\}\ P \longleftrightarrow True$
⟨*proof*⟩

lemma *Bex-insert* [*code func*]:
$Bex\ (insert\ a\ A)\ P \longleftrightarrow P\ a \vee Bex\ A\ P$
⟨*proof*⟩

lemma *Bex-empty* [*code func*]:
$Bex\ \{\}\ P \longleftrightarrow False$
⟨*proof*⟩

3.9.2 Primitive operations

lemma *minus-insert* [*code func*]:
$insert\ (a::'a::eq)\ A - B = (let\ C = A - B\ in\ if\ a \in B\ then\ C\ else\ insert\ a\ C)$
⟨*proof*⟩

lemma *minus-empty1* [*code func*]:
$\{\} - A = \{\}$
⟨*proof*⟩

lemma *minus-empty2* [*code func*]:
$A - \{\} = A$
⟨*proof*⟩

lemma *inter-insert* [*code func*]:
$insert\ a\ A \cap B = (let\ C = A \cap B\ in\ if\ a \in B\ then\ insert\ a\ C\ else\ C)$
⟨*proof*⟩

lemma *inter-empty1* [*code func*]:
$\{\} \cap A = \{\}$
⟨*proof*⟩

lemma *inter-empty2* [*code func*]:
$A \cap \{\} = \{\}$
⟨*proof*⟩

lemma *union-insert* [*code func*]:

insert a $A \cup B$ = (*let* $C = A \cup B$ *in if* $a \in B$ *then* C *else insert a* C)
⟨*proof*⟩

lemma *union-empty1* [*code func*]:
$\{\} \cup A = A$
⟨*proof*⟩

lemma *union-empty2* [*code func*]:
$A \cup \{\} = A$
⟨*proof*⟩

lemma *INTER-insert* [*code func*]:
INTER (*insert a A*) $f = f\ a \cap$ *INTER* $A\ f$
⟨*proof*⟩

lemma *INTER-singleton* [*code func*]:
INTER $\{a\}\ f = f\ a$
⟨*proof*⟩

lemma *UNION-insert* [*code func*]:
UNION (*insert a A*) $f = f\ a \cup$ *UNION* $A\ f$
⟨*proof*⟩

lemma *UNION-empty* [*code func*]:
UNION $\{\}\ f = \{\}$
⟨*proof*⟩

lemma *contents-insert* [*code func*]:
contents (*insert a A*) = *contents* (*insert a* $(A - \{a\})$)
⟨*proof*⟩
declare *contents-eq* [*code func*]

3.9.3 Derived predicates

lemma *in-code* [*code func*]:
$a \in A \longleftrightarrow (\exists x \in A.\ a = x)$
⟨*proof*⟩

instance *set* :: (*eq*) *eq* ⟨*proof*⟩

lemma *eq-set-code* [*code func*]:
fixes $A\ B$ :: *'a::eq set*
shows $A = B \longleftrightarrow A \subseteq B \land B \subseteq A$
⟨*proof*⟩

lemma *subset-eq-code* [*code func*]:
fixes $A\ B$ :: *'a::eq set*
shows $A \subseteq B \longleftrightarrow (\forall x \in A.\ x \in B)$
⟨*proof*⟩

lemma *subset-code* [*code func*]:
fixes $A\ B :: 'a::eq\ set$
shows $A \subset B \longleftrightarrow A \subseteq B \wedge \neg\, B \subseteq A$
⟨*proof*⟩

3.9.4 Derived operations

lemma *image-code* [*code func*]:
$image\ f\ A = UNION\ A\ (\lambda x.\ \{f\ x\})$ ⟨*proof*⟩

definition
$project :: ('a \Rightarrow bool) \Rightarrow 'a\ set \Rightarrow 'a\ set$ **where**
[*code func del*, *code post*]: $project\ P\ A = \{a{\in}A.\ P\ a\}$

lemmas [*symmetric*, *code inline*] = *project-def*

lemma *project-code* [*code func*]:
$project\ P\ A = UNION\ A\ (\lambda a.\ if\ P\ a\ then\ \{a\}\ else\ \{\})$
⟨*proof*⟩

lemma *Inter-code* [*code func*]:
$Inter\ A = INTER\ A\ (\lambda x.\ x)$
⟨*proof*⟩

lemma *Union-code* [*code func*]:
$Union\ A = UNION\ A\ (\lambda x.\ x)$
⟨*proof*⟩

code-reserved *SML union inter*

3.10 Basic ML bindings

⟨*ML*⟩

end

4 Fun: Notions about functions

theory *Fun*
imports *Set*
begin

constdefs
$fun\text{-}upd :: ('a => 'b) => 'a => 'b => ('a => 'b)$
$fun\text{-}upd\ f\ a\ b == \%\ x.\ if\ x{=}a\ then\ b\ else\ f\ x$

nonterminals

updbinds updbind
syntax
-updbind :: [*'a*, *'a*] => *updbind* ((*2*- :=/ -))
:: *updbind* => *updbinds* (-)
-updbinds:: [*updbind*, *updbinds*] => *updbinds* (-,/ -)
-Update :: [*'a*, *updbinds*] => *'a* (-/'((-)') [*1000*,*0*] *900*)

translations
-Update f (*-updbinds b bs*) == *-Update* (*-Update f b*) *bs*
f(*x*:=*y*) == *fun-upd f x y*

definition
override-on :: $('a \Rightarrow 'b) \Rightarrow ('a \Rightarrow 'b) \Rightarrow 'a\ set \Rightarrow 'a \Rightarrow 'b$
where
override-on f g A = (λa. *if* $a \in A$ *then g a else f a*)

definition
id :: $'a \Rightarrow 'a$
where
id = (λx. *x*)

definition
comp :: $('b \Rightarrow 'c) \Rightarrow ('a \Rightarrow 'b) \Rightarrow 'a \Rightarrow 'c$ (**infixl** *o 55*)
where
f o g = (λx. *f* (*g x*))

notation (*xsymbols*)
comp (**infixl** $\circ$ *55*)

notation (*HTML* **output**)
comp (**infixl** $\circ$ *55*)

compatibility

lemmas *o-def* = *comp-def*

constdefs
inj-on :: [*'a* => *'b*, *'a set*] => *bool*
inj-on f A == ! *x*:*A*. ! *y*:*A*. *f*(*x*)=*f*(*y*) --> *x*=*y*

A common special case: functions injective over the entire domain type.

abbreviation
inj f == *inj-on f UNIV*

constdefs
surj :: (*'a* => *'b*) => *bool*
surj f == ! *y*. *?* *x*. *y*=*f*(*x*)

bij :: $('a => 'b) => bool$
bij f == *inj* f & *surj* f

As a simplification rule, it replaces all function equalities by first-order equalities.

lemma *expand-fun-eq*: $f = g \longleftrightarrow (\forall x.\ f\ x = g\ x)$
⟨*proof*⟩

lemma *apply-inverse*:
$[|\ f(x)=u;\ \ !!x.\ P(x) ==> g(f(x)) = x;\ \ P(x)\ |] ==> x=g(u)$
⟨*proof*⟩

The Identity Function: *id*

lemma *id-apply* [*simp*]: *id* $x = x$
⟨*proof*⟩

lemma *inj-on-id*[*simp*]: *inj-on id A*
⟨*proof*⟩

lemma *inj-on-id2*[*simp*]: *inj-on* $(\%x.\ x)$ A
⟨*proof*⟩

lemma *surj-id*[*simp*]: *surj id*
⟨*proof*⟩

lemma *bij-id*[*simp*]: *bij id*
⟨*proof*⟩

4.1 The Composition Operator: $f \circ g$

lemma *o-apply* [*simp*]: $(f\ o\ g)\ x = f\ (g\ x)$
⟨*proof*⟩

lemma *o-assoc*: $f\ o\ (g\ o\ h) = f\ o\ g\ o\ h$
⟨*proof*⟩

lemma *id-o* [*simp*]: *id* $o\ g = g$
⟨*proof*⟩

lemma *o-id* [*simp*]: $f\ o$ *id* $= f$
⟨*proof*⟩

lemma *image-compose*: $(f\ o\ g)$ ` $r = f`(g`r)$
⟨*proof*⟩

lemma *image-eq-UN*: $f`A = (UN\ x{:}A.\ \{f\ x\})$
⟨*proof*⟩

lemma *UN-o*: $UNION\ A\ (g\ o\ f) = UNION\ (f`A)\ g$

⟨*proof*⟩

4.2 The Injectivity Predicate, *inj*

NB: *inj* now just translates to *inj-on*

For Proofs in *Tools/datatype-rep-proofs*

lemma *datatype-injI*:
  $(!!\ x.\ ALL\ y.\ f(x) = f(y) \dashrightarrow x{=}y) ==> inj(f)$
⟨*proof*⟩

theorem *range-ex1-eq*: $inj\ f \Longrightarrow b : range\ f = (EX!\ x.\ b = f\ x)$
⟨*proof*⟩

lemma *injD*: $[|\ inj(f); f(x) = f(y)\ |] ==> x{=}y$
⟨*proof*⟩

lemma *inj-eq*: $inj(f) ==> (f(x) = f(y)) = (x{=}y)$
⟨*proof*⟩

4.3 The Predicate *inj-on*: Injectivity On A Restricted Domain

lemma *inj-onI*:
  $(!!\ x\ y.\ [|\ \ x{:}A;\ \ y{:}A;\ \ f(x) = f(y)\ |] ==> x{=}y) ==> inj\text{-}on\ f\ A$
⟨*proof*⟩

lemma *inj-on-inverseI*: $(!!x.\ x{:}A ==> g(f(x)) = x) ==> inj\text{-}on\ f\ A$
⟨*proof*⟩

lemma *inj-onD*: $[|\ inj\text{-}on\ f\ A;\ \ f(x){=}f(y);\ \ x{:}A;\ \ y{:}A\ |] ==> x{=}y$
⟨*proof*⟩

lemma *inj-on-iff*: $[|\ inj\text{-}on\ f\ A;\ \ x{:}A;\ \ y{:}A\ |] ==> (f(x){=}f(y)) = (x{=}y)$
⟨*proof*⟩

lemma *comp-inj-on*:
  $[|\ inj\text{-}on\ f\ A;\ \ inj\text{-}on\ g\ (f`A)\ |] ==> inj\text{-}on\ (g\ o\ f)\ A$
⟨*proof*⟩

lemma *inj-on-imageI*: $inj\text{-}on\ (g\ o\ f)\ A \Longrightarrow inj\text{-}on\ g\ (f\ `\ A)$
⟨*proof*⟩

lemma *inj-on-image-iff*: $[\![\ ALL\ x{:}A.\ ALL\ y{:}A.\ (g(f\ x) = g(f\ y)) = (g\ x = g\ y);$
$inj\text{-}on\ f\ A\ ]\!] \Longrightarrow inj\text{-}on\ g\ (f\ `\ A) = inj\text{-}on\ g\ A$
⟨*proof*⟩

lemma *inj-on-contraD*: $[|\ inj\text{-}on\ f\ A;\ \ {\sim}x{=}y;\ \ x{:}A;\ \ y{:}A\ |] ==> {\sim}\ f(x){=}f(y)$

⟨*proof*⟩

lemma *inj-singleton*: *inj* (%*s*. {*s*})
⟨*proof*⟩

lemma *inj-on-empty*[*iff*]: *inj-on f* {}
⟨*proof*⟩

lemma *subset-inj-on*: [| *inj-on f B*; *A* <= *B* |] ==> *inj-on f A*
⟨*proof*⟩

lemma *inj-on-Un*:
inj-on f (*A Un B*) =
(*inj-on f A* & *inj-on f B* & *f*‘(*A*−*B*) *Int f*‘(*B*−*A*) = {})
⟨*proof*⟩

lemma *inj-on-insert*[*iff*]:
inj-on f (*insert a A*) = (*inj-on f A* & *f a* ~: *f*‘(*A*−{*a*}))
⟨*proof*⟩

lemma *inj-on-diff*: *inj-on f A* ==> *inj-on f* (*A*−*B*)
⟨*proof*⟩

4.4 The Predicate *surj*: Surjectivity

lemma *surjI*: (!! *x*. *g*(*f x*) = *x*) ==> *surj g*
⟨*proof*⟩

lemma *surj-range*: *surj f* ==> *range f* = *UNIV*
⟨*proof*⟩

lemma *surjD*: *surj f* ==> *EX x*. *y* = *f x*
⟨*proof*⟩

lemma *surjE*: *surj f* ==> (!!*x*. *y* = *f x* ==> *C*) ==> *C*
⟨*proof*⟩

lemma *comp-surj*: [| *surj f*; *surj g* |] ==> *surj* (*g o f*)
⟨*proof*⟩

4.5 The Predicate *bij*: Bijectivity

lemma *bijI*: [| *inj f*; *surj f* |] ==> *bij f*
⟨*proof*⟩

lemma *bij-is-inj*: *bij f* ==> *inj f*
⟨*proof*⟩

lemma *bij-is-surj*: *bij f* ==> *surj f*
⟨*proof*⟩

4.6 Facts About the Identity Function

We seem to need both the *id* forms and the $\lambda x.\ x$ forms. The latter can arise by rewriting, while *id* may be used explicitly.

lemma *image-ident* [*simp*]: $(\%x.\ x)$ ‘ $Y = Y$
⟨*proof*⟩

lemma *image-id* [*simp*]: *id* ‘ $Y = Y$
⟨*proof*⟩

lemma *vimage-ident* [*simp*]: $(\%x.\ x)$ −‘ $Y = Y$
⟨*proof*⟩

lemma *vimage-id* [*simp*]: *id* −‘ $A = A$
⟨*proof*⟩

lemma *vimage-image-eq* [*noatp*]: f −‘ (f ‘ A) = $\{y.\ EX\ x{:}A.\ f\ x = f\ y\}$
⟨*proof*⟩

lemma *image-vimage-subset*: f ‘ (f −‘ A) <= A
⟨*proof*⟩

lemma *image-vimage-eq* [*simp*]: f ‘ (f −‘ A) = A *Int range f*
⟨*proof*⟩

lemma *surj-image-vimage-eq*: *surj f* ==> f ‘ (f −‘ A) = A
⟨*proof*⟩

lemma *inj-vimage-image-eq*: *inj f* ==> f −‘ (f ‘ A) = A
⟨*proof*⟩

lemma *vimage-subsetD*: *surj f* ==> f −‘ B <= A ==> B <= f ‘ A
⟨*proof*⟩

lemma *vimage-subsetI*: *inj f* ==> B <= f ‘ A ==> f −‘ B <= A
⟨*proof*⟩

lemma *vimage-subset-eq*: *bij f* ==> (f −‘ B <= A) = (B <= f ‘ A)
⟨*proof*⟩

lemma *image-Int-subset*: f‘(A *Int* B) <= f‘A *Int* f‘B
⟨*proof*⟩

lemma *image-diff-subset*: f‘A − f‘B <= f‘(A − B)
⟨*proof*⟩

lemma *inj-on-image-Int*:
[| *inj-on f C*; A<=C; B<=C |] ==> f‘(A *Int* B) = f‘A *Int* f‘B
⟨*proof*⟩

lemma *inj-on-image-set-diff*:
[| *inj-on f C*; *A*<=*C*; *B*<=*C* |] ==> *f*‘(*A*−*B*) = *f*‘*A* − *f*‘*B*
⟨*proof*⟩

lemma *image-Int*: *inj f* ==> *f*‘(*A Int B*) = *f*‘*A Int f*‘*B*
⟨*proof*⟩

lemma *image-set-diff*: *inj f* ==> *f*‘(*A*−*B*) = *f*‘*A* − *f*‘*B*
⟨*proof*⟩

lemma *inj-image-mem-iff*: *inj f* ==> (*f a* : *f*‘*A*) = (*a* : *A*)
⟨*proof*⟩

lemma *inj-image-subset-iff*: *inj f* ==> (*f*‘*A* <= *f*‘*B*) = (*A*<=*B*)
⟨*proof*⟩

lemma *inj-image-eq-iff*: *inj f* ==> (*f*‘*A* = *f*‘*B*) = (*A* = *B*)
⟨*proof*⟩

lemma *image-UN*: (*f* ‘ (*UNION A B*)) = (*UN x*:*A*.(*f* ‘ (*B x*)))
⟨*proof*⟩

lemma *image-INT*:
[| *inj-on f C*; *ALL x*:*A*. *B x* <= *C*; *j*:*A* |]
==> *f* ‘ (*INTER A B*) = (*INT x*:*A*. *f* ‘ *B x*)
⟨*proof*⟩

lemma *bij-image-INT*: *bij f* ==> *f* ‘ (*INTER A B*) = (*INT x*:*A*. *f* ‘ *B x*)
⟨*proof*⟩

lemma *surj-Compl-image-subset*: *surj f* ==> −(*f*‘*A*) <= *f*‘(−*A*)
⟨*proof*⟩

lemma *inj-image-Compl-subset*: *inj f* ==> *f*‘(−*A*) <= −(*f*‘*A*)
⟨*proof*⟩

lemma *bij-image-Compl-eq*: *bij f* ==> *f*‘(−*A*) = −(*f*‘*A*)
⟨*proof*⟩

4.7 Function Updating

lemma *fun-upd-idem-iff*: (*f*(*x*:=*y*) = *f*) = (*f x* = *y*)
⟨*proof*⟩

lemmas *fun-upd-idem* = *fun-upd-idem-iff* [*THEN iffD2*, *standard*]

lemmas *fun-upd-triv* = *refl* [*THEN fun-upd-idem*]
declare *fun-upd-triv* [*iff*]

lemma *fun-upd-apply* [*simp*]: $(f(x{:=}y))z = (\textit{if } z{=}x \textit{ then } y \textit{ else } f\ z)$
⟨*proof*⟩

lemma *fun-upd-same*: $(f(x{:=}y))\ x = y$
⟨*proof*⟩

lemma *fun-upd-other*: $z{\sim}{=}x \Longrightarrow (f(x{:=}y))\ z = f\ z$
⟨*proof*⟩

lemma *fun-upd-upd* [*simp*]: $f(x{:=}y,x{:=}z) = f(x{:=}z)$
⟨*proof*⟩

lemma *fun-upd-twist*: $a\ {\sim}{=}\ c \Longrightarrow (m(a{:=}b))(c{:=}d) = (m(c{:=}d))(a{:=}b)$
⟨*proof*⟩

lemma *inj-on-fun-updI*: $[\![\ \textit{inj-on } f\ A;\ y \notin f`A\ ]\!] \Longrightarrow \textit{inj-on}\ (f(x{:=}y))\ A$
⟨*proof*⟩

lemma *fun-upd-image*:
$f(x{:=}y)\ `\ A = (\textit{if } x \in A \textit{ then insert } y\ (f\ `\ (A - \{x\})) \textit{ else } f\ `\ A)$
⟨*proof*⟩

4.8 *override-on*

lemma *override-on-emptyset*[*simp*]: *override-on f g* {} = *f*
⟨*proof*⟩

lemma *override-on-apply-notin*[*simp*]: $a\ {\sim}{:}\ A \Longrightarrow (\textit{override-on } f\ g\ A)\ a = f\ a$
⟨*proof*⟩

lemma *override-on-apply-in*[*simp*]: $a : A \Longrightarrow (\textit{override-on } f\ g\ A)\ a = g\ a$
⟨*proof*⟩

4.9 swap

definition
$swap :: {'a} \Rightarrow {'a} \Rightarrow ({'a} \Rightarrow {'b}) \Rightarrow ({'a} \Rightarrow {'b})$
where
$swap\ a\ b\ f = f\ (a := f\ b,\ b{:=}\ f\ a)$

lemma *swap-self*: $swap\ a\ a\ f = f$
⟨*proof*⟩

lemma *swap-commute*: $swap\ a\ b\ f = swap\ b\ a\ f$

⟨*proof*⟩

lemma *swap-nilpotent* [*simp*]: *swap a b* (*swap a b f*) = *f*
⟨*proof*⟩

lemma *inj-on-imp-inj-on-swap*:
[|*inj-on f A*; $a \in A$; $b \in A$|] ==> *inj-on* (*swap a b f*) *A*
⟨*proof*⟩

lemma *inj-on-swap-iff* [*simp*]:
assumes *A*: $a \in A$ $b \in A$ **shows** *inj-on* (*swap a b f*) *A* = *inj-on f A*
⟨*proof*⟩

lemma *surj-imp-surj-swap*: *surj f* ==> *surj* (*swap a b f*)
⟨*proof*⟩

lemma *surj-swap-iff* [*simp*]: *surj* (*swap a b f*) = *surj f*
⟨*proof*⟩

lemma *bij-swap-iff*: *bij* (*swap a b f*) = *bij f*
⟨*proof*⟩

4.10 Proof tool setup

simplifies terms of the form f(...,x:=y,...,x:=z,...) to f(...,x:=z,...)

⟨*ML*⟩

4.11 Code generator setup

code-const *op* ∘
(*SML* **infixl** *5 o*)
(*Haskell* **infixr** *9* .)

code-const *id*
(*Haskell id*)

4.12 ML legacy bindings

⟨*ML*⟩

end

5 Orderings: Syntactic and abstract orders

theory *Orderings*
imports *Set Fun*
uses

~~/src/Provers/order.ML
begin

5.1 Partial orders

class *order* = *ord* +
assumes *less-le*: $x < y \longleftrightarrow x \leq y \wedge x \neq y$
and *order-refl* [*iff*]: $x \leq x$
and *order-trans*: $x \leq y \Longrightarrow y \leq z \Longrightarrow x \leq z$
assumes *antisym*: $x \leq y \Longrightarrow y \leq x \Longrightarrow x = y$
begin

Reflexivity.

lemma *eq-refl*: $x = y \Longrightarrow x \leq y$
— This form is useful with the classical reasoner.
⟨*proof*⟩

lemma *less-irrefl* [*iff*]: $\neg\ x < x$
⟨*proof*⟩

lemma *le-less*: $x \leq y \longleftrightarrow x < y \vee x = y$
— NOT suitable for iff, since it can cause PROOF FAILED.
⟨*proof*⟩

lemma *le-imp-less-or-eq*: $x \leq y \Longrightarrow x < y \vee x = y$
⟨*proof*⟩

lemma *less-imp-le*: $x < y \Longrightarrow x \leq y$
⟨*proof*⟩

lemma *less-imp-neq*: $x < y \Longrightarrow x \neq y$
⟨*proof*⟩

Useful for simplification, but too risky to include by default.

lemma *less-imp-not-eq*: $x < y \Longrightarrow (x = y) \longleftrightarrow False$
⟨*proof*⟩

lemma *less-imp-not-eq2*: $x < y \Longrightarrow (y = x) \longleftrightarrow False$
⟨*proof*⟩

Transitivity rules for calculational reasoning

lemma *neq-le-trans*: $a \neq b \Longrightarrow a \leq b \Longrightarrow a < b$
⟨*proof*⟩

lemma *le-neq-trans*: $a \leq b \Longrightarrow a \neq b \Longrightarrow a < b$
⟨*proof*⟩

Asymmetry.

lemma *less-not-sym*: $x < y \Longrightarrow \neg\ (y < x)$

⟨*proof*⟩

lemma *less-asym*: $x < y \Longrightarrow (\neg P \Longrightarrow y < x) \Longrightarrow P$
⟨*proof*⟩

lemma *eq-iff*: $x = y \longleftrightarrow x \leq y \wedge y \leq x$
⟨*proof*⟩

lemma *antisym-conv*: $y \leq x \Longrightarrow x \leq y \longleftrightarrow x = y$
⟨*proof*⟩

lemma *less-imp-neq*: $x < y \Longrightarrow x \neq y$
⟨*proof*⟩

Transitivity.

lemma *less-trans*: $x < y \Longrightarrow y < z \Longrightarrow x < z$
⟨*proof*⟩

lemma *le-less-trans*: $x \leq y \Longrightarrow y < z \Longrightarrow x < z$
⟨*proof*⟩

lemma *less-le-trans*: $x < y \Longrightarrow y \leq z \Longrightarrow x < z$
⟨*proof*⟩

Useful for simplification, but too risky to include by default.

lemma *less-imp-not-less*: $x < y \Longrightarrow (\neg\ y < x) \longleftrightarrow \mathit{True}$
⟨*proof*⟩

lemma *less-imp-triv*: $x < y \Longrightarrow (y < x \longrightarrow P) \longleftrightarrow \mathit{True}$
⟨*proof*⟩

Transitivity rules for calculational reasoning

lemma *less-asym'*: $a < b \Longrightarrow b < a \Longrightarrow P$
⟨*proof*⟩

Reverse order

lemma *order-reverse*:
  order $(op \geq)$ $(op >)$
⟨*proof*⟩

end

5.2 Linear (total) orders

class *linorder* = *order* +
  assumes *linear*: $x \leq y \vee y \leq x$
begin

lemma *less-linear*: $x < y \vee x = y \vee y < x$

⟨*proof*⟩

lemma *le-less-linear*: $x \leq y \vee y < x$
⟨*proof*⟩

lemma *le-cases* [*case-names le ge*]:
$(x \leq y \Longrightarrow P) \Longrightarrow (y \leq x \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

lemma *linorder-cases* [*case-names less equal greater*]:
$(x < y \Longrightarrow P) \Longrightarrow (x = y \Longrightarrow P) \Longrightarrow (y < x \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

lemma *not-less*: $\neg\ x < y \longleftrightarrow y \leq x$
⟨*proof*⟩

lemma *not-less-iff-gr-or-eq*:
$\neg(x < y) \longleftrightarrow (x > y \mid x = y)$
⟨*proof*⟩

lemma *not-le*: $\neg\ x \leq y \longleftrightarrow y < x$
⟨*proof*⟩

lemma *neq-iff*: $x \neq y \longleftrightarrow x < y \vee y < x$
⟨*proof*⟩

lemma *neqE*: $x \neq y \Longrightarrow (x < y \Longrightarrow R) \Longrightarrow (y < x \Longrightarrow R) \Longrightarrow R$
⟨*proof*⟩

lemma *antisym-conv1*: $\neg\ x < y \Longrightarrow x \leq y \longleftrightarrow x = y$
⟨*proof*⟩

lemma *antisym-conv2*: $x \leq y \Longrightarrow \neg\ x < y \longleftrightarrow x = y$
⟨*proof*⟩

lemma *antisym-conv3*: $\neg\ y < x \Longrightarrow \neg\ x < y \longleftrightarrow x = y$
⟨*proof*⟩

Replacing the old Nat.leI

lemma *leI*: $\neg\ x < y \Longrightarrow y \leq x$
⟨*proof*⟩

lemma *leD*: $y \leq x \Longrightarrow \neg\ x < y$
⟨*proof*⟩

lemma *not-leE*: $\neg\ y \leq x \Longrightarrow x < y$
⟨*proof*⟩

Reverse order

lemma *linorder-reverse*:
$linorder\ (op\ \geq)\ (op\ >)$
⟨*proof*⟩

min/max

for historic reasons, definitions are done in context ord

definition (**in** *ord*)
$min :: 'a \Rightarrow 'a \Rightarrow 'a$ **where**
$[code\ unfold,\ code\ inline\ del]$: $min\ a\ b = (if\ a \leq b\ then\ a\ else\ b)$

definition (**in** *ord*)
$max :: 'a \Rightarrow 'a \Rightarrow 'a$ **where**
$[code\ unfold,\ code\ inline\ del]$: $max\ a\ b = (if\ a \leq b\ then\ b\ else\ a)$

lemma *min-le-iff-disj*:
$min\ x\ y \leq z \longleftrightarrow x \leq z \lor y \leq z$
⟨*proof*⟩

lemma *le-max-iff-disj*:
$z \leq max\ x\ y \longleftrightarrow z \leq x \lor z \leq y$
⟨*proof*⟩

lemma *min-less-iff-disj*:
$min\ x\ y < z \longleftrightarrow x < z \lor y < z$
⟨*proof*⟩

lemma *less-max-iff-disj*:
$z < max\ x\ y \longleftrightarrow z < x \lor z < y$
⟨*proof*⟩

lemma *min-less-iff-conj* [*simp*]:
$z < min\ x\ y \longleftrightarrow z < x \land z < y$
⟨*proof*⟩

lemma *max-less-iff-conj* [*simp*]:
$max\ x\ y < z \longleftrightarrow x < z \land y < z$
⟨*proof*⟩

lemma *split-min* [*noatp*]:
$P\ (min\ i\ j) \longleftrightarrow (i \leq j \longrightarrow P\ i) \land (\neg\ i \leq j \longrightarrow P\ j)$
⟨*proof*⟩

lemma *split-max* [*noatp*]:
$P\ (max\ i\ j) \longleftrightarrow (i \leq j \longrightarrow P\ j) \land (\neg\ i \leq j \longrightarrow P\ i)$
⟨*proof*⟩

end

5.3 Reasoning tools setup

⟨*ML*⟩

Declarations to set up transitivity reasoner of partial and linear orders.

context *order*
begin

lemmas
[*order add less-reflE*: *order op* = :: ′*a* ⇒ ′*a* ⇒ *bool op* <= *op* <] =
less-irrefl [*THEN notE*]
lemmas
[*order add le-refl*: *order op* = :: ′*a* => ′*a* => *bool op* <= *op* <] =
order-refl
lemmas
[*order add less-imp-le*: *order op* = :: ′*a* => ′*a* => *bool op* <= *op* <] =
less-imp-le
lemmas
[*order add eqI*: *order op* = :: ′*a* => ′*a* => *bool op* <= *op* <] =
antisym
lemmas
[*order add eqD1*: *order op* = :: ′*a* => ′*a* => *bool op* <= *op* <] =
eq-refl
lemmas
[*order add eqD2*: *order op* = :: ′*a* => ′*a* => *bool op* <= *op* <] =
sym [*THEN eq-refl*]
lemmas
[*order add less-trans*: *order op* = :: ′*a* => ′*a* => *bool op* <= *op* <] =
less-trans
lemmas
[*order add less-le-trans*: *order op* = :: ′*a* => ′*a* => *bool op* <= *op* <] =
less-le-trans
lemmas
[*order add le-less-trans*: *order op* = :: ′*a* => ′*a* => *bool op* <= *op* <] =
le-less-trans
lemmas
[*order add le-trans*: *order op* = :: ′*a* => ′*a* => *bool op* <= *op* <] =
order-trans
lemmas
[*order add le-neq-trans*: *order op* = :: ′*a* => ′*a* => *bool op* <= *op* <] =
le-neq-trans
lemmas
[*order add neq-le-trans*: *order op* = :: ′*a* => ′*a* => *bool op* <= *op* <] =
neq-le-trans
lemmas
[*order add less-imp-neq*: *order op* = :: ′*a* => ′*a* => *bool op* <= *op* <] =
less-imp-neq
lemmas

[*order add eq-neq-eq-imp-neq*: *order op* = :: *'a* => *'a* => *bool op* <= *op* <] =
eq-neq-eq-imp-neq
lemmas
[*order add not-sym*: *order op* = :: *'a* => *'a* => *bool op* <= *op* <] =
not-sym

end

context *linorder*
begin

lemmas
[*order del*: *order op* = :: *'a* => *'a* => *bool op* <= *op* <] = -

lemmas
[*order add less-reflE*: *linorder op* = :: *'a* => *'a* => *bool op* <= *op* <] =
less-irrefl [*THEN notE*]
lemmas
[*order add le-refl*: *linorder op* = :: *'a* => *'a* => *bool op* <= *op* <] =
order-refl
lemmas
[*order add less-imp-le*: *linorder op* = :: *'a* => *'a* => *bool op* <= *op* <] =
less-imp-le
lemmas
[*order add not-lessI*: *linorder op* = :: *'a* => *'a* => *bool op* <= *op* <] =
not-less [*THEN iffD2*]
lemmas
[*order add not-leI*: *linorder op* = :: *'a* => *'a* => *bool op* <= *op* <] =
not-le [*THEN iffD2*]
lemmas
[*order add not-lessD*: *linorder op* = :: *'a* => *'a* => *bool op* <= *op* <] =
not-less [*THEN iffD1*]
lemmas
[*order add not-leD*: *linorder op* = :: *'a* => *'a* => *bool op* <= *op* <] =
not-le [*THEN iffD1*]
lemmas
[*order add eqI*: *linorder op* = :: *'a* => *'a* => *bool op* <= *op* <] =
antisym
lemmas
[*order add eqD1*: *linorder op* = :: *'a* => *'a* => *bool op* <= *op* <] =
eq-refl
lemmas
[*order add eqD2*: *linorder op* = :: *'a* => *'a* => *bool op* <= *op* <] =
sym [*THEN eq-refl*]
lemmas
[*order add less-trans*: *linorder op* = :: *'a* => *'a* => *bool op* <= *op* <] =
less-trans
lemmas
[*order add less-le-trans*: *linorder op* = :: *'a* => *'a* => *bool op* <= *op* <] =

less-le-trans
lemmas
[*order add le-less-trans*: *linorder op* = :: *'a* => *'a* => *bool op* <= *op* <] =
le-less-trans
lemmas
[*order add le-trans*: *linorder op* = :: *'a* => *'a* => *bool op* <= *op* <] =
order-trans
lemmas
[*order add le-neq-trans*: *linorder op* = :: *'a* => *'a* => *bool op* <= *op* <] =
le-neq-trans
lemmas
[*order add neq-le-trans*: *linorder op* = :: *'a* => *'a* => *bool op* <= *op* <] =
neq-le-trans
lemmas
[*order add less-imp-neq*: *linorder op* = :: *'a* => *'a* => *bool op* <= *op* <] =
less-imp-neq
lemmas
[*order add eq-neq-eq-imp-neq*: *linorder op* = :: *'a* => *'a* => *bool op* <= *op* <] =
eq-neq-eq-imp-neq
lemmas
[*order add not-sym*: *linorder op* = :: *'a* => *'a* => *bool op* <= *op* <] =
not-sym

end

⟨*ML*⟩

5.4 Dense orders

class *dense-linear-order* = *linorder* +
assumes *gt-ex*: $\exists\, y.\ x < y$
and *lt-ex*: $\exists\, y.\ y < x$
and *dense*: $x < y \Longrightarrow (\exists\, z.\ x < z \wedge z < y)$

begin

lemma *interval-empty-iff* :
$\{y.\ x < y \wedge y < z\} = \{\} \longleftrightarrow \neg\, x < z$
⟨*proof*⟩

end

5.5 Name duplicates

lemmas *order-less-le* = *less-le*
lemmas *order-eq-refl* = *order-class.eq-refl*
lemmas *order-less-irrefl* = *order-class.less-irrefl*
lemmas *order-le-less* = *order-class.le-less*
lemmas *order-le-imp-less-or-eq* = *order-class.le-imp-less-or-eq*

lemmas *order-less-imp-le = order-class.less-imp-le*
lemmas *order-less-imp-not-eq = order-class.less-imp-not-eq*
lemmas *order-less-imp-not-eq2 = order-class.less-imp-not-eq2*
lemmas *order-neq-le-trans = order-class.neq-le-trans*
lemmas *order-le-neq-trans = order-class.le-neq-trans*

lemmas *order-antisym = antisym*
lemmas *order-less-not-sym = order-class.less-not-sym*
lemmas *order-less-asym = order-class.less-asym*
lemmas *order-eq-iff = order-class.eq-iff*
lemmas *order-antisym-conv = order-class.antisym-conv*
lemmas *order-less-trans = order-class.less-trans*
lemmas *order-le-less-trans = order-class.le-less-trans*
lemmas *order-less-le-trans = order-class.less-le-trans*
lemmas *order-less-imp-not-less = order-class.less-imp-not-less*
lemmas *order-less-imp-triv = order-class.less-imp-triv*
lemmas *order-less-asym' = order-class.less-asym'*

lemmas *linorder-linear = linear*
lemmas *linorder-less-linear = linorder-class.less-linear*
lemmas *linorder-le-less-linear = linorder-class.le-less-linear*
lemmas *linorder-le-cases = linorder-class.le-cases*
lemmas *linorder-not-less = linorder-class.not-less*
lemmas *linorder-not-le = linorder-class.not-le*
lemmas *linorder-neq-iff = linorder-class.neq-iff*
lemmas *linorder-neqE = linorder-class.neqE*
lemmas *linorder-antisym-conv1 = linorder-class.antisym-conv1*
lemmas *linorder-antisym-conv2 = linorder-class.antisym-conv2*
lemmas *linorder-antisym-conv3 = linorder-class.antisym-conv3*

5.6 Bounded quantifiers

syntax
-All-less :: [idt, 'a, bool] => bool ((3ALL -<-./ -) [0, 0, 10] 10)
-Ex-less :: [idt, 'a, bool] => bool ((3EX -<-./ -) [0, 0, 10] 10)
-All-less-eq :: [idt, 'a, bool] => bool ((3ALL -<=-./ -) [0, 0, 10] 10)
-Ex-less-eq :: [idt, 'a, bool] => bool ((3EX -<=-./ -) [0, 0, 10] 10)

-All-greater :: [idt, 'a, bool] => bool ((3ALL ->-./ -) [0, 0, 10] 10)
-Ex-greater :: [idt, 'a, bool] => bool ((3EX ->-./ -) [0, 0, 10] 10)
-All-greater-eq :: [idt, 'a, bool] => bool ((3ALL ->=-./ -) [0, 0, 10] 10)
-Ex-greater-eq :: [idt, 'a, bool] => bool ((3EX ->=-./ -) [0, 0, 10] 10)

syntax (*xsymbols*)
-All-less :: [idt, 'a, bool] => bool ((3∀ -<-./ -) [0, 0, 10] 10)
-Ex-less :: [idt, 'a, bool] => bool ((3∃ -<-./ -) [0, 0, 10] 10)
-All-less-eq :: [idt, 'a, bool] => bool ((3∀ -≤-./ -) [0, 0, 10] 10)
-Ex-less-eq :: [idt, 'a, bool] => bool ((3∃ -≤-./ -) [0, 0, 10] 10)

-*All-greater* :: [*idt*, '*a*, *bool*] => *bool* ((*3*∀ ->-./ -) [*0*, *0*, *10*] *10*)
-*Ex-greater* :: [*idt*, '*a*, *bool*] => *bool* ((*3*∃ ->-./ -) [*0*, *0*, *10*] *10*)
-*All-greater-eq* :: [*idt*, '*a*, *bool*] => *bool* ((*3*∀ -≥-./ -) [*0*, *0*, *10*] *10*)
-*Ex-greater-eq* :: [*idt*, '*a*, *bool*] => *bool* ((*3*∃ -≥-./ -) [*0*, *0*, *10*] *10*)

syntax (*HOL*)
-*All-less* :: [*idt*, '*a*, *bool*] => *bool* ((*3*! -<-./ -) [*0*, *0*, *10*] *10*)
-*Ex-less* :: [*idt*, '*a*, *bool*] => *bool* ((*3?* -<-./ -) [*0*, *0*, *10*] *10*)
-*All-less-eq* :: [*idt*, '*a*, *bool*] => *bool* ((*3*! -<=-./ -) [*0*, *0*, *10*] *10*)
-*Ex-less-eq* :: [*idt*, '*a*, *bool*] => *bool* ((*3?* -<=-./ -) [*0*, *0*, *10*] *10*)

syntax (*HTML* **output**)
-*All-less* :: [*idt*, '*a*, *bool*] => *bool* ((*3*∀ -<-./ -) [*0*, *0*, *10*] *10*)
-*Ex-less* :: [*idt*, '*a*, *bool*] => *bool* ((*3*∃ -<-./ -) [*0*, *0*, *10*] *10*)
-*All-less-eq* :: [*idt*, '*a*, *bool*] => *bool* ((*3*∀ -≤-./ -) [*0*, *0*, *10*] *10*)
-*Ex-less-eq* :: [*idt*, '*a*, *bool*] => *bool* ((*3*∃ -≤-./ -) [*0*, *0*, *10*] *10*)

-*All-greater* :: [*idt*, '*a*, *bool*] => *bool* ((*3*∀ ->-./ -) [*0*, *0*, *10*] *10*)
-*Ex-greater* :: [*idt*, '*a*, *bool*] => *bool* ((*3*∃ ->-./ -) [*0*, *0*, *10*] *10*)
-*All-greater-eq* :: [*idt*, '*a*, *bool*] => *bool* ((*3*∀ -≥-./ -) [*0*, *0*, *10*] *10*)
-*Ex-greater-eq* :: [*idt*, '*a*, *bool*] => *bool* ((*3*∃ -≥-./ -) [*0*, *0*, *10*] *10*)

translations
ALL x<y. P => *ALL x. x* < *y* ⟶ *P*
EX x<y. P => *EX x. x* < *y* ∧ *P*
ALL x<=y. P => *ALL x. x* <= *y* ⟶ *P*
EX x<=y. P => *EX x. x* <= *y* ∧ *P*
ALL x>y. P => *ALL x. x* > *y* ⟶ *P*
EX x>y. P => *EX x. x* > *y* ∧ *P*
ALL x>=y. P => *ALL x. x* >= *y* ⟶ *P*
EX x>=y. P => *EX x. x* >= *y* ∧ *P*

⟨*ML*⟩

5.7 Transitivity reasoning

context *ord*
begin

lemma *ord-le-eq-trans*: $a \le b \Longrightarrow b = c \Longrightarrow a \le c$
⟨*proof*⟩

lemma *ord-eq-le-trans*: $a = b \Longrightarrow b \le c \Longrightarrow a \le c$
⟨*proof*⟩

lemma *ord-less-eq-trans*: $a < b \Longrightarrow b = c \Longrightarrow a < c$
⟨*proof*⟩

lemma *ord-eq-less-trans*: $a = b \Longrightarrow b < c \Longrightarrow a < c$

⟨*proof*⟩

end

lemma *order-less-subst2*: $(a::'a::order) < b ==> f\ b < (c::'c::order) ==>$
$(!!x\ y.\ x < y ==> f\ x < f\ y) ==> f\ a < c$
⟨*proof*⟩

lemma *order-less-subst1*: $(a::'a::order) < f\ b ==> (b::'b::order) < c ==>$
$(!!x\ y.\ x < y ==> f\ x < f\ y) ==> a < f\ c$
⟨*proof*⟩

lemma *order-le-less-subst2*: $(a::'a::order) <= b ==> f\ b < (c::'c::order) ==>$
$(!!x\ y.\ x <= y ==> f\ x <= f\ y) ==> f\ a < c$
⟨*proof*⟩

lemma *order-le-less-subst1*: $(a::'a::order) <= f\ b ==> (b::'b::order) < c ==>$
$(!!x\ y.\ x < y ==> f\ x < f\ y) ==> a < f\ c$
⟨*proof*⟩

lemma *order-less-le-subst2*: $(a::'a::order) < b ==> f\ b <= (c::'c::order) ==>$
$(!!x\ y.\ x < y ==> f\ x < f\ y) ==> f\ a < c$
⟨*proof*⟩

lemma *order-less-le-subst1*: $(a::'a::order) < f\ b ==> (b::'b::order) <= c ==>$
$(!!x\ y.\ x <= y ==> f\ x <= f\ y) ==> a < f\ c$
⟨*proof*⟩

lemma *order-subst1*: $(a::'a::order) <= f\ b ==> (b::'b::order) <= c ==>$
$(!!x\ y.\ x <= y ==> f\ x <= f\ y) ==> a <= f\ c$
⟨*proof*⟩

lemma *order-subst2*: $(a::'a::order) <= b ==> f\ b <= (c::'c::order) ==>$
$(!!x\ y.\ x <= y ==> f\ x <= f\ y) ==> f\ a <= c$
⟨*proof*⟩

lemma *ord-le-eq-subst*: $a <= b ==> f\ b = c ==>$
$(!!x\ y.\ x <= y ==> f\ x <= f\ y) ==> f\ a <= c$
⟨*proof*⟩

lemma *ord-eq-le-subst*: $a = f\ b ==> b <= c ==>$
$(!!x\ y.\ x <= y ==> f\ x <= f\ y) ==> a <= f\ c$
⟨*proof*⟩

lemma *ord-less-eq-subst*: $a < b ==> f\ b = c ==>$
$(!!x\ y.\ x < y ==> f\ x < f\ y) ==> f\ a < c$
⟨*proof*⟩

lemma *ord-eq-less-subst*: $a = f\ b ==> b < c ==>$

$(!!x\ y.\ x < y ==> f\ x < f\ y) ==> a < f\ c$
$\langle proof \rangle$

Note that this list of rules is in reverse order of priorities.

lemmas *order-trans-rules* [*trans*] =
order-less-subst2
order-less-subst1
order-le-less-subst2
order-le-less-subst1
order-less-le-subst2
order-less-le-subst1
order-subst2
order-subst1
ord-le-eq-subst
ord-eq-le-subst
ord-less-eq-subst
ord-eq-less-subst
forw-subst
back-subst
rev-mp
mp
order-neq-le-trans
order-le-neq-trans
order-less-trans
order-less-asym'
order-le-less-trans
order-less-le-trans
order-trans
order-antisym
ord-le-eq-trans
ord-eq-le-trans
ord-less-eq-trans
ord-eq-less-trans
trans

These support proving chains of decreasing inequalities a ¿= b ¿= c ... in Isar proofs.

lemma *xt1*:
$a = b ==> b > c ==> a > c$
$a > b ==> b = c ==> a > c$
$a = b ==> b >= c ==> a >= c$
$a >= b ==> b = c ==> a >= c$
$(x::'a::order) >= y ==> y >= x ==> x = y$
$(x::'a::order) >= y ==> y >= z ==> x >= z$
$(x::'a::order) > y ==> y >= z ==> x > z$
$(x::'a::order) >= y ==> y > z ==> x > z$
$(a::'a::order) > b ==> b > a ==> P$
$(x::'a::order) > y ==> y > z ==> x > z$
$(a::'a::order) >= b ==> a\ \tilde{}= b ==> a > b$

$(a::'a::order)$ ~= b ==> a >= b ==> a > b
$a = f\ b$ ==> b > c ==> (!!$x\ y$. x > y ==> $f\ x$ > $f\ y$) ==> a > $f\ c$
a > b ==> $f\ b = c$ ==> (!!$x\ y$. x > y ==> $f\ x$ > $f\ y$) ==> $f\ a$ > c
$a = f\ b$ ==> b >= c ==> (!!$x\ y$. x >= y ==> $f\ x$ >= $f\ y$) ==> a >= $f\ c$
a >= b ==> $f\ b = c$ ==> (!! $x\ y$. x >= y ==> $f\ x$ >= $f\ y$) ==> $f\ a$ >= c
⟨*proof*⟩

lemma *xt2*:
$(a::'a::order)$ >= $f\ b$ ==> b >= c ==> (!!$x\ y$. x >= y ==> $f\ x$ >= $f\ y$) ==> a >= $f\ c$
⟨*proof*⟩

lemma *xt3*: $(a::'a::order)$ >= b ==> $(f\ b::'b::order)$ >= c ==>
(!!$x\ y$. x >= y ==> $f\ x$ >= $f\ y$) ==> $f\ a$ >= c
⟨*proof*⟩

lemma *xt4*: $(a::'a::order)$ > $f\ b$ ==> $(b::'b::order)$ >= c ==>
(!!$x\ y$. x >= y ==> $f\ x$ >= $f\ y$) ==> a > $f\ c$
⟨*proof*⟩

lemma *xt5*: $(a::'a::order)$ > b ==> $(f\ b::'b::order)$ >= c==>
(!!$x\ y$. x > y ==> $f\ x$ > $f\ y$) ==> $f\ a$ > c
⟨*proof*⟩

lemma *xt6*: $(a::'a::order)$ >= $f\ b$ ==> b > c ==>
(!!$x\ y$. x > y ==> $f\ x$ > $f\ y$) ==> a > $f\ c$
⟨*proof*⟩

lemma *xt7*: $(a::'a::order)$ >= b ==> $(f\ b::'b::order)$ > c ==>
(!!$x\ y$. x >= y ==> $f\ x$ >= $f\ y$) ==> $f\ a$ > c
⟨*proof*⟩

lemma *xt8*: $(a::'a::order)$ > $f\ b$ ==> $(b::'b::order)$ > c ==>
(!!$x\ y$. x > y ==> $f\ x$ > $f\ y$) ==> a > $f\ c$
⟨*proof*⟩

lemma *xt9*: $(a::'a::order)$ > b ==> $(f\ b::'b::order)$ > c ==>
(!!$x\ y$. x > y ==> $f\ x$ > $f\ y$) ==> $f\ a$ > c
⟨*proof*⟩

lemmas *xtrans* = *xt1 xt2 xt3 xt4 xt5 xt6 xt7 xt8 xt9*

5.8 Order on bool

instance *bool* :: *order*
le-bool-def: $P \leq Q \equiv P \longrightarrow Q$
less-bool-def: $P < Q \equiv P \leq Q \wedge P \neq Q$
⟨*proof*⟩
lemmas [*code func del*] = *le-bool-def less-bool-def*

lemma *le-boolI*: $(P \Longrightarrow Q) \Longrightarrow P \le Q$
⟨*proof*⟩

lemma *le-boolI'*: $P \longrightarrow Q \Longrightarrow P \le Q$
⟨*proof*⟩

lemma *le-boolE*: $P \le Q \Longrightarrow P \Longrightarrow (Q \Longrightarrow R) \Longrightarrow R$
⟨*proof*⟩

lemma *le-boolD*: $P \le Q \Longrightarrow P \longrightarrow Q$
⟨*proof*⟩

lemma [*code func*]:
$\mathit{False} \le b \longleftrightarrow \mathit{True}$
$\mathit{True} \le b \longleftrightarrow b$
$\mathit{False} < b \longleftrightarrow b$
$\mathit{True} < b \longleftrightarrow \mathit{False}$
⟨*proof*⟩

5.9 Order on sets

instance *set* :: (*type*) *order*
⟨*proof*⟩

lemmas *basic-trans-rules* [*trans*] =
order-trans-rules set-rev-mp set-mp

5.10 Order on functions

instance *fun* :: (*type*, *ord*) *ord*
le-fun-def: $f \le g \equiv \forall x.\ f\ x \le g\ x$
less-fun-def: $f < g \equiv f \le g \land f \ne g$ ⟨*proof*⟩

lemmas [*code func del*] = *le-fun-def less-fun-def*

instance *fun* :: (*type*, *order*) *order*
⟨*proof*⟩

lemma *le-funI*: $(\bigwedge x.\ f\ x \le g\ x) \Longrightarrow f \le g$
⟨*proof*⟩

lemma *le-funE*: $f \le g \Longrightarrow (f\ x \le g\ x \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

lemma *le-funD*: $f \le g \Longrightarrow f\ x \le g\ x$
⟨*proof*⟩

Handy introduction and elimination rules for $\le$ on unary and binary predicates

lemma *predicate1I* [*Pure.intro*!, *intro*!]:
  assumes *PQ*: $\bigwedge x.\ P\ x \Longrightarrow Q\ x$
  shows $P \leq Q$
  ⟨*proof*⟩

lemma *predicate1D* [*Pure.dest*, *dest*]: $P \leq Q \Longrightarrow P\ x \Longrightarrow Q\ x$
  ⟨*proof*⟩

lemma *predicate2I* [*Pure.intro*!, *intro*!]:
  assumes *PQ*: $\bigwedge x\ y.\ P\ x\ y \Longrightarrow Q\ x\ y$
  shows $P \leq Q$
  ⟨*proof*⟩

lemma *predicate2D* [*Pure.dest*, *dest*]: $P \leq Q \Longrightarrow P\ x\ y \Longrightarrow Q\ x\ y$
  ⟨*proof*⟩

lemma *rev-predicate1D*: $P\ x$ ==> P <= Q ==> $Q\ x$
  ⟨*proof*⟩

lemma *rev-predicate2D*: $P\ x\ y$ ==> P <= Q ==> $Q\ x\ y$
  ⟨*proof*⟩

5.11 Monotonicity, least value operator and min/max

context *order*
begin

definition
  mono :: $('a \Rightarrow 'b{::}order) \Rightarrow bool$
where
  $mono\ f \longleftrightarrow (\forall\, x\ y.\ x \leq y \longrightarrow f\ x \leq f\ y)$

lemma *monoI* [*intro?*]:
  fixes f :: $'a \Rightarrow 'b{::}order$
  shows $(\bigwedge x\ y.\ x \leq y \Longrightarrow f\ x \leq f\ y) \Longrightarrow mono\ f$
  ⟨*proof*⟩

lemma *monoD* [*dest?*]:
  fixes f :: $'a \Rightarrow 'b{::}order$
  shows $mono\ f \Longrightarrow x \leq y \Longrightarrow f\ x \leq f\ y$
  ⟨*proof*⟩

end

context *linorder*
begin

lemma *min-of-mono*:
  fixes f :: $'a \Rightarrow 'b{::}linorder$

shows $mono\ f \Longrightarrow min\ (f\ m)\ (f\ n) = f\ (min\ m\ n)$
$\langle proof \rangle$

lemma *max-of-mono*:
fixes $f :: 'a \Rightarrow 'b{::}linorder$
shows $mono\ f \Longrightarrow max\ (f\ m)\ (f\ n) = f\ (max\ m\ n)$
$\langle proof \rangle$

end

lemma *LeastI2-order*:
$[|\ P\ (x{::}'a{::}order);$
$!!y.\ P\ y ==> x <= y;$
$!!x.\ [|\ P\ x;\ ALL\ y.\ P\ y --> x \le y\ |] ==> Q\ x\ |]$
$==> Q\ (Least\ P)$
$\langle proof \rangle$

lemma *Least-mono*:
$mono\ (f{::}'a{::}order => 'b{::}order) ==> EX\ x{:}S.\ ALL\ y{:}S.\ x <= y$
$==> (LEAST\ y.\ y : f\ `\ S) = f\ (LEAST\ x.\ x : S)$
— Courtesy of Stephan Merz
$\langle proof \rangle$

lemma *Least-equality*:
$[|\ P\ (k{::}'a{::}order);\ !!x.\ P\ x ==> k <= x\ |] ==> (LEAST\ x.\ P\ x) = k$
$\langle proof \rangle$

lemma *min-leastL*: $(!!x.\ least <= x) ==> min\ least\ x = least$
$\langle proof \rangle$

lemma *max-leastL*: $(!!x.\ least <= x) ==> max\ least\ x = x$
$\langle proof \rangle$

lemma *min-leastR*: $(\bigwedge x{::}'a{::}order.\ least \le x) \Longrightarrow min\ x\ least = least$
$\langle proof \rangle$

lemma *max-leastR*: $(\bigwedge x{::}'a{::}order.\ least \le x) \Longrightarrow max\ x\ least = x$
$\langle proof \rangle$

end

6 Lattices: Abstract lattices

theory *Lattices*
imports *Orderings*
begin

6.1 Lattices

notation
less-eq (**infix** $\sqsubseteq$ *50*) **and**
less (**infix** $\sqsubset$ *50*)

class *lower-semilattice* = *order* +
fixes *inf* :: $'a \Rightarrow 'a \Rightarrow 'a$ (**infixl** $\sqcap$ *70*)
assumes *inf-le1* [*simp*]: $x \sqcap y \sqsubseteq x$
and *inf-le2* [*simp*]: $x \sqcap y \sqsubseteq y$
and *inf-greatest*: $x \sqsubseteq y \Longrightarrow x \sqsubseteq z \Longrightarrow x \sqsubseteq y \sqcap z$

class *upper-semilattice* = *order* +
fixes *sup* :: $'a \Rightarrow 'a \Rightarrow 'a$ (**infixl** $\sqcup$ *65*)
assumes *sup-ge1* [*simp*]: $x \sqsubseteq x \sqcup y$
and *sup-ge2* [*simp*]: $y \sqsubseteq x \sqcup y$
and *sup-least*: $y \sqsubseteq x \Longrightarrow z \sqsubseteq x \Longrightarrow y \sqcup z \sqsubseteq x$

class *lattice* = *lower-semilattice* + *upper-semilattice*

6.1.1 Intro and elim rules

context *lower-semilattice*
begin

lemma *le-infI1*[*intro*]:
assumes $a \sqsubseteq x$
shows $a \sqcap b \sqsubseteq x$
⟨*proof*⟩
lemmas (**in** −) [*rule del*] = *le-infI1*

lemma *le-infI2*[*intro*]:
assumes $b \sqsubseteq x$
shows $a \sqcap b \sqsubseteq x$
⟨*proof*⟩
lemmas (**in** −) [*rule del*] = *le-infI2*

lemma *le-infI*[*intro*!]: $x \sqsubseteq a \Longrightarrow x \sqsubseteq b \Longrightarrow x \sqsubseteq a \sqcap b$
⟨*proof*⟩
lemmas (**in** −) [*rule del*] = *le-infI*

lemma *le-infE* [*elim*!]: $x \sqsubseteq a \sqcap b \Longrightarrow (x \sqsubseteq a \Longrightarrow x \sqsubseteq b \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩
lemmas (**in** −) [*rule del*] = *le-infE*

lemma *le-inf-iff* [*simp*]:
$x \sqsubseteq y \sqcap z = (x \sqsubseteq y \wedge x \sqsubseteq z)$
⟨*proof*⟩

lemma *le-iff-inf*: $(x \sqsubseteq y) = (x \sqcap y = x)$

⟨*proof*⟩

lemma *mono-inf*:
fixes f :: $'a \Rightarrow 'b$::*lower-semilattice*
shows *mono* $f \Longrightarrow f\ (A \sqcap B) \le f\ A \sqcap f\ B$
⟨*proof*⟩

end

context *upper-semilattice*
begin

lemma *le-supI1*[*intro*]: $x \sqsubseteq a \Longrightarrow x \sqsubseteq a \sqcup b$
⟨*proof*⟩
lemmas (**in** −) [*rule del*] = *le-supI1*

lemma *le-supI2*[*intro*]: $x \sqsubseteq b \Longrightarrow x \sqsubseteq a \sqcup b$
⟨*proof*⟩
lemmas (**in** −) [*rule del*] = *le-supI2*

lemma *le-supI*[*intro!*]: $a \sqsubseteq x \Longrightarrow b \sqsubseteq x \Longrightarrow a \sqcup b \sqsubseteq x$
⟨*proof*⟩
lemmas (**in** −) [*rule del*] = *le-supI*

lemma *le-supE*[*elim!*]: $a \sqcup b \sqsubseteq x \Longrightarrow (a \sqsubseteq x \Longrightarrow b \sqsubseteq x \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩
lemmas (**in** −) [*rule del*] = *le-supE*

lemma *ge-sup-conv*[*simp*]:
$x \sqcup y \sqsubseteq z = (x \sqsubseteq z \wedge y \sqsubseteq z)$
⟨*proof*⟩

lemma *le-iff-sup*: $(x \sqsubseteq y) = (x \sqcup y = y)$
⟨*proof*⟩

lemma *mono-sup*:
fixes f :: $'a \Rightarrow 'b$::*upper-semilattice*
shows *mono* $f \Longrightarrow f\ A \sqcup f\ B \le f\ (A \sqcup B)$
⟨*proof*⟩

end

6.1.2 Equational laws

context *lower-semilattice*
begin

lemma *inf-commute*: $(x \sqcap y) = (y \sqcap x)$
⟨*proof*⟩

lemma *inf-assoc*: $(x \sqcap y) \sqcap z = x \sqcap (y \sqcap z)$
⟨*proof*⟩

lemma *inf-idem*[*simp*]: $x \sqcap x = x$
⟨*proof*⟩

lemma *inf-left-idem*[*simp*]: $x \sqcap (x \sqcap y) = x \sqcap y$
⟨*proof*⟩

lemma *inf-absorb1*: $x \sqsubseteq y \Longrightarrow x \sqcap y = x$
⟨*proof*⟩

lemma *inf-absorb2*: $y \sqsubseteq x \Longrightarrow x \sqcap y = y$
⟨*proof*⟩

lemma *inf-left-commute*: $x \sqcap (y \sqcap z) = y \sqcap (x \sqcap z)$
⟨*proof*⟩

lemmas *inf-ACI* = *inf-commute inf-assoc inf-left-commute inf-left-idem*

end

context *upper-semilattice*
begin

lemma *sup-commute*: $(x \sqcup y) = (y \sqcup x)$
⟨*proof*⟩

lemma *sup-assoc*: $(x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$
⟨*proof*⟩

lemma *sup-idem*[*simp*]: $x \sqcup x = x$
⟨*proof*⟩

lemma *sup-left-idem*[*simp*]: $x \sqcup (x \sqcup y) = x \sqcup y$
⟨*proof*⟩

lemma *sup-absorb1*: $y \sqsubseteq x \Longrightarrow x \sqcup y = x$
⟨*proof*⟩

lemma *sup-absorb2*: $x \sqsubseteq y \Longrightarrow x \sqcup y = y$
⟨*proof*⟩

lemma *sup-left-commute*: $x \sqcup (y \sqcup z) = y \sqcup (x \sqcup z)$
⟨*proof*⟩

lemmas *sup-ACI* = *sup-commute sup-assoc sup-left-commute sup-left-idem*

end

context *lattice*
begin

lemma *inf-sup-absorb*: $x \sqcap (x \sqcup y) = x$
⟨*proof*⟩

lemma *sup-inf-absorb*: $x \sqcup (x \sqcap y) = x$
⟨*proof*⟩

lemmas *ACI* = *inf-ACI sup-ACI*

lemmas *inf-sup-ord* = *inf-le1 inf-le2 sup-ge1 sup-ge2*

Towards distributivity

lemma *distrib-sup-le*: $x \sqcup (y \sqcap z) \sqsubseteq (x \sqcup y) \sqcap (x \sqcup z)$
⟨*proof*⟩

lemma *distrib-inf-le*: $(x \sqcap y) \sqcup (x \sqcap z) \sqsubseteq x \sqcap (y \sqcup z)$
⟨*proof*⟩

If you have one of them, you have them all.

lemma *distrib-imp1*:
assumes *D*: $!!x\ y\ z.\ x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$
shows $x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z)$
⟨*proof*⟩

lemma *distrib-imp2*:
assumes *D*: $!!x\ y\ z.\ x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z)$
shows $x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$
⟨*proof*⟩

lemma *modular-le*: $x \sqsubseteq z \Longrightarrow x \sqcup (y \sqcap z) \sqsubseteq (x \sqcup y) \sqcap z$
⟨*proof*⟩

end

6.2 Distributive lattices

class *distrib-lattice* = *lattice* +
assumes *sup-inf-distrib1*: $x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z)$

context *distrib-lattice*
begin

lemma *sup-inf-distrib2*:

$(y \sqcap z) \sqcup x = (y \sqcup x) \sqcap (z \sqcup x)$
⟨*proof*⟩

lemma *inf-sup-distrib1*:
$x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$
⟨*proof*⟩

lemma *inf-sup-distrib2*:
$(y \sqcup z) \sqcap x = (y \sqcap x) \sqcup (z \sqcap x)$
⟨*proof*⟩

lemmas *distrib* =
sup-inf-distrib1 sup-inf-distrib2 inf-sup-distrib1 inf-sup-distrib2

end

6.3 Uniqueness of inf and sup

lemma (**in** *lower-semilattice*) *inf-unique*:
fixes f (**infixl** $\triangle$ *70*)
assumes *le1*: $\bigwedge x\ y.\ x \triangle y \leq x$ **and** *le2*: $\bigwedge x\ y.\ x \triangle y \leq y$
and *greatest*: $\bigwedge x\ y\ z.\ x \leq y \Longrightarrow x \leq z \Longrightarrow x \leq y \triangle z$
shows $x \sqcap y = x \triangle y$
⟨*proof*⟩

lemma (**in** *upper-semilattice*) *sup-unique*:
fixes f (**infixl** ∇ *70*)
assumes *ge1* [*simp*]: $\bigwedge x\ y.\ x \leq x \nabla y$ **and** *ge2*: $\bigwedge x\ y.\ y \leq x \nabla y$
and *least*: $\bigwedge x\ y\ z.\ y \leq x \Longrightarrow z \leq x \Longrightarrow y \nabla z \leq x$
shows $x \sqcup y = x \nabla y$
⟨*proof*⟩

6.4 *min*/*max* on linear orders as special case of *op* $\sqcap$/*op* $\sqcup$

lemma (**in** *linorder*) *distrib-lattice-min-max*:
distrib-lattice (*op* $\leq$) (*op* $<$) *min max*
⟨*proof*⟩

interpretation *min-max*:
distrib-lattice [*op* $\leq$ :: *'a::linorder* $\Rightarrow$ *'a* $\Rightarrow$ *bool op* $<$ *min max*]
⟨*proof*⟩

lemma *inf-min*: *inf* = (*min* :: *'a*::{*lower-semilattice*, *linorder*} $\Rightarrow$ *'a* $\Rightarrow$ *'a*)
⟨*proof*⟩

lemma *sup-max*: *sup* = (*max* :: *'a*::{*upper-semilattice*, *linorder*} $\Rightarrow$ *'a* $\Rightarrow$ *'a*)
⟨*proof*⟩

lemmas *le-maxI1* = *min-max.sup-ge1*
lemmas *le-maxI2* = *min-max.sup-ge2*

lemmas *max-ac* = *min-max.sup-assoc min-max.sup-commute*
mk-left-commute [*of max*, *OF min-max.sup-assoc min-max.sup-commute*]

lemmas *min-ac* = *min-max.inf-assoc min-max.inf-commute*
mk-left-commute [*of min*, *OF min-max.inf-assoc min-max.inf-commute*]

Now we have inherited antisymmetry as an intro-rule on all linear orders. This is a problem because it applies to bool, which is undesirable.

lemmas [*rule del*] = *min-max.le-infI min-max.le-supI*
min-max.le-supE min-max.le-infE min-max.le-supI1 min-max.le-supI2
min-max.le-infI1 min-max.le-infI2

6.5 Complete lattices

class *complete-lattice* = *lattice* +
fixes *Inf* :: $'a\ set \Rightarrow 'a$ ($\bigsqcap$ - [*900*] *900*)
and *Sup* :: $'a\ set \Rightarrow 'a$ ($\bigsqcup$ - [*900*] *900*)
assumes *Inf-lower*: $x \in A \Longrightarrow \bigsqcap A \sqsubseteq x$
and *Inf-greatest*: $(\bigwedge x.\ x \in A \Longrightarrow z \sqsubseteq x) \Longrightarrow z \sqsubseteq \bigsqcap A$
assumes *Sup-upper*: $x \in A \Longrightarrow x \sqsubseteq \bigsqcup A$
and *Sup-least*: $(\bigwedge x.\ x \in A \Longrightarrow x \sqsubseteq z) \Longrightarrow \bigsqcup A \sqsubseteq z$
begin

lemma *Inf-Sup*: $\bigsqcap A = \bigsqcup \{b.\ \forall\, a \in A.\ b \leq a\}$
⟨*proof*⟩

lemma *Sup-Inf*: $\bigsqcup A = \bigsqcap \{b.\ \forall\, a \in A.\ a \leq b\}$
⟨*proof*⟩

lemma *Inf-Univ*: $\bigsqcap\ UNIV = \bigsqcup \{\}$
⟨*proof*⟩

lemma *Sup-Univ*: $\bigsqcup\ UNIV = \bigsqcap \{\}$
⟨*proof*⟩

lemma *Inf-insert*: $\bigsqcap\ insert\ a\ A = a \sqcap \bigsqcap A$
⟨*proof*⟩

lemma *Sup-insert*: $\bigsqcup\ insert\ a\ A = a \sqcup \bigsqcup A$
⟨*proof*⟩

lemma *Inf-singleton* [*simp*]:
$\bigsqcap \{a\} = a$
⟨*proof*⟩

lemma *Sup-singleton* [*simp*]:
$\bigsqcup \{a\} = a$
⟨*proof*⟩

lemma *Inf-insert-simp*:
$\bigsqcap insert\ a\ A = (if\ A = \{\}\ then\ a\ else\ a \sqcap \bigsqcap A)$
⟨*proof*⟩

lemma *Sup-insert-simp*:
$\bigsqcup insert\ a\ A = (if\ A = \{\}\ then\ a\ else\ a \sqcup \bigsqcup A)$
⟨*proof*⟩

lemma *Inf-binary*:
$\bigsqcap\{a,\ b\} = a \sqcap b$
⟨*proof*⟩

lemma *Sup-binary*:
$\bigsqcup\{a,\ b\} = a \sqcup b$
⟨*proof*⟩

definition
top :: *'a* **where**
$top = \bigsqcap\{\}$

definition
bot :: *'a* **where**
$bot = \bigsqcup\{\}$

lemma *top-greatest* [*simp*]: $x \leq top$
⟨*proof*⟩

lemma *bot-least* [*simp*]: $bot \leq x$
⟨*proof*⟩

definition
SUPR :: $'b\ set \Rightarrow ('b \Rightarrow 'a) \Rightarrow 'a$
where
$SUPR\ A\ f == \bigsqcup\ (f\ `\ A)$

definition
INFI :: $'b\ set \Rightarrow ('b \Rightarrow 'a) \Rightarrow 'a$
where
$INFI\ A\ f == \bigsqcap\ (f\ `\ A)$

end

syntax
-SUP1 :: *pttrns => 'b => 'b* ((*3SUP* -./ -) [*0*, *10*] *10*)
-SUP :: *pttrn => 'a set => 'b => 'b* ((*3SUP* -:-./ -) [*0*, *10*] *10*)
-INF1 :: *pttrns => 'b => 'b* ((*3INF* -./ -) [*0*, *10*] *10*)
-INF :: *pttrn => 'a set => 'b => 'b* ((*3INF* -:-./ -) [*0*, *10*] *10*)

translations
$SUP\ x\ y.\ B\ \ == SUP\ x.\ SUP\ y.\ B$
$SUP\ x.\ B\ \ == CONST\ SUPR\ UNIV\ (\%x.\ B)$
$SUP\ x.\ B\ \ == SUP\ x{:}UNIV.\ B$
$SUP\ x{:}A.\ B\ \ == CONST\ SUPR\ A\ (\%x.\ B)$
$INF\ x\ y.\ B\ \ == INF\ x.\ INF\ y.\ B$
$INF\ x.\ B\ \ == CONST\ INFI\ UNIV\ (\%x.\ B)$
$INF\ x.\ B\ \ == INF\ x{:}UNIV.\ B$
$INF\ x{:}A.\ B\ \ == CONST\ INFI\ A\ (\%x.\ B)$

⟨*ML*⟩

context *complete-lattice*
begin

lemma *le-SUPI*: $i : A \Longrightarrow M\ i \leq (SUP\ i{:}A.\ M\ i)$
⟨*proof*⟩

lemma *SUP-leI*: $(\bigwedge i.\ i : A \Longrightarrow M\ i \leq u) \Longrightarrow (SUP\ i{:}A.\ M\ i) \leq u$
⟨*proof*⟩

lemma *INF-leI*: $i : A \Longrightarrow (INF\ i{:}A.\ M\ i) \leq M\ i$
⟨*proof*⟩

lemma *le-INFI*: $(\bigwedge i.\ i : A \Longrightarrow u \leq M\ i) \Longrightarrow u \leq (INF\ i{:}A.\ M\ i)$
⟨*proof*⟩

lemma *SUP-const*[*simp*]: $A \neq \{\} \Longrightarrow (SUP\ i{:}A.\ M) = M$
⟨*proof*⟩

lemma *INF-const*[*simp*]: $A \neq \{\} \Longrightarrow (INF\ i{:}A.\ M) = M$
⟨*proof*⟩

end

6.6 Bool as lattice

instance *bool* :: *distrib-lattice*
inf-bool-eq: $P \sqcap Q \equiv P \wedge Q$
sup-bool-eq: $P \sqcup Q \equiv P \vee Q$
⟨*proof*⟩

instance *bool* :: *complete-lattice*
Inf-bool-def: $\bigsqcap A \equiv \forall x{\in}A.\ x$
Sup-bool-def: $\bigsqcup A \equiv \exists x{\in}A.\ x$
⟨*proof*⟩

lemma *Inf-empty-bool* [*simp*]:

$\bigsqcap\{\}$
⟨*proof*⟩

lemma *not-Sup-empty-bool* [*simp*]:
$\neg$ *Sup* $\{\}$
⟨*proof*⟩

lemma *top-bool-eq*: *top* = *True*
⟨*proof*⟩

lemma *bot-bool-eq*: *bot* = *False*
⟨*proof*⟩

6.7 Set as lattice

instance *set* :: (*type*) *distrib-lattice*
inf-set-eq: $A \sqcap B \equiv A \cap B$
sup-set-eq: $A \sqcup B \equiv A \cup B$
⟨*proof*⟩

lemmas [*code func del*] = *inf-set-eq sup-set-eq*

lemma *mono-Int*: *mono f* $\Longrightarrow f\ (A \cap B) \subseteq f\ A \cap f\ B$
⟨*proof*⟩

lemma *mono-Un*: *mono f* $\Longrightarrow f\ A \cup f\ B \subseteq f\ (A \cup B)$
⟨*proof*⟩

instance *set* :: (*type*) *complete-lattice*
Inf-set-def: $\bigsqcap S \equiv \bigcap S$
Sup-set-def: $\bigsqcup S \equiv \bigcup S$
⟨*proof*⟩

lemmas [*code func del*] = *Inf-set-def Sup-set-def*

lemma *top-set-eq*: *top* = *UNIV*
⟨*proof*⟩

lemma *bot-set-eq*: *bot* = $\{\}$
⟨*proof*⟩

6.8 Fun as lattice

instance *fun* :: (*type*, *lattice*) *lattice*
inf-fun-eq: $f \sqcap g \equiv (\lambda x.\ f\ x \sqcap g\ x)$
sup-fun-eq: $f \sqcup g \equiv (\lambda x.\ f\ x \sqcup g\ x)$
⟨*proof*⟩

lemmas [*code func del*] = *inf-fun-eq sup-fun-eq*

instance *fun* :: (*type*, *distrib-lattice*) *distrib-lattice*
⟨*proof*⟩

instance *fun* :: (*type*, *complete-lattice*) *complete-lattice*
Inf-fun-def: $\bigsqcap A \equiv (\lambda x.\ \bigsqcap\{y.\ \exists f{\in}A.\ y = f\ x\})$
Sup-fun-def: $\bigsqcup A \equiv (\lambda x.\ \bigsqcup\{y.\ \exists f{\in}A.\ y = f\ x\})$
⟨*proof*⟩

lemmas [*code func del*] = *Inf-fun-def Sup-fun-def*

lemma *Inf-empty-fun*:
$\bigsqcap\{\} = (\lambda\text{-}.\ \bigsqcap\{\})$
⟨*proof*⟩

lemma *Sup-empty-fun*:
$\bigsqcup\{\} = (\lambda\text{-}.\ \bigsqcup\{\})$
⟨*proof*⟩

lemma *top-fun-eq*: *top* = (λ*x*. *top*)
⟨*proof*⟩

lemma *bot-fun-eq*: *bot* = (λ*x*. *bot*)
⟨*proof*⟩

redundant bindings

lemmas *inf-aci* = *inf-ACI*
lemmas *sup-aci* = *sup-ACI*

no-notation
less-eq (**infix** ⊑ *50*) **and**
less (**infix** ⊏ *50*) **and**
inf (**infixl** ⊓ *70*) **and**
sup (**infixl** ⊔ *65*) **and**
Inf (⨅ - [*900*] *900*) **and**
Sup (⨆ - [*900*] *900*)

end

7 Typedef: HOL type definitions

theory *Typedef*
imports *Set*
uses
(*Tools/typedef-package.ML*)
(*Tools/typecopy-package.ML*)
(*Tools/typedef-codegen.ML*)
begin

⟨*ML*⟩

locale *type-definition* =
  fixes *Rep* **and** *Abs* **and** *A*
  assumes *Rep*: *Rep* $x \in A$
    and *Rep-inverse*: *Abs* (*Rep* x) = x
    and *Abs-inverse*: $y \in A$ ==> *Rep* (*Abs* y) = y
  — This will be axiomatized for each typedef!
begin

lemma *Rep-inject*:
  (*Rep* x = *Rep* y) = (x = y)
⟨*proof*⟩

lemma *Abs-inject*:
  assumes x: $x \in A$ **and** y: $y \in A$
  shows (*Abs* x = *Abs* y) = (x = y)
⟨*proof*⟩

lemma *Rep-cases* [*cases set*]:
  assumes y: $y \in A$
    and *hyp*: !!x. y = *Rep* x ==> P
  shows P
⟨*proof*⟩

lemma *Abs-cases* [*cases type*]:
  assumes r: !!y. x = *Abs* y ==> $y \in A$ ==> P
  shows P
⟨*proof*⟩

lemma *Rep-induct* [*induct set*]:
  assumes y: $y \in A$
    and *hyp*: !!x. P (*Rep* x)
  shows P y
⟨*proof*⟩

lemma *Abs-induct* [*induct type*]:
  assumes r: !!y. $y \in A$ ==> P (*Abs* y)
  shows P x
⟨*proof*⟩

lemma *Rep-range*:
  shows *range Rep* = A
⟨*proof*⟩

end

⟨*ML*⟩

end

8 Sum-Type: The Disjoint Sum of Two Types

theory *Sum-Type*
imports *Typedef Fun*
begin

The representations of the two injections

constdefs
  Inl-Rep :: ['a, 'a, 'b, bool] => bool
  Inl-Rep == (%a. %x y p. x=a & p)

  Inr-Rep :: ['b, 'a, 'b, bool] => bool
  Inr-Rep == (%b. %x y p. y=b & ~p)

global

typedef (*Sum*)
  ('a, 'b) +   (**infixr** + *10*)
    = {f. (? a. f = Inl-Rep(a::'a)) | (? b. f = Inr-Rep(b::'b))}
  ⟨*proof*⟩

local

abstract constants and syntax

constdefs
  Inl :: 'a => 'a + 'b
   Inl == (%a. Abs-Sum(Inl-Rep(a)))

  Inr :: 'b => 'a + 'b
   Inr == (%b. Abs-Sum(Inr-Rep(b)))

  Plus :: ['a set, 'b set] => ('a + 'b) set   (**infixr** <+> *65*)
   A <+> B == (Inl'A) Un (Inr'B)
   — disjoint sum for sets; the operator + is overloaded with wrong type!

  Part :: ['a set, 'b => 'a] => 'a set
   Part A h == A Int {x. ? z. x = h(z)}
   — for selecting out the components of a mutually recursive definition

lemma *Inl-RepI*: *Inl-Rep*(a) : *Sum*
⟨*proof*⟩

lemma *Inr-RepI*: *Inr-Rep*(b) : *Sum*
⟨*proof*⟩

lemma *inj-on-Abs-Sum*: *inj-on Abs-Sum Sum*
⟨*proof*⟩

8.1 Freeness Properties for *Inl* and *Inr*

Distinctness

lemma *Inl-Rep-not-Inr-Rep*: *Inl-Rep*(a) ~= *Inr-Rep*(b)
⟨*proof*⟩

lemma *Inl-not-Inr* [*iff*]: *Inl*(a) ~= *Inr*(b)
⟨*proof*⟩

lemmas *Inr-not-Inl* = *Inl-not-Inr* [*THEN not-sym*, *standard*]
declare *Inr-not-Inl* [*iff*]

lemmas *Inl-neq-Inr* = *Inl-not-Inr* [*THEN notE*, *standard*]
lemmas *Inr-neq-Inl* = *sym* [*THEN Inl-neq-Inr*, *standard*]

Injectiveness

lemma *Inl-Rep-inject*: *Inl-Rep*(a) = *Inl-Rep*(c) ==> a=c
⟨*proof*⟩

lemma *Inr-Rep-inject*: *Inr-Rep*(b) = *Inr-Rep*(d) ==> b=d
⟨*proof*⟩

lemma *inj-Inl*: *inj*(*Inl*)
⟨*proof*⟩
lemmas *Inl-inject* = *inj-Inl* [*THEN injD*, *standard*]

lemma *inj-Inr*: *inj*(*Inr*)
⟨*proof*⟩

lemmas *Inr-inject* = *inj-Inr* [*THEN injD*, *standard*]

lemma *Inl-eq* [*iff*]: (*Inl*(x)=*Inl*(y)) = (x=y)
⟨*proof*⟩

lemma *Inr-eq* [*iff*]: (*Inr*(x)=*Inr*(y)) = (x=y)
⟨*proof*⟩

8.2 Projections

definition

sum-case f g x =
(*if* (∃!*y*. *x* = *Inl y*)
then f (*THE y*. *x* = *Inl y*)
else g (*THE y*. *x* = *Inr y*))
definition *Projl x* = *sum-case id arbitrary x*
definition *Projr x* = *sum-case arbitrary id x*

lemma *sum-cases*[*simp*]:
sum-case f g (*Inl x*) = *f x*
sum-case f g (*Inr y*) = *g y*
⟨*proof*⟩

lemma *Projl-Inl*[*simp*]: *Projl* (*Inl x*) = *x*
⟨*proof*⟩

lemma *Projr-Inr*[*simp*]: *Projr* (*Inr x*) = *x*
⟨*proof*⟩

8.3 The Disjoint Sum of Sets

lemma *InlI* [*intro*!]: *a* : *A* ==> *Inl*(*a*) : *A* <+> *B*
⟨*proof*⟩

lemma *InrI* [*intro*!]: *b* : *B* ==> *Inr*(*b*) : *A* <+> *B*
⟨*proof*⟩

lemma *PlusE* [*elim*!]:
[| *u*: *A* <+> *B*;
!!*x*. [| *x*:*A*; *u*=*Inl*(*x*) |] ==> *P*;
!!*y*. [| *y*:*B*; *u*=*Inr*(*y*) |] ==> *P*
|] ==> *P*
⟨*proof*⟩

Exhaustion rule for sums, a degenerate form of induction

lemma *sumE*:
[| !!*x*::'*a*. *s* = *Inl*(*x*) ==> *P*; !!*y*::'*b*. *s* = *Inr*(*y*) ==> *P*
|] ==> *P*
⟨*proof*⟩

lemma *sum-induct*: [| !!*x*. *P* (*Inl x*); !!*x*. *P* (*Inr x*) |] ==> *P x*
⟨*proof*⟩

lemma *UNIV-Plus-UNIV* [*simp*]: *UNIV* <+> *UNIV* = *UNIV*
⟨*proof*⟩

8.4 The *Part* Primitive

lemma *Part-eqI* [*intro*]: [| a : A; a=h(b) |] ==> a : *Part* A h
⟨*proof*⟩

lemmas *PartI* = *Part-eqI* [*OF* - *refl*, *standard*]

lemma *PartE* [*elim*!]: [| a : *Part* A h; !!z. [| a : A; a=h(z) |] ==> P |] ==> P
⟨*proof*⟩

lemma *Part-subset*: *Part* A h <= A
⟨*proof*⟩

lemma *Part-mono*: A<=B ==> *Part* A h <= *Part* B h
⟨*proof*⟩

lemmas *basic-monos* = *basic-monos Part-mono*

lemma *PartD1*: a : *Part* A h ==> a : A
⟨*proof*⟩

lemma *Part-id*: *Part* A (%x. x) = A
⟨*proof*⟩

lemma *Part-Int*: *Part* (A *Int* B) h = (*Part* A h) *Int* (*Part* B h)
⟨*proof*⟩

lemma *Part-Collect*: *Part* (A *Int* {x. P x}) h = (*Part* A h) *Int* {x. P x}
⟨*proof*⟩

8.5 Code generator setup

instance + :: (*eq*, *eq*) *eq* ⟨*proof*⟩

lemma [*code func*]:
  (*Inl* x :: $'a$::*eq* + $'b$::*eq*) = *Inl* y $\longleftrightarrow$ x = y
  ⟨*proof*⟩

lemma [*code func*]:
  (*Inr* x :: $'a$::*eq* + $'b$::*eq*) = *Inr* y $\longleftrightarrow$ x = y
  ⟨*proof*⟩

lemma [*code func*]:
  Inl (x::$'a$::*eq*) = *Inr* (y::$'b$::*eq*) $\longleftrightarrow$ *False*
  ⟨*proof*⟩

lemma [*code func*]:
  Inr (x::$'b$::*eq*) = *Inl* (y::$'a$::*eq*) $\longleftrightarrow$ *False*
  ⟨*proof*⟩

⟨*ML*⟩

end

9 Inductive: Knaster-Tarski Fixpoint Theorem and inductive definitions

theory *Inductive*
imports *Lattices Sum-Type*
uses
(*Tools/inductive-package.ML*)
Tools/dseq.ML
(*Tools/inductive-codegen.ML*)
(*Tools/datatype-aux.ML*)
(*Tools/datatype-prop.ML*)
(*Tools/datatype-rep-proofs.ML*)
(*Tools/datatype-abs-proofs.ML*)
(*Tools/datatype-case.ML*)
(*Tools/datatype-package.ML*)
(*Tools/primrec-package.ML*)
begin

9.1 Least and greatest fixed points

definition
$lfp :: ('a::complete\text{-}lattice \Rightarrow 'a) \Rightarrow 'a$ **where**
$lfp\ f = Inf\ \{u.\ f\ u \leq u\}$ — least fixed point

definition
$gfp :: ('a::complete\text{-}lattice \Rightarrow 'a) \Rightarrow 'a$ **where**
$gfp\ f = Sup\ \{u.\ u \leq f\ u\}$ — greatest fixed point

9.2 Proof of Knaster-Tarski Theorem using *lfp*

$lfp\ f$ is the least upper bound of the set $\{u.\ f\ u \leq u\}$

lemma *lfp-lowerbound*: $f\ A \leq A$ ==> $lfp\ f \leq A$
⟨*proof*⟩

lemma *lfp-greatest*: (!!u. $f\ u \leq u$ ==> $A \leq u$) ==> $A \leq lfp\ f$
⟨*proof*⟩

lemma *lfp-lemma2*: *mono* f ==> $f\ (lfp\ f) \leq lfp\ f$
⟨*proof*⟩

lemma *lfp-lemma3*: *mono* f ==> $lfp\ f \leq f\ (lfp\ f)$
⟨*proof*⟩

lemma *lfp-unfold*: *mono f* ==> *lfp f* = *f* (*lfp f*)
⟨*proof*⟩

lemma *lfp-const*: *lfp* ($\lambda x.\ t$) = *t*
⟨*proof*⟩

9.3 General induction rules for least fixed points

theorem *lfp-induct*:
assumes *mono*: *mono f* **and** *ind*: *f* (*inf* (*lfp f*) *P*) <= *P*
shows *lfp f* <= *P*
⟨*proof*⟩

lemma *lfp-induct-set*:
assumes *lfp*: *a*: *lfp*(*f*)
and *mono*: *mono*(*f*)
and *indhyp*: !!*x*. [| *x*: *f*(*lfp*(*f*) *Int* {*x*. *P*(*x*)}) |] ==> *P*(*x*)
shows *P*(*a*)
⟨*proof*⟩

lemma *lfp-ordinal-induct*:
assumes *mono*: *mono f*
and *P-f*: !!*S*. *P S* ==> *P*(*f S*)
and *P-Union*: !!*M*. !*S*:*M*. *P S* ==> *P*(*Union M*)
shows *P*(*lfp f*)
⟨*proof*⟩

Definition forms of *lfp-unfold* and *lfp-induct*, to control unfolding

lemma *def-lfp-unfold*: [| *h*==*lfp*(*f*); *mono*(*f*) |] ==> *h* = *f*(*h*)
⟨*proof*⟩

lemma *def-lfp-induct*:
[| *A* == *lfp*(*f*); *mono*(*f*);
f (*inf A P*) ≤ *P*
|] ==> *A* ≤ *P*
⟨*proof*⟩

lemma *def-lfp-induct-set*:
[| *A* == *lfp*(*f*); *mono*(*f*); *a*:*A*;
!!*x*. [| *x*: *f*(*A Int* {*x*. *P*(*x*)}) |] ==> *P*(*x*)
|] ==> *P*(*a*)
⟨*proof*⟩

lemma *lfp-mono*: (!!*Z*. *f Z* ≤ *g Z*) ==> *lfp f* ≤ *lfp g*
⟨*proof*⟩

9.4 Proof of Knaster-Tarski Theorem using *gfp*

gfp f is the greatest lower bound of the set $\{u.\ u \leq f\ u\}$

lemma *gfp-upperbound*: $X \leq f\ X \ {==}{>}\ X \leq \mathit{gfp}\ f$
⟨*proof*⟩

lemma *gfp-least*: $(!!u.\ u \leq f\ u \ {==}{>}\ u \leq X) \ {==}{>}\ \mathit{gfp}\ f \leq X$
⟨*proof*⟩

lemma *gfp-lemma2*: $\mathit{mono}\ f \ {==}{>}\ \mathit{gfp}\ f \leq f\ (\mathit{gfp}\ f)$
⟨*proof*⟩

lemma *gfp-lemma3*: $\mathit{mono}\ f \ {==}{>}\ f\ (\mathit{gfp}\ f) \leq \mathit{gfp}\ f$
⟨*proof*⟩

lemma *gfp-unfold*: $\mathit{mono}\ f \ {==}{>}\ \mathit{gfp}\ f = f\ (\mathit{gfp}\ f)$
⟨*proof*⟩

9.5 Coinduction rules for greatest fixed points

weak version

lemma *weak-coinduct*: $[|\ a{:}\ X;\ \ X \subseteq f(X)\ |] \ {==}{>}\ a : \mathit{gfp}(f)$
⟨*proof*⟩

lemma *weak-coinduct-image*: $!!X.\ [|\ a : X;\ g`X \subseteq f\ (g`X)\ |] \ {==}{>}\ g\ a : \mathit{gfp}\ f$
⟨*proof*⟩

lemma *coinduct-lemma*:
$[|\ X \leq f\ (\mathit{sup}\ X\ (\mathit{gfp}\ f));\ \ \mathit{mono}\ f\ |] \ {==}{>}\ \mathit{sup}\ X\ (\mathit{gfp}\ f) \leq f\ (\mathit{sup}\ X\ (\mathit{gfp}\ f))$
⟨*proof*⟩

strong version, thanks to Coen and Frost

lemma *coinduct-set*: $[|\ \mathit{mono}(f);\ \ a{:}\ X;\ \ X \subseteq f(X\ \mathit{Un}\ \mathit{gfp}(f))\ |] \ {==}{>}\ a : \mathit{gfp}(f)$
⟨*proof*⟩

lemma *coinduct*: $[|\ \mathit{mono}(f);\ X \leq f\ (\mathit{sup}\ X\ (\mathit{gfp}\ f))\ |] \ {==}{>}\ X \leq \mathit{gfp}(f)$
⟨*proof*⟩

lemma *gfp-fun-UnI2*: $[|\ \mathit{mono}(f);\ \ a{:}\ \mathit{gfp}(f)\ |] \ {==}{>}\ a{:}\ f(X\ \mathit{Un}\ \mathit{gfp}(f))$
⟨*proof*⟩

9.6 Even Stronger Coinduction Rule, by Martin Coen

Weakens the condition $X \subseteq f\ X$ to one expressed using both *lfp* and *gfp*

lemma *coinduct3-mono-lemma*: $\mathit{mono}(f) \ {==}{>}\ \mathit{mono}(\%x.\ f(x)\ \mathit{Un}\ X\ \mathit{Un}\ B)$
⟨*proof*⟩

lemma *coinduct3-lemma*:

$[| X \subseteq f(lfp(\%x.\ f(x)\ Un\ X\ Un\ gfp(f)));\ mono(f)\ |]$
$\implies lfp(\%x.\ f(x)\ Un\ X\ Un\ gfp(f)) \subseteq f(lfp(\%x.\ f(x)\ Un\ X\ Un\ gfp(f)))$
⟨*proof*⟩

lemma *coinduct3*:
$[| mono(f);\ a{:}X;\ X \subseteq f(lfp(\%x.\ f(x)\ Un\ X\ Un\ gfp(f)))\ |] \implies a : gfp(f)$
⟨*proof*⟩

Definition forms of *gfp-unfold* and *coinduct*, to control unfolding

lemma *def-gfp-unfold*: $[| A{=}{=}gfp(f);\ mono(f)\ |] \implies A = f(A)$
⟨*proof*⟩

lemma *def-coinduct*:
$[| A{=}{=}gfp(f);\ mono(f);\ X \leq f(sup\ X\ A)\ |] \implies X \leq A$
⟨*proof*⟩

lemma *def-coinduct-set*:
$[| A{=}{=}gfp(f);\ mono(f);\ a{:}X;\ X \subseteq f(X\ Un\ A)\ |] \implies a{:}\ A$
⟨*proof*⟩

lemma *def-Collect-coinduct*:
$[| A == gfp(\%w.\ Collect(P(w)));\ mono(\%w.\ Collect(P(w)));$
$a{:}\ X;\ !!z.\ z{:}\ X \implies P\ (X\ Un\ A)\ z\ |] \implies$
$a : A$
⟨*proof*⟩

lemma *def-coinduct3*:
$[| A{=}{=}gfp(f); mono(f);\ a{:}X;\ X \subseteq f(lfp(\%x.\ f(x)\ Un\ X\ Un\ A))\ |] \implies a{:}\ A$
⟨*proof*⟩

Monotonicity of *gfp*!

lemma *gfp-mono*: $(!!Z.\ f\ Z \leq g\ Z) \implies gfp\ f \leq gfp\ g$
⟨*proof*⟩

9.7 Inductive predicates and sets

Inversion of injective functions.

constdefs
$myinv :: ('a => 'b) => ('b => 'a)$
$myinv\ (f :: 'a => 'b) == \lambda y.\ THE\ x.\ f\ x = y$

lemma *myinv-f-f*: $inj\ f \implies myinv\ f\ (f\ x) = x$
⟨*proof*⟩

lemma *f-myinv-f*: $inj\ f \implies y \in range\ f \implies f\ (myinv\ f\ y) = y$
⟨*proof*⟩

hide *const myinv*

Package setup.

theorems *basic-monos* =
  subset-refl imp-refl disj-mono conj-mono ex-mono all-mono if-bool-eq-conj
  Collect-mono in-mono vimage-mono
  imp-conv-disj not-not de-Morgan-disj de-Morgan-conj
  not-all not-ex
  Ball-def Bex-def
  induct-rulify-fallback

⟨*ML*⟩

theorems [*mono*] =
  imp-refl disj-mono conj-mono ex-mono all-mono if-bool-eq-conj
  imp-conv-disj not-not de-Morgan-disj de-Morgan-conj
  not-all not-ex
  Ball-def Bex-def
  induct-rulify-fallback

9.8 Inductive datatypes and primitive recursion

Package setup.

⟨*ML*⟩

Lambda-abstractions with pattern matching:

syntax
  -lam-pats-syntax :: *cases-syn* => *'a* => *'b*   ((%-) *10*)
syntax (*xsymbols*)
  -lam-pats-syntax :: *cases-syn* => *'a* => *'b*   ((λ-) *10*)

⟨*ML*⟩

end

10 Product-Type: Cartesian products

theory *Product-Type*
imports *Inductive*
uses
  (*Tools/split-rule.ML*)
  (*Tools/inductive-set-package.ML*)
  (*Tools/inductive-realizer.ML*)
  (*Tools/datatype-realizer.ML*)
begin

10.1 *bool* is a datatype

rep-datatype *bool*
  distinct *True-not-False False-not-True*
  induction *bool-induct*

declare *case-split* [*cases type*: *bool*]
  — prefer plain propositional version

10.2 Unit

typedef *unit* = {*True*}
⟨*proof*⟩

definition
  Unity :: *unit*   ('('))
where
  () = *Abs-unit True*

lemma *unit-eq* [*noatp*]: *u* = ()
  ⟨*proof*⟩

Simplification procedure for *unit-eq*. Cannot use this rule directly — it loops!

⟨*ML*⟩

lemma *unit-induct* [*noatp*,*induct type*: *unit*]: *P* () ==> *P x*
  ⟨*proof*⟩

rep-datatype *unit*
  induction *unit-induct*

lemma *unit-all-eq1*: (!!*x*::*unit*. *PROP P x*) == *PROP P* ()
  ⟨*proof*⟩

lemma *unit-all-eq2*: (!!*x*::*unit*. *PROP P*) == *PROP P*
  ⟨*proof*⟩

This rewrite counters the effect of *unit-eq-proc* on %*u*::*unit*. *f u*, replacing it by *f* rather than by %*u*. *f* ().

lemma *unit-abs-eta-conv* [*simp*,*noatp*]: (%*u*::*unit*. *f* ()) = *f*
  ⟨*proof*⟩

10.3 Pairs

10.3.1 Type definition

constdefs
  Pair-Rep :: ['*a*, '*b*] => ['*a*, '*b*] => *bool*
  Pair-Rep == (%*a b*. %*x y*. *x*=*a* & *y*=*b*)

global

typedef (*Prod*)
  ($'a$, $'b$) $*$ (**infixr** $*$ *20*)
    = $\{f.\ EX\ a\ b.\ f = Pair\text{-}Rep\ (a{::}'a)\ (b{::}'b)\}$
⟨*proof*⟩

syntax (*xsymbols*)
  $*$ :: [*type*, *type*] => *type*   ((- ×/ -) [*21*, *20*] *20*)
syntax (*HTML* **output**)
  $*$ :: [*type*, *type*] => *type*   ((- ×/ -) [*21*, *20*] *20*)

local

10.3.2 Definitions

global

consts
  fst   :: $'a * 'b => 'a$
  snd   :: $'a * 'b => 'b$
  split   :: $[['a, 'b] => 'c, 'a * 'b] => 'c$
  curry   :: $['a * 'b => 'c, 'a, 'b] => 'c$
  prod-fun :: $['a => 'b, 'c => 'd, 'a * 'c] => 'b * 'd$
  Pair   :: $['a, 'b] => 'a * 'b$
  Sigma   :: $['a\ set, 'a => 'b\ set] => ('a * 'b)\ set$

local

defs
  Pair-def: *Pair a b* == *Abs-Prod* (*Pair-Rep a b*)
  fst-def: *fst p* == *THE a. EX b. p* = *Pair a b*
  snd-def: *snd p* == *THE b. EX a. p* = *Pair a b*
  split-def: *split* == (%*c p. c* (*fst p*) (*snd p*))
  curry-def: *curry* == (%*c x y. c* (*Pair x y*))
  prod-fun-def: *prod-fun f g* == *split* (%*x y. Pair* (*f x*) (*g y*))
  Sigma-def [*code func*]:   *Sigma A B* == *UN x*:*A*. *UN y*:*B x*. {*Pair x y*}

abbreviation
  Times :: $['a\ set, 'b\ set] => ('a * 'b)\ set$
    (**infixr** <$*$> *80*) **where**
  A <$*$> *B* == *Sigma A* (%-. *B*)

notation (*xsymbols*)
  Times (**infixr** × *80*)

notation (*HTML* **output**)
  Times (**infixr** × *80*)

10.3.3 Concrete syntax

Patterns – extends pre-defined type *pttrn* used in abstractions.

nonterminals
tuple-args patterns

syntax
```
-tuple      :: 'a => tuple-args => 'a * 'b        ((1'(-,/ -')))
-tuple-arg  :: 'a => tuple-args                   (-)
-tuple-args :: 'a => tuple-args => tuple-args     (-,/ -)
-pattern    :: [pttrn, patterns] => pttrn         ('(-,/ -'))
            :: pttrn => patterns                  (-)
-patterns   :: [pttrn, patterns] => patterns      (-,/ -)
@Sigma ::[pttrn, 'a set, 'b set] => ('a * 'b) set ((3SIGMA -:-./ -) [0, 0, 10] 10)
```

translations
```
(x, y)       == Pair x y
-tuple x (-tuple-args y z) == -tuple x (-tuple-arg (-tuple y z))
%(x,y,zs).b  == split(%x (y,zs).b)
%(x,y).b     == split(%x y. b)
-abs (Pair x y) t => %(x,y).t

SIGMA x:A. B == Sigma A (%x. B)
```

⟨*ML*⟩

10.3.4 Lemmas and proof tool setup

lemma *ProdI*: *Pair-Rep a b* : *Prod*
⟨*proof*⟩

lemma *Pair-Rep-inject*: *Pair-Rep a b* = *Pair-Rep a′ b′* ==> *a* = *a′* & *b* = *b′*
⟨*proof*⟩

lemma *inj-on-Abs-Prod*: *inj-on Abs-Prod Prod*
⟨*proof*⟩

lemma *Pair-inject*:
assumes (*a*, *b*) = (*a′*, *b′*)
and *a* = *a′* ==> *b* = *b′* ==> *R*
shows *R*
⟨*proof*⟩

lemma *Pair-eq* [*iff*]: ((*a*, *b*) = (*a′*, *b′*)) = (*a* = *a′* & *b* = *b′*)
⟨*proof*⟩

lemma *fst-conv* [*simp*, *code*]: *fst* (*a*, *b*) = *a*

⟨*proof*⟩

lemma *snd-conv* [*simp*, *code*]: *snd* (*a*, *b*) = *b*
⟨*proof*⟩

lemma *fst-eqD*: *fst* (*x*, *y*) = *a* ==> *x* = *a*
⟨*proof*⟩

lemma *snd-eqD*: *snd* (*x*, *y*) = *a* ==> *y* = *a*
⟨*proof*⟩

lemma *PairE-lemma*: *EX x y*. *p* = (*x*, *y*)
⟨*proof*⟩

lemma *PairE* [*cases type*: ∗]: (!!*x y*. *p* = (*x*, *y*) ==> *Q*) ==> *Q*
⟨*proof*⟩

⟨*ML*⟩

lemma *surjective-pairing*: *p* = (*fst p*, *snd p*)
— Do not add as rewrite rule: invalidates some proofs in IMP
⟨*proof*⟩

lemmas *pair-collapse* = *surjective-pairing* [*symmetric*]
declare *pair-collapse* [*simp*]

lemma *surj-pair* [*simp*]: *EX x y*. *z* = (*x*, *y*)
⟨*proof*⟩

lemma *split-paired-all*: (!!*x*. *PROP P x*) == (!!*a b*. *PROP P* (*a*, *b*))
⟨*proof*⟩

lemmas *split-tupled-all* = *split-paired-all unit-all-eq2*

The rule *split-paired-all* does not work with the Simplifier because it also affects premises in congrence rules, where this can lead to premises of the form !!*a b*. ... = *?P*(*a*, *b*) which cannot be solved by reflexivity.

⟨*ML*⟩

lemma *split-paired-All* [*simp*]: (*ALL x*. *P x*) = (*ALL a b*. *P* (*a*, *b*))
— [*iff*] is not a good idea because it makes *blast* loop
⟨*proof*⟩

lemma *curry-split* [*simp*]: *curry* (*split f*) = *f*
⟨*proof*⟩

lemma *split-curry* [*simp*]: *split* (*curry f*) = *f*
⟨*proof*⟩

lemma *curryI* [*intro*!]: *f* (*a*,*b*) ==> *curry f a b*
⟨*proof*⟩

lemma *curryD* [*dest*!]: *curry f a b* ==> *f* (*a*,*b*)
⟨*proof*⟩

lemma *curryE*: [| *curry f a b* ; *f* (*a*,*b*) ==> *Q* |] ==> *Q*
⟨*proof*⟩

lemma *curry-conv* [*simp*, *code func*]: *curry f a b* = *f* (*a*,*b*)
⟨*proof*⟩

lemma *prod-induct* [*induct type*: ∗]: !!*x*. (!!*a b*. *P* (*a*, *b*)) ==> *P x*
⟨*proof*⟩

rep-datatype *prod*
inject *Pair-eq*
induction *prod-induct*

lemma *split-paired-Ex* [*simp*]: (*EX x*. *P x*) = (*EX a b*. *P* (*a*, *b*))
⟨*proof*⟩

lemma *split-conv* [*simp*, *code func*]: *split c* (*a*, *b*) = *c a b*
⟨*proof*⟩

lemmas *split* = *split-conv* — for backwards compatibility

lemmas *splitI* = *split-conv* [*THEN iffD2*, *standard*]
lemmas *splitD* = *split-conv* [*THEN iffD1*, *standard*]

lemma *split-Pair-apply*: *split* (%*x y*. *f* (*x*, *y*)) = *f*
— Subsumes the old *split-Pair* when *f* is the identity function.
⟨*proof*⟩

lemma *split-paired-The*: (*THE x*. *P x*) = (*THE* (*a*, *b*). *P* (*a*, *b*))
— Can't be added to simpset: loops!
⟨*proof*⟩

lemma *The-split*: *The* (*split P*) = (*THE xy*. *P* (*fst xy*) (*snd xy*))
⟨*proof*⟩

lemma *Pair-fst-snd-eq*: !!*s t*. (*s* = *t*) = (*fst s* = *fst t* & *snd s* = *snd t*)
⟨*proof*⟩

lemma *prod-eqI* [*intro?*]: *fst p* = *fst q* ==> *snd p* = *snd q* ==> *p* = *q*
⟨*proof*⟩

lemma *split-weak-cong*: *p* = *q* ==> *split c p* = *split c q*
— Prevents simplification of *c*: much faster

⟨*proof*⟩

lemma *split-eta*: $(\%(x,\ y).\ f\ (x,\ y)) = f$
⟨*proof*⟩

lemma *cond-split-eta*: $(!!x\ y.\ f\ x\ y = g\ (x,\ y))$ ==> $(\%(x,\ y).\ f\ x\ y) = g$
⟨*proof*⟩

Simplification procedure for *cond-split-eta*. Using *split-eta* as a rewrite rule is not general enough, and using *cond-split-eta* directly would render some existing proofs very inefficient; similarly for *split-beta*.

⟨*ML*⟩

lemma *split-beta*: $(\%(x,\ y).\ P\ x\ y)\ z = P\ (fst\ z)\ (snd\ z)$
⟨*proof*⟩

lemma *split-split* [*noatp*]: $R(split\ c\ p) = (ALL\ x\ y.\ p = (x,\ y)$ --> $R(c\ x\ y))$
— For use with *split* and the Simplifier.
⟨*proof*⟩

split-split could be declared as [*split*] done after the Splitter has been speeded up significantly; precompute the constants involved and don't do anything unless the current goal contains one of those constants.

lemma *split-split-asm* [*noatp*]: $R\ (split\ c\ p) = (\sim(EX\ x\ y.\ p = (x,\ y)\ \&\ (\sim R\ (c\ x\ y))))$
⟨*proof*⟩

split used as a logical connective or set former.

These rules are for use with *blast*; could instead call *simp* using *split* as rewrite.

lemma *splitI2*: $!!p.$ [| $!!a\ b.\ p = (a,\ b)$ ==> $c\ a\ b$ |] ==> $split\ c\ p$
⟨*proof*⟩

lemma *splitI2'*: $!!p.$ [| $!!a\ b.\ (a,\ b) = p$ ==> $c\ a\ b\ x$ |] ==> $split\ c\ p\ x$
⟨*proof*⟩

lemma *splitE*: $split\ c\ p$ ==> $(!!x\ y.\ p = (x,\ y)$ ==> $c\ x\ y$ ==> $Q)$ ==> Q
⟨*proof*⟩

lemma *splitE'*: $split\ c\ p\ z$ ==> $(!!x\ y.\ p = (x,\ y)$ ==> $c\ x\ y\ z$ ==> $Q)$ ==> Q
⟨*proof*⟩

lemma *splitE2*:
[| $Q\ (split\ P\ z)$; $!!x\ y.$ [|$z = (x,\ y)$; $Q\ (P\ x\ y)$|] ==> R |] ==> R
⟨*proof*⟩

lemma *splitD'*: $split\ R\ (a,b)\ c$ ==> $R\ a\ b\ c$

⟨*proof*⟩

lemma *mem-splitI*: *z*: *c a b* ==> *z*: *split c* (*a*, *b*)
⟨*proof*⟩

lemma *mem-splitI2*: !!*p*. [| !!*a b*. *p* = (*a*, *b*) ==> *z*: *c a b* |] ==> *z*: *split c p*
⟨*proof*⟩

lemma *mem-splitE*:
assumes *major*: *z*: *split c p*
and *cases*: !!*x y*. [| *p* = (*x*,*y*); *z*: *c x y* |] ==> *Q*
shows *Q*
⟨*proof*⟩

declare *mem-splitI2* [*intro*!] *mem-splitI* [*intro*!] *splitI2'* [*intro*!] *splitI2* [*intro*!] *splitI* [*intro*!]
declare *mem-splitE* [*elim*!] *splitE'* [*elim*!] *splitE* [*elim*!]

⟨*ML*⟩

lemma *split-eta-SetCompr* [*simp*,*noatp*]: (%*u*. *EX x y*. *u* = (*x*, *y*) & *P* (*x*, *y*)) = *P*
⟨*proof*⟩

lemma *split-eta-SetCompr2* [*simp*,*noatp*]: (%*u*. *EX x y*. *u* = (*x*, *y*) & *P x y*) = *split P*
⟨*proof*⟩

lemma *split-part* [*simp*]: (%(*a*,*b*). *P* & *Q a b*) = (%*ab*. *P* & *split Q ab*)
— Allows simplifications of nested splits in case of independent predicates.
⟨*proof*⟩

lemma *split-comp-eq*:
fixes *f* :: *'a* => *'b* => *'c* **and** *g* :: *'d* => *'a*
shows (%*u*. *f* (*g* (*fst u*)) (*snd u*)) = (*split* (%*x*. *f* (*g x*)))
⟨*proof*⟩

lemma *The-split-eq* [*simp*]: (*THE* (*x'*,*y'*). *x* = *x'* & *y* = *y'*) = (*x*, *y*)
⟨*proof*⟩

lemma *injective-fst-snd*: !!*x y*. [|*fst x* = *fst y*; *snd x* = *snd y*|] ==> *x* = *y*
⟨*proof*⟩

prod-fun — action of the product functor upon functions.

lemma *prod-fun* [*simp*, *code func*]: *prod-fun f g* (*a*, *b*) = (*f a*, *g b*)

⟨*proof*⟩

lemma *prod-fun-compose*: *prod-fun* (*f1 o f2*) (*g1 o g2*) = (*prod-fun f1 g1 o prod-fun f2 g2*)
⟨*proof*⟩

lemma *prod-fun-ident* [*simp*]: *prod-fun* (%*x*. *x*) (%*y*. *y*) = (%*z*. *z*)
⟨*proof*⟩

lemma *prod-fun-imageI* [*intro*]: (*a*, *b*) : *r* ==> (*f a*, *g b*) : *prod-fun f g* ' *r*
⟨*proof*⟩

lemma *prod-fun-imageE* [*elim*!]:
assumes *major*: *c*: (*prod-fun f g*)'*r*
and *cases*: !!*x y*. [| *c*=(*f*(*x*),*g*(*y*)); (*x*,*y*):*r* |] ==> *P*
shows *P*
⟨*proof*⟩

definition
upd-fst :: ('*a* ⇒ '*c*) ⇒ '*a* × '*b* ⇒ '*c* × '*b*
where
[*code func del*]: *upd-fst f* = *prod-fun f id*

definition
upd-snd :: ('*b* ⇒ '*c*) ⇒ '*a* × '*b* ⇒ '*a* × '*c*
where
[*code func del*]: *upd-snd f* = *prod-fun id f*

lemma *upd-fst-conv* [*simp*, *code*]:
upd-fst f (*x*, *y*) = (*f x*, *y*)
⟨*proof*⟩

lemma *upd-snd-conv* [*simp*, *code*]:
upd-snd f (*x*, *y*) = (*x*, *f y*)
⟨*proof*⟩

Disjoint union of a family of sets – Sigma.

lemma *SigmaI* [*intro*!]: [| *a*:*A*; *b*:*B*(*a*) |] ==> (*a*,*b*) : *Sigma A B*
⟨*proof*⟩

lemma *SigmaE* [*elim*!]:
[| *c*: *Sigma A B*;
!!*x y*.[| *x*:*A*; *y*:*B*(*x*); *c*=(*x*,*y*) |] ==> *P*
|] ==> *P*
— The general elimination rule.
⟨*proof*⟩

Elimination of $(a, b) \in A \times B$ – introduces no eigenvariables.

lemma *SigmaD1*: $(a,\ b)$: *Sigma A B* ==> *a* : *A*
⟨*proof*⟩

lemma *SigmaD2*: $(a,\ b)$: *Sigma A B* ==> *b* : *B a*
⟨*proof*⟩

lemma *SigmaE2*:
  [| $(a,\ b)$: *Sigma A B*;
    [| *a*:*A*; *b*:*B*(*a*) |] ==> *P*
  |] ==> *P*
⟨*proof*⟩

lemma *Sigma-cong*:
  $[\![A = B; !!x.\ x \in B \Longrightarrow C\ x = D\ x]\!]$
  $\Longrightarrow$ (*SIGMA x*: *A*. *C x*) = (*SIGMA x*: *B*. *D x*)
⟨*proof*⟩

lemma *Sigma-mono*: [| *A* <= *C*; !!*x*. *x*:*A* ==> *B x* <= *D x* |] ==> *Sigma A B* <= *Sigma C D*
⟨*proof*⟩

lemma *Sigma-empty1* [*simp*]: *Sigma* {} *B* = {}
⟨*proof*⟩

lemma *Sigma-empty2* [*simp*]: *A* <∗> {} = {}
⟨*proof*⟩

lemma *UNIV-Times-UNIV* [*simp*]: *UNIV* <∗> *UNIV* = *UNIV*
⟨*proof*⟩

lemma *Compl-Times-UNIV1* [*simp*]: − (*UNIV* <∗> *A*) = *UNIV* <∗> (−*A*)
⟨*proof*⟩

lemma *Compl-Times-UNIV2* [*simp*]: − (*A* <∗> *UNIV*) = (−*A*) <∗> *UNIV*
⟨*proof*⟩

lemma *mem-Sigma-iff* [*iff*]: ((*a*,*b*): *Sigma A B*) = (*a*:*A* & *b*:*B*(*a*))
⟨*proof*⟩

lemma *Times-subset-cancel2*: *x*:*C* ==> (*A* <∗> *C* <= *B* <∗> *C*) = (*A* <= *B*)
⟨*proof*⟩

lemma *Times-eq-cancel2*: *x*:*C* ==> (*A* <∗> *C* = *B* <∗> *C*) = (*A* = *B*)
⟨*proof*⟩

lemma *SetCompr-Sigma-eq*:
  Collect (*split* (%*x y*. *P x* & *Q x y*)) = (*SIGMA x*:*Collect P*. *Collect* (*Q x*))
⟨*proof*⟩

Complex rules for Sigma.

lemma *Collect-split* [*simp*]: {(*a*,*b*). *P a* & *Q b*} = *Collect P* <∗> *Collect Q*
⟨*proof*⟩

lemma *UN-Times-distrib*:
(*UN* (*a*,*b*):(*A* <∗> *B*). *E a* <∗> *F b*) = (*UNION A E*) <∗> (*UNION B F*)
— Suggested by Pierre Chartier
⟨*proof*⟩

lemma *split-paired-Ball-Sigma* [*simp*,*noatp*]:
(*ALL z*: *Sigma A B*. *P z*) = (*ALL x*:*A*. *ALL y*: *B x*. *P*(*x*,*y*))
⟨*proof*⟩

lemma *split-paired-Bex-Sigma* [*simp*,*noatp*]:
(*EX z*: *Sigma A B*. *P z*) = (*EX x*:*A*. *EX y*: *B x*. *P*(*x*,*y*))
⟨*proof*⟩

lemma *Sigma-Un-distrib1*: (*SIGMA i*:*I Un J*. *C*(*i*)) = (*SIGMA i*:*I*. *C*(*i*)) *Un* (*SIGMA j*:*J*. *C*(*j*))
⟨*proof*⟩

lemma *Sigma-Un-distrib2*: (*SIGMA i*:*I*. *A*(*i*) *Un B*(*i*)) = (*SIGMA i*:*I*. *A*(*i*)) *Un* (*SIGMA i*:*I*. *B*(*i*))
⟨*proof*⟩

lemma *Sigma-Int-distrib1*: (*SIGMA i*:*I Int J*. *C*(*i*)) = (*SIGMA i*:*I*. *C*(*i*)) *Int* (*SIGMA j*:*J*. *C*(*j*))
⟨*proof*⟩

lemma *Sigma-Int-distrib2*: (*SIGMA i*:*I*. *A*(*i*) *Int B*(*i*)) = (*SIGMA i*:*I*. *A*(*i*)) *Int* (*SIGMA i*:*I*. *B*(*i*))
⟨*proof*⟩

lemma *Sigma-Diff-distrib1*: (*SIGMA i*:*I* − *J*. *C*(*i*)) = (*SIGMA i*:*I*. *C*(*i*)) − (*SIGMA j*:*J*. *C*(*j*))
⟨*proof*⟩

lemma *Sigma-Diff-distrib2*: (*SIGMA i*:*I*. *A*(*i*) − *B*(*i*)) = (*SIGMA i*:*I*. *A*(*i*)) − (*SIGMA i*:*I*. *B*(*i*))
⟨*proof*⟩

lemma *Sigma-Union*: *Sigma* (*Union X*) *B* = (*UN A*:*X*. *Sigma A B*)
⟨*proof*⟩

Non-dependent versions are needed to avoid the need for higher-order matching, especially when the rules are re-oriented.

lemma *Times-Un-distrib1*: (*A Un B*) <∗> *C* = (*A* <∗> *C*) *Un* (*B* <∗> *C*)
⟨*proof*⟩

lemma *Times-Int-distrib1*: (A Int B) <*> C = (A <*> C) Int (B <*> C)
  ⟨*proof*⟩

lemma *Times-Diff-distrib1*: (A − B) <*> C = (A <*> C) − (B <*> C)
  ⟨*proof*⟩

lemma *pair-imageI* [*intro*]: (a, b) : A ==> f a b : (%(a, b). f a b) ‘ A
  ⟨*proof*⟩

Setup of internal *split-rule*.

constdefs
  internal-split :: ('a => 'b => 'c) => 'a * 'b => 'c
  internal-split == *split*

lemmas [*code func del*] = *internal-split-def*

lemma *internal-split-conv*: *internal-split* c (a, b) = c a b
  ⟨*proof*⟩

hide *const internal-split*

⟨*ML*⟩

lemmas *prod-caseI* = *prod.cases* [*THEN iffD2*, *standard*]

lemma *prod-caseI2*: !!p. [| !!a b. p = (a, b) ==> c a b |] ==> *prod-case* c p
  ⟨*proof*⟩

lemma *prod-caseI2'*: !!p. [| !!a b. (a, b) = p ==> c a b x |] ==> *prod-case* c p x
  ⟨*proof*⟩

lemma *prod-caseE*: *prod-case* c p ==> (!!x y. p = (x, y) ==> c x y ==> Q)
==> Q
  ⟨*proof*⟩

lemma *prod-caseE'*: *prod-case* c p z ==> (!!x y. p = (x, y) ==> c x y z ==>
Q) ==> Q
  ⟨*proof*⟩

lemma *prod-case-unfold*: *prod-case* = (%c p. c (*fst* p) (*snd* p))
  ⟨*proof*⟩

declare *prod-caseI2'* [*intro*!] *prod-caseI2* [*intro*!] *prod-caseI* [*intro*!]
declare *prod-caseE'* [*elim*!] *prod-caseE* [*elim*!]

lemma *prod-case-split*:

prod-case = *split*
⟨*proof*⟩

10.4 Further cases/induct rules for tuples

lemma *prod-cases3* [*cases type*]:
obtains (*fields*) *a b c* **where** *y* = (*a*, *b*, *c*)
⟨*proof*⟩

lemma *prod-induct3* [*case-names fields*, *induct type*]:
(!!*a b c*. *P* (*a*, *b*, *c*)) ==> *P x*
⟨*proof*⟩

lemma *prod-cases4* [*cases type*]:
obtains (*fields*) *a b c d* **where** *y* = (*a*, *b*, *c*, *d*)
⟨*proof*⟩

lemma *prod-induct4* [*case-names fields*, *induct type*]:
(!!*a b c d*. *P* (*a*, *b*, *c*, *d*)) ==> *P x*
⟨*proof*⟩

lemma *prod-cases5* [*cases type*]:
obtains (*fields*) *a b c d e* **where** *y* = (*a*, *b*, *c*, *d*, *e*)
⟨*proof*⟩

lemma *prod-induct5* [*case-names fields*, *induct type*]:
(!!*a b c d e*. *P* (*a*, *b*, *c*, *d*, *e*)) ==> *P x*
⟨*proof*⟩

lemma *prod-cases6* [*cases type*]:
obtains (*fields*) *a b c d e f* **where** *y* = (*a*, *b*, *c*, *d*, *e*, *f*)
⟨*proof*⟩

lemma *prod-induct6* [*case-names fields*, *induct type*]:
(!!*a b c d e f*. *P* (*a*, *b*, *c*, *d*, *e*, *f*)) ==> *P x*
⟨*proof*⟩

lemma *prod-cases7* [*cases type*]:
obtains (*fields*) *a b c d e f g* **where** *y* = (*a*, *b*, *c*, *d*, *e*, *f*, *g*)
⟨*proof*⟩

lemma *prod-induct7* [*case-names fields*, *induct type*]:
(!!*a b c d e f g*. *P* (*a*, *b*, *c*, *d*, *e*, *f*, *g*)) ==> *P x*
⟨*proof*⟩

10.5 Further lemmas

lemma
split-Pair: *split Pair x* = *x*
⟨*proof*⟩

lemma
split-comp: $split\ (f \circ g)\ x = f\ (g\ (fst\ x))\ (snd\ x)$
$\langle proof \rangle$

10.6 Code generator setup

instance *unit* :: *eq* $\langle proof \rangle$

lemma [*code func*]:
$(u{::}unit) = v \longleftrightarrow True$ $\langle proof \rangle$

code-type *unit*
(*SML unit*)
(*OCaml unit*)
(*Haskell* ())

code-instance *unit* :: *eq*
(*Haskell* −)

code-const $op = :: unit \Rightarrow unit \Rightarrow bool$
(*Haskell* **infixl** *4* ==)

code-const *Unity*
(*SML* ())
(*OCaml* ())
(*Haskell* ())

code-reserved *SML*
unit

code-reserved *OCaml*
unit

instance $*$ :: (*eq*, *eq*) *eq* $\langle proof \rangle$

lemma [*code func*]:
$(x1{::}'a{::}eq,\ y1{::}'b{::}eq) = (x2,\ y2) \longleftrightarrow x1 = x2 \wedge y1 = y2$ $\langle proof \rangle$

lemma *split-case-cert*:
assumes $CASE \equiv split\ f$
shows $CASE\ (a,\ b) \equiv f\ a\ b$
$\langle proof \rangle$

$\langle ML \rangle$

code-type $*$
(*SML* **infix** *2* $*$)
(*OCaml* **infix** *2* $*$)

(*Haskell* !((-),/ (-)))

code-instance $*$ :: *eq*
(*Haskell* $-$)

code-const *op* $=$ :: $'a::eq \times 'b::eq \Rightarrow 'a \times 'b \Rightarrow bool$
(*Haskell* **infixl** *4* ==)

code-const *Pair*
(*SML* !((-),/ (-)))
(*OCaml* !((-),/ (-)))
(*Haskell* !((-),/ (-)))

code-const *fst* **and** *snd*
(*Haskell* *fst* **and** *snd*)

types-code
$*$ ((- $*$/ -))
attach (*term-of*) ⟪
fun term-of-id-42 f T g U (x, y) = HOLogic.pair-const T U $ f x $ g y;
⟫
attach (*test*) ⟪
fun gen-id-42 aG bG i = (aG i, bG i);
⟫

consts-code
Pair ((-,/ -))

⟨*ML*⟩

10.7 Legacy bindings

⟨*ML*⟩

10.8 Further inductive packages

⟨*ML*⟩

end

11 Relation: Relations

theory *Relation*
imports *Product-Type*
begin

11.1 Definitions

definition
converse :: $('a * 'b)$ *set* => $('b * 'a)$ *set*
((-^−1) [1000] 999) **where**
r^−1 == {(y, x). (x, y) : r}

notation (*xsymbols*)
converse ((-$^{-1}$) [1000] 999)

definition
rel-comp :: [('b * 'c) *set*, ('a * 'b) *set*] => ('a * 'c) *set*
(**infixr** *O 75*) **where**
r O s == {(x,z). EX y. (x, y) : s & (y, z) : r}

definition
Image :: [('a * 'b) *set*, 'a *set*] => 'b *set*
(**infixl** `` 90) **where**
r `` *s* == {y. EX x:s. (x,y):r}

definition
Id :: ('a * 'a) *set* **where** — the identity relation
Id == {p. EX x. p = (x,x)}

definition
diag :: 'a *set* => ('a * 'a) *set* **where** — diagonal: identity over a set
diag A == $\bigcup x{\in}A$. {(x,x)}

definition
Domain :: ('a * 'b) *set* => 'a *set* **where**
Domain r == {x. EX y. (x,y):r}

definition
Range :: ('a * 'b) *set* => 'b *set* **where**
Range r == *Domain*(r^−1)

definition
Field :: ('a * 'a) *set* => 'a *set* **where**
Field r == *Domain r* ∪ *Range r*

definition
refl :: ['a *set*, ('a * 'a) *set*] => *bool* **where** — reflexivity over a set
refl A r == $r \subseteq A \times A$ & (ALL x: A. (x,x) : r)

definition
sym :: ('a * 'a) *set* => *bool* **where** — symmetry predicate
sym r == ALL x y. (x,y): r --> (y,x): r

definition
antisym :: ('a * 'a) *set* => *bool* **where** — antisymmetry predicate

antisym r == ALL x y. (x,y):r --> (y,x):r --> x=y

definition
trans :: (*'a * 'a*) *set => bool* **where** — transitivity predicate
trans r == (ALL x y z. (x,y):r --> (y,z):r --> (x,z):r)

definition
single-valued :: (*'a * 'b*) *set => bool* **where**
single-valued r == ALL x y. (x,y):r --> (ALL z. (x,z):r --> y=z)

definition
inv-image :: (*'b * 'b*) *set =>* (*'a => 'b*) *=>* (*'a * 'a*) *set* **where**
inv-image r f == {(*x*, *y*). (*f x*, *f y*) : *r*}

abbreviation
reflexive :: (*'a * 'a*) *set => bool* **where** — reflexivity over a type
reflexive == refl UNIV

11.2 The identity relation

lemma *IdI* [*intro*]: (*a*, *a*) : *Id*
⟨*proof*⟩

lemma *IdE* [*elim*!]: *p* : *Id ==>* (!!*x*. *p* = (*x*, *x*) *==> P*) *==> P*
⟨*proof*⟩

lemma *pair-in-Id-conv* [*iff*]: ((*a*, *b*) : *Id*) = (*a* = *b*)
⟨*proof*⟩

lemma *reflexive-Id*: *reflexive Id*
⟨*proof*⟩

lemma *antisym-Id*: *antisym Id*
— A strange result, since *Id* is also symmetric.
⟨*proof*⟩

lemma *sym-Id*: *sym Id*
⟨*proof*⟩

lemma *trans-Id*: *trans Id*
⟨*proof*⟩

11.3 Diagonal: identity over a set

lemma *diag-empty* [*simp*]: *diag* {} = {}
⟨*proof*⟩

lemma *diag-eqI*: *a* = *b ==> a* : *A ==>* (*a*, *b*) : *diag A*
⟨*proof*⟩

lemma *diagI* [*intro*!,*noatp*]: $a : A ==> (a,\ a) : diag\ A$
⟨*proof*⟩

lemma *diagE* [*elim*!]:
$c : diag\ A ==> (!!x.\ x : A ==> c = (x,\ x) ==> P) ==> P$
— The general elimination rule.
⟨*proof*⟩

lemma *diag-iff*: $((x,\ y) : diag\ A) = (x = y\ \&\ x : A)$
⟨*proof*⟩

lemma *diag-subset-Times*: $diag\ A \subseteq A \times A$
⟨*proof*⟩

11.4 Composition of two relations

lemma *rel-compI* [*intro*]:
$(a,\ b) : s ==> (b,\ c) : r ==> (a,\ c) : r\ O\ s$
⟨*proof*⟩

lemma *rel-compE* [*elim*!]: $xz : r\ O\ s ==>$
$(!!x\ y\ z.\ xz = (x,\ z) ==> (x,\ y) : s ==> (y,\ z) : r\ \ ==> P) ==> P$
⟨*proof*⟩

lemma *rel-compEpair*:
$(a,\ c) : r\ O\ s ==> (!!y.\ (a,\ y) : s ==> (y,\ c) : r ==> P) ==> P$
⟨*proof*⟩

lemma *R-O-Id* [*simp*]: $R\ O\ Id = R$
⟨*proof*⟩

lemma *Id-O-R* [*simp*]: $Id\ O\ R = R$
⟨*proof*⟩

lemma *rel-comp-empty1*[*simp*]: $\{\}\ O\ R = \{\}$
⟨*proof*⟩

lemma *rel-comp-empty2*[*simp*]: $R\ O\ \{\} = \{\}$
⟨*proof*⟩

lemma *O-assoc*: $(R\ O\ S)\ O\ T = R\ O\ (S\ O\ T)$
⟨*proof*⟩

lemma *trans-O-subset*: $trans\ r ==> r\ O\ r \subseteq r$
⟨*proof*⟩

lemma *rel-comp-mono*: $r' \subseteq r ==> s' \subseteq s ==> (r'\ O\ s') \subseteq (r\ O\ s)$
⟨*proof*⟩

lemma *rel-comp-subset-Sigma*:
$s \subseteq A \times B$ ==> $r \subseteq B \times C$ ==> $(r\ O\ s) \subseteq A \times C$
⟨*proof*⟩

11.5 Reflexivity

lemma *reflI*: $r \subseteq A \times A$ ==> (!!x. x : A ==> (x, x) : r) ==> *refl* A r
⟨*proof*⟩

lemma *reflD*: *refl* A r ==> a : A ==> (a, a) : r
⟨*proof*⟩

lemma *reflD1*: *refl* A r ==> (x, y) : r ==> x : A
⟨*proof*⟩

lemma *reflD2*: *refl* A r ==> (x, y) : r ==> y : A
⟨*proof*⟩

lemma *refl-Int*: *refl* A r ==> *refl* B s ==> *refl* $(A \cap B)$ $(r \cap s)$
⟨*proof*⟩

lemma *refl-Un*: *refl* A r ==> *refl* B s ==> *refl* $(A \cup B)$ $(r \cup s)$
⟨*proof*⟩

lemma *refl-INTER*:
ALL x:S. *refl* $(A\ x)$ $(r\ x)$ ==> *refl* (*INTER* S A) (*INTER* S r)
⟨*proof*⟩

lemma *refl-UNION*:
ALL x:S. *refl* $(A\ x)$ $(r\ x)$ $\Longrightarrow$ *refl* (*UNION* S A) (*UNION* S r)
⟨*proof*⟩

lemma *refl-diag*: *refl* A (*diag* A)
⟨*proof*⟩

11.6 Antisymmetry

lemma *antisymI*:
(!!x y. (x, y) : r ==> (y, x) : r ==> x=y) ==> *antisym* r
⟨*proof*⟩

lemma *antisymD*: *antisym* r ==> (a, b) : r ==> (b, a) : r ==> $a = b$
⟨*proof*⟩

lemma *antisym-subset*: $r \subseteq s$ ==> *antisym* s ==> *antisym* r
⟨*proof*⟩

lemma *antisym-empty* [*simp*]: *antisym* {}
⟨*proof*⟩

lemma *antisym-diag* [*simp*]: *antisym* (*diag A*)
⟨*proof*⟩

11.7 Symmetry

lemma *symI*: (!!*a b*. (*a*, *b*) : *r* ==> (*b*, *a*) : *r*) ==> *sym r*
⟨*proof*⟩

lemma *symD*: *sym r* ==> (*a*, *b*) : *r* ==> (*b*, *a*) : *r*
⟨*proof*⟩

lemma *sym-Int*: *sym r* ==> *sym s* ==> *sym* (*r* ∩ *s*)
⟨*proof*⟩

lemma *sym-Un*: *sym r* ==> *sym s* ==> *sym* (*r* ∪ *s*)
⟨*proof*⟩

lemma *sym-INTER*: *ALL x*:*S*. *sym* (*r x*) ==> *sym* (*INTER S r*)
⟨*proof*⟩

lemma *sym-UNION*: *ALL x*:*S*. *sym* (*r x*) ==> *sym* (*UNION S r*)
⟨*proof*⟩

lemma *sym-diag* [*simp*]: *sym* (*diag A*)
⟨*proof*⟩

11.8 Transitivity

lemma *transI*:
(!!*x y z*. (*x*, *y*) : *r* ==> (*y*, *z*) : *r* ==> (*x*, *z*) : *r*) ==> *trans r*
⟨*proof*⟩

lemma *transD*: *trans r* ==> (*a*, *b*) : *r* ==> (*b*, *c*) : *r* ==> (*a*, *c*) : *r*
⟨*proof*⟩

lemma *trans-Int*: *trans r* ==> *trans s* ==> *trans* (*r* ∩ *s*)
⟨*proof*⟩

lemma *trans-INTER*: *ALL x*:*S*. *trans* (*r x*) ==> *trans* (*INTER S r*)
⟨*proof*⟩

lemma *trans-diag* [*simp*]: *trans* (*diag A*)
⟨*proof*⟩

11.9 Converse

lemma *converse-iff* [*iff*]: ((*a*,*b*): *r*^−*1*) = ((*b*,*a*) : *r*)
⟨*proof*⟩

lemma *converseI*[*sym*]: (*a*, *b*) : *r* ==> (*b*, *a*) : *r*^−*1*

⟨*proof*⟩

lemma *converseD*[*sym*]: $(a,b) : r\hat{}-1 ==> (b, a) : r$
⟨*proof*⟩

lemma *converseE* [*elim!*]:
$yx : r\hat{}-1 ==> (!!x\ y.\ yx = (y, x) ==> (x, y) : r ==> P) ==> P$
— More general than *converseD*, as it “splits” the member of the relation.
⟨*proof*⟩

lemma *converse-converse* [*simp*]: $(r\hat{}-1)\hat{}-1 = r$
⟨*proof*⟩

lemma *converse-rel-comp*: $(r\ O\ s)\hat{}-1 = s\hat{}-1\ O\ r\hat{}-1$
⟨*proof*⟩

lemma *converse-Int*: $(r \cap s)\hat{}-1 = r\hat{}-1 \cap s\hat{}-1$
⟨*proof*⟩

lemma *converse-Un*: $(r \cup s)\hat{}-1 = r\hat{}-1 \cup s\hat{}-1$
⟨*proof*⟩

lemma *converse-INTER*: $(INTER\ S\ r)\hat{}-1 = (INT\ x{:}S.\ (r\ x)\hat{}-1)$
⟨*proof*⟩

lemma *converse-UNION*: $(UNION\ S\ r)\hat{}-1 = (UN\ x{:}S.\ (r\ x)\hat{}-1)$
⟨*proof*⟩

lemma *converse-Id* [*simp*]: $Id\hat{}-1 = Id$
⟨*proof*⟩

lemma *converse-diag* [*simp*]: $(diag\ A)\hat{}-1 = diag\ A$
⟨*proof*⟩

lemma *refl-converse* [*simp*]: *refl A* (*converse r*) = *refl A r*
⟨*proof*⟩

lemma *sym-converse* [*simp*]: *sym* (*converse r*) = *sym r*
⟨*proof*⟩

lemma *antisym-converse* [*simp*]: *antisym* (*converse r*) = *antisym r*
⟨*proof*⟩

lemma *trans-converse* [*simp*]: *trans* (*converse r*) = *trans r*
⟨*proof*⟩

lemma *sym-conv-converse-eq*: $sym\ r = (r\hat{}-1 = r)$
⟨*proof*⟩

lemma *sym-Un-converse*: $sym\ (r \cup r\hat{\ }{-1})$
⟨*proof*⟩

lemma *sym-Int-converse*: $sym\ (r \cap r\hat{\ }{-1})$
⟨*proof*⟩

11.10 Domain

declare *Domain-def* [*noatp*]

lemma *Domain-iff*: $(a : Domain\ r) = (EX\ y.\ (a,\ y) : r)$
⟨*proof*⟩

lemma *DomainI* [*intro*]: $(a,\ b) : r ==> a : Domain\ r$
⟨*proof*⟩

lemma *DomainE* [*elim*!]:
$a : Domain\ r ==> (!!y.\ (a,\ y) : r ==> P) ==> P$
⟨*proof*⟩

lemma *Domain-empty* [*simp*]: $Domain\ \{\} = \{\}$
⟨*proof*⟩

lemma *Domain-insert*: $Domain\ (insert\ (a,\ b)\ r) = insert\ a\ (Domain\ r)$
⟨*proof*⟩

lemma *Domain-Id* [*simp*]: $Domain\ Id = UNIV$
⟨*proof*⟩

lemma *Domain-diag* [*simp*]: $Domain\ (diag\ A) = A$
⟨*proof*⟩

lemma *Domain-Un-eq*: $Domain(A \cup B) = Domain(A) \cup Domain(B)$
⟨*proof*⟩

lemma *Domain-Int-subset*: $Domain(A \cap B) \subseteq Domain(A) \cap Domain(B)$
⟨*proof*⟩

lemma *Domain-Diff-subset*: $Domain(A) - Domain(B) \subseteq Domain(A - B)$
⟨*proof*⟩

lemma *Domain-Union*: $Domain\ (Union\ S) = (\bigcup A{\in}S.\ Domain\ A)$
⟨*proof*⟩

lemma *Domain-mono*: $r \subseteq s ==> Domain\ r \subseteq Domain\ s$
⟨*proof*⟩

lemma *fst-eq-Domain*: $fst\ `\ R = Domain\ R$
⟨*proof*⟩

11.11 Range

lemma *Range-iff*: $(a : Range\ r) = (EX\ y.\ (y,\ a) : r)$
⟨*proof*⟩

lemma *RangeI* [*intro*]: $(a,\ b) : r ==> b : Range\ r$
⟨*proof*⟩

lemma *RangeE* [*elim*!]: $b : Range\ r ==> (!!x.\ (x,\ b) : r ==> P) ==> P$
⟨*proof*⟩

lemma *Range-empty* [*simp*]: $Range\ \{\} = \{\}$
⟨*proof*⟩

lemma *Range-insert*: $Range\ (insert\ (a,\ b)\ r) = insert\ b\ (Range\ r)$
⟨*proof*⟩

lemma *Range-Id* [*simp*]: $Range\ Id = UNIV$
⟨*proof*⟩

lemma *Range-diag* [*simp*]: $Range\ (diag\ A) = A$
⟨*proof*⟩

lemma *Range-Un-eq*: $Range(A \cup B) = Range(A) \cup Range(B)$
⟨*proof*⟩

lemma *Range-Int-subset*: $Range(A \cap B) \subseteq Range(A) \cap Range(B)$
⟨*proof*⟩

lemma *Range-Diff-subset*: $Range(A) - Range(B) \subseteq Range(A - B)$
⟨*proof*⟩

lemma *Range-Union*: $Range\ (Union\ S) = (\bigcup A{\in}S.\ Range\ A)$
⟨*proof*⟩

lemma *snd-eq-Range*: $snd\ `\ R = Range\ R$
⟨*proof*⟩

11.12 Image of a set under a relation

declare *Image-def* [*noatp*]

lemma *Image-iff*: $(b : r``A) = (EX\ x{:}A.\ (x,\ b) : r)$
⟨*proof*⟩

lemma *Image-singleton*: $r``\{a\} = \{b.\ (a,\ b) : r\}$
⟨*proof*⟩

lemma *Image-singleton-iff* [*iff*]: $(b : r``\{a\}) = ((a,\ b) : r)$
⟨*proof*⟩

lemma *ImageI* [*intro*,*noatp*]: $(a,\ b) : r ==> a : A ==> b : r``A$
⟨*proof*⟩

lemma *ImageE* [*elim*!]:
$b : r `` A ==> (!!x.\ (x,\ b) : r ==> x : A ==> P) ==> P$
⟨*proof*⟩

lemma *rev-ImageI*: $a : A ==> (a,\ b) : r ==> b : r `` A$
— This version's more effective when we already have the required a
⟨*proof*⟩

lemma *Image-empty* [*simp*]: $R``\{\} = \{\}$
⟨*proof*⟩

lemma *Image-Id* [*simp*]: $Id `` A = A$
⟨*proof*⟩

lemma *Image-diag* [*simp*]: $diag\ A `` B = A \cap B$
⟨*proof*⟩

lemma *Image-Int-subset*: $R `` (A \cap B) \subseteq R `` A \cap R `` B$
⟨*proof*⟩

lemma *Image-Int-eq*:
$single\text{-}valued\ (converse\ R) ==> R `` (A \cap B) = R `` A \cap R `` B$
⟨*proof*⟩

lemma *Image-Un*: $R `` (A \cup B) = R `` A \cup R `` B$
⟨*proof*⟩

lemma *Un-Image*: $(R \cup S) `` A = R `` A \cup S `` A$
⟨*proof*⟩

lemma *Image-subset*: $r \subseteq A \times B ==> r``C \subseteq B$
⟨*proof*⟩

lemma *Image-eq-UN*: $r``B = (\bigcup y\in B.\ r``\{y\})$
— NOT suitable for rewriting
⟨*proof*⟩

lemma *Image-mono*: $r' \subseteq r ==> A' \subseteq A ==> (r' `` A') \subseteq (r `` A)$
⟨*proof*⟩

lemma *Image-UN*: $(r `` (UNION\ A\ B)) = (\bigcup x\in A.\ r `` (B\ x))$
⟨*proof*⟩

lemma *Image-INT-subset*: $(r `` INTER\ A\ B) \subseteq (\bigcap x\in A.\ r `` (B\ x))$
⟨*proof*⟩

Converse inclusion requires some assumptions

lemma *Image-INT-eq*:
$[|single\text{-}valued\ (r^{-1});\ A \neq \{\}|] \implies r \text{ `` } INTER\ A\ B = (\bigcap x \in A.\ r \text{ `` } B\ x)$
⟨*proof*⟩

lemma *Image-subset-eq*: $(r``A \subseteq B) = (A \subseteq -\ ((r\hat{}-1)\ ``\ (-B)))$
⟨*proof*⟩

11.13 Single valued relations

lemma *single-valuedI*:
$ALL\ x\ y.\ (x,y){:}r \longrightarrow (ALL\ z.\ (x,z){:}r \longrightarrow y{=}z) \implies single\text{-}valued\ r$
⟨*proof*⟩

lemma *single-valuedD*:
$single\text{-}valued\ r \implies (x,\ y) : r \implies (x,\ z) : r \implies y = z$
⟨*proof*⟩

lemma *single-valued-rel-comp*:
$single\text{-}valued\ r \implies single\text{-}valued\ s \implies single\text{-}valued\ (r\ O\ s)$
⟨*proof*⟩

lemma *single-valued-subset*:
$r \subseteq s \implies single\text{-}valued\ s \implies single\text{-}valued\ r$
⟨*proof*⟩

lemma *single-valued-Id* [*simp*]: *single-valued Id*
⟨*proof*⟩

lemma *single-valued-diag* [*simp*]: *single-valued* (*diag A*)
⟨*proof*⟩

11.14 Graphs given by *Collect*

lemma *Domain-Collect-split* [*simp*]: $Domain\{(x,y).\ P\ x\ y\} = \{x.\ EX\ y.\ P\ x\ y\}$
⟨*proof*⟩

lemma *Range-Collect-split* [*simp*]: $Range\{(x,y).\ P\ x\ y\} = \{y.\ EX\ x.\ P\ x\ y\}$
⟨*proof*⟩

lemma *Image-Collect-split* [*simp*]: $\{(x,y).\ P\ x\ y\}\ ``\ A = \{y.\ EX\ x{:}A.\ P\ x\ y\}$
⟨*proof*⟩

11.15 Inverse image

lemma *sym-inv-image*: $sym\ r \implies sym\ (inv\text{-}image\ r\ f)$
⟨*proof*⟩

lemma *trans-inv-image*: $trans\ r \implies trans\ (inv\text{-}image\ r\ f)$
⟨*proof*⟩

11.16 Version of *lfp-induct* for binary relations

lemmas *lfp-induct2* =
  lfp-induct-set [*of* (*a*, *b*), *split-format* (*complete*)]

end

12 Predicate: Predicates

theory *Predicate*
imports *Inductive Relation*
begin

12.1 Equality and Subsets

lemma *pred-equals-eq* [*pred-set-conv*]: $((\lambda x.\ x \in R) = (\lambda x.\ x \in S)) = (R = S)$
  ⟨*proof*⟩

lemma *pred-equals-eq2* [*pred-set-conv*]: $((\lambda x\ y.\ (x, y) \in R) = (\lambda x\ y.\ (x, y) \in S)) = (R = S)$
  ⟨*proof*⟩

lemma *pred-subset-eq* [*pred-set-conv*]: $((\lambda x.\ x \in R) <= (\lambda x.\ x \in S)) = (R <= S)$
  ⟨*proof*⟩

lemma *pred-subset-eq2* [*pred-set-conv*]: $((\lambda x\ y.\ (x, y) \in R) <= (\lambda x\ y.\ (x, y) \in S)) = (R <= S)$
  ⟨*proof*⟩

12.2 Top and bottom elements

lemma *top1I* [*intro*!]: *top x*
  ⟨*proof*⟩

lemma *top2I* [*intro*!]: *top x y*
  ⟨*proof*⟩

lemma *bot1E* [*elim*!]: $bot\ x \Longrightarrow P$
  ⟨*proof*⟩

lemma *bot2E* [*elim*!]: $bot\ x\ y \Longrightarrow P$
  ⟨*proof*⟩

12.3 The empty set

lemma *bot-empty-eq*: $bot = (\lambda x.\ x \in \{\})$
  ⟨*proof*⟩

lemma *bot-empty-eq2*: $bot = (\lambda x\ y.\ (x,\ y) \in \{\})$
⟨*proof*⟩

12.4 Binary union

lemma *sup1-iff* [*simp*]: $sup\ A\ B\ x \longleftrightarrow A\ x \mid B\ x$
⟨*proof*⟩

lemma *sup2-iff* [*simp*]: $sup\ A\ B\ x\ y \longleftrightarrow A\ x\ y \mid B\ x\ y$
⟨*proof*⟩

lemma *sup-Un-eq* [*pred-set-conv*]: $sup\ (\lambda x.\ x \in R)\ (\lambda x.\ x \in S) = (\lambda x.\ x \in R \cup S)$
⟨*proof*⟩

lemma *sup-Un-eq2* [*pred-set-conv*]: $sup\ (\lambda x\ y.\ (x,\ y) \in R)\ (\lambda x\ y.\ (x,\ y) \in S) = (\lambda x\ y.\ (x,\ y) \in R \cup S)$
⟨*proof*⟩

lemma *sup1I1* [*elim?*]: $A\ x \Longrightarrow sup\ A\ B\ x$
⟨*proof*⟩

lemma *sup2I1* [*elim?*]: $A\ x\ y \Longrightarrow sup\ A\ B\ x\ y$
⟨*proof*⟩

lemma *sup1I2* [*elim?*]: $B\ x \Longrightarrow sup\ A\ B\ x$
⟨*proof*⟩

lemma *sup2I2* [*elim?*]: $B\ x\ y \Longrightarrow sup\ A\ B\ x\ y$
⟨*proof*⟩

Classical introduction rule: no commitment to A vs B.

lemma *sup1CI* [*intro*!]: $(\sim\ B\ x ==> A\ x) ==> sup\ A\ B\ x$
⟨*proof*⟩

lemma *sup2CI* [*intro*!]: $(\sim\ B\ x\ y ==> A\ x\ y) ==> sup\ A\ B\ x\ y$
⟨*proof*⟩

lemma *sup1E* [*elim*!]: $sup\ A\ B\ x ==> (A\ x ==> P) ==> (B\ x ==> P) ==> P$
⟨*proof*⟩

lemma *sup2E* [*elim*!]: $sup\ A\ B\ x\ y ==> (A\ x\ y ==> P) ==> (B\ x\ y ==> P) ==> P$
⟨*proof*⟩

12.5 Binary intersection

lemma *inf1-iff* [*simp*]: $inf\ A\ B\ x \longleftrightarrow A\ x \wedge B\ x$

⟨*proof*⟩

lemma *inf2-iff* [*simp*]: $inf\ A\ B\ x\ y \longleftrightarrow A\ x\ y \wedge B\ x\ y$
⟨*proof*⟩

lemma *inf-Int-eq* [*pred-set-conv*]: $inf\ (\lambda x.\ x \in R)\ (\lambda x.\ x \in S) = (\lambda x.\ x \in R \cap S)$
⟨*proof*⟩

lemma *inf-Int-eq2* [*pred-set-conv*]: $inf\ (\lambda x\ y.\ (x,\ y) \in R)\ (\lambda x\ y.\ (x,\ y) \in S) = (\lambda x\ y.\ (x,\ y) \in R \cap S)$
⟨*proof*⟩

lemma *inf1I* [*intro*!]: $A\ x ==> B\ x ==> inf\ A\ B\ x$
⟨*proof*⟩

lemma *inf2I* [*intro*!]: $A\ x\ y ==> B\ x\ y ==> inf\ A\ B\ x\ y$
⟨*proof*⟩

lemma *inf1D1*: $inf\ A\ B\ x ==> A\ x$
⟨*proof*⟩

lemma *inf2D1*: $inf\ A\ B\ x\ y ==> A\ x\ y$
⟨*proof*⟩

lemma *inf1D2*: $inf\ A\ B\ x ==> B\ x$
⟨*proof*⟩

lemma *inf2D2*: $inf\ A\ B\ x\ y ==> B\ x\ y$
⟨*proof*⟩

lemma *inf1E* [*elim*!]: $inf\ A\ B\ x ==> (A\ x ==> B\ x ==> P) ==> P$
⟨*proof*⟩

lemma *inf2E* [*elim*!]: $inf\ A\ B\ x\ y ==> (A\ x\ y ==> B\ x\ y ==> P) ==> P$
⟨*proof*⟩

12.6 Unions of families

lemma *SUP1-iff* [*simp*]: $(SUP\ x{:}A.\ B\ x)\ b = (EX\ x{:}A.\ B\ x\ b)$
⟨*proof*⟩

lemma *SUP2-iff* [*simp*]: $(SUP\ x{:}A.\ B\ x)\ b\ c = (EX\ x{:}A.\ B\ x\ b\ c)$
⟨*proof*⟩

lemma *SUP1-I* [*intro*]: $a : A ==> B\ a\ b ==> (SUP\ x{:}A.\ B\ x)\ b$
⟨*proof*⟩

lemma *SUP2-I* [*intro*]: $a : A ==> B\ a\ b\ c ==> (SUP\ x{:}A.\ B\ x)\ b\ c$
⟨*proof*⟩

lemma *SUP1-E* [*elim*!]: (*SUP* x:A. B x) b ==> (!!x. x : A ==> B x b ==> R) ==> R
⟨*proof*⟩

lemma *SUP2-E* [*elim*!]: (*SUP* x:A. B x) b c ==> (!!x. x : A ==> B x b c ==> R) ==> R
⟨*proof*⟩

lemma *SUP-UN-eq*: (*SUP* i. (λx. $x \in r$ i)) = (λx. $x \in$ (*UN* i. r i))
⟨*proof*⟩

lemma *SUP-UN-eq2*: (*SUP* i. (λx y. (x, y) $\in r$ i)) = (λx y. (x, y) $\in$ (*UN* i. r i))
⟨*proof*⟩

12.7 Intersections of families

lemma *INF1-iff* [*simp*]: (*INF* x:A. B x) b = (*ALL* x:A. B x b)
⟨*proof*⟩

lemma *INF2-iff* [*simp*]: (*INF* x:A. B x) b c = (*ALL* x:A. B x b c)
⟨*proof*⟩

lemma *INF1-I* [*intro*!]: (!!x. x : A ==> B x b) ==> (*INF* x:A. B x) b
⟨*proof*⟩

lemma *INF2-I* [*intro*!]: (!!x. x : A ==> B x b c) ==> (*INF* x:A. B x) b c
⟨*proof*⟩

lemma *INF1-D* [*elim*]: (*INF* x:A. B x) b ==> a : A ==> B a b
⟨*proof*⟩

lemma *INF2-D* [*elim*]: (*INF* x:A. B x) b c ==> a : A ==> B a b c
⟨*proof*⟩

lemma *INF1-E* [*elim*]: (*INF* x:A. B x) b ==> (B a b ==> R) ==> (a ~: A ==> R) ==> R
⟨*proof*⟩

lemma *INF2-E* [*elim*]: (*INF* x:A. B x) b c ==> (B a b c ==> R) ==> (a ~: A ==> R) ==> R
⟨*proof*⟩

lemma *INF-INT-eq*: (*INF* i. (λx. $x \in r$ i)) = (λx. $x \in$ (*INT* i. r i))
⟨*proof*⟩

lemma *INF-INT-eq2*: (*INF* i. (λx y. (x, y) $\in r$ i)) = (λx y. (x, y) $\in$ (*INT* i. r i))
⟨*proof*⟩

12.8 Composition of two relations

inductive
pred-comp :: ['b => 'c => bool, 'a => 'b => bool] => 'a => 'c => bool
(**infixr** *OO 75*)
for *r* :: 'b => 'c => bool **and** *s* :: 'a => 'b => bool
where
pred-compI [*intro*]: *s a b* ==> *r b c* ==> (*r OO s*) *a c*

inductive-cases *pred-compE* [*elim*!]: (*r OO s*) *a c*

lemma *pred-comp-rel-comp-eq* [*pred-set-conv*]:
$((\lambda x\ y.\ (x,\ y) \in r)\ OO\ (\lambda x\ y.\ (x,\ y) \in s)) = (\lambda x\ y.\ (x,\ y) \in r\ O\ s)$
⟨*proof*⟩

12.9 Converse

inductive
conversep :: ('a => 'b => bool) => 'b => 'a => bool
((-^--1) [1000] 1000)
for *r* :: 'a => 'b => bool
where
conversepI: *r a b* ==> *r*^--1 *b a*

notation (*xsymbols*)
conversep $((\text{-}^{-1-1})\ [1000]\ 1000)$

lemma *conversepD*:
assumes *ab*: *r*^--1 *a b*
shows *r b a* ⟨*proof*⟩

lemma *conversep-iff* [*iff*]: *r*^--1 *a b* = *r b a*
⟨*proof*⟩

lemma *conversep-converse-eq* [*pred-set-conv*]:
$(\lambda x\ y.\ (x,\ y) \in r)$^--1 = $(\lambda x\ y.\ (x,\ y) \in r$^-1)
⟨*proof*⟩

lemma *conversep-conversep* [*simp*]: (*r*^--1)^--1 = *r*
⟨*proof*⟩

lemma *converse-pred-comp*: (*r OO s*)^--1 = *s*^--1 *OO r*^--1
⟨*proof*⟩

lemma *converse-meet*: (*inf r s*)^--1 = *inf r*^--1 *s*^--1
⟨*proof*⟩

lemma *converse-join*: (*sup r s*)^--1 = *sup r*^--1 *s*^--1
⟨*proof*⟩

lemma *conversep-noteq* [*simp*]: (*op* ~=)^--1 = *op* ~=
⟨*proof*⟩

lemma *conversep-eq* [*simp*]: (*op* =)^--1 = *op* =
⟨*proof*⟩

12.10 Domain

inductive
DomainP :: ('a => 'b => bool) => 'a => bool
for *r* :: 'a => 'b => bool
where
DomainPI [*intro*]: *r a b* ==> *DomainP r a*

inductive-cases *DomainPE* [*elim*!]: *DomainP r a*

lemma *DomainP-Domain-eq* [*pred-set-conv*]: *DomainP* (λx *y*. (*x*, *y*) $\in$ *r*) = (λx. *x* $\in$ *Domain r*)
⟨*proof*⟩

12.11 Range

inductive
RangeP :: ('a => 'b => bool) => 'b => bool
for *r* :: 'a => 'b => bool
where
RangePI [*intro*]: *r a b* ==> *RangeP r b*

inductive-cases *RangePE* [*elim*!]: *RangeP r b*

lemma *RangeP-Range-eq* [*pred-set-conv*]: *RangeP* (λx *y*. (*x*, *y*) $\in$ *r*) = (λx. *x* $\in$ *Range r*)
⟨*proof*⟩

12.12 Inverse image

definition
inv-imagep :: ('b => 'b => bool) => ('a => 'b) => 'a => 'a => bool **where**
inv-imagep r f == %*x y*. *r* (*f x*) (*f y*)

lemma [*pred-set-conv*]: *inv-imagep* (λx *y*. (*x*, *y*) $\in$ *r*) *f* = (λx *y*. (*x*, *y*) $\in$ *inv-image r f*)
⟨*proof*⟩

lemma *in-inv-imagep* [*simp*]: *inv-imagep r f x y* = *r* (*f x*) (*f y*)
⟨*proof*⟩

12.13 The Powerset operator

definition *Powp* :: ('a $\Rightarrow$ *bool*) $\Rightarrow$ 'a *set* $\Rightarrow$ *bool* **where**

Powp A == $\lambda B.\ \forall x \in B.\ A\ x$

lemma *Powp-Pow-eq* [*pred-set-conv*]: *Powp* ($\lambda x.\ x \in A$) = ($\lambda x.\ x \in$ *Pow A*)
⟨*proof*⟩

12.14 Properties of relations - predicate versions

abbreviation *antisymP* :: (*'a* => *'a* => *bool*) => *bool* **where**
antisymP r == *antisym* {(*x*, *y*). *r x y*}

abbreviation *transP* :: (*'a* => *'a* => *bool*) => *bool* **where**
transP r == *trans* {(*x*, *y*). *r x y*}

abbreviation *single-valuedP* :: (*'a* => *'b* => *bool*) => *bool* **where**
single-valuedP r == *single-valued* {(*x*, *y*). *r x y*}

end

13 Transitive-Closure: Reflexive and Transitive closure of a relation

theory *Transitive-Closure*
imports *Predicate*
uses ~~/*src*/*Provers*/*trancl.ML*
begin

rtrancl is reflexive/transitive closure, *trancl* is transitive closure, *reflcl* is reflexive closure.

These postfix operators have *maximum priority*, forcing their operands to be atomic.

inductive-set
rtrancl :: (*'a* × *'a*) *set* ⇒ (*'a* × *'a*) *set* ((-^*) [*1000*] *999*)
for *r* :: (*'a* × *'a*) *set*
where
rtrancl-refl [*intro*!, *Pure.intro*!, *simp*]: (*a*, *a*) : *r*^*
| *rtrancl-into-rtrancl* [*Pure.intro*]: (*a*, *b*) : *r*^* ==> (*b*, *c*) : *r* ==> (*a*, *c*) : *r*^*

inductive-set
trancl :: (*'a* × *'a*) *set* ⇒ (*'a* × *'a*) *set* ((-^+) [*1000*] *999*)
for *r* :: (*'a* × *'a*) *set*
where
r-into-trancl [*intro*, *Pure.intro*]: (*a*, *b*) : *r* ==> (*a*, *b*) : *r*^+
| *trancl-into-trancl* [*Pure.intro*]: (*a*, *b*) : *r*^+ ==> (*b*, *c*) : *r* ==> (*a*, *c*) : *r*^+

notation
rtranclp ((-^**) [*1000*] *1000*) **and**
tranclp ((-^++) [*1000*] *1000*)

abbreviation
reflclp :: ('a => 'a => bool) => 'a => 'a => bool ((-^==) [1000] 1000)
where
r^== == sup r op =

abbreviation
reflcl :: ('a × 'a) set => ('a × 'a) set ((-^=) [1000] 999) **where**
r^= == r ∪ Id

notation (*xsymbols*)
rtranclp ((-**) [1000] 1000) **and**
tranclp ((-$^{++}$) [1000] 1000) **and**
reflclp ((-$^{==}$) [1000] 1000) **and**
rtrancl ((-*) [1000] 999) **and**
trancl ((-$^{+}$) [1000] 999) **and**
reflcl ((-$^{=}$) [1000] 999)

notation (*HTML* **output**)
rtranclp ((-**) [1000] 1000) **and**
tranclp ((-$^{++}$) [1000] 1000) **and**
reflclp ((-$^{==}$) [1000] 1000) **and**
rtrancl ((-*) [1000] 999) **and**
trancl ((-$^{+}$) [1000] 999) **and**
reflcl ((-$^{=}$) [1000] 999)

13.1 Reflexive-transitive closure

lemma *reflcl-set-eq* [*pred-set-conv*]: (*sup* (λx y. (x, y) ∈ r) *op* =) = (λx y. (x, y) ∈ r Un Id)
⟨*proof*⟩

lemma *r-into-rtrancl* [*intro*]: !!p. p ∈ r ==> p ∈ r^*
— *rtrancl* of *r* contains *r*
⟨*proof*⟩

lemma *r-into-rtranclp* [*intro*]: r x y ==> r^** x y
— *rtrancl* of *r* contains *r*
⟨*proof*⟩

lemma *rtranclp-mono*: r ≤ s ==> r^** ≤ s^**
— monotonicity of *rtrancl*
⟨*proof*⟩

lemmas *rtrancl-mono* = *rtranclp-mono* [*to-set*]

theorem *rtranclp-induct* [*consumes 1*, *induct set*: *rtranclp*]:
assumes *a*: r^** a b
and *cases*: P a !!y z. [| r^** a y; r y z; P y |] ==> P z

shows $P\ b$
⟨*proof*⟩

lemmas *rtrancl-induct* [*induct set*: *rtrancl*] = *rtranclp-induct* [*to-set*]

lemmas *rtranclp-induct2* =
rtranclp-induct[*of* - (*ax*,*ay*) (*bx*,*by*), *split-rule*,
consumes 1, *case-names refl step*]

lemmas *rtrancl-induct2* =
rtrancl-induct[*of* (*ax*,*ay*) (*bx*,*by*), *split-format* (*complete*),
consumes 1, *case-names refl step*]

lemma *reflexive-rtrancl*: *reflexive* $(r\hat{}*)$
⟨*proof*⟩

lemma *trans-rtrancl*: *trans*$(r\hat{}*)$
— transitivity of transitive closure!! – by induction
⟨*proof*⟩

lemmas *rtrancl-trans* = *trans-rtrancl* [*THEN transD*, *standard*]

lemma *rtranclp-trans*:
assumes *xy*: $r\hat{}{**}\ x\ y$
and *yz*: $r\hat{}{**}\ y\ z$
shows $r\hat{}{**}\ x\ z$ ⟨*proof*⟩

lemma *rtranclE*:
assumes *major*: $(a{::}'a,b) : r\hat{}*$
and *cases*: $(a = b) ==> P$
$!!y.\ [|\ (a,y) : r\hat{}*;\ (y,b) : r\ |] ==> P$
shows P
— elimination of *rtrancl* – by induction on a special formula
⟨*proof*⟩

lemma *rtrancl-Int-subset*: $[|\ Id \subseteq s;\ r\ O\ (r\hat{}* \cap s) \subseteq s|] ==> r\hat{}* \subseteq s$
⟨*proof*⟩

lemma *converse-rtranclp-into-rtranclp*:
$r\ a\ b \Longrightarrow r^{**}\ b\ c \Longrightarrow r^{**}\ a\ c$
⟨*proof*⟩

lemmas *converse-rtrancl-into-rtrancl* = *converse-rtranclp-into-rtranclp* [*to-set*]

More r^* equations and inclusions.

lemma *rtranclp-idemp* [*simp*]: $(r\hat{}{**})\hat{}{**} = r\hat{}{**}$
⟨*proof*⟩

lemmas *rtrancl-idemp* [*simp*] = *rtranclp-idemp* [*to-set*]

lemma *rtrancl-idemp-self-comp* [*simp*]: $R\hat{}* \ O \ R\hat{}* = R\hat{}*$
⟨*proof*⟩

lemma *rtrancl-subset-rtrancl*: $r \subseteq s\hat{}* ==> r\hat{}* \subseteq s\hat{}*$
⟨*proof*⟩

lemma *rtranclp-subset*: $R \leq S ==> S \leq R\hat{}** ==> S\hat{}** = R\hat{}**$
⟨*proof*⟩

lemmas *rtrancl-subset* = *rtranclp-subset* [*to-set*]

lemma *rtranclp-sup-rtranclp*: $(sup\ (R\hat{}**)\ (S\hat{}**))\hat{}** = (sup\ R\ S)\hat{}**$
⟨*proof*⟩

lemmas *rtrancl-Un-rtrancl* = *rtranclp-sup-rtranclp* [*to-set*]

lemma *rtranclp-reflcl* [*simp*]: $(R\hat{}==)\hat{}** = R\hat{}**$
⟨*proof*⟩

lemmas *rtrancl-reflcl* [*simp*] = *rtranclp-reflcl* [*to-set*]

lemma *rtrancl-r-diff-Id*: $(r - Id)\hat{}* = r\hat{}*$
⟨*proof*⟩

lemma *rtranclp-r-diff-Id*: $(inf\ r\ op\ \tilde{}=)\hat{}** = r\hat{}**$
⟨*proof*⟩

theorem *rtranclp-converseD*:
assumes r: $(r\hat{}--1)\hat{}**\ x\ y$
shows $r\hat{}**\ y\ x$
⟨*proof*⟩

lemmas *rtrancl-converseD* = *rtranclp-converseD* [*to-set*]

theorem *rtranclp-converseI*:
assumes r: $r\hat{}**\ y\ x$
shows $(r\hat{}--1)\hat{}**\ x\ y$
⟨*proof*⟩

lemmas *rtrancl-converseI* = *rtranclp-converseI* [*to-set*]

lemma *rtrancl-converse*: $(r\hat{}-1)\hat{}* = (r\hat{}*)\hat{}-1$
⟨*proof*⟩

lemma *sym-rtrancl*: $sym\ r ==> sym\ (r\hat{}*)$
⟨*proof*⟩

theorem *converse-rtranclp-induct*[*consumes 1*]:

assumes *major*: $r\hat{}** \ a \ b$
and *cases*: $P \ b \ !!y \ z. \ [| \ r \ y \ z; \ r\hat{}** \ z \ b; \ P \ z \ |] ==> P \ y$
shows $P \ a$
⟨*proof*⟩

lemmas *converse-rtrancl-induct* = *converse-rtranclp-induct* [*to-set*]

lemmas *converse-rtranclp-induct2* =
converse-rtranclp-induct[*of* - (*ax*,*ay*) (*bx*,*by*), *split-rule*,
consumes 1, *case-names refl step*]

lemmas *converse-rtrancl-induct2* =
converse-rtrancl-induct[*of* (*ax*,*ay*) (*bx*,*by*), *split-format* (*complete*),
consumes 1, *case-names refl step*]

lemma *converse-rtranclpE*:
assumes *major*: $r\hat{}** \ x \ z$
and *cases*: $x=z ==> P$
$!!y. \ [| \ r \ x \ y; \ r\hat{}** \ y \ z \ |] ==> P$
shows P
⟨*proof*⟩

lemmas *converse-rtranclE* = *converse-rtranclpE* [*to-set*]

lemmas *converse-rtranclpE2* = *converse-rtranclpE* [*of* - (*xa*,*xb*) (*za*,*zb*), *split-rule*]

lemmas *converse-rtranclE2* = *converse-rtranclE* [*of* (*xa*,*xb*) (*za*,*zb*), *split-rule*]

lemma *r-comp-rtrancl-eq*: $r \ O \ r\hat{}* = r\hat{}* \ O \ r$
⟨*proof*⟩

lemma *rtrancl-unfold*: $r\hat{}* = Id \ Un \ r \ O \ r\hat{}*$
⟨*proof*⟩

13.2 Transitive closure

lemma *trancl-mono*: $!!p. \ p \in r\hat{}+ ==> r \subseteq s ==> p \in s\hat{}+$
⟨*proof*⟩

lemma *r-into-trancl'*: $!!p. \ p : r ==> p : r\hat{}+$
⟨*proof*⟩

Conversions between *trancl* and *rtrancl*.

lemma *tranclp-into-rtranclp*: $r\hat{}++ \ a \ b ==> r\hat{}** \ a \ b$
⟨*proof*⟩

lemmas *trancl-into-rtrancl* = *tranclp-into-rtranclp* [*to-set*]

lemma *rtranclp-into-tranclp1*: **assumes** *r*: $r\hat{}** \ a \ b$

shows !!*c*. *r b c* ==> *r*ˆ++ *a c* ⟨*proof*⟩

lemmas *rtrancl-into-trancl1* = *rtranclp-into-tranclp1* [*to-set*]

lemma *rtranclp-into-tranclp2*: [| *r a b*; *r*ˆ** *b c* |] ==> *r*ˆ++ *a c*
— intro rule from *r* and *rtrancl*
⟨*proof*⟩

lemmas *rtrancl-into-trancl2* = *rtranclp-into-tranclp2* [*to-set*]

lemma *tranclp-induct* [*consumes 1*, *induct set*: *tranclp*]:
assumes *a*: *r*ˆ++ *a b*
and *cases*: !!*y*. *r a y* ==> *P y*
!!*y z*. *r*ˆ++ *a y* ==> *r y z* ==> *P y* ==> *P z*
shows *P b*
— Nice induction rule for *trancl*
⟨*proof*⟩

lemmas *trancl-induct* [*induct set*: *trancl*] = *tranclp-induct* [*to-set*]

lemmas *tranclp-induct2* =
tranclp-induct[*of* - (*ax*,*ay*) (*bx*,*by*), *split-rule*,
consumes 1, *case-names base step*]

lemmas *trancl-induct2* =
trancl-induct[*of* (*ax*,*ay*) (*bx*,*by*), *split-format* (*complete*),
consumes 1, *case-names base step*]

lemma *tranclp-trans-induct*:
assumes *major*: *r*ˆ++ *x y*
and *cases*: !!*x y*. *r x y* ==> *P x y*
!!*x y z*. [| *r*ˆ++ *x y*; *P x y*; *r*ˆ++ *y z*; *P y z* |] ==> *P x z*
shows *P x y*
— Another induction rule for trancl, incorporating transitivity
⟨*proof*⟩

lemmas *trancl-trans-induct* = *tranclp-trans-induct* [*to-set*]

inductive-cases *tranclE*: (*a*, *b*) : *r*ˆ+

lemma *trancl-Int-subset*: [| *r* ⊆ *s*; *r O* (*r*ˆ+ ∩ *s*) ⊆ *s*|] ==> *r*ˆ+ ⊆ *s*
⟨*proof*⟩

lemma *trancl-unfold*: *r*ˆ+ = *r Un r O r*ˆ+
⟨*proof*⟩

lemma *trans-trancl*[*simp*]: *trans*(*r*ˆ+)
— Transitivity of r^+
⟨*proof*⟩

lemmas *trancl-trans* = *trans-trancl* [*THEN transD*, *standard*]

lemma *tranclp-trans*:
  assumes *xy*: $r\hat{}{++}\ x\ y$
  and *yz*: $r\hat{}{++}\ y\ z$
  shows $r\hat{}{++}\ x\ z$ ⟨*proof*⟩

lemma *trancl-id*[*simp*]: $trans\ r \Longrightarrow r\hat{}{+} = r$
⟨*proof*⟩

lemma *rtranclp-tranclp-tranclp*: **assumes** *r*: $r\hat{}{**}\ x\ y$
  shows $!!z.\ r\hat{}{++}\ y\ z ==> r\hat{}{++}\ x\ z$ ⟨*proof*⟩

lemmas *rtrancl-trancl-trancl* = *rtranclp-tranclp-tranclp* [*to-set*]

lemma *tranclp-into-tranclp2*: $r\ a\ b ==> r\hat{}{++}\ b\ c ==> r\hat{}{++}\ a\ c$
  ⟨*proof*⟩

lemmas *trancl-into-trancl2* = *tranclp-into-tranclp2* [*to-set*]

lemma *trancl-insert*:
  $(insert\ (y,\ x)\ r)\hat{}{+} = r\hat{}{+} \cup \{(a,\ b).\ (a,\ y) \in r\hat{}{*} \wedge (x,\ b) \in r\hat{}{*}\}$
  — primitive recursion for *trancl* over finite relations
  ⟨*proof*⟩

lemma *tranclp-converseI*: $(r\hat{}{++})\hat{}{--1}\ x\ y ==> (r\hat{}{--1})\hat{}{++}\ x\ y$
  ⟨*proof*⟩

lemmas *trancl-converseI* = *tranclp-converseI* [*to-set*]

lemma *tranclp-converseD*: $(r\hat{}{--1})\hat{}{++}\ x\ y ==> (r\hat{}{++})\hat{}{--1}\ x\ y$
  ⟨*proof*⟩

lemmas *trancl-converseD* = *tranclp-converseD* [*to-set*]

lemma *tranclp-converse*: $(r\hat{}{--1})\hat{}{++} = (r\hat{}{++})\hat{}{--1}$
  ⟨*proof*⟩

lemmas *trancl-converse* = *tranclp-converse* [*to-set*]

lemma *sym-trancl*: $sym\ r ==> sym\ (r\hat{}{+})$
  ⟨*proof*⟩

lemma *converse-tranclp-induct*:
  assumes *major*: $r\hat{}{++}\ a\ b$
    and *cases*: $!!y.\ r\ y\ b ==> P(y)$
      $!!y\ z.[|\ r\ y\ z;\ \ r\hat{}{++}\ z\ b;\ \ P(z)\ |] ==> P(y)$
  shows $P\ a$

⟨*proof*⟩

lemmas *converse-trancl-induct* = *converse-tranclp-induct* [*to-set*]

lemma *tranclpD*: $R\hat{}{++}\ x\ y$ ==> $EX\ z.\ R\ x\ z \wedge R\hat{}{**}\ z\ y$
⟨*proof*⟩

lemmas *tranclD* = *tranclpD* [*to-set*]

lemma *tranclD2*:
$(x, y) \in R^+ \Longrightarrow \exists z. (x, z) \in R^* \wedge (z, y) \in R$
⟨*proof*⟩

lemma *irrefl-tranclI*: $r\hat{}{-1} \cap r\hat{}{*} = \{\}$ ==> $(x, x) \notin r\hat{}{+}$
⟨*proof*⟩

lemma *irrefl-trancl-rD*: $!!X.\ ALL\ x.\ (x, x) \notin r\hat{}{+}$ ==> $(x, y) \in r$ ==> $x \neq y$
⟨*proof*⟩

lemma *trancl-subset-Sigma-aux*:
$(a, b) \in r\hat{}{*}$ ==> $r \subseteq A \times A$ ==> $a = b \vee a \in A$
⟨*proof*⟩

lemma *trancl-subset-Sigma*: $r \subseteq A \times A$ ==> $r\hat{}{+} \subseteq A \times A$
⟨*proof*⟩

lemma *reflcl-tranclp* [*simp*]: $(r\hat{}{++})\hat{}{==} = r\hat{}{**}$
⟨*proof*⟩

lemmas *reflcl-trancl* [*simp*] = *reflcl-tranclp* [*to-set*]

lemma *trancl-reflcl* [*simp*]: $(r\hat{}{=})\hat{}{+} = r\hat{}{*}$
⟨*proof*⟩

lemma *trancl-empty* [*simp*]: $\{\}\hat{}{+} = \{\}$
⟨*proof*⟩

lemma *rtrancl-empty* [*simp*]: $\{\}\hat{}{*} = Id$
⟨*proof*⟩

lemma *rtranclpD*: $R\hat{}{**}\ a\ b$ ==> $a = b \vee a \neq b \wedge R\hat{}{++}\ a\ b$
⟨*proof*⟩

lemmas *rtranclD* = *rtranclpD* [*to-set*]

lemma *rtrancl-eq-or-trancl*:
$(x,y) \in R^* = (x=y \vee x\neq y \wedge (x,y) \in R^+)$
⟨*proof*⟩

Domain and *Range*

lemma *Domain-rtrancl* [*simp*]: *Domain* $(R\hat{\ }*)$ = *UNIV*
⟨*proof*⟩

lemma *Range-rtrancl* [*simp*]: *Range* $(R\hat{\ }*)$ = *UNIV*
⟨*proof*⟩

lemma *rtrancl-Un-subset*: $(R\hat{\ }* \cup S\hat{\ }*) \subseteq (R\ Un\ S)\hat{\ }*$
⟨*proof*⟩

lemma *in-rtrancl-UnI*: $x \in R\hat{\ }* \vee x \in S\hat{\ }* ==> x \in (R \cup S)\hat{\ }*$
⟨*proof*⟩

lemma *trancl-domain* [*simp*]: *Domain* $(r\hat{\ }+)$ = *Domain r*
⟨*proof*⟩

lemma *trancl-range* [*simp*]: *Range* $(r\hat{\ }+)$ = *Range r*
⟨*proof*⟩

lemma *Not-Domain-rtrancl*:
$x\ \tilde{}\!:\ Domain\ R ==> ((x,\ y) : R\hat{\ }*) = (x = y)$
⟨*proof*⟩

More about converse *rtrancl* and *trancl*, should be merged with main body.

lemma *single-valued-confluent*:
$[\![$ *single-valued* r; $(x,y) \in r\hat{\ }*$; $(x,z) \in r\hat{\ }*$ $]\!]$
$\Longrightarrow (y,z) \in r\hat{\ }* \vee (z,y) \in r\hat{\ }*$
⟨*proof*⟩

lemma *r-r-into-trancl*: $(a,\ b) \in R ==> (b,\ c) \in R ==> (a,\ c) \in R\hat{\ }+$
⟨*proof*⟩

lemma *trancl-into-trancl* [*rule-format*]:
$(a,\ b) \in r^{+} ==> (b,\ c) \in r --> (a,c) \in r^{+}$
⟨*proof*⟩

lemma *tranclp-rtranclp-tranclp*:
$r^{++}\ a\ b ==> r^{**}\ b\ c ==> r^{++}\ a\ c$
⟨*proof*⟩

lemmas *trancl-rtrancl-trancl* = *tranclp-rtranclp-tranclp* [*to-set*]

lemmas *transitive-closure-trans* [*trans*] =
r-r-into-trancl trancl-trans rtrancl-trans
trancl.trancl-into-trancl trancl-into-trancl2
rtrancl.rtrancl-into-rtrancl converse-rtrancl-into-rtrancl
rtrancl-trancl-trancl trancl-rtrancl-trancl

lemmas *transitive-closurep-trans′* [*trans*] =
tranclp-trans rtranclp-trans

tranclp.trancl-into-trancl tranclp-into-tranclp2
rtranclp.rtrancl-into-rtrancl converse-rtranclp-into-rtranclp
rtranclp-tranclp-tranclp tranclp-rtranclp-tranclp

declare *trancl-into-rtrancl* [*elim*]

declare *rtranclE* [*cases set*: *rtrancl*]
declare *tranclE* [*cases set*: *trancl*]

13.3 Setup of transitivity reasoner

⟨*ML*⟩

end

14 Wellfounded-Recursion: Well-founded Recursion

theory *Wellfounded-Recursion*
imports *Transitive-Closure*
begin

inductive
```
  wfrec-rel :: ('a * 'a) set => (('a => 'b) => 'a => 'b) => 'a => 'b => bool
  for R :: ('a * 'a) set
  and F :: ('a => 'b) => 'a => 'b
where
  wfrecI: ALL z. (z, x) : R --> wfrec-rel R F z (g z) ==>
          wfrec-rel R F x (F g x)
```

constdefs
```
  wf       :: ('a * 'a)set => bool
  wf(r) == (!P. (!x. (!y. (y,x):r --> P(y)) --> P(x)) --> (!x. P(x)))

  wfP :: ('a => 'a => bool) => bool
  wfP r == wf {(x, y). r x y}

  acyclic :: ('a*'a)set => bool
  acyclic r == !x. (x,x) ~: r^+

  cut      :: ('a => 'b) => ('a * 'a)set => 'a => 'a => 'b
  cut f r x == (%y. if (y,x):r then f y else arbitrary)

  adm-wf :: ('a * 'a) set => (('a => 'b) => 'a => 'b) => bool
  adm-wf R F == ALL f g x.
    (ALL z. (z, x) : R --> f z = g z) --> F f x = F g x
```

wfrec :: ('*a* ∗ '*a*) *set* => (('*a* => '*b*) => '*a* => '*b*) => '*a* => '*b*
[*code func del*]: *wfrec R F* == %*x*. *THE y*. *wfrec-rel R* (%*f x*. *F* (*cut f R x*) *x*) *x y*

abbreviation *acyclicP* :: ('*a* => '*a* => *bool*) => *bool* **where**
acyclicP r == *acyclic* {(*x*, *y*). *r x y*}

class *wellorder* = *linorder* +
assumes *wf*: *wf* {(*x*, *y*). *x* < *y*}

lemma *wfP-wf-eq* [*pred-set-conv*]: *wfP* (λ*x y*. (*x*, *y*) ∈ *r*) = *wf r*
⟨*proof*⟩

lemma *wfUNIVI*:
(!!*P x*. (*ALL x*. (*ALL y*. (*y*,*x*) : *r* --> *P*(*y*)) --> *P*(*x*)) ==> *P*(*x*)) ==> *wf*(*r*)
⟨*proof*⟩

lemmas *wfPUNIVI* = *wfUNIVI* [*to-pred*]

Restriction to domain A and range B. If r is well-founded over their intersection, then *wf r*

lemma *wfI*:
[| *r* ⊆ *A* <∗> *B*;
!!*x P*. [|∀ *x*. (∀ *y*. (*y*,*x*) : *r* --> *P y*) --> *P x*; *x* : *A*; *x* : *B* |] ==> *P x* |]
==> *wf r*
⟨*proof*⟩

lemma *wf-induct*:
[| *wf*(*r*);
!!*x*.[| *ALL y*. (*y*,*x*): *r* --> *P*(*y*) |] ==> *P*(*x*)
|] ==> *P*(*a*)
⟨*proof*⟩

lemmas *wfP-induct* = *wf-induct* [*to-pred*]

lemmas *wf-induct-rule* = *wf-induct* [*rule-format*, *consumes 1*, *case-names less*, *induct set*: *wf*]

lemmas *wfP-induct-rule* = *wf-induct-rule* [*to-pred*, *induct set*: *wfP*]

lemma *wf-not-sym* [*rule-format*]: *wf*(*r*) ==> *ALL x*. (*a*,*x*):*r* --> (*x*,*a*)~:*r*
⟨*proof*⟩

lemmas *wf-asym* = *wf-not-sym* [*elim-format*]

lemma *wf-not-refl* [*simp*]: *wf*(*r*) ==> (*a*,*a*) ~: *r*

⟨*proof*⟩

lemmas *wf-irrefl* = *wf-not-refl* [*elim-format*]

transitive closure of a well-founded relation is well-founded!

lemma *wf-trancl*: *wf*(*r*) ==> *wf*(*r*ˆ+)
⟨*proof*⟩

lemmas *wfP-trancl* = *wf-trancl* [*to-pred*]

lemma *wf-converse-trancl*: *wf* (*r*ˆ−*1*) ==> *wf* ((*r*ˆ+)ˆ−*1*)
⟨*proof*⟩

14.0.1 Other simple well-foundedness results

Minimal-element characterization of well-foundedness

lemma *wf-eq-minimal*: *wf r* = (∀ *Q x*. *x*∈*Q* --> (∃ *z*∈*Q*. ∀ *y*. (*y*,*z*)∈*r* --> *y*∉*Q*))
⟨*proof*⟩

lemma *wfE-min*:
assumes *p*:*wf R x* ∈ *Q*
obtains *z* **where** *z* ∈ *Q* ⋀*y*. (*y*, *z*) ∈ *R* ⟹ *y* ∉ *Q*
⟨*proof*⟩

lemma *wfI-min*:
(⋀*x Q*. *x* ∈ *Q* ⟹ ∃ *z*∈*Q*. ∀ *y*. (*y*, *z*) ∈ *R* ⟶ *y* ∉ *Q*)
⟹ *wf R*
⟨*proof*⟩

lemmas *wfP-eq-minimal* = *wf-eq-minimal* [*to-pred*]

Well-foundedness of subsets

lemma *wf-subset*: [| *wf*(*r*); *p*<=*r* |] ==> *wf*(*p*)
⟨*proof*⟩

lemmas *wfP-subset* = *wf-subset* [*to-pred*]

Well-foundedness of the empty relation

lemma *wf-empty* [*iff*]: *wf*({})
⟨*proof*⟩

lemmas *wfP-empty* [*iff*] =
wf-empty [*to-pred bot-empty-eq2*, *simplified bot-fun-eq bot-bool-eq*]

lemma *wf-Int1*: *wf r* ==> *wf* (*r Int r*′)
⟨*proof*⟩

lemma *wf-Int2*: *wf r ==> wf (r′ Int r)*
⟨*proof*⟩

Well-foundedness of insert

lemma *wf-insert* [*iff*]: *wf(insert (y,x) r) = (wf(r) & (x,y) ~: r^*)*
⟨*proof*⟩

Well-foundedness of image

lemma *wf-prod-fun-image*: [| *wf r*; *inj f* |] ==> *wf(prod-fun f f ‘ r)*
⟨*proof*⟩

14.0.2 Well-Foundedness Results for Unions

Well-foundedness of indexed union with disjoint domains and ranges

lemma *wf-UN*: [| *ALL i:I. wf(r i)*;
ALL i:I. ALL j:I. r i ~= r j --> Domain(r i) Int Range(r j) = {}
|] ==> *wf(UN i:I. r i)*
⟨*proof*⟩

lemmas *wfP-SUP* = *wf-UN* [**where** *I=UNIV* **and** *r=λi. {(x, y). r i x y}*,
to-pred SUP-UN-eq2 bot-empty-eq, *simplified*, *standard*]

lemma *wf-Union*:
[| *ALL r:R. wf r*;
ALL r:R. ALL s:R. r ~= s --> Domain r Int Range s = {}
|] ==> *wf(Union R)*
⟨*proof*⟩

lemma *wf-Un*:
[| *wf r*; *wf s*; *Domain r Int Range s = {}* |] ==> *wf(r Un s)*
⟨*proof*⟩

lemma *wf-union-merge*:
wf (R ∪ S) = wf (R O R ∪ R O S ∪ S) (**is** *wf ?A = wf ?B*)
⟨*proof*⟩

lemma *wf-comp-self*: *wf R = wf (R O R)*
⟨*proof*⟩

14.0.3 acyclic

lemma *acyclicI*: *ALL x. (x, x) ~: r^+ ==> acyclic r*
⟨*proof*⟩

lemma *wf-acyclic*: *wf r ==> acyclic r*
⟨*proof*⟩

lemmas *wfP-acyclicP = wf-acyclic [to-pred]*

lemma *acyclic-insert [iff]*:
acyclic(insert (y,x) r) = (acyclic r & (x,y) ~: r^*)
⟨*proof*⟩

lemma *acyclic-converse [iff]*: *acyclic(r^−1) = acyclic r*
⟨*proof*⟩

lemmas *acyclicP-converse [iff] = acyclic-converse [to-pred]*

lemma *acyclic-impl-antisym-rtrancl*: *acyclic r ==> antisym(r^*)*
⟨*proof*⟩

lemma *acyclic-subset*: *[| acyclic s; r <= s |] ==> acyclic r*
⟨*proof*⟩

14.1 Well-Founded Recursion

cut

lemma *cuts-eq*: *(cut f r x = cut g r x) = (ALL y. (y,x):r --> f(y)=g(y))*
⟨*proof*⟩

lemma *cut-apply*: *(x,a):r ==> (cut f r a)(x) = f(x)*
⟨*proof*⟩

Inductive characterization of wfrec combinator; for details see: John Harrison, "Inductive definitions: automation and application"

lemma *wfrec-unique*: *[| adm-wf R F; wf R |] ==> EX! y. wfrec-rel R F x y*
⟨*proof*⟩

lemma *adm-lemma*: *adm-wf R (%f x. F (cut f R x) x)*
⟨*proof*⟩

lemma *wfrec*: *wf(r) ==> wfrec r H a = H (cut (wfrec r H) r a) a*
⟨*proof*⟩

* This form avoids giant explosions in proofs. NOTE USE OF ==

lemma *def-wfrec*: *[| f==wfrec r H; wf(r) |] ==> f(a) = H (cut f r a) a*
⟨*proof*⟩

14.2 Code generator setup

consts-code
wfrec (⟨**module**⟩*wfrec?*)

attach ⟪
fun wfrec f x = f (wfrec f) x;
⟫

14.3 Variants for TFL: the Recdef Package

lemma *tfl-wf-induct*: *ALL R. wf R -->*
(ALL P. (ALL x. (ALL y. (y,x):R --> P y) --> P x) --> (ALL x. P x))
⟨*proof*⟩

lemma *tfl-cut-apply*: *ALL f R. (x,a):R --> (cut f R a)(x) = f(x)*
⟨*proof*⟩

lemma *tfl-wfrec*:
ALL M R f. (f=wfrec R M) --> wf R --> (ALL x. f x = M (cut f R x) x)
⟨*proof*⟩

14.4 LEAST and wellorderings

See also *wf-linord-ex-has-least* and its consequences in *Wellfounded-Relations.ML*

lemma *wellorder-Least-lemma* [*rule-format*]:
P (k::'a::wellorder) --> P (LEAST x. P(x)) & (LEAST x. P(x)) <= k
⟨*proof*⟩

lemmas *LeastI* = *wellorder-Least-lemma* [*THEN conjunct1*, *standard*]
lemmas *Least-le* = *wellorder-Least-lemma* [*THEN conjunct2*, *standard*]

— The following 3 lemmas are due to Brian Huffman
lemma *LeastI-ex*: *EX x::'a::wellorder. P x ==> P (Least P)*
⟨*proof*⟩

lemma *LeastI2*:
[| P (a::'a::wellorder); !!x. P x ==> Q x |] ==> Q (Least P)
⟨*proof*⟩

lemma *LeastI2-ex*:
[| EX a::'a::wellorder. P a; !!x. P x ==> Q x |] ==> Q (Least P)
⟨*proof*⟩

lemma *not-less-Least*: *[| k < (LEAST x. P x) |] ==> ~P (k::'a::wellorder)*
⟨*proof*⟩

⟨*ML*⟩

end

15 OrderedGroup: Ordered Groups

theory *OrderedGroup*
imports *Lattices*
uses ~~/*src*/*Provers*/*Arith*/*abel-cancel.ML*
begin

The theory of partially ordered groups is taken from the books:

- *Lattice Theory* by Garret Birkhoff, American Mathematical Society 1979
- *Partially Ordered Algebraic Systems*, Pergamon Press 1963

Most of the used notions can also be looked up in

- http://www.mathworld.com by Eric Weisstein et. al.
- *Algebra I* by van der Waerden, Springer.

15.1 Semigroups and Monoids

class *semigroup-add* = *plus* +
  assumes *add-assoc*: $(a + b) + c = a + (b + c)$

class *ab-semigroup-add* = *semigroup-add* +
  assumes *add-commute*: $a + b = b + a$
begin

lemma *add-left-commute*: $a + (b + c) = b + (a + c)$
  ⟨*proof*⟩

theorems *add-ac* = *add-assoc add-commute add-left-commute*

end

theorems *add-ac* = *add-assoc add-commute add-left-commute*

class *semigroup-mult* = *times* +
  assumes *mult-assoc*: $(a * b) * c = a * (b * c)$

class *ab-semigroup-mult* = *semigroup-mult* +
  assumes *mult-commute*: $a * b = b * a$
begin

lemma *mult-left-commute*: $a * (b * c) = b * (a * c)$
  ⟨*proof*⟩

theorems *mult-ac* = *mult-assoc mult-commute mult-left-commute*

end

theorems *mult-ac* = *mult-assoc mult-commute mult-left-commute*

class *monoid-add* = *zero* + *semigroup-add* +
  assumes *add-0-left* [*simp*]: $0 + a = a$
    and *add-0-right* [*simp*]: $a + 0 = a$

class *comm-monoid-add* = *zero* + *ab-semigroup-add* +
  assumes *add-0*: $0 + a = a$
begin

subclass *monoid-add*
  ⟨*proof*⟩

end

class *monoid-mult* = *one* + *semigroup-mult* +
  assumes *mult-1-left* [*simp*]: $1 * a = a$
  assumes *mult-1-right* [*simp*]: $a * 1 = a$

class *comm-monoid-mult* = *one* + *ab-semigroup-mult* +
  assumes *mult-1*: $1 * a = a$
begin

subclass *monoid-mult*
  ⟨*proof*⟩

end

class *cancel-semigroup-add* = *semigroup-add* +
  assumes *add-left-imp-eq*: $a + b = a + c \Longrightarrow b = c$
  assumes *add-right-imp-eq*: $b + a = c + a \Longrightarrow b = c$

class *cancel-ab-semigroup-add* = *ab-semigroup-add* +
  assumes *add-imp-eq*: $a + b = a + c \Longrightarrow b = c$
begin

subclass *cancel-semigroup-add*
⟨*proof*⟩

end

context *cancel-ab-semigroup-add*
begin

lemma *add-left-cancel* [*simp*]:
  $a + b = a + c \longleftrightarrow b = c$

⟨*proof*⟩

lemma *add-right-cancel* [*simp*]:
$b + a = c + a \longleftrightarrow b = c$
⟨*proof*⟩

end

15.2 Groups

class *group-add* = *minus* + *monoid-add* +
assumes *left-minus* [*simp*]: $- a + a = 0$
assumes *diff-minus*: $a - b = a + (- b)$
begin

lemma *minus-add-cancel*: $- a + (a + b) = b$
⟨*proof*⟩

lemma *minus-zero* [*simp*]: $- 0 = 0$
⟨*proof*⟩

lemma *minus-minus* [*simp*]: $- (- a) = a$
⟨*proof*⟩

lemma *right-minus* [*simp*]: $a + - a = 0$
⟨*proof*⟩

lemma *right-minus-eq*: $a - b = 0 \longleftrightarrow a = b$
⟨*proof*⟩

lemma *equals-zero-I*:
assumes $a + b = 0$
shows $- a = b$
⟨*proof*⟩

lemma *diff-self* [*simp*]: $a - a = 0$
⟨*proof*⟩

lemma *diff-0* [*simp*]: $0 - a = - a$
⟨*proof*⟩

lemma *diff-0-right* [*simp*]: $a - 0 = a$
⟨*proof*⟩

lemma *diff-minus-eq-add* [*simp*]: $a - - b = a + b$
⟨*proof*⟩

lemma *neg-equal-iff-equal* [*simp*]:
$- a = - b \longleftrightarrow a = b$

⟨*proof*⟩

lemma *neg-equal-0-iff-equal* [*simp*]:
$- a = 0 \longleftrightarrow a = 0$
⟨*proof*⟩

lemma *neg-0-equal-iff-equal* [*simp*]:
$0 = - a \longleftrightarrow 0 = a$
⟨*proof*⟩

The next two equations can make the simplifier loop!

lemma *equation-minus-iff*:
$a = - b \longleftrightarrow b = - a$
⟨*proof*⟩

lemma *minus-equation-iff*:
$- a = b \longleftrightarrow - b = a$
⟨*proof*⟩

end

class *ab-group-add* = *minus* + *comm-monoid-add* +
assumes *ab-left-minus*: $- a + a = 0$
assumes *ab-diff-minus*: $a - b = a + (- b)$
begin

subclass *group-add*
⟨*proof*⟩

subclass *cancel-ab-semigroup-add*
⟨*proof*⟩

lemma *uminus-add-conv-diff*:
$- a + b = b - a$
⟨*proof*⟩

lemma *minus-add-distrib* [*simp*]:
$- (a + b) = - a + - b$
⟨*proof*⟩

lemma *minus-diff-eq* [*simp*]:
$- (a - b) = b - a$
⟨*proof*⟩

lemma *add-diff-eq*: $a + (b - c) = (a + b) - c$
⟨*proof*⟩

lemma *diff-add-eq*: $(a - b) + c = (a + c) - b$
⟨*proof*⟩

lemma *diff-eq-eq*: $a - b = c \longleftrightarrow a = c + b$
⟨*proof*⟩

lemma *eq-diff-eq*: $a = c - b \longleftrightarrow a + b = c$
⟨*proof*⟩

lemma *diff-diff-eq*: $(a - b) - c = a - (b + c)$
⟨*proof*⟩

lemma *diff-diff-eq2*: $a - (b - c) = (a + c) - b$
⟨*proof*⟩

lemma *diff-add-cancel*: $a - b + b = a$
⟨*proof*⟩

lemma *add-diff-cancel*: $a + b - b = a$
⟨*proof*⟩

lemmas *compare-rls* =
diff-minus [*symmetric*]
add-diff-eq diff-add-eq diff-diff-eq diff-diff-eq2
diff-eq-eq eq-diff-eq

lemma *eq-iff-diff-eq-0*: $a = b \longleftrightarrow a - b = 0$
⟨*proof*⟩

end

15.3 (Partially) Ordered Groups

class *pordered-ab-semigroup-add* = *order* + *ab-semigroup-add* +
assumes *add-left-mono*: $a \leq b \Longrightarrow c + a \leq c + b$
begin

lemma *add-right-mono*:
$a \leq b \Longrightarrow a + c \leq b + c$
⟨*proof*⟩

non-strict, in both arguments

lemma *add-mono*:
$a \leq b \Longrightarrow c \leq d \Longrightarrow a + c \leq b + d$
⟨*proof*⟩

end

class *pordered-cancel-ab-semigroup-add* =
pordered-ab-semigroup-add + *cancel-ab-semigroup-add*
begin

lemma *add-strict-left-mono*:
$a < b \Longrightarrow c + a < c + b$
⟨*proof*⟩

lemma *add-strict-right-mono*:
$a < b \Longrightarrow a + c < b + c$
⟨*proof*⟩

Strict monotonicity in both arguments

lemma *add-strict-mono*:
$a < b \Longrightarrow c < d \Longrightarrow a + c < b + d$
⟨*proof*⟩

lemma *add-less-le-mono*:
$a < b \Longrightarrow c \leq d \Longrightarrow a + c < b + d$
⟨*proof*⟩

lemma *add-le-less-mono*:
$a \leq b \Longrightarrow c < d \Longrightarrow a + c < b + d$
⟨*proof*⟩

end

class *pordered-ab-semigroup-add-imp-le* $=$
pordered-cancel-ab-semigroup-add $+$
assumes *add-le-imp-le-left*: $c + a \leq c + b \Longrightarrow a \leq b$
begin

lemma *add-less-imp-less-left*:
assumes *less*: $c + a < c + b$
shows $a < b$
⟨*proof*⟩

lemma *add-less-imp-less-right*:
$a + c < b + c \Longrightarrow a < b$
⟨*proof*⟩

lemma *add-less-cancel-left* [*simp*]:
$c + a < c + b \longleftrightarrow a < b$
⟨*proof*⟩

lemma *add-less-cancel-right* [*simp*]:
$a + c < b + c \longleftrightarrow a < b$
⟨*proof*⟩

lemma *add-le-cancel-left* [*simp*]:
$c + a \leq c + b \longleftrightarrow a \leq b$
⟨*proof*⟩

lemma *add-le-cancel-right* [*simp*]:
$a + c \leq b + c \longleftrightarrow a \leq b$
⟨*proof*⟩

lemma *add-le-imp-le-right*:
$a + c \leq b + c \Longrightarrow a \leq b$
⟨*proof*⟩

lemma *max-add-distrib-left*:
$\mathit{max}\ x\ y + z = \mathit{max}\ (x + z)\ (y + z)$
⟨*proof*⟩

lemma *min-add-distrib-left*:
$\mathit{min}\ x\ y + z = \mathit{min}\ (x + z)\ (y + z)$
⟨*proof*⟩

end

15.4 Support for reasoning about signs

class *pordered-comm-monoid-add* =
pordered-cancel-ab-semigroup-add + *comm-monoid-add*
begin

lemma *add-pos-nonneg*:
assumes $0 < a$ **and** $0 \leq b$
shows $0 < a + b$
⟨*proof*⟩

lemma *add-pos-pos*:
assumes $0 < a$ **and** $0 < b$
shows $0 < a + b$
⟨*proof*⟩

lemma *add-nonneg-pos*:
assumes $0 \leq a$ **and** $0 < b$
shows $0 < a + b$
⟨*proof*⟩

lemma *add-nonneg-nonneg*:
assumes $0 \leq a$ **and** $0 \leq b$
shows $0 \leq a + b$
⟨*proof*⟩

lemma *add-neg-nonpos*:
assumes $a < 0$ **and** $b \leq 0$
shows $a + b < 0$
⟨*proof*⟩

lemma *add-neg-neg*:
assumes $a < 0$ **and** $b < 0$
shows $a + b < 0$
⟨*proof*⟩

lemma *add-nonpos-neg*:
assumes $a \leq 0$ **and** $b < 0$
shows $a + b < 0$
⟨*proof*⟩

lemma *add-nonpos-nonpos*:
assumes $a \leq 0$ **and** $b \leq 0$
shows $a + b \leq 0$
⟨*proof*⟩

end

class *pordered-ab-group-add* =
ab-group-add + *pordered-ab-semigroup-add*
begin

subclass *pordered-cancel-ab-semigroup-add*
⟨*proof*⟩

subclass *pordered-ab-semigroup-add-imp-le*
⟨*proof*⟩

subclass *pordered-comm-monoid-add*
⟨*proof*⟩

lemma *max-diff-distrib-left*:
shows $max\ x\ y - z = max\ (x - z)\ (y - z)$
⟨*proof*⟩

lemma *min-diff-distrib-left*:
shows $min\ x\ y - z = min\ (x - z)\ (y - z)$
⟨*proof*⟩

lemma *le-imp-neg-le*:
assumes $a \leq b$
shows $-b \leq -a$
⟨*proof*⟩

lemma *neg-le-iff-le* [*simp*]: $- b \leq - a \longleftrightarrow a \leq b$
⟨*proof*⟩

lemma *neg-le-0-iff-le* [*simp*]: $- a \leq 0 \longleftrightarrow 0 \leq a$
⟨*proof*⟩

lemma *neg-0-le-iff-le* [*simp*]: $0 \leq - a \longleftrightarrow a \leq 0$
⟨*proof*⟩

lemma *neg-less-iff-less* [*simp*]: $- b < - a \longleftrightarrow a < b$
⟨*proof*⟩

lemma *neg-less-0-iff-less* [*simp*]: $- a < 0 \longleftrightarrow 0 < a$
⟨*proof*⟩

lemma *neg-0-less-iff-less* [*simp*]: $0 < - a \longleftrightarrow a < 0$
⟨*proof*⟩

The next several equations can make the simplifier loop!

lemma *less-minus-iff*: $a < - b \longleftrightarrow b < - a$
⟨*proof*⟩

lemma *minus-less-iff*: $- a < b \longleftrightarrow - b < a$
⟨*proof*⟩

lemma *le-minus-iff*: $a \leq - b \longleftrightarrow b \leq - a$
⟨*proof*⟩

lemma *minus-le-iff*: $- a \leq b \longleftrightarrow - b \leq a$
⟨*proof*⟩

lemma *less-iff-diff-less-0*: $a < b \longleftrightarrow a - b < 0$
⟨*proof*⟩

lemma *diff-less-eq*: $a - b < c \longleftrightarrow a < c + b$
⟨*proof*⟩

lemma *less-diff-eq*: $a < c - b \longleftrightarrow a + b < c$
⟨*proof*⟩

lemma *diff-le-eq*: $a - b \leq c \longleftrightarrow a \leq c + b$
⟨*proof*⟩

lemma *le-diff-eq*: $a \leq c - b \longleftrightarrow a + b \leq c$
⟨*proof*⟩

lemmas *compare-rls* =
diff-minus [*symmetric*]
add-diff-eq diff-add-eq diff-diff-eq diff-diff-eq2
diff-less-eq less-diff-eq diff-le-eq le-diff-eq
diff-eq-eq eq-diff-eq

This list of rewrites simplifies (in)equalities by bringing subtractions to the top and then moving negative terms to the other side. Use with *add-ac*

lemmas (**in** −) *compare-rls* =
diff-minus [*symmetric*]
add-diff-eq diff-add-eq diff-diff-eq diff-diff-eq2
diff-less-eq less-diff-eq diff-le-eq le-diff-eq
diff-eq-eq eq-diff-eq

lemma *le-iff-diff-le-0*: $a \le b \longleftrightarrow a - b \le 0$
⟨*proof*⟩

lemmas *group-simps* =
add-ac
add-diff-eq diff-add-eq diff-diff-eq diff-diff-eq2
diff-eq-eq eq-diff-eq diff-minus [*symmetric*] *uminus-add-conv-diff*
diff-less-eq less-diff-eq diff-le-eq le-diff-eq

end

lemmas *group-simps* =
mult-ac
add-ac
add-diff-eq diff-add-eq diff-diff-eq diff-diff-eq2
diff-eq-eq eq-diff-eq diff-minus [*symmetric*] *uminus-add-conv-diff*
diff-less-eq less-diff-eq diff-le-eq le-diff-eq

class *ordered-ab-semigroup-add* =
linorder + *pordered-ab-semigroup-add*

class *ordered-cancel-ab-semigroup-add* =
linorder + *pordered-cancel-ab-semigroup-add*
begin

subclass *ordered-ab-semigroup-add*
⟨*proof*⟩

subclass *pordered-ab-semigroup-add-imp-le*
⟨*proof*⟩

end

class *ordered-ab-group-add* =
linorder + *pordered-ab-group-add*
begin

subclass *ordered-cancel-ab-semigroup-add*
⟨*proof*⟩

lemma *neg-less-eq-nonneg*:
$- a \le a \longleftrightarrow 0 \le a$
⟨*proof*⟩

lemma *less-eq-neg-nonpos*:
  $a \leq - a \longleftrightarrow a \leq 0$
⟨*proof*⟩

lemma *equal-neg-zero*:
  $a = - a \longleftrightarrow a = 0$
⟨*proof*⟩

lemma *neg-equal-zero*:
  $- a = a \longleftrightarrow a = 0$
  ⟨*proof*⟩

end

— FIXME localize the following

lemma *add-increasing*:
  fixes c :: $'a$::{*pordered-ab-semigroup-add-imp-le*, *comm-monoid-add*}
  shows $[|0 \leq a;\ b \leq c|] ==> b \leq a + c$
⟨*proof*⟩

lemma *add-increasing2*:
  fixes c :: $'a$::{*pordered-ab-semigroup-add-imp-le*, *comm-monoid-add*}
  shows $[|0 \leq c;\ b \leq a|] ==> b \leq a + c$
⟨*proof*⟩

lemma *add-strict-increasing*:
  fixes c :: $'a$::{*pordered-ab-semigroup-add-imp-le*, *comm-monoid-add*}
  shows $[|0 < a;\ b \leq c|] ==> b < a + c$
⟨*proof*⟩

lemma *add-strict-increasing2*:
  fixes c :: $'a$::{*pordered-ab-semigroup-add-imp-le*, *comm-monoid-add*}
  shows $[|0 \leq a;\ b < c|] ==> b < a + c$
⟨*proof*⟩

class *pordered-ab-group-add-abs* = *pordered-ab-group-add* + *abs* +
  assumes *abs-ge-zero* [*simp*]: $|a| \geq 0$
    and *abs-ge-self*: $a \leq |a|$
    and *abs-leI*: $a \leq b \Longrightarrow - a \leq b \Longrightarrow |a| \leq b$
    and *abs-minus-cancel* [*simp*]: $|-a| = |a|$
    and *abs-triangle-ineq*: $|a + b| \leq |a| + |b|$
begin

lemma *abs-minus-le-zero*: $- |a| \leq 0$
  ⟨*proof*⟩

lemma *abs-of-nonneg* [*simp*]:
assumes *nonneg*: $0 \le a$
shows $|a| = a$
⟨*proof*⟩

lemma *abs-idempotent* [*simp*]: $||a|| = |a|$
⟨*proof*⟩

lemma *abs-eq-0* [*simp*]: $|a| = 0 \longleftrightarrow a = 0$
⟨*proof*⟩

lemma *abs-zero* [*simp*]: $|0| = 0$
⟨*proof*⟩

lemma *abs-0-eq* [*simp, noatp*]: $0 = |a| \longleftrightarrow a = 0$
⟨*proof*⟩

lemma *abs-le-zero-iff* [*simp*]: $|a| \le 0 \longleftrightarrow a = 0$
⟨*proof*⟩

lemma *zero-less-abs-iff* [*simp*]: $0 < |a| \longleftrightarrow a \ne 0$
⟨*proof*⟩

lemma *abs-not-less-zero* [*simp*]: $\neg\ |a| < 0$
⟨*proof*⟩

lemma *abs-ge-minus-self*: $-\ a \le |a|$
⟨*proof*⟩

lemma *abs-minus-commute*:
$|a - b| = |b - a|$
⟨*proof*⟩

lemma *abs-of-pos*: $0 < a \Longrightarrow |a| = a$
⟨*proof*⟩

lemma *abs-of-nonpos* [*simp*]:
assumes $a \le 0$
shows $|a| = -\ a$
⟨*proof*⟩

lemma *abs-of-neg*: $a < 0 \Longrightarrow |a| = -\ a$
⟨*proof*⟩

lemma *abs-le-D1*: $|a| \le b \Longrightarrow a \le b$
⟨*proof*⟩

lemma *abs-le-D2*: $|a| \le b \Longrightarrow -\ a \le b$
⟨*proof*⟩

lemma *abs-le-iff*: $|a| \leq b \longleftrightarrow a \leq b \wedge - a \leq b$
⟨*proof*⟩

lemma *abs-triangle-ineq2*: $|a| - |b| \leq |a - b|$
⟨*proof*⟩

lemma *abs-triangle-ineq3*: $||a| - |b|| \leq |a - b|$
⟨*proof*⟩

lemma *abs-triangle-ineq4*: $|a - b| \leq |a| + |b|$
⟨*proof*⟩

lemma *abs-diff-triangle-ineq*: $|a + b - (c + d)| \leq |a - c| + |b - d|$
⟨*proof*⟩

lemma *abs-add-abs* [*simp*]:
$||a| + |b|| = |a| + |b|$ (**is** $?L = ?R$)
⟨*proof*⟩

end

15.5 Lattice Ordered (Abelian) Groups

class *lordered-ab-group-add-meet* = *pordered-ab-group-add* + *lower-semilattice*
begin

lemma *add-inf-distrib-left*:
$a + inf\ b\ c = inf\ (a + b)\ (a + c)$
⟨*proof*⟩

lemma *add-inf-distrib-right*:
$inf\ a\ b + c = inf\ (a + c)\ (b + c)$
⟨*proof*⟩

end

class *lordered-ab-group-add-join* = *pordered-ab-group-add* + *upper-semilattice*
begin

lemma *add-sup-distrib-left*:
$a + sup\ b\ c = sup\ (a + b)\ (a + c)$
⟨*proof*⟩

lemma *add-sup-distrib-right*:
$sup\ a\ b + c = sup\ (a+c)\ (b+c)$
⟨*proof*⟩

end

class *lordered-ab-group-add* = *pordered-ab-group-add* + *lattice*
begin

subclass *lordered-ab-group-add-meet* ⟨*proof*⟩
subclass *lordered-ab-group-add-join* ⟨*proof*⟩

lemmas *add-sup-inf-distribs* = *add-inf-distrib-right add-inf-distrib-left add-sup-distrib-right add-sup-distrib-left*

lemma *inf-eq-neg-sup*: $inf\ a\ b = -\ sup\ (-a)\ (-b)$
⟨*proof*⟩

lemma *sup-eq-neg-inf*: $sup\ a\ b = -\ inf\ (-a)\ (-b)$
⟨*proof*⟩

lemma *neg-inf-eq-sup*: $-\ inf\ a\ b = sup\ (-a)\ (-b)$
⟨*proof*⟩

lemma *neg-sup-eq-inf*: $-\ sup\ a\ b = inf\ (-a)\ (-b)$
⟨*proof*⟩

lemma *add-eq-inf-sup*: $a + b = sup\ a\ b + inf\ a\ b$
⟨*proof*⟩

15.6 Positive Part, Negative Part, Absolute Value

definition
$nprt :: 'a \Rightarrow 'a$ **where**
$nprt\ x = inf\ x\ 0$

definition
$pprt :: 'a \Rightarrow 'a$ **where**
$pprt\ x = sup\ x\ 0$

lemma *pprt-neg*: $pprt\ (-\ x) = -\ nprt\ x$
⟨*proof*⟩

lemma *nprt-neg*: $nprt\ (-\ x) = -\ pprt\ x$
⟨*proof*⟩

lemma *prts*: $a = pprt\ a + nprt\ a$
⟨*proof*⟩

lemma *zero-le-pprt*[*simp*]: $0 \leq pprt\ a$
⟨*proof*⟩

lemma *nprt-le-zero*[*simp*]: $nprt\ a \leq 0$
⟨*proof*⟩

lemma *le-eq-neg*: $a \le - b \longleftrightarrow a + b \le 0$ (**is** $?l = ?r$)
⟨*proof*⟩

lemma *pprt-0*[*simp*]: $pprt\ 0 = 0$ ⟨*proof*⟩
lemma *nprt-0*[*simp*]: $nprt\ 0 = 0$ ⟨*proof*⟩

lemma *pprt-eq-id* [*simp*, *noatp*]: $0 \le x \Longrightarrow pprt\ x = x$
⟨*proof*⟩

lemma *nprt-eq-id* [*simp*, *noatp*]: $x \le 0 \Longrightarrow nprt\ x = x$
⟨*proof*⟩

lemma *pprt-eq-0* [*simp*, *noatp*]: $x \le 0 \Longrightarrow pprt\ x = 0$
⟨*proof*⟩

lemma *nprt-eq-0* [*simp*, *noatp*]: $0 \le x \Longrightarrow nprt\ x = 0$
⟨*proof*⟩

lemma *sup-0-imp-0*: $sup\ a\ (-\ a) = 0 \Longrightarrow a = 0$
⟨*proof*⟩

lemma *inf-0-imp-0*: $inf\ a\ (-a) = 0 \Longrightarrow a = 0$
⟨*proof*⟩

lemma *inf-0-eq-0* [*simp*, *noatp*]: $inf\ a\ (-\ a) = 0 \longleftrightarrow a = 0$
⟨*proof*⟩

lemma *sup-0-eq-0* [*simp*, *noatp*]: $sup\ a\ (-\ a) = 0 \longleftrightarrow a = 0$
⟨*proof*⟩

lemma *zero-le-double-add-iff-zero-le-single-add* [*simp*]:
$0 \le a + a \longleftrightarrow 0 \le a$
⟨*proof*⟩

lemma *double-zero*: $a + a = 0 \longleftrightarrow a = 0$
⟨*proof*⟩

lemma *zero-less-double-add-iff-zero-less-single-add*:
$0 < a + a \longleftrightarrow 0 < a$
⟨*proof*⟩

lemma *double-add-le-zero-iff-single-add-le-zero* [*simp*]:
$a + a \le 0 \longleftrightarrow a \le 0$
⟨*proof*⟩

lemma *double-add-less-zero-iff-single-less-zero* [*simp*]:
$a + a < 0 \longleftrightarrow a < 0$
⟨*proof*⟩

declare *neg-inf-eq-sup* [*simp*] *neg-sup-eq-inf* [*simp*]

lemma *le-minus-self-iff*: $a \le - a \longleftrightarrow a \le 0$
⟨*proof*⟩

lemma *minus-le-self-iff*: $- a \le a \longleftrightarrow 0 \le a$
⟨*proof*⟩

lemma *zero-le-iff-zero-nprt*: $0 \le a \longleftrightarrow nprt\ a = 0$
⟨*proof*⟩

lemma *le-zero-iff-zero-pprt*: $a \le 0 \longleftrightarrow pprt\ a = 0$
⟨*proof*⟩

lemma *le-zero-iff-pprt-id*: $0 \le a \longleftrightarrow pprt\ a = a$
⟨*proof*⟩

lemma *zero-le-iff-nprt-id*: $a \le 0 \longleftrightarrow nprt\ a = a$
⟨*proof*⟩

lemma *pprt-mono* [*simp*, *noatp*]: $a \le b \Longrightarrow pprt\ a \le pprt\ b$
⟨*proof*⟩

lemma *nprt-mono* [*simp*, *noatp*]: $a \le b \Longrightarrow nprt\ a \le nprt\ b$
⟨*proof*⟩

end

lemmas *add-sup-inf-distribs* = *add-inf-distrib-right add-inf-distrib-left add-sup-distrib-right add-sup-distrib-left*

class *lordered-ab-group-add-abs* = *lordered-ab-group-add* + *abs* +
assumes *abs-lattice*: $|a| = sup\ a\ (- a)$
begin

lemma *abs-prts*: $|a| = pprt\ a - nprt\ a$
⟨*proof*⟩

subclass *pordered-ab-group-add-abs*
⟨*proof*⟩

end

lemma *sup-eq-if*:
fixes $a :: 'a::\{lordered\text{-}ab\text{-}group\text{-}add,\ linorder\}$
shows $sup\ a\ (- a) = (if\ a < 0\ then\ - a\ else\ a)$
⟨*proof*⟩

lemma *abs-if-lattice*:
  fixes a :: $'a$::{*lordered-ab-group-add-abs*, *linorder*}
  shows $|a| = (if\ a < 0\ then - a\ else\ a)$
  ⟨*proof*⟩

Needed for abelian cancellation simprocs:

lemma *add-cancel-21*: $((x::'a::ab\text{-}group\text{-}add) + (y + z) = y + u) = (x + z = u)$
⟨*proof*⟩

lemma *add-cancel-end*: $(x + (y + z) = y) = (x = - (z::'a::ab\text{-}group\text{-}add))$
⟨*proof*⟩

lemma *less-eqI*: $(x::'a::pordered\text{-}ab\text{-}group\text{-}add) - y = x' - y' \Longrightarrow (x < y) = (x' < y')$
⟨*proof*⟩

lemma *le-eqI*: $(x::'a::pordered\text{-}ab\text{-}group\text{-}add) - y = x' - y' \Longrightarrow (y <= x) = (y' <= x')$
⟨*proof*⟩

lemma *eq-eqI*: $(x::'a::ab\text{-}group\text{-}add) - y = x' - y' \Longrightarrow (x = y) = (x' = y')$
⟨*proof*⟩

lemma *diff-def*: $(x::'a::ab\text{-}group\text{-}add) - y == x + (-y)$
⟨*proof*⟩

lemma *add-minus-cancel*: $(a::'a::ab\text{-}group\text{-}add) + (-a + b) = b$
⟨*proof*⟩

lemma *le-add-right-mono*:
  assumes
  $a <= b + (c::'a::pordered\text{-}ab\text{-}group\text{-}add)$
  $c <= d$
  shows $a <= b + d$
  ⟨*proof*⟩

lemma *estimate-by-abs*:
  $a + b <= (c::'a::lordered\text{-}ab\text{-}group\text{-}add\text{-}abs) \Longrightarrow a <= c + abs\ b$
⟨*proof*⟩

15.7 Tools setup

lemma *add-mono-thms-ordered-semiring* [*noatp*]:
  fixes $i\ j\ k$ :: $'a::pordered\text{-}ab\text{-}semigroup\text{-}add$
  shows $i \le j \wedge k \le l \Longrightarrow i + k \le j + l$
    and $i = j \wedge k \le l \Longrightarrow i + k \le j + l$
    and $i \le j \wedge k = l \Longrightarrow i + k \le j + l$
    and $i = j \wedge k = l \Longrightarrow i + k = j + l$

⟨*proof*⟩

lemma *add-mono-thms-ordered-field* [*noatp*]:
  fixes $i\ j\ k$ :: $'a$::*pordered-cancel-ab-semigroup-add*
  shows $i < j \wedge k = l \Longrightarrow i + k < j + l$
    and $i = j \wedge k < l \Longrightarrow i + k < j + l$
    and $i < j \wedge k \leq l \Longrightarrow i + k < j + l$
    and $i \leq j \wedge k < l \Longrightarrow i + k < j + l$
    and $i < j \wedge k < l \Longrightarrow i + k < j + l$
⟨*proof*⟩

Simplification of $x - y < (0::'a)$, etc.

lemmas *diff-less-0-iff-less* [*simp*] = *less-iff-diff-less-0* [*symmetric*]
lemmas *diff-eq-0-iff-eq* [*simp*, *noatp*] = *eq-iff-diff-eq-0* [*symmetric*]
lemmas *diff-le-0-iff-le* [*simp*] = *le-iff-diff-le-0* [*symmetric*]

⟨*ML*⟩

end

16 Ring-and-Field: (Ordered) Rings and Fields

theory *Ring-and-Field*
imports *OrderedGroup*
begin

The theory of partially ordered rings is taken from the books:

- *Lattice Theory* by Garret Birkhoff, American Mathematical Society 1979
- *Partially Ordered Algebraic Systems*, Pergamon Press 1963

Most of the used notions can also be looked up in

- http://www.mathworld.com by Eric Weisstein et. al.
- *Algebra I* by van der Waerden, Springer.

class *semiring* = *ab-semigroup-add* + *semigroup-mult* +
  assumes *left-distrib*: $(a + b) * c = a * c + b * c$
  assumes *right-distrib*: $a * (b + c) = a * b + a * c$
begin

For the *combine-numerals* simproc

lemma *combine-common-factor*:
  $a * e + (b * e + c) = (a + b) * e + c$

⟨*proof*⟩

end

class *mult-zero* = *times* + *zero* +
assumes *mult-zero-left* [*simp*]: $0 * a = 0$
assumes *mult-zero-right* [*simp*]: $a * 0 = 0$

class *semiring-0* = *semiring* + *comm-monoid-add* + *mult-zero*

class *semiring-0-cancel* = *semiring* + *comm-monoid-add* + *cancel-ab-semigroup-add*
begin

subclass *semiring-0*
⟨*proof*⟩

end

class *comm-semiring* = *ab-semigroup-add* + *ab-semigroup-mult* +
assumes *distrib*: $(a + b) * c = a * c + b * c$
begin

subclass *semiring*
⟨*proof*⟩

end

class *comm-semiring-0* = *comm-semiring* + *comm-monoid-add* + *mult-zero*
begin

subclass *semiring-0* ⟨*proof*⟩

end

class *comm-semiring-0-cancel* = *comm-semiring* + *comm-monoid-add* + *cancel-ab-semigroup-add*
begin

subclass *semiring-0-cancel* ⟨*proof*⟩

end

class *zero-neq-one* = *zero* + *one* +
assumes *zero-neq-one* [*simp*]: $0 \neq 1$

class *semiring-1* = *zero-neq-one* + *semiring-0* + *monoid-mult*

class *comm-semiring-1* = *zero-neq-one* + *comm-semiring-0* + *comm-monoid-mult*

begin

subclass *semiring-1* ⟨*proof*⟩

end

class *no-zero-divisors* = *zero* + *times* +
assumes *no-zero-divisors*: $a \neq 0 \Longrightarrow b \neq 0 \Longrightarrow a * b \neq 0$

class *semiring-1-cancel* = *semiring* + *comm-monoid-add* + *zero-neq-one* + *cancel-ab-semigroup-add* + *monoid-mult*
begin

subclass *semiring-0-cancel* ⟨*proof*⟩

subclass *semiring-1* ⟨*proof*⟩

end

class *comm-semiring-1-cancel* = *comm-semiring* + *comm-monoid-add* + *comm-monoid-mult* + *zero-neq-one* + *cancel-ab-semigroup-add*
begin

subclass *semiring-1-cancel* ⟨*proof*⟩
subclass *comm-semiring-0-cancel* ⟨*proof*⟩
subclass *comm-semiring-1* ⟨*proof*⟩

end

class *ring* = *semiring* + *ab-group-add*
begin

subclass *semiring-0-cancel* ⟨*proof*⟩

Distribution rules

lemma *minus-mult-left*: $- (a * b) = - a * b$
⟨*proof*⟩

lemma *minus-mult-right*: $- (a * b) = a * - b$
⟨*proof*⟩

lemma *minus-mult-minus* [*simp*]: $- a * - b = a * b$
⟨*proof*⟩

lemma *minus-mult-commute*: $- a * b = a * - b$
⟨*proof*⟩

lemma *right-diff-distrib*: $a * (b - c) = a * b - a * c$
⟨*proof*⟩

lemma *left-diff-distrib*: $(a - b) * c = a * c - b * c$
⟨*proof*⟩

lemmas *ring-distribs* =
right-distrib left-distrib left-diff-distrib right-diff-distrib

lemmas *ring-simps* =
add-ac
add-diff-eq diff-add-eq diff-diff-eq diff-diff-eq2
diff-eq-eq eq-diff-eq diff-minus [*symmetric*] *uminus-add-conv-diff*
ring-distribs

lemma *eq-add-iff1*:
$a * e + c = b * e + d \longleftrightarrow (a - b) * e + c = d$
⟨*proof*⟩

lemma *eq-add-iff2*:
$a * e + c = b * e + d \longleftrightarrow c = (b - a) * e + d$
⟨*proof*⟩

end

lemmas *ring-distribs* =
right-distrib left-distrib left-diff-distrib right-diff-distrib

class *comm-ring* = *comm-semiring* + *ab-group-add*
begin

subclass *ring* ⟨*proof*⟩
subclass *comm-semiring-0* ⟨*proof*⟩

end

class *ring-1* = *ring* + *zero-neq-one* + *monoid-mult*
begin

subclass *semiring-1-cancel* ⟨*proof*⟩

end

class *comm-ring-1* = *comm-ring* + *zero-neq-one* + *comm-monoid-mult*

begin

subclass *ring-1* ⟨*proof*⟩
subclass *comm-semiring-1-cancel* ⟨*proof*⟩

end

class *ring-no-zero-divisors* = *ring* + *no-zero-divisors*
begin

lemma *mult-eq-0-iff* [*simp*]:
  shows $a * b = 0 \longleftrightarrow (a = 0 \lor b = 0)$
⟨*proof*⟩

end

class *ring-1-no-zero-divisors* = *ring-1* + *ring-no-zero-divisors*

class *idom* = *comm-ring-1* + *no-zero-divisors*
begin

subclass *ring-1-no-zero-divisors* ⟨*proof*⟩

end

class *division-ring* = *ring-1* + *inverse* +
  assumes *left-inverse* [*simp*]: $a \neq 0 \Longrightarrow inverse\ a * a = 1$
  assumes *right-inverse* [*simp*]: $a \neq 0 \Longrightarrow a * inverse\ a = 1$
begin

subclass *ring-1-no-zero-divisors*
⟨*proof*⟩

end

class *field* = *comm-ring-1* + *inverse* +
  assumes *field-inverse*: $a \neq 0 \Longrightarrow inverse\ a * a = 1$
  assumes *divide-inverse*: $a\ /\ b = a * inverse\ b$
begin

subclass *division-ring*
⟨*proof*⟩

subclass *idom* ⟨*proof*⟩

lemma *right-inverse-eq*: $b \neq 0 \Longrightarrow a\ /\ b = 1 \longleftrightarrow a = b$
⟨*proof*⟩

lemma *nonzero-inverse-eq-divide*: $a \neq 0 \Longrightarrow inverse\ a = 1\ /\ a$
  ⟨*proof*⟩

lemma *divide-self* [*simp*]: $a \neq 0 \Longrightarrow a\ /\ a = 1$
  ⟨*proof*⟩

lemma *divide-zero-left* [*simp*]: $0\ /\ a = 0$
  ⟨*proof*⟩

lemma *inverse-eq-divide*: $inverse\ a = 1\ /\ a$
⟨*proof*⟩

lemma *add-divide-distrib*: $(a+b)\ /\ c = a/c + b/c$
⟨*proof*⟩

end

class *division-by-zero* = *zero* + *inverse* +
assumes *inverse-zero* [*simp*]: $inverse\ 0 = 0$

lemma *divide-zero* [*simp*]:
$a\ /\ 0 = (0::'a::\{field,division\text{-}by\text{-}zero\})$
⟨*proof*⟩

lemma *divide-self-if* [*simp*]:
$a\ /\ (a::'a::\{field,division\text{-}by\text{-}zero\}) = (if\ a{=}0\ then\ 0\ else\ 1)$
⟨*proof*⟩

class *mult-mono* = *times* + *zero* + *ord* +
assumes *mult-left-mono*: $a \le b \Longrightarrow 0 \le c \Longrightarrow c * a \le c * b$
assumes *mult-right-mono*: $a \le b \Longrightarrow 0 \le c \Longrightarrow a * c \le b * c$

class *pordered-semiring* = *mult-mono* + *semiring-0* + *pordered-ab-semigroup-add*

begin

lemma *mult-mono*:
$a \le b \Longrightarrow c \le d \Longrightarrow 0 \le b \Longrightarrow 0 \le c$
$\Longrightarrow a * c \le b * d$
⟨*proof*⟩

lemma *mult-mono'*:
$a \le b \Longrightarrow c \le d \Longrightarrow 0 \le a \Longrightarrow 0 \le c$
$\Longrightarrow a * c \le b * d$
⟨*proof*⟩

end

class *pordered-cancel-semiring* = *mult-mono* + *pordered-ab-semigroup-add*
+ *semiring* + *comm-monoid-add* + *cancel-ab-semigroup-add*
begin

subclass *semiring-0-cancel* ⟨*proof*⟩
subclass *pordered-semiring* ⟨*proof*⟩

lemma *mult-nonneg-nonneg*: $0 \le a \Longrightarrow 0 \le b \Longrightarrow 0 \le a * b$
⟨*proof*⟩

lemma *mult-nonneg-nonpos*: $0 \le a \Longrightarrow b \le 0 \Longrightarrow a * b \le 0$
⟨*proof*⟩

lemma *mult-nonneg-nonpos2*: $0 \le a \Longrightarrow b \le 0 \Longrightarrow b * a \le 0$
⟨*proof*⟩

lemma *split-mult-neg-le*: $(0 \le a \ \& \ b \le 0) \mid (a \le 0 \ \& \ 0 \le b) \Longrightarrow a * b \le$ $(0::-::pordered\text{-}cancel\text{-}semiring)$
⟨*proof*⟩

end

class *ordered-semiring* = *semiring* + *comm-monoid-add* + *ordered-cancel-ab-semigroup-add* + *mult-mono*
begin

subclass *pordered-cancel-semiring* ⟨*proof*⟩

subclass *pordered-comm-monoid-add* ⟨*proof*⟩

lemma *mult-left-less-imp-less*:
$c * a < c * b \Longrightarrow 0 \le c \Longrightarrow a < b$
⟨*proof*⟩

lemma *mult-right-less-imp-less*:
$a * c < b * c \Longrightarrow 0 \le c \Longrightarrow a < b$
⟨*proof*⟩

end

class *ordered-semiring-strict* = *semiring* + *comm-monoid-add* + *ordered-cancel-ab-semigroup-add* +
assumes *mult-strict-left-mono*: $a < b \Longrightarrow 0 < c \Longrightarrow c * a < c * b$
assumes *mult-strict-right-mono*: $a < b \Longrightarrow 0 < c \Longrightarrow a * c < b * c$
begin

subclass *semiring-0-cancel* ⟨*proof*⟩

subclass *ordered-semiring*
⟨*proof*⟩

lemma *mult-left-le-imp-le*:
$c * a \le c * b \Longrightarrow 0 < c \Longrightarrow a \le b$
⟨*proof*⟩

lemma *mult-right-le-imp-le*:
$a * c \le b * c \Longrightarrow 0 < c \Longrightarrow a \le b$
⟨*proof*⟩

lemma *mult-pos-pos*:
$0 < a \Longrightarrow 0 < b \Longrightarrow 0 < a * b$
⟨*proof*⟩

lemma *mult-pos-neg*:
$0 < a \Longrightarrow b < 0 \Longrightarrow a * b < 0$
⟨*proof*⟩

lemma *mult-pos-neg2*:
$0 < a \Longrightarrow b < 0 \Longrightarrow b * a < 0$
⟨*proof*⟩

lemma *zero-less-mult-pos*:
$0 < a * b \Longrightarrow 0 < a \Longrightarrow 0 < b$
⟨*proof*⟩

lemma *zero-less-mult-pos2*:
$0 < b * a \Longrightarrow 0 < a \Longrightarrow 0 < b$
⟨*proof*⟩

end

class *mult-mono1* = *times* + *zero* + *ord* +
assumes *mult-mono1*: $a \leq b \Longrightarrow 0 \leq c \Longrightarrow c * a \leq c * b$

class *pordered-comm-semiring* = *comm-semiring-0*
+ *pordered-ab-semigroup-add* + *mult-mono1*
begin

subclass *pordered-semiring*
⟨*proof*⟩

end

class *pordered-cancel-comm-semiring* = *comm-semiring-0-cancel*
+ *pordered-ab-semigroup-add* + *mult-mono1*
begin

subclass *pordered-comm-semiring* ⟨*proof*⟩
subclass *pordered-cancel-semiring* ⟨*proof*⟩

end

class *ordered-comm-semiring-strict* = *comm-semiring-0* + *ordered-cancel-ab-semigroup-add*
+
assumes *mult-strict-mono*: $a < b \Longrightarrow 0 < c \Longrightarrow c * a < c * b$
begin

subclass *ordered-semiring-strict*
⟨*proof*⟩

subclass *pordered-cancel-comm-semiring*
⟨*proof*⟩

end

class *pordered-ring* = *ring* + *pordered-cancel-semiring*
begin

subclass *pordered-ab-group-add* ⟨*proof*⟩

lemmas *ring-simps* = *ring-simps group-simps*

lemma *less-add-iff1*:
$a * e + c < b * e + d \longleftrightarrow (a - b) * e + c < d$
⟨*proof*⟩

lemma *less-add-iff2*:
$a * e + c < b * e + d \longleftrightarrow c < (b - a) * e + d$
⟨*proof*⟩

lemma *le-add-iff1*:
$a * e + c \leq b * e + d \longleftrightarrow (a - b) * e + c \leq d$
⟨*proof*⟩

lemma *le-add-iff2*:
$a * e + c \leq b * e + d \longleftrightarrow c \leq (b - a) * e + d$
⟨*proof*⟩

lemma *mult-left-mono-neg*:
$b \leq a \Longrightarrow c \leq 0 \Longrightarrow c * a \leq c * b$
⟨*proof*⟩

lemma *mult-right-mono-neg*:
$b \leq a \Longrightarrow c \leq 0 \Longrightarrow a * c \leq b * c$
⟨*proof*⟩

lemma *mult-nonpos-nonpos*:
$a \leq 0 \Longrightarrow b \leq 0 \Longrightarrow 0 \leq a * b$
⟨*proof*⟩

lemma *split-mult-pos-le*:
$(0 \leq a \wedge 0 \leq b) \vee (a \leq 0 \wedge b \leq 0) \Longrightarrow 0 \leq a * b$
⟨*proof*⟩

end

class *abs-if* = *minus* + *ord* + *zero* + *abs* +
  assumes *abs-if*: $|a| = (\mathit{if}\ a < 0\ \mathit{then}\ (-\ a)\ \mathit{else}\ a)$

class *sgn-if* = *sgn* + *zero* + *one* + *minus* + *ord* +
  assumes *sgn-if*: $\mathit{sgn}\ x = (\mathit{if}\ x = 0\ \mathit{then}\ 0\ \mathit{else}\ \mathit{if}\ 0 < x\ \mathit{then}\ 1\ \mathit{else}\ -\ 1)$

class *ordered-ring* = *ring* + *ordered-semiring*
  + *ordered-ab-group-add* + *abs-if*
begin

subclass *pordered-ring* ⟨*proof*⟩

subclass *pordered-ab-group-add-abs*
⟨*proof*⟩

end

class *ordered-ring-strict* = *ring* + *ordered-semiring-strict*
  + *ordered-ab-group-add* + *abs-if*
begin

subclass *ordered-ring* ⟨*proof*⟩

lemma *mult-strict-left-mono-neg*:
  $b < a \Longrightarrow c < 0 \Longrightarrow c * a < c * b$
  ⟨*proof*⟩

lemma *mult-strict-right-mono-neg*:
  $b < a \Longrightarrow c < 0 \Longrightarrow a * c < b * c$
  ⟨*proof*⟩

lemma *mult-neg-neg*:
  $a < 0 \Longrightarrow b < 0 \Longrightarrow 0 < a * b$
  ⟨*proof*⟩

end

instance *ordered-ring-strict* ⊆ *ring-no-zero-divisors*
⟨*proof*⟩

lemma *zero-less-mult-iff*:
  fixes $a :: 'a::\mathit{ordered\text{-}ring\text{-}strict}$
  shows $0 < a * b \longleftrightarrow 0 < a \wedge 0 < b \vee a < 0 \wedge b < 0$
  ⟨*proof*⟩

lemma *zero-le-mult-iff*:
    $((0::'a::\mathit{ordered\text{-}ring\text{-}strict}) \leq a{*}b) = (0 \leq a\ \&\ 0 \leq b \mid a \leq 0\ \&\ b \leq 0)$
⟨*proof*⟩

lemma *mult-less-0-iff*:
$(a{*}b < (0::'a::ordered\text{-}ring\text{-}strict)) = (0 < a \,\&\, b < 0 \mid a < 0 \,\&\, 0 < b)$
⟨*proof*⟩

lemma *mult-le-0-iff*:
$(a{*}b \leq (0::'a::ordered\text{-}ring\text{-}strict)) = (0 \leq a \,\&\, b \leq 0 \mid a \leq 0 \,\&\, 0 \leq b)$
⟨*proof*⟩

lemma *zero-le-square*[*simp*]: $(0::'a::ordered\text{-}ring\text{-}strict) \leq a{*}a$
⟨*proof*⟩

lemma *not-square-less-zero*[*simp*]: $\neg\ (a * a < (0::'a::ordered\text{-}ring\text{-}strict))$
⟨*proof*⟩

This list of rewrites simplifies ring terms by multiplying everything out and bringing sums and products into a canonical form (by ordered rewriting). As a result it decides ring equalities but also helps with inequalities.

lemmas *ring-simps* = *group-simps ring-distribs*

class *pordered-comm-ring* = *comm-ring* + *pordered-comm-semiring*
begin

subclass *pordered-ring* ⟨*proof*⟩
subclass *pordered-cancel-comm-semiring* ⟨*proof*⟩

end

class *ordered-semidom* = *comm-semiring-1-cancel* + *ordered-comm-semiring-strict* +

assumes *zero-less-one* [*simp*]: $0 < 1$
begin

lemma *pos-add-strict*:
shows $0 < a \Longrightarrow b < c \Longrightarrow b < a + c$
⟨*proof*⟩

end

class *ordered-idom* =
comm-ring-1 +
ordered-comm-semiring-strict +
ordered-ab-group-add +
abs-if + *sgn-if*

instance *ordered-idom* ⊆ *ordered-ring-strict* ⟨*proof*⟩

instance *ordered-idom* ⊆ *pordered-comm-ring* ⟨*proof*⟩

class *ordered-field* = *field* + *ordered-idom*

lemma *linorder-neqE-ordered-idom*:
fixes $x\ y :: 'a :: ordered\text{-}idom$
assumes $x \neq y$ **obtains** $x < y \mid y < x$
⟨*proof*⟩

Proving axiom *zero-less-one* makes all *ordered-semidom* theorems available to members of *ordered-idom*

instance *ordered-idom* ⊆ *ordered-semidom*
⟨*proof*⟩

instance *ordered-idom* ⊆ *idom* ⟨*proof*⟩

All three types of comparision involving 0 and 1 are covered.

lemmas *one-neq-zero* = *zero-neq-one* [*THEN not-sym*]
declare *one-neq-zero* [*simp*]

lemma *zero-le-one* [*simp*]: $(0::'a::ordered\text{-}semidom) \leq 1$
⟨*proof*⟩

lemma *not-one-le-zero* [*simp*]: $\sim (1::'a::ordered\text{-}semidom) \leq 0$
⟨*proof*⟩

lemma *not-one-less-zero* [*simp*]: $\sim (1::'a::ordered\text{-}semidom) < 0$
⟨*proof*⟩

16.1 More Monotonicity

Strict monotonicity in both arguments

lemma *mult-strict-mono*:
$[|a<b;\ c<d;\ 0<b;\ 0\leq c|] \implies a * c < b * (d::'a::ordered\text{-}semiring\text{-}strict)$
⟨*proof*⟩

This weaker variant has more natural premises

lemma *mult-strict-mono'*:
$[|\ a<b;\ c<d;\ 0 \leq a;\ 0 \leq c|] \implies a * c < b * (d::'a::ordered\text{-}semiring\text{-}strict)$
⟨*proof*⟩

lemma *less-1-mult*: $[|\ 1 < m;\ 1 < n\ |] \implies 1 < m*(n::'a::ordered\text{-}semidom)$
⟨*proof*⟩

lemma *mult-less-le-imp-less*: $(a::'a::ordered\text{-}semiring\text{-}strict) < b \implies$
$c <= d \implies 0 <= a \implies 0 < c \implies a * c < b * d$
⟨*proof*⟩

lemma *mult-le-less-imp-less*: $(a::'a::ordered\text{-}semiring\text{-}strict) <= b ==>$
$c < d ==> 0 < a ==> 0 <= c ==> a * c < b * d$
⟨*proof*⟩

16.2 Cancellation Laws for Relationships With a Common Factor

Cancellation laws for $c * a < c * b$ and $a * c < b * c$, also with the relations $\leq$ and equality.

These "disjunction" versions produce two cases when the comparison is an assumption, but effectively four when the comparison is a goal.

lemma *mult-less-cancel-right-disj*:
$(a*c < b*c) = ((0 < c \ \& \ a < b) \mid (c < 0 \ \& \ b < (a::'a::ordered\text{-}ring\text{-}strict)))$
⟨*proof*⟩

lemma *mult-less-cancel-left-disj*:
$(c*a < c*b) = ((0 < c \ \& \ a < b) \mid (c < 0 \ \& \ b < (a::'a::ordered\text{-}ring\text{-}strict)))$
⟨*proof*⟩

The "conjunction of implication" lemmas produce two cases when the comparison is a goal, but give four when the comparison is an assumption.

lemma *mult-less-cancel-right*:
fixes $c :: 'a :: ordered\text{-}ring\text{-}strict$
shows $(a*c < b*c) = ((0 \leq c \ --> a < b) \ \& \ (c \leq 0 \ --> b < a))$
⟨*proof*⟩

lemma *mult-less-cancel-left*:
fixes $c :: 'a :: ordered\text{-}ring\text{-}strict$
shows $(c*a < c*b) = ((0 \leq c \ --> a < b) \ \& \ (c \leq 0 \ --> b < a))$
⟨*proof*⟩

lemma *mult-le-cancel-right*:
$(a*c \leq b*c) = ((0<c \ --> a\leq b) \ \& \ (c<0 \ --> b \leq (a::'a::ordered\text{-}ring\text{-}strict)))$
⟨*proof*⟩

lemma *mult-le-cancel-left*:
$(c*a \leq c*b) = ((0<c \ --> a\leq b) \ \& \ (c<0 \ --> b \leq (a::'a::ordered\text{-}ring\text{-}strict)))$
⟨*proof*⟩

lemma *mult-less-imp-less-left*:
assumes *less*: $c*a < c*b$ **and** *nonneg*: $0 \leq c$
shows $a < (b::'a::ordered\text{-}semiring\text{-}strict)$
⟨*proof*⟩

lemma *mult-less-imp-less-right*:
assumes *less*: $a*c < b*c$ **and** *nonneg*: $0 <= c$

shows $a < (b{::}'a{::}\mathit{ordered\text{-}semiring\text{-}strict})$
⟨*proof*⟩

Cancellation of equalities with a common factor

lemma *mult-cancel-right* [*simp*,*noatp*]:
fixes $a\ b\ c :: 'a{::}\mathit{ring\text{-}no\text{-}zero\text{-}divisors}$
shows $(a * c = b * c) = (c = 0 \vee a = b)$
⟨*proof*⟩

lemma *mult-cancel-left* [*simp*,*noatp*]:
fixes $a\ b\ c :: 'a{::}\mathit{ring\text{-}no\text{-}zero\text{-}divisors}$
shows $(c * a = c * b) = (c = 0 \vee a = b)$
⟨*proof*⟩

16.2.1 Special Cancellation Simprules for Multiplication

These also produce two cases when the comparison is a goal.

lemma *mult-le-cancel-right1*:
fixes $c :: 'a :: \mathit{ordered\text{-}idom}$
shows $(c \leq b{*}c) = ((0{<}c \dashrightarrow 1{\leq}b)\ \&\ (c{<}0 \dashrightarrow b \leq 1))$
⟨*proof*⟩

lemma *mult-le-cancel-right2*:
fixes $c :: 'a :: \mathit{ordered\text{-}idom}$
shows $(a{*}c \leq c) = ((0{<}c \dashrightarrow a{\leq}1)\ \&\ (c{<}0 \dashrightarrow 1 \leq a))$
⟨*proof*⟩

lemma *mult-le-cancel-left1*:
fixes $c :: 'a :: \mathit{ordered\text{-}idom}$
shows $(c \leq c{*}b) = ((0{<}c \dashrightarrow 1{\leq}b)\ \&\ (c{<}0 \dashrightarrow b \leq 1))$
⟨*proof*⟩

lemma *mult-le-cancel-left2*:
fixes $c :: 'a :: \mathit{ordered\text{-}idom}$
shows $(c{*}a \leq c) = ((0{<}c \dashrightarrow a{\leq}1)\ \&\ (c{<}0 \dashrightarrow 1 \leq a))$
⟨*proof*⟩

lemma *mult-less-cancel-right1*:
fixes $c :: 'a :: \mathit{ordered\text{-}idom}$
shows $(c < b{*}c) = ((0 \leq c \dashrightarrow 1{<}b)\ \&\ (c \leq 0 \dashrightarrow b < 1))$
⟨*proof*⟩

lemma *mult-less-cancel-right2*:
fixes $c :: 'a :: \mathit{ordered\text{-}idom}$
shows $(a{*}c < c) = ((0 \leq c \dashrightarrow a{<}1)\ \&\ (c \leq 0 \dashrightarrow 1 < a))$
⟨*proof*⟩

lemma *mult-less-cancel-left1*:
fixes $c :: 'a :: \mathit{ordered\text{-}idom}$

shows $(c < c*b) = ((0 \le c \dashrightarrow 1{<}b)\ \&\ (c \le 0 \dashrightarrow b < 1))$
⟨*proof*⟩

lemma *mult-less-cancel-left2*:
fixes $c :: 'a :: ordered\text{-}idom$
shows $(c*a < c) = ((0 \le c \dashrightarrow a{<}1)\ \&\ (c \le 0 \dashrightarrow 1 < a))$
⟨*proof*⟩

lemma *mult-cancel-right1* [*simp*]:
fixes $c :: 'a :: ring\text{-}1\text{-}no\text{-}zero\text{-}divisors$
shows $(c = b*c) = (c = 0 \mid b{=}1)$
⟨*proof*⟩

lemma *mult-cancel-right2* [*simp*]:
fixes $c :: 'a :: ring\text{-}1\text{-}no\text{-}zero\text{-}divisors$
shows $(a*c = c) = (c = 0 \mid a{=}1)$
⟨*proof*⟩

lemma *mult-cancel-left1* [*simp*]:
fixes $c :: 'a :: ring\text{-}1\text{-}no\text{-}zero\text{-}divisors$
shows $(c = c*b) = (c = 0 \mid b{=}1)$
⟨*proof*⟩

lemma *mult-cancel-left2* [*simp*]:
fixes $c :: 'a :: ring\text{-}1\text{-}no\text{-}zero\text{-}divisors$
shows $(c*a = c) = (c = 0 \mid a{=}1)$
⟨*proof*⟩

Simprules for comparisons where common factors can be cancelled.

lemmas *mult-compare-simps* =
mult-le-cancel-right mult-le-cancel-left
mult-le-cancel-right1 mult-le-cancel-right2
mult-le-cancel-left1 mult-le-cancel-left2
mult-less-cancel-right mult-less-cancel-left
mult-less-cancel-right1 mult-less-cancel-right2
mult-less-cancel-left1 mult-less-cancel-left2
mult-cancel-right mult-cancel-left
mult-cancel-right1 mult-cancel-right2
mult-cancel-left1 mult-cancel-left2

lemma *nonzero-imp-inverse-nonzero*:
$a \ne 0 \Longrightarrow inverse\ a \ne (0{::}'a{::}division\text{-}ring)$
⟨*proof*⟩

16.3 Basic Properties of *inverse*

lemma *inverse-zero-imp-zero*: $inverse\ a = 0 ==> a = (0::'a::division\text{-}ring)$
⟨*proof*⟩

lemma *inverse-nonzero-imp-nonzero*:
$inverse\ a = 0 ==> a = (0::'a::division\text{-}ring)$
⟨*proof*⟩

lemma *inverse-nonzero-iff-nonzero* [*simp*]:
$(inverse\ a = 0) = (a = (0::'a::\{division\text{-}ring,division\text{-}by\text{-}zero\}))$
⟨*proof*⟩

lemma *nonzero-inverse-minus-eq*:
assumes [*simp*]: $a\neq 0$
shows $inverse(-a) = -inverse(a::'a::division\text{-}ring)$
⟨*proof*⟩

lemma *inverse-minus-eq* [*simp*]:
$inverse(-a) = -inverse(a::'a::\{division\text{-}ring,division\text{-}by\text{-}zero\})$
⟨*proof*⟩

lemma *nonzero-inverse-eq-imp-eq*:
assumes *inveq*: $inverse\ a = inverse\ b$
and *anz*: $a \neq 0$
and *bnz*: $b \neq 0$
shows $a = (b::'a::division\text{-}ring)$
⟨*proof*⟩

lemma *inverse-eq-imp-eq*:
$inverse\ a = inverse\ b ==> a = (b::'a::\{division\text{-}ring,division\text{-}by\text{-}zero\})$
⟨*proof*⟩

lemma *inverse-eq-iff-eq* [*simp*]:
$(inverse\ a = inverse\ b) = (a = (b::'a::\{division\text{-}ring,division\text{-}by\text{-}zero\}))$
⟨*proof*⟩

lemma *nonzero-inverse-inverse-eq*:
assumes [*simp*]: $a \neq 0$
shows $inverse(inverse\ (a::'a::division\text{-}ring)) = a$
⟨*proof*⟩

lemma *inverse-inverse-eq* [*simp*]:
$inverse(inverse\ (a::'a::\{division\text{-}ring,division\text{-}by\text{-}zero\})) = a$
⟨*proof*⟩

lemma *inverse-1* [*simp*]: $inverse\ 1 = (1::'a::division\text{-}ring)$
⟨*proof*⟩

lemma *inverse-unique*:

assumes ab: $a*b = 1$
shows $inverse\ a = (b::'a::division\text{-}ring)$
$\langle proof \rangle$

lemma *nonzero-inverse-mult-distrib*:
assumes anz: $a \neq 0$
and bnz: $b \neq 0$
shows $inverse(a*b) = inverse(b) * inverse(a::'a::division\text{-}ring)$
$\langle proof \rangle$

This version builds in division by zero while also re-orienting the right-hand side.

lemma *inverse-mult-distrib* [*simp*]:
$inverse(a*b) = inverse(a) * inverse(b::'a::\{field,division\text{-}by\text{-}zero\})$
$\langle proof \rangle$

lemma *division-ring-inverse-add*:
$[|(a::'a::division\text{-}ring) \neq 0;\ b \neq 0|]$
$==> inverse\ a + inverse\ b = inverse\ a * (a+b) * inverse\ b$
$\langle proof \rangle$

lemma *division-ring-inverse-diff*:
$[|(a::'a::division\text{-}ring) \neq 0;\ b \neq 0|]$
$==> inverse\ a - inverse\ b = inverse\ a * (b-a) * inverse\ b$
$\langle proof \rangle$

There is no slick version using division by zero.

lemma *inverse-add*:
$[|a \neq 0;\ \ b \neq 0|]$
$==> inverse\ a + inverse\ b = (a+b) * inverse\ a * inverse\ (b::'a::field)$
$\langle proof \rangle$

lemma *inverse-divide* [*simp*]:
$inverse\ (a/b) = b\ /\ (a::'a::\{field,division\text{-}by\text{-}zero\})$
$\langle proof \rangle$

16.4 Calculations with fractions

There is a whole bunch of simp-rules just for class *field* but none for class *field* and *nonzero-divides* because the latter are covered by a simproc.

lemma *nonzero-mult-divide-mult-cancel-left*[*simp,noatp*]:
assumes [*simp*]: $b\neq 0$ **and** [*simp*]: $c\neq 0$ **shows** $(c*a)/(c*b) = a/(b::'a::field)$
$\langle proof \rangle$

lemma *mult-divide-mult-cancel-left*:
$c\neq 0 ==> (c*a)\ /\ (c*b) = a\ /\ (b::'a::\{field,division\text{-}by\text{-}zero\})$
$\langle proof \rangle$

lemma *nonzero-mult-divide-mult-cancel-right* [*noatp*]:
$[|b \neq 0;\ c \neq 0|] ==> (a*c)\ /\ (b*c) = a/(b::'a::field)$
⟨*proof*⟩

lemma *mult-divide-mult-cancel-right*:
$c \neq 0 ==> (a*c)\ /\ (b*c) = a\ /\ (b::'a::\{field,division\text{-}by\text{-}zero\})$
⟨*proof*⟩

lemma *divide-1* [*simp*]: $a/1 = (a::'a::field)$
⟨*proof*⟩

lemma *times-divide-eq-right*: $a * (b/c) = (a*b)\ /\ (c::'a::field)$
⟨*proof*⟩

lemma *times-divide-eq-left*: $(b/c) * a = (b*a)\ /\ (c::'a::field)$
⟨*proof*⟩

lemmas *times-divide-eq = times-divide-eq-right times-divide-eq-left*

lemma *divide-divide-eq-right* [*simp*,*noatp*]:
$a\ /\ (b/c) = (a*c)\ /\ (b::'a::\{field,division\text{-}by\text{-}zero\})$
⟨*proof*⟩

lemma *divide-divide-eq-left* [*simp*,*noatp*]:
$(a\ /\ b)\ /\ (c::'a::\{field,division\text{-}by\text{-}zero\}) = a\ /\ (b*c)$
⟨*proof*⟩

lemma *add-frac-eq*: $(y::'a::field)\ \tilde{}= 0 ==> z\ \tilde{}= 0 ==>$
$x\ /\ y + w\ /\ z = (x * z + w * y)\ /\ (y * z)$
⟨*proof*⟩

16.4.1 Special Cancellation Simprules for Division

lemma *mult-divide-mult-cancel-left-if*[*simp*,*noatp*]:
fixes $c :: 'a :: \{field,division\text{-}by\text{-}zero\}$
shows $(c*a)\ /\ (c*b) = (if\ c=0\ then\ 0\ else\ a/b)$
⟨*proof*⟩

lemma *nonzero-mult-divide-cancel-right*[*simp*,*noatp*]:
$b \neq 0 \Longrightarrow a * b\ /\ b = (a::'a::field)$
⟨*proof*⟩

lemma *nonzero-mult-divide-cancel-left*[*simp*,*noatp*]:
$a \neq 0 \Longrightarrow a * b\ /\ a = (b::'a::field)$
⟨*proof*⟩

lemma *nonzero-divide-mult-cancel-right*[*simp*,*noatp*]:
$[\![\ a \neq 0;\ b \neq 0\ ]\!] \Longrightarrow b\ /\ (a * b) = 1/(a::'a::field)$

⟨*proof*⟩

lemma *nonzero-divide-mult-cancel-left*[*simp*,*noatp*]:
$[\![\ a{\neq}0;\ b{\neq}0\ ]\!] \Longrightarrow a\ /\ (a * b) = 1/(b{::}'a{::}field)$
⟨*proof*⟩

lemma *nonzero-mult-divide-mult-cancel-left2*[*simp*,*noatp*]:
$[|b{\neq}0;\ c{\neq}0|] ==> (c{*}a)\ /\ (b{*}c) = a/(b{::}'a{::}field)$
⟨*proof*⟩

lemma *nonzero-mult-divide-mult-cancel-right2*[*simp*,*noatp*]:
$[|b{\neq}0;\ c{\neq}0|] ==> (a{*}c)\ /\ (c{*}b) = a/(b{::}'a{::}field)$
⟨*proof*⟩

16.5 Division and Unary Minus

lemma *nonzero-minus-divide-left*: $b \neq 0 ==> -\ (a/b) = (-a)\ /\ (b{::}'a{::}field)$
⟨*proof*⟩

lemma *nonzero-minus-divide-right*: $b \neq 0 ==> -\ (a/b) = a\ /\ -(b{::}'a{::}field)$
⟨*proof*⟩

lemma *nonzero-minus-divide-divide*: $b \neq 0 ==> (-a)/(-b) = a\ /\ (b{::}'a{::}field)$
⟨*proof*⟩

lemma *minus-divide-left*: $-\ (a/b) = (-a)\ /\ (b{::}'a{::}field)$
⟨*proof*⟩

lemma *minus-divide-right*: $-\ (a/b) = a\ /\ -(b{::}'a{::}\{field,division\text{-}by\text{-}zero\})$
⟨*proof*⟩

The effect is to extract signs from divisions

lemmas *divide-minus-left* = *minus-divide-left* [*symmetric*]
lemmas *divide-minus-right* = *minus-divide-right* [*symmetric*]
declare *divide-minus-left* [*simp*] *divide-minus-right* [*simp*]

Also, extract signs from products

lemmas *mult-minus-left* = *minus-mult-left* [*symmetric*]
lemmas *mult-minus-right* = *minus-mult-right* [*symmetric*]
declare *mult-minus-left* [*simp*] *mult-minus-right* [*simp*]

lemma *minus-divide-divide* [*simp*]:
$(-a)/(-b) = a\ /\ (b{::}'a{::}\{field,division\text{-}by\text{-}zero\})$
⟨*proof*⟩

lemma *diff-divide-distrib*: $(a{-}b)/(c{::}'a{::}field) = a/c - b/c$
⟨*proof*⟩

lemma *add-divide-eq-iff*:
$(z::'a::field) \neq 0 \Longrightarrow x + y/z = (z*x + y)/z$
⟨*proof*⟩

lemma *divide-add-eq-iff*:
$(z::'a::field) \neq 0 \Longrightarrow x/z + y = (x + z*y)/z$
⟨*proof*⟩

lemma *diff-divide-eq-iff*:
$(z::'a::field) \neq 0 \Longrightarrow x - y/z = (z*x - y)/z$
⟨*proof*⟩

lemma *divide-diff-eq-iff*:
$(z::'a::field) \neq 0 \Longrightarrow x/z - y = (x - z*y)/z$
⟨*proof*⟩

lemma *nonzero-eq-divide-eq*: $c\neq 0 ==> ((a::'a::field) = b/c) = (a*c = b)$
⟨*proof*⟩

lemma *nonzero-divide-eq-eq*: $c\neq 0 ==> (b/c = (a::'a::field)) = (b = a*c)$
⟨*proof*⟩

lemma *eq-divide-eq*:
$((a::'a::\{field,division\text{-}by\text{-}zero\}) = b/c) = (if\ c\neq 0\ then\ a*c = b\ else\ a=0)$
⟨*proof*⟩

lemma *divide-eq-eq*:
$(b/c = (a::'a::\{field,division\text{-}by\text{-}zero\})) = (if\ c\neq 0\ then\ b = a*c\ else\ a=0)$
⟨*proof*⟩

lemma *divide-eq-imp*: $(c::'a::\{division\text{-}by\text{-}zero,field\})\ \tilde{}= 0 ==>$
$b = a * c ==> b\ /\ c = a$
⟨*proof*⟩

lemma *eq-divide-imp*: $(c::'a::\{division\text{-}by\text{-}zero,field\})\ \tilde{}= 0 ==>$
$a * c = b ==> a = b\ /\ c$
⟨*proof*⟩

lemmas *field-eq-simps* = *ring-simps*

add-divide-eq-iff divide-add-eq-iff
diff-divide-eq-iff divide-diff-eq-iff

nonzero-eq-divide-eq nonzero-divide-eq-eq

An example:

lemma fixes $a\ b\ c\ d\ e\ f :: 'a::field$
shows $[\![a\neq b;\ c\neq d;\ e\neq f\ ]\!] \Longrightarrow ((a-b)*(c-d)*(e-f))/((c-d)*(e-f)*(a-b)) = 1$

⟨*proof*⟩

lemma *diff-frac-eq*: $(y::'a::field) \sim= 0 ==> z \sim= 0 ==>$
$x \,/\, y - w \,/\, z = (x * z - w * y) \,/\, (y * z)$
⟨*proof*⟩

lemma *frac-eq-eq*: $(y::'a::field) \sim= 0 ==> z \sim= 0 ==>$
$(x \,/\, y = w \,/\, z) = (x * z = w * y)$
⟨*proof*⟩

16.6 Ordered Fields

lemma *positive-imp-inverse-positive*:
assumes *a-gt-0*: $0 < a$ **shows** $0 < inverse\ (a::'a::ordered\text{-}field)$
⟨*proof*⟩

lemma *negative-imp-inverse-negative*:
$a < 0 ==> inverse\ a < (0::'a::ordered\text{-}field)$
⟨*proof*⟩

lemma *inverse-le-imp-le*:
assumes *invle*: $inverse\ a \le inverse\ b$ **and** *apos*: $0 < a$
shows $b \le (a::'a::ordered\text{-}field)$
⟨*proof*⟩

lemma *inverse-positive-imp-positive*:
assumes *inv-gt-0*: $0 < inverse\ a$ **and** *nz*: $a \neq 0$
shows $0 < (a::'a::ordered\text{-}field)$
⟨*proof*⟩

lemma *inverse-positive-iff-positive* [*simp*]:
$(0 < inverse\ a) = (0 < (a::'a::\{ordered\text{-}field,division\text{-}by\text{-}zero\}))$
⟨*proof*⟩

lemma *inverse-negative-imp-negative*:
assumes *inv-less-0*: $inverse\ a < 0$ **and** *nz*: $a \neq 0$
shows $a < (0::'a::ordered\text{-}field)$
⟨*proof*⟩

lemma *inverse-negative-iff-negative* [*simp*]:
$(inverse\ a < 0) = (a < (0::'a::\{ordered\text{-}field,division\text{-}by\text{-}zero\}))$
⟨*proof*⟩

lemma *inverse-nonnegative-iff-nonnegative* [*simp*]:
$(0 \le inverse\ a) = (0 \le (a::'a::\{ordered\text{-}field,division\text{-}by\text{-}zero\}))$
⟨*proof*⟩

lemma *inverse-nonpositive-iff-nonpositive* [*simp*]:

$(inverse\ a \le 0) = (a \le (0::'a::\{ordered\text{-}field,division\text{-}by\text{-}zero\}))$
⟨*proof*⟩

lemma *ordered-field-no-lb*: $\forall\ x.\ \exists y.\ y < (x::'a::ordered\text{-}field)$
⟨*proof*⟩

lemma *ordered-field-no-ub*: $\forall\ x.\ \exists y.\ y > (x::'a::ordered\text{-}field)$
⟨*proof*⟩

16.7 Anti-Monotonicity of *inverse*

lemma *less-imp-inverse-less*:
assumes *less*: $a < b$ **and** *apos*: $0 < a$
shows $inverse\ b < inverse\ (a::'a::ordered\text{-}field)$
⟨*proof*⟩

lemma *inverse-less-imp-less*:
$[|inverse\ a < inverse\ b;\ 0 < a|] ==> b < (a::'a::ordered\text{-}field)$
⟨*proof*⟩

Both premises are essential. Consider -1 and 1.

lemma *inverse-less-iff-less* [*simp*,*noatp*]:
$[|0 < a;\ 0 < b|] ==> (inverse\ a < inverse\ b) = (b < (a::'a::ordered\text{-}field))$
⟨*proof*⟩

lemma *le-imp-inverse-le*:
$[|a \le b;\ 0 < a|] ==> inverse\ b \le inverse\ (a::'a::ordered\text{-}field)$
⟨*proof*⟩

lemma *inverse-le-iff-le* [*simp*,*noatp*]:
$[|0 < a;\ 0 < b|] ==> (inverse\ a \le inverse\ b) = (b \le (a::'a::ordered\text{-}field))$
⟨*proof*⟩

These results refer to both operands being negative. The opposite-sign case is trivial, since inverse preserves signs.

lemma *inverse-le-imp-le-neg*:
$[|inverse\ a \le inverse\ b;\ b < 0|] ==> b \le (a::'a::ordered\text{-}field)$
⟨*proof*⟩

lemma *less-imp-inverse-less-neg*:
$[|a < b;\ b < 0|] ==> inverse\ b < inverse\ (a::'a::ordered\text{-}field)$
⟨*proof*⟩

lemma *inverse-less-imp-less-neg*:
$[|inverse\ a < inverse\ b;\ b < 0|] ==> b < (a::'a::ordered\text{-}field)$
⟨*proof*⟩

lemma *inverse-less-iff-less-neg* [*simp*,*noatp*]:
$[|a < 0;\ b < 0|] ==> (inverse\ a < inverse\ b) = (b < (a::'a::ordered\text{-}field))$

⟨*proof*⟩

lemma *le-imp-inverse-le-neg*:
$[|a \leq b;\ b < 0|] \implies inverse\ b \leq inverse\ (a::'a::ordered\text{-}field)$
⟨*proof*⟩

lemma *inverse-le-iff-le-neg* [*simp*,*noatp*]:
$[|a < 0;\ b < 0|] \implies (inverse\ a \leq inverse\ b) = (b \leq (a::'a::ordered\text{-}field))$
⟨*proof*⟩

16.8 Inverses and the Number One

lemma *one-less-inverse-iff*:
$(1 < inverse\ x) = (0 < x\ \&\ x < (1::'a::\{ordered\text{-}field,division\text{-}by\text{-}zero\}))$
⟨*proof*⟩

lemma *inverse-eq-1-iff* [*simp*]:
$(inverse\ x = 1) = (x = (1::'a::\{field,division\text{-}by\text{-}zero\}))$
⟨*proof*⟩

lemma *one-le-inverse-iff*:
$(1 \leq inverse\ x) = (0 < x\ \&\ x \leq (1::'a::\{ordered\text{-}field,division\text{-}by\text{-}zero\}))$
⟨*proof*⟩

lemma *inverse-less-1-iff*:
$(inverse\ x < 1) = (x \leq 0\ |\ 1 < (x::'a::\{ordered\text{-}field,division\text{-}by\text{-}zero\}))$
⟨*proof*⟩

lemma *inverse-le-1-iff*:
$(inverse\ x \leq 1) = (x \leq 0\ |\ 1 \leq (x::'a::\{ordered\text{-}field,division\text{-}by\text{-}zero\}))$
⟨*proof*⟩

16.9 Simplification of Inequalities Involving Literal Divisors

lemma *pos-le-divide-eq*: $0 < (c::'a::ordered\text{-}field) \implies (a \leq b/c) = (a*c \leq b)$
⟨*proof*⟩

lemma *neg-le-divide-eq*: $c < (0::'a::ordered\text{-}field) \implies (a \leq b/c) = (b \leq a*c)$
⟨*proof*⟩

lemma *le-divide-eq*:
$(a \leq b/c) =$
$(if\ 0 < c\ then\ a*c \leq b$
$\quad else\ if\ c < 0\ then\ b \leq a*c$
$\quad else\ \ a \leq (0::'a::\{ordered\text{-}field,division\text{-}by\text{-}zero\}))$
⟨*proof*⟩

lemma *pos-divide-le-eq*: $0 < (c::'a::ordered\text{-}field) \implies (b/c \leq a) = (b \leq a*c)$
⟨*proof*⟩

lemma *neg-divide-le-eq*: $c < (0::'a::ordered\text{-}field) ==> (b/c \le a) = (a*c \le b)$
⟨*proof*⟩

lemma *divide-le-eq*:
$(b/c \le a) =$
$(if\ 0 < c\ then\ b \le a*c$
$else\ if\ c < 0\ then\ a*c \le b$
$else\ 0 \le (a::'a::\{ordered\text{-}field,division\text{-}by\text{-}zero\}))$
⟨*proof*⟩

lemma *pos-less-divide-eq*:
$0 < (c::'a::ordered\text{-}field) ==> (a < b/c) = (a*c < b)$
⟨*proof*⟩

lemma *neg-less-divide-eq*:
$c < (0::'a::ordered\text{-}field) ==> (a < b/c) = (b < a*c)$
⟨*proof*⟩

lemma *less-divide-eq*:
$(a < b/c) =$
$(if\ 0 < c\ then\ a*c < b$
$else\ if\ c < 0\ then\ b < a*c$
$else\ \ a < (0::'a::\{ordered\text{-}field,division\text{-}by\text{-}zero\}))$
⟨*proof*⟩

lemma *pos-divide-less-eq*:
$0 < (c::'a::ordered\text{-}field) ==> (b/c < a) = (b < a*c)$
⟨*proof*⟩

lemma *neg-divide-less-eq*:
$c < (0::'a::ordered\text{-}field) ==> (b/c < a) = (a*c < b)$
⟨*proof*⟩

lemma *divide-less-eq*:
$(b/c < a) =$
$(if\ 0 < c\ then\ b < a*c$
$else\ if\ c < 0\ then\ a*c < b$
$else\ 0 < (a::'a::\{ordered\text{-}field,division\text{-}by\text{-}zero\}))$
⟨*proof*⟩

16.10 Field simplification

Lemmas *field-simps* multiply with denominators in in(equations) if they can be proved to be non-zero (for equations) or positive/negative (for inequations).

lemmas *field-simps* = *field-eq-simps*

pos-divide-less-eq neg-divide-less-eq
pos-less-divide-eq neg-less-divide-eq

pos-divide-le-eq neg-divide-le-eq
pos-le-divide-eq neg-le-divide-eq

Lemmas *sign-simps* is a first attempt to automate proofs of positivity/negativity needed for *field-simps*. Have not added *sign-simps* to *field-simps* because the former can lead to case explosions.

lemmas *sign-simps* = *group-simps*
zero-less-mult-iff mult-less-0-iff

16.11 Division and Signs

lemma *zero-less-divide-iff*:
$((0::'a::\{ordered\text{-}field,division\text{-}by\text{-}zero\}) < a/b) = (0 < a \;\&\; 0 < b \mid a < 0 \;\& \; b < 0)$
⟨*proof*⟩

lemma *divide-less-0-iff*:
$(a/b < (0::'a::\{ordered\text{-}field,division\text{-}by\text{-}zero\})) =$
$(0 < a \;\&\; b < 0 \mid a < 0 \;\&\; 0 < b)$
⟨*proof*⟩

lemma *zero-le-divide-iff*:
$((0::'a::\{ordered\text{-}field,division\text{-}by\text{-}zero\}) \leq a/b) =$
$(0 \leq a \;\&\; 0 \leq b \mid a \leq 0 \;\&\; b \leq 0)$
⟨*proof*⟩

lemma *divide-le-0-iff*:
$(a/b \leq (0::'a::\{ordered\text{-}field,division\text{-}by\text{-}zero\})) =$
$(0 \leq a \;\&\; b \leq 0 \mid a \leq 0 \;\&\; 0 \leq b)$
⟨*proof*⟩

lemma *divide-eq-0-iff* [*simp*,*noatp*]:
$(a/b = 0) = (a=0 \mid b=(0::'a::\{field,division\text{-}by\text{-}zero\}))$
⟨*proof*⟩

lemma *divide-pos-pos*:
$0 < (x::'a::ordered\text{-}field) ==> 0 < y ==> 0 < x / y$
⟨*proof*⟩

lemma *divide-nonneg-pos*:
$0 <= (x::'a::ordered\text{-}field) ==> 0 < y ==> 0 <= x / y$
⟨*proof*⟩

lemma *divide-neg-pos*:
$(x::'a::ordered\text{-}field) < 0 ==> 0 < y ==> x / y < 0$
⟨*proof*⟩

lemma *divide-nonpos-pos*:

$(x::'a::ordered\text{-}field) <= 0 ==> 0 < y ==> x \ / \ y <= 0$
⟨proof⟩

lemma *divide-pos-neg*:
$0 < (x::'a::ordered\text{-}field) ==> y < 0 ==> x \ / \ y < 0$
⟨proof⟩

lemma *divide-nonneg-neg*:
$0 <= (x::'a::ordered\text{-}field) ==> y < 0 ==> x \ / \ y <= 0$
⟨proof⟩

lemma *divide-neg-neg*:
$(x::'a::ordered\text{-}field) < 0 ==> y < 0 ==> 0 < x \ / \ y$
⟨proof⟩

lemma *divide-nonpos-neg*:
$(x::'a::ordered\text{-}field) <= 0 ==> y < 0 ==> 0 <= x \ / \ y$
⟨proof⟩

16.12 Cancellation Laws for Division

lemma *divide-cancel-right* [*simp*,*noatp*]:
$(a/c = b/c) = (c = 0 \mid a = (b::'a::\{field,division\text{-}by\text{-}zero\}))$
⟨proof⟩

lemma *divide-cancel-left* [*simp*,*noatp*]:
$(c/a = c/b) = (c = 0 \mid a = (b::'a::\{field,division\text{-}by\text{-}zero\}))$
⟨proof⟩

16.13 Division and the Number One

Simplify expressions equated with 1

lemma *divide-eq-1-iff* [*simp*,*noatp*]:
$(a/b = 1) = (b \neq 0 \ \& \ a = (b::'a::\{field,division\text{-}by\text{-}zero\}))$
⟨proof⟩

lemma *one-eq-divide-iff* [*simp*,*noatp*]:
$(1 = a/b) = (b \neq 0 \ \& \ a = (b::'a::\{field,division\text{-}by\text{-}zero\}))$
⟨proof⟩

lemma *zero-eq-1-divide-iff* [*simp*,*noatp*]:
$((0::'a::\{ordered\text{-}field,division\text{-}by\text{-}zero\}) = 1/a) = (a = 0)$
⟨proof⟩

lemma *one-divide-eq-0-iff* [*simp*,*noatp*]:
$(1/a = (0::'a::\{ordered\text{-}field,division\text{-}by\text{-}zero\})) = (a = 0)$
⟨proof⟩

Simplify expressions such as $0 < 1/x$ to $0 < x$

lemmas *zero-less-divide-1-iff* = *zero-less-divide-iff* [*of 1*, *simplified*]
lemmas *divide-less-0-1-iff* = *divide-less-0-iff* [*of 1*, *simplified*]
lemmas *zero-le-divide-1-iff* = *zero-le-divide-iff* [*of 1*, *simplified*]
lemmas *divide-le-0-1-iff* = *divide-le-0-iff* [*of 1*, *simplified*]

declare *zero-less-divide-1-iff* [*simp*]
declare *divide-less-0-1-iff* [*simp*,*noatp*]
declare *zero-le-divide-1-iff* [*simp*]
declare *divide-le-0-1-iff* [*simp*,*noatp*]

16.14 Ordering Rules for Division

lemma *divide-strict-right-mono*:
$[|a < b;\ 0 < c|] ==> a\ /\ c < b\ /\ (c::'a::ordered\text{-}field)$
⟨*proof*⟩

lemma *divide-right-mono*:
$[|a \leq b;\ 0 \leq c|] ==> a/c \leq b/(c::'a::\{ordered\text{-}field,division\text{-}by\text{-}zero\})$
⟨*proof*⟩

lemma *divide-right-mono-neg*: $(a::'a::\{division\text{-}by\text{-}zero,ordered\text{-}field\}) <= b$
$==> c <= 0 ==> b\ /\ c <= a\ /\ c$
⟨*proof*⟩

lemma *divide-strict-right-mono-neg*:
$[|b < a;\ c < 0|] ==> a\ /\ c < b\ /\ (c::'a::ordered\text{-}field)$
⟨*proof*⟩

The last premise ensures that a and b have the same sign

lemma *divide-strict-left-mono*:
$[|b < a;\ 0 < c;\ 0 < a*b|] ==> c\ /\ a < c\ /\ (b::'a::ordered\text{-}field)$
⟨*proof*⟩

lemma *divide-left-mono*:
$[|b \leq a;\ 0 \leq c;\ 0 < a*b|] ==> c\ /\ a \leq c\ /\ (b::'a::ordered\text{-}field)$
⟨*proof*⟩

lemma *divide-left-mono-neg*: $(a::'a::\{division\text{-}by\text{-}zero,ordered\text{-}field\}) <= b$
$==> c <= 0 ==> 0 < a * b ==> c\ /\ a <= c\ /\ b$
⟨*proof*⟩

lemma *divide-strict-left-mono-neg*:
$[|a < b;\ c < 0;\ 0 < a*b|] ==> c\ /\ a < c\ /\ (b::'a::ordered\text{-}field)$
⟨*proof*⟩

Simplify quotients that are compared with the value 1.

lemma *le-divide-eq-1* [*noatp*]:
fixes $a :: 'a :: \{ordered\text{-}field,division\text{-}by\text{-}zero\}$
shows $(1 \leq b\ /\ a) = ((0 < a\ \&\ a \leq b) \mid (a < 0\ \&\ b \leq a))$

⟨*proof*⟩

lemma *divide-le-eq-1* [*noatp*]:
fixes $a :: 'a :: \{ordered\text{-}field, division\text{-}by\text{-}zero\}$
shows $(b\ /\ a \le 1) = ((0 < a \ \&\ b \le a) \mid (a < 0 \ \&\ a \le b) \mid a{=}0)$
⟨*proof*⟩

lemma *less-divide-eq-1* [*noatp*]:
fixes $a :: 'a :: \{ordered\text{-}field, division\text{-}by\text{-}zero\}$
shows $(1 < b\ /\ a) = ((0 < a \ \&\ a < b) \mid (a < 0 \ \&\ b < a))$
⟨*proof*⟩

lemma *divide-less-eq-1* [*noatp*]:
fixes $a :: 'a :: \{ordered\text{-}field, division\text{-}by\text{-}zero\}$
shows $(b\ /\ a < 1) = ((0 < a \ \&\ b < a) \mid (a < 0 \ \&\ a < b) \mid a{=}0)$
⟨*proof*⟩

16.15 Conditional Simplification Rules: No Case Splits

lemma *le-divide-eq-1-pos* [*simp*,*noatp*]:
fixes $a :: 'a :: \{ordered\text{-}field, division\text{-}by\text{-}zero\}$
shows $0 < a \Longrightarrow (1 \le b/a) = (a \le b)$
⟨*proof*⟩

lemma *le-divide-eq-1-neg* [*simp*,*noatp*]:
fixes $a :: 'a :: \{ordered\text{-}field, division\text{-}by\text{-}zero\}$
shows $a < 0 \Longrightarrow (1 \le b/a) = (b \le a)$
⟨*proof*⟩

lemma *divide-le-eq-1-pos* [*simp*,*noatp*]:
fixes $a :: 'a :: \{ordered\text{-}field, division\text{-}by\text{-}zero\}$
shows $0 < a \Longrightarrow (b/a \le 1) = (b \le a)$
⟨*proof*⟩

lemma *divide-le-eq-1-neg* [*simp*,*noatp*]:
fixes $a :: 'a :: \{ordered\text{-}field, division\text{-}by\text{-}zero\}$
shows $a < 0 \Longrightarrow (b/a \le 1) = (a \le b)$
⟨*proof*⟩

lemma *less-divide-eq-1-pos* [*simp*,*noatp*]:
fixes $a :: 'a :: \{ordered\text{-}field, division\text{-}by\text{-}zero\}$
shows $0 < a \Longrightarrow (1 < b/a) = (a < b)$
⟨*proof*⟩

lemma *less-divide-eq-1-neg* [*simp*,*noatp*]:
fixes $a :: 'a :: \{ordered\text{-}field, division\text{-}by\text{-}zero\}$
shows $a < 0 \Longrightarrow (1 < b/a) = (b < a)$
⟨*proof*⟩

lemma *divide-less-eq-1-pos* [*simp*,*noatp*]:
  fixes a :: $'a$ :: {*ordered-field*,*division-by-zero*}
  shows $0 < a \Longrightarrow (b/a < 1) = (b < a)$
⟨*proof*⟩

lemma *divide-less-eq-1-neg* [*simp*,*noatp*]:
  fixes a :: $'a$ :: {*ordered-field*,*division-by-zero*}
  shows $a < 0 \Longrightarrow b/a < 1 <-> a < b$
⟨*proof*⟩

lemma *eq-divide-eq-1* [*simp*,*noatp*]:
  fixes a :: $'a$ :: {*ordered-field*,*division-by-zero*}
  shows $(1 = b/a) = ((a \neq 0\ \&\ a = b))$
⟨*proof*⟩

lemma *divide-eq-eq-1* [*simp*,*noatp*]:
  fixes a :: $'a$ :: {*ordered-field*,*division-by-zero*}
  shows $(b/a = 1) = ((a \neq 0\ \&\ a = b))$
⟨*proof*⟩

16.16 Reasoning about inequalities with division

lemma *mult-right-le-one-le*: $0 <= (x{::}'a{::}ordered\text{-}idom) ==> 0 <= y ==> y <= 1$
  $==> x * y <= x$
  ⟨*proof*⟩

lemma *mult-left-le-one-le*: $0 <= (x{::}'a{::}ordered\text{-}idom) ==> 0 <= y ==> y <= 1$
  $==> y * x <= x$
  ⟨*proof*⟩

lemma *mult-imp-div-pos-le*: $0 < (y{::}'a{::}ordered\text{-}field) ==> x <= z * y ==>$
  $x\ /\ y <= z$
  ⟨*proof*⟩

lemma *mult-imp-le-div-pos*: $0 < (y{::}'a{::}ordered\text{-}field) ==> z * y <= x ==>$
  $z <= x\ /\ y$
⟨*proof*⟩

lemma *mult-imp-div-pos-less*: $0 < (y{::}'a{::}ordered\text{-}field) ==> x < z * y ==>$
  $x\ /\ y < z$
⟨*proof*⟩

lemma *mult-imp-less-div-pos*: $0 < (y{::}'a{::}ordered\text{-}field) ==> z * y < x ==>$
  $z < x\ /\ y$
⟨*proof*⟩

lemma *frac-le*: $(0{::}'a{::}ordered\text{-}field) <= x ==>$

  x <= y ==> 0 < w ==> w <= z ==> x / z <= y / w
 ⟨*proof*⟩

lemma *frac-less*: (*0*::*'a*::*ordered-field*) <= *x* ==>
  x < y ==> 0 < w ==> w <= z ==> x / z < y / w
 ⟨*proof*⟩

lemma *frac-less2*: (*0*::*'a*::*ordered-field*) < *x* ==>
  x <= y ==> 0 < w ==> w < z ==> x / z < y / w
 ⟨*proof*⟩

It's not obvious whether these should be simprules or not. Their effect is to gather terms into one big fraction, like a*b*c / x*y*z. The rationale for that is unclear, but many proofs seem to need them.

declare *times-divide-eq* [*simp*]

16.17 Ordered Fields are Dense

context *ordered-semidom*
begin

lemma *less-add-one*: *a < a + 1*
⟨*proof*⟩

lemma *zero-less-two*: *0 < 1 + 1*
 ⟨*proof*⟩

end

lemma *less-half-sum*: *a < b ==> a < (a+b) / (1+1::'a::ordered-field)*
⟨*proof*⟩

lemma *gt-half-sum*: *a < b ==> (a+b)/(1+1::'a::ordered-field) < b*
⟨*proof*⟩

instance *ordered-field < dense-linear-order*
⟨*proof*⟩

16.18 Absolute Value

context *ordered-idom*
begin

lemma *mult-sgn-abs*: *sgn x * abs x = x*
 ⟨*proof*⟩

end

lemma *abs-one* [*simp*]: *abs 1 = (1::'a::ordered-idom)*

⟨*proof*⟩

class *pordered-ring-abs* = *pordered-ring* + *pordered-ab-group-add-abs* +
assumes *abs-eq-mult*:
$(0 \le a \lor a \le 0) \land (0 \le b \lor b \le 0) \Longrightarrow |a * b| = |a| * |b|$

class *lordered-ring* = *pordered-ring* + *lordered-ab-group-add-abs*
begin

subclass *lordered-ab-group-add-meet* ⟨*proof*⟩
subclass *lordered-ab-group-add-join* ⟨*proof*⟩

end

lemma *abs-le-mult*: $abs\ (a * b) \le (abs\ a) * (abs\ (b{::}'a{::}lordered\text{-}ring))$
⟨*proof*⟩

instance *lordered-ring* ⊆ *pordered-ring-abs*
⟨*proof*⟩

instance *ordered-idom* ⊆ *pordered-ring-abs*
⟨*proof*⟩

lemma *abs-mult*: $abs\ (a * b) = abs\ a * abs\ (b{::}'a{::}ordered\text{-}idom)$
⟨*proof*⟩

lemma *abs-mult-self*: $abs\ a * abs\ a = a * (a{::}'a{::}ordered\text{-}idom)$
⟨*proof*⟩

lemma *nonzero-abs-inverse*:
$a \ne 0 ==> abs\ (inverse\ (a{::}'a{::}ordered\text{-}field)) = inverse\ (abs\ a)$
⟨*proof*⟩

lemma *abs-inverse* [*simp*]:
$abs\ (inverse\ (a{::}'a{::}\{ordered\text{-}field,division\text{-}by\text{-}zero\})) =$
$inverse\ (abs\ a)$
⟨*proof*⟩

lemma *nonzero-abs-divide*:
$b \ne 0 ==> abs\ (a\ /\ (b{::}'a{::}ordered\text{-}field)) = abs\ a\ /\ abs\ b$
⟨*proof*⟩

lemma *abs-divide* [*simp*]:
$abs\ (a\ /\ (b{::}'a{::}\{ordered\text{-}field,division\text{-}by\text{-}zero\})) = abs\ a\ /\ abs\ b$
⟨*proof*⟩

lemma *abs-mult-less*:
$[|\ abs\ a < c;\ abs\ b < d\ |] ==> abs\ a * abs\ b < c*(d{::}'a{::}ordered\text{-}idom)$

⟨*proof*⟩

lemmas *eq-minus-self-iff* = *equal-neg-zero*

lemma *less-minus-self-iff*: $(a < -a) = (a < (0::'a::ordered\text{-}idom))$
⟨*proof*⟩

lemma *abs-less-iff*: $(abs\ a < b) = (a < b\ \&\ -a < (b::'a::ordered\text{-}idom))$
⟨*proof*⟩

lemma *abs-mult-pos*: $(0::'a::ordered\text{-}idom) <= x ==>$
$(abs\ y) * x = abs\ (y * x)$
⟨*proof*⟩

lemma *abs-div-pos*: $(0::'a::\{division\text{-}by\text{-}zero,ordered\text{-}field\}) < y ==>$
$abs\ x\ /\ y = abs\ (x\ /\ y)$
⟨*proof*⟩

16.19 Bounds of products via negative and positive Part

lemma *mult-le-prts*:
assumes
$a1 <= (a::'a::lordered\text{-}ring)$
$a <= a2$
$b1 <= b$
$b <= b2$
shows
$a * b <= pprt\ a2 * pprt\ b2 + pprt\ a1 * nprt\ b2 + nprt\ a2 * pprt\ b1 + nprt\ a1 * nprt\ b1$
⟨*proof*⟩

lemma *mult-ge-prts*:
assumes
$a1 <= (a::'a::lordered\text{-}ring)$
$a <= a2$
$b1 <= b$
$b <= b2$
shows
$a * b >= nprt\ a1 * pprt\ b2 + nprt\ a2 * nprt\ b2 + pprt\ a1 * pprt\ b1 + pprt\ a2 * nprt\ b1$
⟨*proof*⟩

end

17 Nat: Natural numbers

theory *Nat*
imports *Wellfounded-Recursion Ring-and-Field*

uses
  ~~/src/Tools/rat.ML
  ~~/src/Provers/Arith/cancel-sums.ML
  (*arith-data.ML*)
  ~~/src/Provers/Arith/fast-lin-arith.ML
  (*Tools/lin-arith.ML*)
  (*Tools/function-package/size.ML*)
begin

17.1 Type *ind*

typedecl *ind*

axiomatization
  Zero-Rep :: *ind* **and**
  Suc-Rep :: *ind* => *ind*
where
  — the axiom of infinity in 2 parts
  inj-Suc-Rep: *inj Suc-Rep* **and**
  Suc-Rep-not-Zero-Rep: *Suc-Rep* $x \neq$ *Zero-Rep*

17.2 Type nat

Type definition

inductive-set *Nat* :: *ind set*
where
  Zero-RepI: *Zero-Rep* : *Nat*
  | *Suc-RepI*: *i* : *Nat* ==> *Suc-Rep i* : *Nat*

global

typedef (**open** *Nat*)
  nat = *Nat*
⟨*proof*⟩

consts
  Suc :: *nat* => *nat*

local

instance *nat* :: *zero*
  Zero-nat-def: *0* == *Abs-Nat Zero-Rep* ⟨*proof*⟩
lemmas [*code func del*] = *Zero-nat-def*

defs
  Suc-def: *Suc* == (%*n*. *Abs-Nat* (*Suc-Rep* (*Rep-Nat n*)))

theorem *nat-induct*: *P 0* ==> (!!*n*. *P n* ==> *P* (*Suc n*)) ==> *P n*
  ⟨*proof*⟩

lemma *Suc-not-Zero* [*iff*]: *Suc m* $\neq$ *0*
⟨*proof*⟩

lemma *Zero-not-Suc* [*iff*]: *0* $\neq$ *Suc m*
⟨*proof*⟩

lemma *inj-Suc*[*simp*]: *inj-on Suc N*
⟨*proof*⟩

lemma *Suc-Suc-eq* [*iff*]: (*Suc m* = *Suc n*) = (*m* = *n*)
⟨*proof*⟩

rep-datatype *nat*
distinct *Suc-not-Zero Zero-not-Suc*
inject *Suc-Suc-eq*
induction *nat-induct*

declare *nat.induct* [*case-names 0 Suc*, *induct type*: *nat*]
declare *nat.exhaust* [*case-names 0 Suc*, *cases type*: *nat*]

lemmas *nat-rec-0* = *nat.recs*(*1*)
and *nat-rec-Suc* = *nat.recs*(*2*)

lemmas *nat-case-0* = *nat.cases*(*1*)
and *nat-case-Suc* = *nat.cases*(*2*)

Injectiveness and distinctness lemmas

lemma *Suc-neq-Zero*: *Suc m* = *0* ==> *R*
⟨*proof*⟩

lemma *Zero-neq-Suc*: *0* = *Suc m* ==> *R*
⟨*proof*⟩

lemma *Suc-inject*: *Suc x* = *Suc y* ==> *x* = *y*
⟨*proof*⟩

lemma *nat-not-singleton*: (∀ *x*. *x* = (*0*::*nat*)) = *False*
⟨*proof*⟩

lemma *n-not-Suc-n*: *n* $\neq$ *Suc n*
⟨*proof*⟩

lemma *Suc-n-not-n*: *Suc t* $\neq$ *t*
⟨*proof*⟩

A special form of induction for reasoning about $m < n$ and $m - n$

theorem *diff-induct*: (!!*x*. *P x 0*) ==> (!!*y*. *P 0* (*Suc y*)) ==>
(!!*x y*. *P x y* ==> *P* (*Suc x*) (*Suc y*)) ==> *P m n*

⟨*proof*⟩

17.3 Arithmetic operators

instance *nat* :: {*one*, *plus*, *minus*, *times*}
One-nat-def [*simp*]: *1* == *Suc 0* ⟨*proof*⟩

primrec
add-0: *0* + *n* = *n*
add-Suc: *Suc m* + *n* = *Suc* (*m* + *n*)

primrec
diff-0: *m* − *0* = *m*
diff-Suc: *m* − *Suc n* = (*case m* − *n of 0* => *0* | *Suc k* => *k*)

primrec
mult-0: *0* ∗ *n* = *0*
mult-Suc: *Suc m* ∗ *n* = *n* + (*m* ∗ *n*)

17.4 Orders on *nat*

definition
pred-nat :: (*nat* ∗ *nat*) *set* **where**
pred-nat = {(*m*, *n*). *n* = *Suc m*}

instance *nat* :: *ord*
less-def: *m* < *n* == (*m*, *n*) : *pred-nat*ˆ+
le-def: *m* ≤ (*n*::*nat*) == ~ (*n* < *m*) ⟨*proof*⟩

lemmas [*code func del*] = *less-def le-def*

lemma *wf-pred-nat*: *wf pred-nat*
⟨*proof*⟩

lemma *wf-less*: *wf* {(*x*, *y*::*nat*). *x* < *y*}
⟨*proof*⟩

lemma *less-eq*: ((*m*, *n*) : *pred-nat*ˆ+) = (*m* < *n*)
⟨*proof*⟩

17.4.1 Introduction properties

lemma *less-trans*: *i* < *j* ==> *j* < *k* ==> *i* < (*k*::*nat*)
⟨*proof*⟩

lemma *lessI* [*iff*]: *n* < *Suc n*
⟨*proof*⟩

lemma *less-SucI*: *i* < *j* ==> *i* < *Suc j*
⟨*proof*⟩

lemma *zero-less-Suc* [*iff*]: *0* < *Suc n*
⟨*proof*⟩

17.4.2 Elimination properties

lemma *less-not-sym*: *n* < *m* ==> ~ *m* < (*n*::*nat*)
⟨*proof*⟩

lemma *less-asym*:
assumes *h1*: (*n*::*nat*) < *m* **and** *h2*: ~ *P* ==> *m* < *n* **shows** *P*
⟨*proof*⟩

lemma *less-not-refl*: ~ *n* < (*n*::*nat*)
⟨*proof*⟩

lemma *less-irrefl* [*elim*!]: (*n*::*nat*) < *n* ==> *R*
⟨*proof*⟩

lemma *less-not-refl2*: *n* < *m* ==> *m* ≠ (*n*::*nat*) ⟨*proof*⟩

lemma *less-not-refl3*: (*s*::*nat*) < *t* ==> *s* ≠ *t*
⟨*proof*⟩

lemma *lessE*:
assumes *major*: *i* < *k*
and *p1*: *k* = *Suc i* ==> *P* **and** *p2*: !!*j*. *i* < *j* ==> *k* = *Suc j* ==> *P*
shows *P*
⟨*proof*⟩

lemma *not-less0* [*iff*]: ~ *n* < (*0*::*nat*)
⟨*proof*⟩

lemma *less-zeroE*: (*n*::*nat*) < *0* ==> *R*
⟨*proof*⟩

lemma *less-SucE*: **assumes** *major*: *m* < *Suc n*
and *less*: *m* < *n* ==> *P* **and** *eq*: *m* = *n* ==> *P* **shows** *P*
⟨*proof*⟩

lemma *less-Suc-eq*: (*m* < *Suc n*) = (*m* < *n* | *m* = *n*)
⟨*proof*⟩

lemma *less-one* [*iff*,*noatp*]: (*n* < (*1*::*nat*)) = (*n* = *0*)
⟨*proof*⟩

lemma *less-Suc0* [*iff*]: (*n* < *Suc 0*) = (*n* = *0*)
⟨*proof*⟩

lemma *Suc-mono*: $m < n ==> Suc\ m < Suc\ n$
⟨*proof*⟩

"Less than" is a linear ordering

lemma *less-linear*: $m < n \mid m = n \mid n < (m::nat)$
⟨*proof*⟩

"Less than" is antisymmetric, sort of

lemma *less-antisym*: $[\![\neg\ n < m;\ n < Suc\ m]\!] \Longrightarrow m = n$
⟨*proof*⟩

lemma *nat-neq-iff*: $((m::nat) \neq n) = (m < n \mid n < m)$
⟨*proof*⟩

lemma *nat-less-cases*: **assumes** *major*: $(m::nat) < n ==> P\ n\ m$
and *eqCase*: $m = n ==> P\ n\ m$ **and** *lessCase*: $n<m ==> P\ n\ m$
shows $P\ n\ m$
⟨*proof*⟩

17.4.3 Inductive (?) properties

lemma *Suc-lessI*: $m < n ==> Suc\ m \neq n ==> Suc\ m < n$
⟨*proof*⟩

lemma *Suc-lessD*: $Suc\ m < n ==> m < n$
⟨*proof*⟩

lemma *Suc-lessE*: **assumes** *major*: $Suc\ i < k$
and *minor*: $!!j.\ i < j ==> k = Suc\ j ==> P$ **shows** P
⟨*proof*⟩

lemma *Suc-less-SucD*: $Suc\ m < Suc\ n ==> m < n$
⟨*proof*⟩

lemma *Suc-less-eq* [*iff*, *code*]: $(Suc\ m < Suc\ n) = (m < n)$
⟨*proof*⟩

lemma *less-trans-Suc*:
assumes *le*: $i < j$ **shows** $j < k ==> Suc\ i < k$
⟨*proof*⟩

lemma [*code*]: $((n::nat) < 0) = False$ ⟨*proof*⟩
lemma [*code*]: $(0 < Suc\ n) = True$ ⟨*proof*⟩

Can be used with *less-Suc-eq* to get $n = m \vee n < m$

lemma *not-less-eq*: $(\sim m < n) = (n < Suc\ m)$
⟨*proof*⟩

Complete induction, aka course-of-values induction

lemma *nat-less-induct*:
assumes *prem*: $!!n.\ \forall\, m{::}nat.\ m < n \dashrightarrow P\ m \Longrightarrow P\ n$ **shows** $P\ n$
⟨*proof*⟩

lemmas *less-induct* = *nat-less-induct* [*rule-format*, *case-names less*]

Properties of "less than or equal"

Was *le-eq-less-Suc*, but this orientation is more useful

lemma *less-Suc-eq-le*: $(m < Suc\ n) = (m \leq n)$
⟨*proof*⟩

lemma *le-imp-less-Suc*: $m \leq n \Longrightarrow m < Suc\ n$
⟨*proof*⟩

lemma *le0* [*iff*]: $(0{::}nat) \leq n$
⟨*proof*⟩

lemma *Suc-n-not-le-n*: $\sim\ Suc\ n \leq n$
⟨*proof*⟩

lemma *le-0-eq* [*iff*]: $((i{::}nat) \leq 0) = (i = 0)$
⟨*proof*⟩

lemma *le-Suc-eq*: $(m \leq Suc\ n) = (m \leq n \mid m = Suc\ n)$
⟨*proof*⟩

lemma *le-SucE*: $m \leq Suc\ n \Longrightarrow (m \leq n \Longrightarrow R) \Longrightarrow (m = Suc\ n \Longrightarrow R) \Longrightarrow R$
⟨*proof*⟩

lemma *Suc-leI*: $m < n \Longrightarrow Suc(m) \leq n$
⟨*proof*⟩

lemma *Suc-leD*: $Suc(m) \leq n \Longrightarrow m \leq n$
⟨*proof*⟩

Stronger version of *Suc-leD*

lemma *Suc-le-lessD*: $Suc\ m \leq n \Longrightarrow m < n$
⟨*proof*⟩

lemma *Suc-le-eq*: $(Suc\ m \leq n) = (m < n)$
⟨*proof*⟩

lemma *le-SucI*: $m \leq n \Longrightarrow m \leq Suc\ n$
⟨*proof*⟩

lemma *less-imp-le*: $m < n \Longrightarrow m \leq (n{::}nat)$
⟨*proof*⟩

For instance, $(Suc\ m < Suc\ n) = (Suc\ m \leq n) = (m < n)$

lemmas *le-simps* = *less-imp-le less-Suc-eq-le Suc-le-eq*

Equivalence of $m \leq n$ and $m < n \lor m = n$

lemma *le-imp-less-or-eq*: $m \leq n ==> m < n \mid m = (n::nat)$
⟨*proof*⟩

lemma *less-or-eq-imp-le*: $m < n \mid m = n ==> m \leq (n::nat)$
⟨*proof*⟩

lemma *le-eq-less-or-eq*: $(m \leq (n::nat)) = (m < n \mid m{=}n)$
⟨*proof*⟩

Useful with *blast*.

lemma *eq-imp-le*: $(m::nat) = n ==> m \leq n$
⟨*proof*⟩

lemma *le-refl*: $n \leq (n::nat)$
⟨*proof*⟩

lemma *le-less-trans*: $[\mid i \leq j;\ j < k \mid] ==> i < (k::nat)$
⟨*proof*⟩

lemma *less-le-trans*: $[\mid i < j;\ j \leq k \mid] ==> i < (k::nat)$
⟨*proof*⟩

lemma *le-trans*: $[\mid i \leq j;\ j \leq k \mid] ==> i \leq (k::nat)$
⟨*proof*⟩

lemma *le-anti-sym*: $[\mid m \leq n;\ n \leq m \mid] ==> m = (n::nat)$
⟨*proof*⟩

lemma *Suc-le-mono* [*iff*]: $(Suc\ n \leq Suc\ m) = (n \leq m)$
⟨*proof*⟩

Axiom *order-less-le* of class *order*:

lemma *nat-less-le*: $((m::nat) < n) = (m \leq n\ \&\ m \neq n)$
⟨*proof*⟩

lemma *le-neq-implies-less*: $(m::nat) \leq n ==> m \neq n ==> m < n$
⟨*proof*⟩

Axiom *linorder-linear* of class *linorder*:

lemma *nat-le-linear*: $(m::nat) \leq n \mid n \leq m$
⟨*proof*⟩

Type *nat* is a wellfounded linear order

instance *nat* :: *wellorder*

⟨*proof*⟩

lemmas *linorder-neqE-nat* = *linorder-neqE* [**where** *'a* = *nat*]

lemma *not-less-less-Suc-eq*: ~ *n* < *m* ==> (*n* < *Suc m*) = (*n* = *m*)
⟨*proof*⟩

Rewrite $n < Suc\ m$ to $n = m$ if $\neg\ n < m$ or $m \leq n$ hold. Not suitable as default simprules because they often lead to looping

lemma *le-less-Suc-eq*: *m* ≤ *n* ==> (*n* < *Suc m*) = (*n* = *m*)
⟨*proof*⟩

lemmas *not-less-simps* = *not-less-less-Suc-eq le-less-Suc-eq*

Re-orientation of the equations $0 = x$ and $1 = x$. No longer added as simprules (they loop) but via *reorient-simproc* in Bin

Polymorphic, not just for *nat*

lemma *zero-reorient*: (*0* = *x*) = (*x* = *0*)
⟨*proof*⟩

lemma *one-reorient*: (*1* = *x*) = (*x* = *1*)
⟨*proof*⟩

These two rules ease the use of primitive recursion. NOTE USE OF ==

lemma *def-nat-rec-0*: (!!*n*. *f n* == *nat-rec c h n*) ==> *f 0* = *c*
⟨*proof*⟩

lemma *def-nat-rec-Suc*: (!!*n*. *f n* == *nat-rec c h n*) ==> *f* (*Suc n*) = *h n* (*f n*)
⟨*proof*⟩

lemma *not0-implies-Suc*: *n* ≠ *0* ==> ∃ *m*. *n* = *Suc m*
⟨*proof*⟩

lemma *gr0-implies-Suc*: *n* > *0* ==> ∃ *m*. *n* = *Suc m*
⟨*proof*⟩

lemma *gr-implies-not0*: **fixes** *n* :: *nat* **shows** *m*<*n* ==> *n* ≠ *0*
⟨*proof*⟩

lemma *neq0-conv*[*iff*]: **fixes** *n* :: *nat* **shows** (*n* ≠ *0*) = (*0* < *n*)
⟨*proof*⟩

This theorem is useful with *blast*

lemma *gr0I*: ((*n*::*nat*) = *0* ==> *False*) ==> *0* < *n*
⟨*proof*⟩

lemma *gr0-conv-Suc*: (*0* < *n*) = (∃ *m*. *n* = *Suc m*)

⟨*proof*⟩

lemma *not-gr0* [*iff*,*noatp*]: $!!n::nat.\ (\sim\ (0 < n)) = (n = 0)$
⟨*proof*⟩

lemma *Suc-le-D*: $(Suc\ n \le m') ==> (?\ m.\ m' = Suc\ m)$
⟨*proof*⟩

Useful in certain inductive arguments

lemma *less-Suc-eq-0-disj*: $(m < Suc\ n) = (m = 0 \mid (\exists j.\ m = Suc\ j\ \&\ j < n))$
⟨*proof*⟩

lemma *nat-induct2*: $[|P\ 0;\ P\ (Suc\ 0);\ !!k.\ P\ k ==> P\ (Suc\ (Suc\ k))|] ==> P\ n$
⟨*proof*⟩

17.5 *LEAST* theorems for type *nat*

lemma *Least-Suc*:
$[|\ P\ n;\ \sim\ P\ 0\ |] ==> (LEAST\ n.\ P\ n) = Suc\ (LEAST\ m.\ P(Suc\ m))$
⟨*proof*⟩

lemma *Least-Suc2*:
$[|P\ n;\ Q\ m;\ \sim P\ 0;\ !k.\ P\ (Suc\ k) = Q\ k|] ==> Least\ P = Suc\ (Least\ Q)$
⟨*proof*⟩

17.6 *min* and *max*

lemma *mono-Suc*: *mono Suc*
⟨*proof*⟩

lemma *min-0L* [*simp*]: $min\ 0\ n = (0::nat)$
⟨*proof*⟩

lemma *min-0R* [*simp*]: $min\ n\ 0 = (0::nat)$
⟨*proof*⟩

lemma *min-Suc-Suc* [*simp*]: $min\ (Suc\ m)\ (Suc\ n) = Suc\ (min\ m\ n)$
⟨*proof*⟩

lemma *min-Suc1*:
$min\ (Suc\ n)\ m = (case\ m\ of\ 0 => 0 \mid Suc\ m' => Suc(min\ n\ m'))$
⟨*proof*⟩

lemma *min-Suc2*:
$min\ m\ (Suc\ n) = (case\ m\ of\ 0 => 0 \mid Suc\ m' => Suc(min\ m'\ n))$
⟨*proof*⟩

lemma *max-0L* [*simp*]: $max\ 0\ n = (n::nat)$
⟨*proof*⟩

lemma *max-0R* [*simp*]: $max\ n\ 0 = (n::nat)$
⟨*proof*⟩

lemma *max-Suc-Suc* [*simp*]: $max\ (Suc\ m)\ (Suc\ n) = Suc(max\ m\ n)$
⟨*proof*⟩

lemma *max-Suc1*:
$max\ (Suc\ n)\ m = (case\ m\ of\ 0 => Suc\ n \mid Suc\ m' => Suc(max\ n\ m'))$
⟨*proof*⟩

lemma *max-Suc2*:
$max\ m\ (Suc\ n) = (case\ m\ of\ 0 => Suc\ n \mid Suc\ m' => Suc(max\ m'\ n))$
⟨*proof*⟩

17.7 Basic rewrite rules for the arithmetic operators

Difference

lemma *diff-0-eq-0* [*simp*, *code*]: $0 - n = (0::nat)$
⟨*proof*⟩

lemma *diff-Suc-Suc* [*simp*, *code*]: $Suc(m) - Suc(n) = m - n$
⟨*proof*⟩

Could be (and is, below) generalized in various ways However, none of the generalizations are currently in the simpset, and I dread to think what happens if I put them in

lemma *Suc-pred* [*simp*]: $n>0 ==> Suc\ (n - Suc\ 0) = n$
⟨*proof*⟩

declare *diff-Suc* [*simp del*, *code del*]

17.8 Addition

lemma *add-0-right* [*simp*]: $m + 0 = (m::nat)$
⟨*proof*⟩

lemma *add-Suc-right* [*simp*]: $m + Suc\ n = Suc\ (m + n)$
⟨*proof*⟩

lemma *add-Suc-shift* [*code*]: $Suc\ m + n = m + Suc\ n$
⟨*proof*⟩

Associative law for addition

lemma *nat-add-assoc*: $(m + n) + k = m + ((n + k)::nat)$
⟨*proof*⟩

Commutative law for addition

lemma *nat-add-commute*: $m + n = n + (m{::}nat)$
⟨*proof*⟩

lemma *nat-add-left-commute*: $x + (y + z) = y + ((x + z){::}nat)$
⟨*proof*⟩

lemma *nat-add-left-cancel* [*simp*]: $(k + m = k + n) = (m = (n{::}nat))$
⟨*proof*⟩

lemma *nat-add-right-cancel* [*simp*]: $(m + k = n + k) = (m{=}(n{::}nat))$
⟨*proof*⟩

lemma *nat-add-left-cancel-le* [*simp*]: $(k + m \leq k + n) = (m{\leq}(n{::}nat))$
⟨*proof*⟩

lemma *nat-add-left-cancel-less* [*simp*]: $(k + m < k + n) = (m{<}(n{::}nat))$
⟨*proof*⟩

Reasoning about $m + 0 = 0$, etc.

lemma *add-is-0* [*iff*]: **fixes** $m :: nat$ **shows** $(m + n = 0) = (m = 0 \ \& \ n = 0)$
⟨*proof*⟩

lemma *add-is-1*: $(m{+}n{=}\ Suc\ 0) = (m{=}\ Suc\ 0\ \&\ n{=}0\ |\ m{=}0\ \&\ n{=}\ Suc\ 0)$
⟨*proof*⟩

lemma *one-is-add*: $(Suc\ 0 = m + n) = (m = Suc\ 0\ \&\ n = 0\ |\ m = 0\ \&\ n = Suc\ 0)$
⟨*proof*⟩

lemma *add-gr-0* [*iff*]: $!!m{::}nat.\ (m + n > 0) = (m{>}0\ |\ n{>}0)$
⟨*proof*⟩

lemma *add-eq-self-zero*: $!!m{::}nat.\ m + n = m \Longrightarrow n = 0$
⟨*proof*⟩

lemma *inj-on-add-nat*[*simp*]: $inj\text{-}on\ (\%n{::}nat.\ n{+}k)\ N$
⟨*proof*⟩

17.9 Multiplication

right annihilation in product

lemma *mult-0-right* [*simp*]: $(m{::}nat) * 0 = 0$
⟨*proof*⟩

right successor law for multiplication

lemma *mult-Suc-right* [*simp*]: $m * Suc\ n = m + (m * n)$
⟨*proof*⟩

Commutative law for multiplication

lemma *nat-mult-commute*: $m * n = n * (m::nat)$
⟨*proof*⟩

addition distributes over multiplication

lemma *add-mult-distrib*: $(m + n) * k = (m * k) + ((n * k)::nat)$
⟨*proof*⟩

lemma *add-mult-distrib2*: $k * (m + n) = (k * m) + ((k * n)::nat)$
⟨*proof*⟩

Associative law for multiplication

lemma *nat-mult-assoc*: $(m * n) * k = m * ((n * k)::nat)$
⟨*proof*⟩

The naturals form a *comm-semiring-1-cancel*

instance *nat* :: *comm-semiring-1-cancel*
⟨*proof*⟩

lemma *mult-is-0* [*simp*]: $((m::nat) * n = 0) = (m=0 \mid n=0)$
⟨*proof*⟩

17.10 Monotonicity of Addition

strict, in 1st argument

lemma *add-less-mono1*: $i < j ==> i + k < j + (k::nat)$
⟨*proof*⟩

strict, in both arguments

lemma *add-less-mono*: $[|i < j; k < l|] ==> i + k < j + (l::nat)$
⟨*proof*⟩

Deleted *less-natE*; use *less-imp-Suc-add RS exE*

lemma *less-imp-Suc-add*: $m < n ==> (\exists k. n = Suc (m + k))$
⟨*proof*⟩

strict, in 1st argument; proof is by induction on $k > 0$

lemma *mult-less-mono2*: $(i::nat) < j ==> 0<k ==> k * i < k * j$
⟨*proof*⟩

The naturals form an ordered *comm-semiring-1-cancel*

instance *nat* :: *ordered-semidom*
⟨*proof*⟩

lemma *nat-mult-1*: $(1::nat) * n = n$
⟨*proof*⟩

lemma *nat-mult-1-right*: $n * (1::nat) = n$
⟨*proof*⟩

17.11 Additional theorems about "less than"

An induction rule for estabilishing binary relations

lemma *less-Suc-induct*:
assumes *less*: $i < j$
and *step*: $!!i.\ P\ i\ (Suc\ i)$
and *trans*: $!!i\ j\ k.\ P\ i\ j \ ==>\ P\ j\ k \ ==>\ P\ i\ k$
shows $P\ i\ j$
⟨*proof*⟩

The method of infinite descent, frequently used in number theory. Provided by Roelof Oosterhuis. $P(n)$ is true for all $n \in \mathbb{N}$ if

- case "0": given $n = 0$ prove $P(n)$,
- case "smaller": given $n > 0$ and $\neg P(n)$ prove there exists a smaller integer m such that $\neg P(m)$.

lemma *infinite-descent0*[*case-names 0 smaller*]:
$[\![\ P\ 0;\ !!n.\ n{>}0 \Longrightarrow \neg\ P\ n \Longrightarrow (\exists\, m{::}nat.\ m < n \wedge \neg P\ m)\ ]\!] \Longrightarrow P\ n$
⟨*proof*⟩

A compact version without explicit base case:

lemma *infinite-descent*:
$[\![\ !!n{::}nat.\ \neg\ P\ n \Longrightarrow\ \exists\, m{<}n.\ \neg\ \ P\ m\ ]\!] \Longrightarrow\ \ P\ n$
⟨*proof*⟩

Infinite descent using a mapping to $\mathbb{N}$: $P(x)$ is true for all $x \in D$ if there exists a $V : D \to \mathbb{N}$ and

- case "0": given $V(x) = 0$ prove $P(x)$,
- case "smaller": given $V(x) > 0$ and $\neg P(x)$ prove there exists a $y \in D$ such that $V(y) < V(x)$ and $\neg P(y)$.

NB: the proof also shows how to use the previous lemma.

corollary *infinite-descent0-measure*[*case-names 0 smaller*]:
assumes *0*: $!!x.\ V\ x = (0{::}nat) \Longrightarrow P\ x$
and *1*: $!!x.\ V\ x > 0 \Longrightarrow \neg P\ x \Longrightarrow (\exists\, y.\ V\ y < V\ x \wedge \neg P\ y)$
shows $P\ x$
⟨*proof*⟩

Again, without explicit base case:

lemma *infinite-descent-measure*:
assumes $!!x.\ \neg\ P\ x \Longrightarrow \exists\, y.\ (V{::}'a{\Rightarrow}nat)\ y < \ V\ x \wedge \neg\ P\ y$ **shows** $P\ x$
⟨*proof*⟩

A [clumsy] way of lifting $<$ monotonicity to $\leq$ monotonicity

lemma *less-mono-imp-le-mono*:
$[\![\; !!i\ j{::}nat.\ i < j \Longrightarrow f\,i < f\,j;\ i \le j \;]\!] \Longrightarrow f\,i \le ((f\,j){::}nat)$
⟨*proof*⟩

non-strict, in 1st argument

lemma *add-le-mono1*: $i \le j ==> i + k \le j + (k{::}nat)$
⟨*proof*⟩

non-strict, in both arguments

lemma *add-le-mono*: $[|\ i \le j;\ \ k \le l\ |] ==> i + k \le j + (l{::}nat)$
⟨*proof*⟩

lemma *le-add2*: $n \le ((m + n){::}nat)$
⟨*proof*⟩

lemma *le-add1*: $n \le ((n + m){::}nat)$
⟨*proof*⟩

lemma *less-add-Suc1*: $i < Suc\ (i + m)$
⟨*proof*⟩

lemma *less-add-Suc2*: $i < Suc\ (m + i)$
⟨*proof*⟩

lemma *less-iff-Suc-add*: $(m < n) = (\exists\,k.\ n = Suc\ (m + k))$
⟨*proof*⟩

lemma *trans-le-add1*: $(i{::}nat) \le j ==> i \le j + m$
⟨*proof*⟩

lemma *trans-le-add2*: $(i{::}nat) \le j ==> i \le m + j$
⟨*proof*⟩

lemma *trans-less-add1*: $(i{::}nat) < j ==> i < j + m$
⟨*proof*⟩

lemma *trans-less-add2*: $(i{::}nat) < j ==> i < m + j$
⟨*proof*⟩

lemma *add-lessD1*: $i + j < (k{::}nat) ==> i < k$
⟨*proof*⟩

lemma *not-add-less1* [*iff*]: $\sim\ (i + j < (i{::}nat))$
⟨*proof*⟩

lemma *not-add-less2* [*iff*]: $\sim\ (j + i < (i{::}nat))$
⟨*proof*⟩

lemma *add-leD1*: $m + k \le n ==> m \le (n{::}nat)$

⟨*proof*⟩

lemma *add-leD2*: $m + k \le n ==> k \le (n::nat)$
⟨*proof*⟩

lemma *add-leE*: $(m::nat) + k \le n ==> (m \le n ==> k \le n ==> R) ==> R$
⟨*proof*⟩

needs !!k for *add-ac* to work

lemma *less-add-eq-less*: $!!k::nat.\ k < l ==> m + l = k + n ==> m < n$
⟨*proof*⟩

17.12 Difference

lemma *diff-self-eq-0* [*simp*]: $(m::nat) - m = 0$
⟨*proof*⟩

Addition is the inverse of subtraction: if $n \le m$ then $n + (m - n) = m$.

lemma *add-diff-inverse*: $\sim\ m < n ==> n + (m - n) = (m::nat)$
⟨*proof*⟩

lemma *le-add-diff-inverse* [*simp*]: $n \le m ==> n + (m - n) = (m::nat)$
⟨*proof*⟩

lemma *le-add-diff-inverse2* [*simp*]: $n \le m ==> (m - n) + n = (m::nat)$
⟨*proof*⟩

17.13 More results about difference

lemma *Suc-diff-le*: $n \le m ==> Suc\ m - n = Suc\ (m - n)$
⟨*proof*⟩

lemma *diff-less-Suc*: $m - n < Suc\ m$
⟨*proof*⟩

lemma *diff-le-self* [*simp*]: $m - n \le (m::nat)$
⟨*proof*⟩

lemma *less-imp-diff-less*: $(j::nat) < k ==> j - n < k$
⟨*proof*⟩

lemma *diff-diff-left*: $(i::nat) - j - k = i - (j + k)$
⟨*proof*⟩

lemma *Suc-diff-diff* [*simp*]: $(Suc\ m - n) - Suc\ k = m - n - k$
⟨*proof*⟩

lemma *diff-Suc-less* [*simp*]: $0<n ==> n - Suc\ i < n$
⟨*proof*⟩

This and the next few suggested by Florian Kammueller

lemma *diff-commute*: $(i::nat) - j - k = i - k - j$
⟨*proof*⟩

lemma *diff-add-assoc*: $k \leq (j::nat) ==> (i + j) - k = i + (j - k)$
⟨*proof*⟩

lemma *diff-add-assoc2*: $k \leq (j::nat) ==> (j + i) - k = (j - k) + i$
⟨*proof*⟩

lemma *diff-add-inverse*: $(n + m) - n = (m::nat)$
⟨*proof*⟩

lemma *diff-add-inverse2*: $(m + n) - n = (m::nat)$
⟨*proof*⟩

lemma *le-imp-diff-is-add*: $i \leq (j::nat) ==> (j - i = k) = (j = k + i)$
⟨*proof*⟩

lemma *diff-is-0-eq* [*simp*]: $((m::nat) - n = 0) = (m \leq n)$
⟨*proof*⟩

lemma *diff-is-0-eq′* [*simp*]: $m \leq n ==> (m::nat) - n = 0$
⟨*proof*⟩

lemma *zero-less-diff* [*simp*]: $(0 < n - (m::nat)) = (m < n)$
⟨*proof*⟩

lemma *less-imp-add-positive*:
assumes $i < j$
shows $\exists k::nat.\ 0 < k\ \&\ i + k = j$
⟨*proof*⟩

lemma *diff-cancel*: $(k + m) - (k + n) = m - (n::nat)$
⟨*proof*⟩

lemma *diff-cancel2*: $(m + k) - (n + k) = m - (n::nat)$
⟨*proof*⟩

lemma *diff-add-0*: $n - (n + m) = (0::nat)$
⟨*proof*⟩

Difference distributes over multiplication

lemma *diff-mult-distrib*: $((m::nat) - n) * k = (m * k) - (n * k)$
⟨*proof*⟩

lemma *diff-mult-distrib2*: $k * ((m::nat) - n) = (k * m) - (k * n)$
⟨*proof*⟩

lemmas *nat-distrib* =
add-mult-distrib add-mult-distrib2 diff-mult-distrib diff-mult-distrib2

17.14 Monotonicity of Multiplication

lemma *mult-le-mono1*: $i \leq (j::nat) ==> i * k \leq j * k$
⟨*proof*⟩

lemma *mult-le-mono2*: $i \leq (j::nat) ==> k * i \leq k * j$
⟨*proof*⟩

$\leq$ monotonicity, BOTH arguments

lemma *mult-le-mono*: $i \leq (j::nat) ==> k \leq l ==> i * k \leq j * l$
⟨*proof*⟩

lemma *mult-less-mono1*: $(i::nat) < j ==> 0 < k ==> i * k < j * k$
⟨*proof*⟩

Differs from the standard *zero-less-mult-iff* in that there are no negative numbers.

lemma *nat-0-less-mult-iff* [*simp*]: $(0 < (m::nat) * n) = (0 < m \ \& \ 0 < n)$
⟨*proof*⟩

lemma *one-le-mult-iff* [*simp*]: $(Suc\ 0 \leq m * n) = (1 \leq m \ \& \ 1 \leq n)$
⟨*proof*⟩

lemma *mult-eq-1-iff* [*simp*]: $(m * n = Suc\ 0) = (m = 1 \ \& \ n = 1)$
⟨*proof*⟩

lemma *one-eq-mult-iff* [*simp,noatp*]: $(Suc\ 0 = m * n) = (m = 1 \ \& \ n = 1)$
⟨*proof*⟩

lemma *mult-less-cancel2* [*simp*]: $((m::nat) * k < n * k) = (0 < k \ \& \ m < n)$
⟨*proof*⟩

lemma *mult-less-cancel1* [*simp*]: $(k * (m::nat) < k * n) = (0 < k \ \& \ m < n)$
⟨*proof*⟩

lemma *mult-le-cancel1* [*simp*]: $(k * (m::nat) \leq k * n) = (0 < k --> m \leq n)$
⟨*proof*⟩

lemma *mult-le-cancel2* [*simp*]: $((m::nat) * k \leq n * k) = (0 < k --> m \leq n)$
⟨*proof*⟩

lemma *mult-cancel2* [*simp*]: $(m * k = n * k) = (m = n \mid (k = (0::nat)))$
⟨*proof*⟩

lemma *mult-cancel1* [*simp*]: $(k * m = k * n) = (m = n \mid (k = (0::nat)))$
⟨*proof*⟩

lemma *Suc-mult-less-cancel1*: $(Suc\ k * m < Suc\ k * n) = (m < n)$
⟨*proof*⟩

lemma *Suc-mult-le-cancel1*: $(Suc\ k * m \le Suc\ k * n) = (m \le n)$
⟨*proof*⟩

lemma *Suc-mult-cancel1*: $(Suc\ k * m = Suc\ k * n) = (m = n)$
⟨*proof*⟩

Lemma for *gcd*

lemma *mult-eq-self-implies-10*: $(m{::}nat) = m * n ==> n = 1 \mid m = 0$
⟨*proof*⟩

17.15 size of a datatype value

class *size* = *type* +
fixes *size* :: $'a \Rightarrow nat$

⟨*ML*⟩

lemma *nat-size* [*simp*, *code func*]: $size\ (n{::}nat) = n$
⟨*proof*⟩

lemma *size-bool* [*code func*]:
$size\ (b{::}bool) = 0$ ⟨*proof*⟩

declare $*$.*size* [*noatp*]

17.16 Code generator setup

instance *nat* :: *eq* ⟨*proof*⟩

lemma [*code func*]:
$(0{::}nat) = 0 \longleftrightarrow True$
$Suc\ n = Suc\ m \longleftrightarrow n = m$
$Suc\ n = 0 \longleftrightarrow False$
$0 = Suc\ m \longleftrightarrow False$
⟨*proof*⟩

lemma [*code func*]:
$(0{::}nat) \le m \longleftrightarrow True$
$Suc\ (n{::}nat) \le m \longleftrightarrow n < m$
$(n{::}nat) < 0 \longleftrightarrow False$
$(n{::}nat) < Suc\ m \longleftrightarrow n \le m$
⟨*proof*⟩

17.17 Embedding of the Naturals into any *semiring-1*: *of-nat*

context *semiring-1*

begin

definition
of-nat-def: *of-nat* = *nat-rec* *0* (λ-. (*op* +) *1*)

lemma *of-nat-simps* [*simp*, *code*]:
shows *of-nat-0*: *of-nat 0* = *0*
and *of-nat-Suc*: *of-nat* (*Suc m*) = *1* + *of-nat m*
⟨*proof*⟩

lemma *of-nat-1* [*simp*]: *of-nat 1* = *1*
⟨*proof*⟩

lemma *of-nat-add* [*simp*]: *of-nat* (*m* + *n*) = *of-nat m* + *of-nat n*
⟨*proof*⟩

lemma *of-nat-mult*: *of-nat* (*m* ∗ *n*) = *of-nat m* ∗ *of-nat n*
⟨*proof*⟩

end

context *ordered-semidom*
begin

lemma *zero-le-imp-of-nat*: *0* ≤ *of-nat m*
⟨*proof*⟩

lemma *less-imp-of-nat-less*: *m* < *n* ⟹ *of-nat m* < *of-nat n*
⟨*proof*⟩

lemma *of-nat-less-imp-less*: *of-nat m* < *of-nat n* ⟹ *m* < *n*
⟨*proof*⟩

lemma *of-nat-less-iff* [*simp*]: *of-nat m* < *of-nat n* ⟷ *m* < *n*
⟨*proof*⟩

Special cases where either operand is zero

lemma *of-nat-0-less-iff* [*simp*]: *0* < *of-nat n* ⟷ *0* < *n*
⟨*proof*⟩

lemma *of-nat-less-0-iff* [*simp*]: ¬ *of-nat m* < *0*
⟨*proof*⟩

lemma *of-nat-le-iff* [*simp*]:
of-nat m ≤ *of-nat n* ⟷ *m* ≤ *n*
⟨*proof*⟩

Special cases where either operand is zero

lemma *of-nat-0-le-iff* [*simp*]: *0* ≤ *of-nat n*

⟨*proof*⟩

lemma *of-nat-le-0-iff* [*simp*, *noatp*]: *of-nat m* $\leq$ *0* $\longleftrightarrow$ *m* = *0*
⟨*proof*⟩

end

lemma *of-nat-id* [*simp*]: *of-nat n* = *n*
⟨*proof*⟩

lemma *of-nat-eq-id* [*simp*]: *of-nat* = *id*
⟨*proof*⟩

Class for unital semirings with characteristic zero. Includes non-ordered rings like the complex numbers.

class *semiring-char-0* = *semiring-1* +
assumes *of-nat-eq-iff* [*simp*]: *of-nat m* = *of-nat n* $\longleftrightarrow$ *m* = *n*

Every *ordered-semidom* has characteristic zero.

subclass (**in** *ordered-semidom*) *semiring-char-0*
⟨*proof*⟩

context *semiring-char-0*
begin

Special cases where either operand is zero

lemma *of-nat-0-eq-iff* [*simp*, *noatp*]: *0* = *of-nat n* $\longleftrightarrow$ *0* = *n*
⟨*proof*⟩

lemma *of-nat-eq-0-iff* [*simp*, *noatp*]: *of-nat m* = *0* $\longleftrightarrow$ *m* = *0*
⟨*proof*⟩

lemma *inj-of-nat*: *inj of-nat*
⟨*proof*⟩

end

17.18 Further Arithmetic Facts Concerning the Natural Numbers

lemma *subst-equals*:
assumes *1*: *t* = *s* **and** *2*: *u* = *t*
shows *u* = *s*
⟨*proof*⟩

⟨*ML*⟩

The following proofs may rely on the arithmetic proof procedures.

lemma *le-iff-add*: $(m::nat) \leq n = (\exists k.\ n = m + k)$
⟨*proof*⟩

lemma *pred-nat-trancl-eq-le*: $((m,\ n) : pred\text{-}nat\hat{\ }*) = (m \leq n)$
⟨*proof*⟩

lemma *nat-diff-split*:
$P(a - b::nat) = ((a<b \ {-}{-}{>}\ P\ 0)\ \&\ (ALL\ d.\ a = b + d \ {-}{-}{>}\ P\ d))$
— elimination of $-$ on *nat*
⟨*proof*⟩

context *ring-1*
begin

lemma *of-nat-diff*: $n \leq m \Longrightarrow of\text{-}nat\ (m - n) = of\text{-}nat\ m - of\text{-}nat\ n$
⟨*proof*⟩

end

lemma *abs-of-nat* [*simp*]: $|of\text{-}nat\ n::'a::ordered\text{-}idom| = of\text{-}nat\ n$
⟨*proof*⟩

lemma *nat-diff-split-asm*:
$P(a - b::nat) = (\tilde{}\ (a < b\ \&\ \tilde{}\ P\ 0\ |\ (EX\ d.\ a = b + d\ \&\ \tilde{}\ P\ d)))$
— elimination of $-$ on *nat* in assumptions
⟨*proof*⟩

lemmas [*arith-split*] = *nat-diff-split split-min split-max*

lemma *le-square*: $m \leq m * (m::nat)$
⟨*proof*⟩

lemma *le-cube*: $(m::nat) \leq m * (m * m)$
⟨*proof*⟩

Subtraction laws, mostly by Clemens Ballarin

lemma *diff-less-mono*: $[|\ a < (b::nat);\ c \leq a\ |] ==> a-c < b-c$
⟨*proof*⟩

lemma *less-diff-conv*: $(i < j-k) = (i+k < (j::nat))$
⟨*proof*⟩

lemma *le-diff-conv*: $(j-k \leq (i::nat)) = (j \leq i+k)$
⟨*proof*⟩

lemma *le-diff-conv2*: $k \leq j ==> (i \leq j-k) = (i+k \leq (j::nat))$
⟨*proof*⟩

lemma *diff-diff-cancel* [*simp*]: $i \leq (n::nat) ==> n - (n - i) = i$
⟨*proof*⟩

lemma *le-add-diff*: $k \leq (n::nat) ==> m \leq n + m - k$
⟨*proof*⟩

lemma *diff-less*[*simp*]: $!!m::nat.$ [| $0<n$; $0<m$ |] $==> m - n < m$
⟨*proof*⟩

lemma *diff-diff-eq*: [| $k \leq m$; $k \leq (n::nat)$ |] $==> ((m-k) - (n-k)) = (m-n)$
⟨*proof*⟩

lemma *eq-diff-iff*: [| $k \leq m$; $k \leq (n::nat)$ |] $==> (m-k = n-k) = (m=n)$
⟨*proof*⟩

lemma *less-diff-iff*: [| $k \leq m$; $k \leq (n::nat)$ |] $==> (m-k < n-k) = (m<n)$
⟨*proof*⟩

lemma *le-diff-iff*: [| $k \leq m$; $k \leq (n::nat)$ |] $==> (m-k \leq n-k) = (m\leq n)$
⟨*proof*⟩

(Anti)Monotonicity of subtraction – by Stephan Merz

lemma *diff-le-mono*: $m \leq (n::nat) ==> (m-l) \leq (n-l)$
⟨*proof*⟩

lemma *diff-le-mono2*: $m \leq (n::nat) ==> (l-n) \leq (l-m)$
⟨*proof*⟩

lemma *diff-less-mono2*: [| $m < (n::nat)$; $m<l$ |] $==> (l-n) < (l-m)$
⟨*proof*⟩

lemma *diffs0-imp-equal*: $!!m::nat.$ [| $m-n = 0$; $n-m = 0$ |] $==> m=n$
⟨*proof*⟩

Lemmas for ex/Factorization

lemma *one-less-mult*: [| $Suc\ 0 < n$; $Suc\ 0 < m$ |] $==> Suc\ 0 < m*n$
⟨*proof*⟩

lemma *n-less-m-mult-n*: [| $Suc\ 0 < n$; $Suc\ 0 < m$ |] $==> n<m*n$
⟨*proof*⟩

lemma *n-less-n-mult-m*: [| $Suc\ 0 < n$; $Suc\ 0 < m$ |] $==> n<n*m$
⟨*proof*⟩

Specialized induction principles that work "backwards":

lemma *inc-induct*[*consumes 1*, *case-names base step*]:
assumes *less*: $i <= j$
assumes *base*: $P\ j$
assumes *step*: $!!i.\ [|\ i < j;\ P\ (Suc\ i)\ |] ==> P\ i$
shows $P\ i$
⟨*proof*⟩

lemma *strict-inc-induct*[*consumes 1*, *case-names base step*]:
assumes *less*: $i < j$
assumes *base*: $!!i.\ j = Suc\ i ==> P\ i$
assumes *step*: $!!i.\ [|\ i < j;\ P\ (Suc\ i)\ |] ==> P\ i$
shows $P\ i$
⟨*proof*⟩

lemma *zero-induct-lemma*: $P\ k ==> (!!n.\ P\ (Suc\ n) ==> P\ n) ==> P\ (k - i)$
⟨*proof*⟩

lemma *zero-induct*: $P\ k ==> (!!n.\ P\ (Suc\ n) ==> P\ n) ==> P\ 0$
⟨*proof*⟩

Rewriting to pull differences out

lemma *diff-diff-right* [*simp*]: $k \leq j --> i - (j - k) = i + (k::nat) - j$
⟨*proof*⟩

lemma *diff-Suc-diff-eq1* [*simp*]: $k \leq j ==> m - Suc\ (j - k) = m + k - Suc\ j$
⟨*proof*⟩

lemma *diff-Suc-diff-eq2* [*simp*]: $k \leq j ==> Suc\ (j - k) - m = Suc\ j - (k + m)$
⟨*proof*⟩

lemmas *add-diff-assoc* = *diff-add-assoc* [*symmetric*]
lemmas *add-diff-assoc2* = *diff-add-assoc2*[*symmetric*]
declare *diff-diff-left* [*simp*] *add-diff-assoc* [*simp*] *add-diff-assoc2*[*simp*]

At present we prove no analogue of *not-less-Least* or *Least-Suc*, since there appears to be no need.

17.19 The Set of Natural Numbers

context *semiring-1*
begin

definition
Nats :: *'a set* **where**
Nats = *range of-nat*

notation (*xsymbols*)
Nats (ℕ)

end

context *semiring-1*
begin

lemma *of-nat-in-Nats* [*simp*]: *of-nat* $n \in \mathbb{N}$
⟨*proof*⟩

lemma *Nats-0* [*simp*]: $0 \in \mathbb{N}$
⟨*proof*⟩

lemma *Nats-1* [*simp*]: $1 \in \mathbb{N}$
⟨*proof*⟩

lemma *Nats-add* [*simp*]: $a \in \mathbb{N} \Longrightarrow b \in \mathbb{N} \Longrightarrow a + b \in \mathbb{N}$
⟨*proof*⟩

lemma *Nats-mult* [*simp*]: $a \in \mathbb{N} \Longrightarrow b \in \mathbb{N} \Longrightarrow a * b \in \mathbb{N}$
⟨*proof*⟩

end

the lattice order on *nat*

instance *nat* :: *distrib-lattice*
inf ≡ *min*
sup ≡ *max*
⟨*proof*⟩

17.20 legacy bindings

⟨*ML*⟩

end

18 Power: Exponentiation

theory *Power*
imports *Nat*
begin

class *power* = *type* +
fixes *power* :: $'a \Rightarrow nat \Rightarrow 'a$ (**infixr** ˆ *80*)

18.1 Powers for Arbitrary Monoids

class *recpower* = *monoid-mult* + *power* +
assumes *power-0* [*simp*]: a ˆ 0 = 1

assumes *power-Suc*: $a \;\hat{}\; Suc\ n = a * (a \;\hat{}\; n)$

lemma *power-0-Suc* [*simp*]: $(0::'a::\{recpower,semiring\text{-}0\}) \;\hat{}\; (Suc\ n) = 0$
⟨*proof*⟩

It looks plausible as a simprule, but its effect can be strange.

lemma *power-0-left*: $0\hat{}n = (if\ n{=}0\ then\ 1\ else\ (0::'a::\{recpower,semiring\text{-}0\}))$
⟨*proof*⟩

lemma *power-one* [*simp*]: $1\hat{}n = (1::'a::recpower)$
⟨*proof*⟩

lemma *power-one-right* [*simp*]: $(a::'a::recpower) \;\hat{}\; 1 = a$
⟨*proof*⟩

lemma *power-commutes*: $(a::'a::recpower) \;\hat{}\; n * a = a * a \;\hat{}\; n$
⟨*proof*⟩

lemma *power-add*: $(a::'a::recpower) \;\hat{}\; (m+n) = (a\hat{}m) * (a\hat{}n)$
⟨*proof*⟩

lemma *power-mult*: $(a::'a::recpower) \;\hat{}\; (m*n) = (a\hat{}m) \;\hat{}\; n$
⟨*proof*⟩

lemma *power-mult-distrib*: $((a::'a::\{recpower,comm\text{-}monoid\text{-}mult\}) * b) \;\hat{}\; n = (a\hat{}n) * (b\hat{}n)$
⟨*proof*⟩

lemma *zero-less-power*:
$0 < (a::'a::\{ordered\text{-}semidom,recpower\}) ==> 0 < a\hat{}n$
⟨*proof*⟩

lemma *zero-le-power*:
$0 \le (a::'a::\{ordered\text{-}semidom,recpower\}) ==> 0 \le a\hat{}n$
⟨*proof*⟩

lemma *one-le-power*:
$1 \le (a::'a::\{ordered\text{-}semidom,recpower\}) ==> 1 \le a\hat{}n$
⟨*proof*⟩

lemma *gt1-imp-ge0*: $1 < a ==> 0 \le (a::'a::ordered\text{-}semidom)$
⟨*proof*⟩

lemma *power-gt1-lemma*:
assumes *gt1*: $1 < (a::'a::\{ordered\text{-}semidom,recpower\})$
shows $1 < a * a\hat{}n$
⟨*proof*⟩

lemma *one-less-power*:

$[\![1 < (a::'a::\{ordered\text{-}semidom,recpower\});\ 0 < n]\!] \Longrightarrow 1 < a \hat{}\ n$
⟨*proof*⟩

lemma *power-gt1*:
$1 < (a::'a::\{ordered\text{-}semidom,recpower\}) ==> 1 < a \hat{}\ (Suc\ n)$
⟨*proof*⟩

lemma *power-le-imp-le-exp*:
assumes *gt1*: $(1::'a::\{recpower,ordered\text{-}semidom\}) < a$
shows $!!n.\ a\hat{}m \le a\hat{}n ==> m \le n$
⟨*proof*⟩

Surely we can strengthen this? It holds for $0<a<1$ too.

lemma *power-inject-exp* [*simp*]:
$1 < (a::'a::\{ordered\text{-}semidom,recpower\}) ==> (a\hat{}m = a\hat{}n) = (m=n)$
⟨*proof*⟩

Can relax the first premise to $(0::'a) < a$ in the case of the natural numbers.

lemma *power-less-imp-less-exp*:
$[|\ (1::'a::\{recpower,ordered\text{-}semidom\}) < a;\ a\hat{}m < a\hat{}n\ |] ==> m < n$
⟨*proof*⟩

lemma *power-mono*:
$[|a \le b;\ (0::'a::\{recpower,ordered\text{-}semidom\}) \le a|] ==> a\hat{}n \le b\hat{}n$
⟨*proof*⟩

lemma *power-strict-mono* [*rule-format*]:
$[|a < b;\ (0::'a::\{recpower,ordered\text{-}semidom\}) \le a|]$
$==> 0 < n --> a\hat{}n < b\hat{}n$
⟨*proof*⟩

lemma *power-eq-0-iff* [*simp*]:
$(a\hat{}n = 0) = (a = (0::'a::\{ring\text{-}1\text{-}no\text{-}zero\text{-}divisors,recpower\})\ \&\ n>0)$
⟨*proof*⟩

lemma *field-power-not-zero*:
$a \ne (0::'a::\{ring\text{-}1\text{-}no\text{-}zero\text{-}divisors,recpower\}) ==> a\hat{}n \ne 0$
⟨*proof*⟩

lemma *nonzero-power-inverse*:
fixes $a :: 'a::\{division\text{-}ring,recpower\}$
shows $a \ne 0 ==> inverse\ (a \hat{}\ n) = (inverse\ a) \hat{}\ n$
⟨*proof*⟩

Perhaps these should be simprules.

lemma *power-inverse*:
fixes $a :: 'a::\{division\text{-}ring,division\text{-}by\text{-}zero,recpower\}$
shows $inverse\ (a \hat{}\ n) = (inverse\ a) \hat{}\ n$

⟨*proof*⟩

lemma *power-one-over*: $1\ /\ (a::'a::\{field,division\text{-}by\text{-}zero,recpower\})\hat{\ }n = (1\ /\ a)\hat{\ }n$
⟨*proof*⟩

lemma *nonzero-power-divide*:
$b \neq 0 ==> (a/b)\ \hat{\ }\ n = ((a::'a::\{field,recpower\})\ \hat{\ }\ n)\ /\ (b\ \hat{\ }\ n)$
⟨*proof*⟩

lemma *power-divide*:
$(a/b)\ \hat{\ }\ n = ((a::'a::\{field,division\text{-}by\text{-}zero,recpower\})\ \hat{\ }\ n\ /\ b\ \hat{\ }\ n)$
⟨*proof*⟩

lemma *power-abs*: $abs(a\ \hat{\ }\ n) = abs(a::'a::\{ordered\text{-}idom,recpower\})\ \hat{\ }\ n$
⟨*proof*⟩

lemma *zero-less-power-abs-iff* [*simp*,*noatp*]:
$(0 < (abs\ a)\hat{\ }n) = (a \neq (0::'a::\{ordered\text{-}idom,recpower\})\ |\ n=0)$
⟨*proof*⟩

lemma *zero-le-power-abs* [*simp*]:
$(0::'a::\{ordered\text{-}idom,recpower\}) \leq (abs\ a)\hat{\ }n$
⟨*proof*⟩

lemma *power-minus*: $(-a)\ \hat{\ }\ n = (-\ 1)\hat{\ }n * (a::'a::\{comm\text{-}ring\text{-}1,recpower\})\ \hat{\ }\ n$
⟨*proof*⟩

Lemma for *power-strict-decreasing*

lemma *power-Suc-less*:
$[|(0::'a::\{ordered\text{-}semidom,recpower\}) < a;\ a < 1|]$
$==> a * a\hat{\ }n < a\hat{\ }n$
⟨*proof*⟩

lemma *power-strict-decreasing*:
$[|n < N;\ 0 < a;\ a < (1::'a::\{ordered\text{-}semidom,recpower\})|]$
$==> a\hat{\ }N < a\hat{\ }n$
⟨*proof*⟩

Proof resembles that of *power-strict-decreasing*

lemma *power-decreasing*:
$[|n \leq N;\ 0 \leq a;\ a \leq (1::'a::\{ordered\text{-}semidom,recpower\})|]$
$==> a\hat{\ }N \leq a\hat{\ }n$
⟨*proof*⟩

lemma *power-Suc-less-one*:
$[|\ 0 < a;\ a < (1::'a::\{ordered\text{-}semidom,recpower\})\ |] ==> a\ \hat{\ }\ Suc\ n < 1$
⟨*proof*⟩

Proof again resembles that of *power-strict-decreasing*

lemma *power-increasing*:
$[|n \le N;\ (1::'a::\{ordered\text{-}semidom,recpower\}) \le a|] ==> a\hat{}n \le a\hat{}N$
⟨*proof*⟩

Lemma for *power-strict-increasing*

lemma *power-less-power-Suc*:
$(1::'a::\{ordered\text{-}semidom,recpower\}) < a ==> a\hat{}n < a * a\hat{}n$
⟨*proof*⟩

lemma *power-strict-increasing*:
$[|n < N;\ (1::'a::\{ordered\text{-}semidom,recpower\}) < a|] ==> a\hat{}n < a\hat{}N$
⟨*proof*⟩

lemma *power-increasing-iff* [*simp*]:
$1 < (b::'a::\{ordered\text{-}semidom,recpower\}) ==> (b\ \hat{}\ x \le b\ \hat{}\ y) = (x \le y)$
⟨*proof*⟩

lemma *power-strict-increasing-iff* [*simp*]:
$1 < (b::'a::\{ordered\text{-}semidom,recpower\}) ==> (b\ \hat{}\ x < b\ \hat{}\ y) = (x < y)$
⟨*proof*⟩

lemma *power-le-imp-le-base*:
assumes *le*: $a\ \hat{}\ Suc\ n \le b\ \hat{}\ Suc\ n$
and *ynonneg*: $(0::'a::\{ordered\text{-}semidom,recpower\}) \le b$
shows $a \le b$
⟨*proof*⟩

lemma *power-less-imp-less-base*:
fixes $a\ b :: 'a::\{ordered\text{-}semidom,recpower\}$
assumes *less*: $a\ \hat{}\ n < b\ \hat{}\ n$
assumes *nonneg*: $0 \le b$
shows $a < b$
⟨*proof*⟩

lemma *power-inject-base*:
$[|\ a\ \hat{}\ Suc\ n = b\ \hat{}\ Suc\ n;\ 0 \le a;\ 0 \le b\ |]$
$==> a = (b::'a::\{ordered\text{-}semidom,recpower\})$
⟨*proof*⟩

lemma *power-eq-imp-eq-base*:
fixes $a\ b :: 'a::\{ordered\text{-}semidom,recpower\}$
shows $[\![a\ \hat{}\ n = b\ \hat{}\ n;\ 0 \le a;\ 0 \le b;\ 0 < n]\!] \Longrightarrow a = b$
⟨*proof*⟩

18.2 Exponentiation for the Natural Numbers

instance *nat* :: *power* ⟨*proof*⟩

primrec (*power*)

p ˆ 0 = 1
p ˆ (Suc n) = (p::nat) ∗ (p ˆ n)

instance *nat* :: *recpower*
⟨*proof*⟩

lemma *of-nat-power*:
of-nat (m ˆ n) = (of-nat m::'a::{semiring-1,recpower}) ˆ n
⟨*proof*⟩

lemma *nat-one-le-power* [*simp*]: *1 ≤ i ==> Suc 0 ≤ iˆn*
⟨*proof*⟩

lemma *nat-zero-less-power-iff* [*simp*]: *(xˆn > 0) = (x > (0::nat) | n=0)*
⟨*proof*⟩

Valid for the naturals, but what if *0<i<1*? Premises cannot be weakened: consider the case where *i = (0::'a)*, *m = (1::'a)* and *n = (0::'a)*.

lemma *nat-power-less-imp-less*:
assumes *nonneg*: *0 < (i::nat)*
assumes *less*: *iˆm < iˆn*
shows *m < n*
⟨*proof*⟩

lemma *power-diff*:
assumes *nz*: *a ~= 0*
shows *n <= m ==> (a::'a::{recpower, field}) ˆ (m−n) = (aˆm) / (aˆn)*
⟨*proof*⟩

ML bindings for the general exponentiation theorems

⟨*ML*⟩

ML bindings for the remaining theorems

⟨*ML*⟩

end

19 Divides: The division operators div, mod and the divides relation "dvd"

theory *Divides*
imports *Power*
uses *~~/src/Provers/Arith/cancel-div-mod.ML*
begin

class *div = times +*

fixes *div* :: $'a \Rightarrow 'a \Rightarrow 'a$ (**infixl** *div 70*)
fixes *mod* :: $'a \Rightarrow 'a \Rightarrow 'a$ (**infixl** *mod 70*)

instance *nat* :: *Divides.div*
div-def: *m div n == wfrec (pred-nat^+)*
(%f j. if j<n | n=0 then 0 else Suc (f (j−n))) m
mod-def: *m mod n == wfrec (pred-nat^+)*
(%f j. if j<n | n=0 then j else f (j−n)) m ⟨*proof*⟩

definition (**in** *div*)
dvd :: $'a \Rightarrow 'a \Rightarrow$ *bool* (**infixl** *dvd 50*)
where
[*code func del*]: *m dvd n* ⟷ (∃ *k*. *n* = *m* ∗ *k*)

class *dvd-mod* = *div* + *zero* + — for code generation
assumes *dvd-def-mod* [*code func*]: *x dvd y* ⟷ *y mod x* = *0*

definition
quorem :: (*nat*∗*nat*) ∗ (*nat*∗*nat*) => *bool* **where**

quorem = (%((*a*,*b*), (*q*,*r*)).
a = *b*∗*q* + *r* &
(*if 0<b then 0≤r & r<b else b<r & r ≤0*))

19.1 Initial Lemmas

lemmas *wf-less-trans* =
def-wfrec [*THEN trans*, *OF eq-reflection wf-pred-nat* [*THEN wf-trancl*],
standard]

lemma *mod-eq*: (%*m*. *m mod n*) =
wfrec (pred-nat^+) (%f j. if j<n | n=0 then j else f (j−n))
⟨*proof*⟩

lemma *div-eq*: (%*m*. *m div n*) = *wfrec (pred-nat^+)*
(%f j. if j<n | n=0 then 0 else Suc (f (j−n)))
⟨*proof*⟩

lemma *DIVISION-BY-ZERO-DIV* [*simp*]: *a div 0* = (*0*::*nat*)
⟨*proof*⟩

lemma *DIVISION-BY-ZERO-MOD* [*simp*]: *a mod 0* = (*a*::*nat*)
⟨*proof*⟩

19.2 Remainder

lemma *mod-less* [*simp*]: *m<n ==> m mod n* = (*m*::*nat*)

⟨*proof*⟩

lemma *mod-geq*: ~ $m < (n::nat) ==> m\ mod\ n = (m-n)\ mod\ n$
⟨*proof*⟩

lemma *le-mod-geq*: $(n::nat) \leq m ==> m\ mod\ n = (m-n)\ mod\ n$
⟨*proof*⟩

lemma *mod-if*: $m\ mod\ (n::nat) = (if\ m<n\ then\ m\ else\ (m-n)\ mod\ n)$
⟨*proof*⟩

lemma *mod-1* [*simp*]: $m\ mod\ Suc\ 0 = 0$
⟨*proof*⟩

lemma *mod-self* [*simp*]: $n\ mod\ n = (0::nat)$
⟨*proof*⟩

lemma *mod-add-self2* [*simp*]: $(m+n)\ mod\ n = m\ mod\ (n::nat)$
⟨*proof*⟩

lemma *mod-add-self1* [*simp*]: $(n+m)\ mod\ n = m\ mod\ (n::nat)$
⟨*proof*⟩

lemma *mod-mult-self1* [*simp*]: $(m + k*n)\ mod\ n = m\ mod\ (n::nat)$
⟨*proof*⟩

lemma *mod-mult-self2* [*simp*]: $(m + n*k)\ mod\ n = m\ mod\ (n::nat)$
⟨*proof*⟩

lemma *mod-mult-distrib*: $(m\ mod\ n) * (k::nat) = (m*k)\ mod\ (n*k)$
⟨*proof*⟩

lemma *mod-mult-distrib2*: $(k::nat) * (m\ mod\ n) = (k*m)\ mod\ (k*n)$
⟨*proof*⟩

lemma *mod-mult-self-is-0* [*simp*]: $(m*n)\ mod\ n = (0::nat)$
⟨*proof*⟩

lemma *mod-mult-self1-is-0* [*simp*]: $(n*m)\ mod\ n = (0::nat)$
⟨*proof*⟩

19.3 Quotient

lemma *div-less* [*simp*]: $m<n ==> m\ div\ n = (0::nat)$
⟨*proof*⟩

lemma *div-geq*: [| $0<n$; ~$m<n$ |] $==> m\ div\ n = Suc((m-n)\ div\ n)$
⟨*proof*⟩

lemma *le-div-geq*: $[|\ 0<n;\ \ n \le m\ |] ==> m\ div\ n = Suc((m-n)\ div\ n)$
⟨*proof*⟩

lemma *div-if*: $0<n ==> m\ div\ n = (if\ m<n\ then\ 0\ else\ Suc((m-n)\ div\ n))$
⟨*proof*⟩

lemma *mod-div-equality*: $(m\ div\ n)*n + m\ mod\ n = (m::nat)$
⟨*proof*⟩

lemma *mod-div-equality2*: $n * (m\ div\ n) + m\ mod\ n = (m::nat)$
⟨*proof*⟩

19.4 Simproc for Cancelling Div and Mod

lemma *div-mod-equality*: $((m\ div\ n)*n + m\ mod\ n) + k = (m::nat) + k$
⟨*proof*⟩

lemma *div-mod-equality2*: $(n*(m\ div\ n) + m\ mod\ n) + k = (m::nat) + k$
⟨*proof*⟩

⟨*ML*⟩

lemma *mult-div-cancel*: $(n::nat) * (m\ div\ n) = m - (m\ mod\ n)$
⟨*proof*⟩

lemma *mod-less-divisor* [*simp*]: $0<n ==> m\ mod\ n < (n::nat)$
⟨*proof*⟩

lemma *mod-le-divisor*[*simp*]: $0 < n \Longrightarrow m\ mod\ n \le (n::nat)$
⟨*proof*⟩

lemma *div-mult-self-is-m* [*simp*]: $0<n ==> (m*n)\ div\ n = (m::nat)$
⟨*proof*⟩

lemma *div-mult-self1-is-m* [*simp*]: $0<n ==> (n*m)\ div\ n = (m::nat)$
⟨*proof*⟩

19.5 Proving facts about Quotient and Remainder

lemma *unique-quotient-lemma*:
$[|\ b*q' + r' \ \le b*q + r;\ \ x < b;\ \ r < b\ |]$
$==> q' \le (q::nat)$
⟨*proof*⟩

lemma *unique-quotient*:
$[| \ quorem\ ((a,b),\ (q,r));\ \ quorem\ ((a,b),\ (q',r'));\ \ 0 < b\ |]$
$==> q = q'$
⟨*proof*⟩

lemma *unique-remainder*:
$[| \ quorem\ ((a,b),\ (q,r));\ \ quorem\ ((a,b),\ (q',r'));\ \ 0 < b\ |]$
$==> r = r'$
⟨*proof*⟩

lemma *quorem-div-mod*: $b > 0 ==> quorem\ ((a,\ b),\ (a\ div\ b,\ a\ mod\ b))$
⟨*proof*⟩

lemma *quorem-div*: $[| \ quorem((a,b),(q,r));\ \ b > 0\ |] ==> a\ div\ b = q$
⟨*proof*⟩

lemma *quorem-mod*: $[| \ quorem((a,b),(q,r));\ \ b > 0\ |] ==> a\ mod\ b = r$
⟨*proof*⟩

lemma *div-0* [*simp*]: $0\ div\ m = (0::nat)$
⟨*proof*⟩

lemma *mod-0* [*simp*]: $0\ mod\ m = (0::nat)$
⟨*proof*⟩

lemma *quorem-mult1-eq*:
$[| \ quorem((b,c),(q,r));\ c > 0\ |]$
$==> quorem\ ((a*b,\ c),\ (a*q + a*r\ div\ c,\ a*r\ mod\ c))$
⟨*proof*⟩

lemma *div-mult1-eq*: $(a*b)\ div\ c = a*(b\ div\ c) + a*(b\ mod\ c)\ div\ (c::nat)$
⟨*proof*⟩

lemma *mod-mult1-eq*: $(a*b)\ mod\ c = a*(b\ mod\ c)\ mod\ (c::nat)$
⟨*proof*⟩

lemma *mod-mult1-eq'*: $(a*b)\ mod\ (c::nat) = ((a\ mod\ c) * b)\ mod\ c$
⟨*proof*⟩

lemma *mod-mult-distrib-mod*:
$(a*b)\ mod\ (c::nat) = ((a\ mod\ c) * (b\ mod\ c))\ mod\ c$
⟨*proof*⟩

lemma *quorem-add1-eq*:
[| *quorem*((*a*,*c*),(*aq*,*ar*)); *quorem*((*b*,*c*),(*bq*,*br*)); *c* > *0* |]
==> *quorem* ((*a*+*b*, *c*), (*aq* + *bq* + (*ar*+*br*) *div c*, (*ar*+*br*) *mod c*))
⟨*proof*⟩

lemma *div-add1-eq*:
(*a*+*b*) *div* (*c*::*nat*) = *a div c* + *b div c* + ((*a mod c* + *b mod c*) *div c*)
⟨*proof*⟩

lemma *mod-add1-eq*: (*a*+*b*) *mod* (*c*::*nat*) = (*a mod c* + *b mod c*) *mod c*
⟨*proof*⟩

19.6 Proving $a\ div\ (b * c) = a\ div\ b\ div\ c$

lemma *mod-lemma*: [| (*0*::*nat*) < *c*; *r* < *b* |] ==> *b* * (*q mod c*) + *r* < *b* * *c*
⟨*proof*⟩

lemma *quorem-mult2-eq*: [| *quorem* ((*a*,*b*), (*q*,*r*)); *0* < *b*; *0* < *c* |]
==> *quorem* ((*a*, *b**c*), (*q div c*, *b**(*q mod c*) + *r*))
⟨*proof*⟩

lemma *div-mult2-eq*: *a div* (*b**c*) = (*a div b*) *div* (*c*::*nat*)
⟨*proof*⟩

lemma *mod-mult2-eq*: *a mod* (*b**c*) = *b**(*a div b mod c*) + *a mod* (*b*::*nat*)
⟨*proof*⟩

19.7 Cancellation of Common Factors in Division

lemma *div-mult-mult-lemma*:
[| (*0*::*nat*) < *b*; *0* < *c* |] ==> (*c***a*) *div* (*c***b*) = *a div b*
⟨*proof*⟩

lemma *div-mult-mult1* [*simp*]: (*0*::*nat*) < *c* ==> (*c***a*) *div* (*c***b*) = *a div b*
⟨*proof*⟩

lemma *div-mult-mult2* [*simp*]: (*0*::*nat*) < *c* ==> (*a***c*) *div* (*b***c*) = *a div b*
⟨*proof*⟩

19.8 Further Facts about Quotient and Remainder

lemma *div-1* [*simp*]: *m div Suc 0* = *m*
⟨*proof*⟩

lemma *div-self* [*simp*]: *0*<*n* ==> *n div n* = (*1*::*nat*)
⟨*proof*⟩

lemma *div-add-self2*: *0*<*n* ==> (*m*+*n*) *div n* = *Suc* (*m div n*)
⟨*proof*⟩

lemma *div-add-self1*: $0<n \Longrightarrow (n+m)\ div\ n = Suc\ (m\ div\ n)$
⟨*proof*⟩

lemma *div-mult-self1* [*simp*]: $!!n::nat.\ 0<n \Longrightarrow (m + k*n)\ div\ n = k + m\ div\ n$
⟨*proof*⟩

lemma *div-mult-self2* [*simp*]: $0<n \Longrightarrow (m + n*k)\ div\ n = k + m\ div\ (n::nat)$
⟨*proof*⟩

lemma *div-le-mono* [*rule-format* (*no-asm*)]:
$\forall m::nat.\ m \leq n \longrightarrow (m\ div\ k) \leq (n\ div\ k)$
⟨*proof*⟩

lemma *div-le-mono2*: $!!m::nat.\ [|\ 0<m;\ m\leq n\ |] \Longrightarrow (k\ div\ n) \leq (k\ div\ m)$
⟨*proof*⟩

lemma *div-le-dividend* [*simp*]: $m\ div\ n \leq (m::nat)$
⟨*proof*⟩

lemma *div-less-dividend* [*rule-format*]:
$!!n::nat.\ 1<n \Longrightarrow 0 < m \longrightarrow m\ div\ n < m$
⟨*proof*⟩

declare *div-less-dividend* [*simp*]

A fact for the mutilated chess board

lemma *mod-Suc*: $Suc(m)\ mod\ n = (if\ Suc(m\ mod\ n) = n\ then\ 0\ else\ Suc(m\ mod\ n))$
⟨*proof*⟩

lemma *nat-mod-div-trivial* [*simp*]: $m\ mod\ n\ div\ n = (0 :: nat)$
⟨*proof*⟩

lemma *nat-mod-mod-trivial* [*simp*]: $m\ mod\ n\ mod\ n = (m\ mod\ n :: nat)$
⟨*proof*⟩

19.9 The Divides Relation

lemma *dvdI* [*intro?*]: $n = m * k \Longrightarrow m\ dvd\ n$
⟨*proof*⟩

lemma *dvdE* [*elim?*]: $!!P.\ [|m\ dvd\ n;\ \ !!k.\ n = m*k \Longrightarrow P|] \Longrightarrow P$
⟨*proof*⟩

lemma *dvd-0-right* [*iff*]: *m dvd* (*0*::*nat*)
⟨*proof*⟩

lemma *dvd-0-left*: *0 dvd m* ==> *m* = (*0*::*nat*)
⟨*proof*⟩

lemma *dvd-0-left-iff* [*iff*]: (*0 dvd* (*m*::*nat*)) = (*m* = *0*)
⟨*proof*⟩

declare *dvd-0-left-iff* [*noatp*]

lemma *dvd-1-left* [*iff*]: *Suc 0 dvd k*
⟨*proof*⟩

lemma *dvd-1-iff-1* [*simp*]: (*m dvd Suc 0*) = (*m* = *Suc 0*)
⟨*proof*⟩

lemma *dvd-refl* [*simp*]: *m dvd* (*m*::*nat*)
⟨*proof*⟩

lemma *dvd-trans* [*trans*]: [| *m dvd n*; *n dvd p* |] ==> *m dvd* (*p*::*nat*)
⟨*proof*⟩

lemma *dvd-anti-sym*: [| *m dvd n*; *n dvd m* |] ==> *m* = (*n*::*nat*)
⟨*proof*⟩

op dvd is a partial order

interpretation *dvd*: *order* [*op dvd* $\lambda n\ m$ :: *nat*. *n dvd m* $\wedge$ *m* $\neq$ *n*]
⟨*proof*⟩

lemma *dvd-add*: [| *k dvd m*; *k dvd n* |] ==> *k dvd* (*m*+*n* :: *nat*)
⟨*proof*⟩

lemma *dvd-diff*: [| *k dvd m*; *k dvd n* |] ==> *k dvd* (*m*−*n* :: *nat*)
⟨*proof*⟩

lemma *dvd-diffD*: [| *k dvd m*−*n*; *k dvd n*; *n*≤*m* |] ==> *k dvd* (*m*::*nat*)
⟨*proof*⟩

lemma *dvd-diffD1*: [| *k dvd m*−*n*; *k dvd m*; *n*≤*m* |] ==> *k dvd* (*n*::*nat*)
⟨*proof*⟩

lemma *dvd-mult*: *k dvd n* ==> *k dvd* (*m*∗*n* :: *nat*)
⟨*proof*⟩

lemma *dvd-mult2*: *k dvd m* ==> *k dvd* (*m*∗*n* :: *nat*)
⟨*proof*⟩

lemma *dvd-triv-right* [*iff*]: $k\ dvd\ (m*k :: nat)$
⟨*proof*⟩

lemma *dvd-triv-left* [*iff*]: $k\ dvd\ (k*m :: nat)$
⟨*proof*⟩

lemma *dvd-reduce*: $(k\ dvd\ n + k) = (k\ dvd\ (n::nat))$
⟨*proof*⟩

lemma *dvd-mod*: $!!n::nat.\ [|\ f\ dvd\ m;\ f\ dvd\ n\ |] ==> f\ dvd\ m\ mod\ n$
⟨*proof*⟩

lemma *dvd-mod-imp-dvd*: $[|\ (k::nat)\ dvd\ m\ mod\ n;\ \ k\ dvd\ n\ |] ==> k\ dvd\ m$
⟨*proof*⟩

lemma *dvd-mod-iff*: $k\ dvd\ n ==> ((k::nat)\ dvd\ m\ mod\ n) = (k\ dvd\ m)$
⟨*proof*⟩

lemma *dvd-mult-cancel*: $!!k::nat.\ [|\ k*m\ dvd\ k*n;\ 0<k\ |] ==> m\ dvd\ n$
⟨*proof*⟩

lemma *dvd-mult-cancel1*: $0<m ==> (m*n\ dvd\ m) = (n = (1::nat))$
⟨*proof*⟩

lemma *dvd-mult-cancel2*: $0<m ==> (n*m\ dvd\ m) = (n = (1::nat))$
⟨*proof*⟩

lemma *mult-dvd-mono*: $[|\ i\ dvd\ m;\ j\ dvd\ n|] ==> i*j\ dvd\ (m*n :: nat)$
⟨*proof*⟩

lemma *dvd-mult-left*: $(i*j :: nat)\ dvd\ k ==> i\ dvd\ k$
⟨*proof*⟩

lemma *dvd-mult-right*: $(i*j :: nat)\ dvd\ k ==> j\ dvd\ k$
⟨*proof*⟩

lemma *dvd-imp-le*: $[|\ k\ dvd\ n;\ 0 < n\ |] ==> k \leq (n::nat)$
⟨*proof*⟩

lemma *dvd-eq-mod-eq-0*: $!!k::nat.\ (k\ dvd\ n) = (n\ mod\ k = 0)$
⟨*proof*⟩

lemma *dvd-mult-div-cancel*: $n\ dvd\ m ==> n * (m\ div\ n) = (m::nat)$
⟨*proof*⟩

lemma *le-imp-power-dvd*: $!!i::nat.\ m \leq n ==> i\hat{}m\ dvd\ i\hat{}n$
⟨*proof*⟩

lemma *nat-zero-less-power-iff* [*simp*]: $(x\hat{}n > 0) = (x > (0::nat)\ |\ n{=}0)$

⟨*proof*⟩

lemma *power-le-dvd* [*rule-format*]: $k\hat{}j$ *dvd* n $-->$ $i \le j$ $-->$ $k\hat{}i$ *dvd* $(n::nat)$
⟨*proof*⟩

lemma *power-dvd-imp-le*: $[|i\hat{}m$ *dvd* $i\hat{}n$; $(1::nat) < i|]$ $==>$ $m \le n$
⟨*proof*⟩

lemma *mod-eq-0-iff*: $(m \ mod \ d = 0) = (\exists\, q::nat.\ m = d*q)$
⟨*proof*⟩

lemmas *mod-eq-0D* [*dest*!] = *mod-eq-0-iff* [*THEN iffD1*]

lemma *mod-eqD*: $(m \ mod \ d = r)$ $==>$ $\exists\, q::nat.\ m = r + q*d$
⟨*proof*⟩

lemma *split-div*:
$P(n \ div \ k :: nat) =$
$((k = 0 \longrightarrow P\ 0) \wedge (k \neq 0 \longrightarrow (!i.\ !j<k.\ n = k*i + j \longrightarrow P\ i)))$
(**is** $?P = ?Q$ **is** $\_ = (\_ \wedge (\_ \longrightarrow ?R))$)
⟨*proof*⟩

lemma *split-div-lemma*:
$0 < n \Longrightarrow (n * q \le m \wedge m < n * (Suc\ q)) = (q = ((m::nat)\ div\ n))$
⟨*proof*⟩

theorem *split-div'*:
$P\ ((m::nat)\ div\ n) = ((n = 0 \wedge P\ 0) \vee$
$(\exists\, q.\ (n * q \le m \wedge m < n * (Suc\ q)) \wedge P\ q))$
⟨*proof*⟩

lemma *split-mod*:
$P(n \ mod \ k :: nat) =$
$((k = 0 \longrightarrow P\ n) \wedge (k \neq 0 \longrightarrow (!i.\ !j<k.\ n = k*i + j \longrightarrow P\ j)))$
(**is** $?P = ?Q$ **is** $\_ = (\_ \wedge (\_ \longrightarrow ?R))$)
⟨*proof*⟩

theorem *mod-div-equality'*: $(m::nat)\ mod\ n = m - (m\ div\ n) * n$
⟨*proof*⟩

lemma *div-mod-equality'*:
fixes $m\ n :: nat$
shows $m\ div\ n * n = m - m\ mod\ n$
⟨*proof*⟩

19.10 An "induction" law for modulus arithmetic.

lemma *mod-induct-0*:
  assumes *step*: $\forall\, i<p.\ P\ i \longrightarrow P\ ((Suc\ i)\ mod\ p)$
  and *base*: $P\ i$ **and** *i*: $i<p$
  shows $P\ 0$
⟨*proof*⟩

lemma *mod-induct*:
  assumes *step*: $\forall\, i<p.\ P\ i \longrightarrow P\ ((Suc\ i)\ mod\ p)$
  and *base*: $P\ i$ **and** *i*: $i<p$ **and** *j*: $j<p$
  shows $P\ j$
⟨*proof*⟩

lemma *mod-add-left-eq*: $((a::nat) + b)\ mod\ c = (a\ mod\ c + b)\ mod\ c$
  ⟨*proof*⟩

lemma *mod-add-right-eq*: $(a+b)\ mod\ (c::nat) = (a + (b\ mod\ c))\ mod\ c$
  ⟨*proof*⟩

lemma *mod-div-decomp*:
  fixes $n\ k :: nat$
  obtains $m\ q$ **where** $m = n\ div\ k$ **and** $q = n\ mod\ k$
    and $n = m * k + q$
⟨*proof*⟩

19.11 Code generation for div, mod and dvd on nat

definition [*code func del*]:
  $divmod\ (m::nat)\ n = (m\ div\ n,\ m\ mod\ n)$

lemma *divmod-zero* [*code*]: $divmod\ m\ 0 = (0,\ m)$
  ⟨*proof*⟩

lemma *divmod-succ* [*code*]:
  $divmod\ m\ (Suc\ k) = (if\ m < Suc\ k\ then\ (0,\ m)\ else$
    let
      $(p,\ q) = divmod\ (m - Suc\ k)\ (Suc\ k)$
    $in\ (Suc\ p,\ q))$
  ⟨*proof*⟩

lemma *div-divmod* [*code*]: $m\ div\ n = fst\ (divmod\ m\ n)$
  ⟨*proof*⟩

lemma *mod-divmod* [*code*]: $m\ mod\ n = snd\ (divmod\ m\ n)$
  ⟨*proof*⟩

instance $nat :: dvd\text{-}mod$
  ⟨*proof*⟩

code-modulename *SML*
Divides Nat

code-modulename *OCaml*
Divides Nat

code-modulename *Haskell*
Divides Nat

hide (**open**) *const divmod*

end

20 Record: Extensible records with structural subtyping

theory *Record*
imports *Product-Type*
uses (*Tools/record-package.ML*)
begin

lemma *prop-subst*: $s = t \Longrightarrow PROP\ P\ t \Longrightarrow PROP\ P\ s$
⟨*proof*⟩

lemma *rec-UNIV-I*: $\bigwedge x.\ x \in UNIV \equiv True$
⟨*proof*⟩

lemma *rec-True-simp*: $(True \Longrightarrow PROP\ P) \equiv PROP\ P$
⟨*proof*⟩

constdefs
K-record:: $'a \Rightarrow 'b \Rightarrow 'a$
K-record-apply [*simp*, *code func*]: $K\text{-}record\ c\ x \equiv c$

lemma *K-record-comp* [*simp*]: $(K\text{-}record\ c \circ f) = K\text{-}record\ c$
⟨*proof*⟩

lemma *K-record-cong* [*cong*]: $K\text{-}record\ c\ x = K\text{-}record\ c\ x$
⟨*proof*⟩

20.1 Concrete record syntax

nonterminals
ident field-type field-types field fields update updates
syntax
-constify :: *id* => *ident* (-)

```
-constify            :: longid => ident                  (-)

-field-type          :: [ident, type] => field-type       ((2- ::/ -))
                     :: field-type => field-types          (-)
-field-types         :: [field-type, field-types] => field-types    (-,/ -)
-record-type         :: field-types => type               ((3'(| - |')))
-record-type-scheme :: [field-types, type] => type        ((3'(| -,/ (2... ::/ -) |')))

-field               :: [ident, 'a] => field              ((2- =/ -))
                     :: field => fields                   (-)
-fields              :: [field, fields] => fields         (-,/ -)
-record              :: fields => 'a                      ((3'(| - |')))
-record-scheme       :: [fields, 'a] => 'a                ((3'(| -,/ (2... =/ -) |')))

-update-name         :: idt
-update              :: [ident, 'a] => update             ((2- :=/ -))
                     :: update => updates                 (-)
-updates             :: [update, updates] => updates      (-,/ -)
-record-update       :: ['a, updates] => 'b               (-/(3'(| - |')) [900,0] 900)
```

syntax (*xsymbols*)

```
-record-type         :: field-types => type               ((3⦇-⦈))
-record-type-scheme :: [field-types, type] => type        ((3⦇-,/ (2... ::/ -)⦈))
-record              :: fields => 'a                      ((3⦇-⦈))
-record-scheme       :: [fields, 'a] => 'a                ((3⦇-,/ (2... =/ -)⦈))
-record-update       :: ['a, updates] => 'b               (-/(3⦇-⦈) [900,0] 900)
```

⟨*ML*⟩

end

21 Hilbert-Choice: Hilbert's Epsilon-Operator and the Axiom of Choice

theory *Hilbert-Choice*
imports *Nat*
uses (*Tools/meson.ML*) (*Tools/specification-package.ML*)
begin

21.1 Hilbert's epsilon

axiomatization
Eps :: ('*a* => *bool*) => '*a*
where
someI: *P x* ==> *P* (*Eps P*)

syntax (*epsilon*)

-Eps :: [*pttrn*, *bool*] => ′*a* ((*3ϵ -./ -*) [*0*, *10*] *10*)
syntax (*HOL*)
-Eps :: [*pttrn*, *bool*] => ′*a* ((*3@ -./ -*) [*0*, *10*] *10*)
syntax
-Eps :: [*pttrn*, *bool*] => ′*a* ((*3SOME -./ -*) [*0*, *10*] *10*)
translations
SOME x. P == *CONST Eps* (%*x. P*)

⟨*ML*⟩

constdefs
inv :: (′*a* => ′*b*) => (′*b* => ′*a*)
inv(*f* :: ′*a* => ′*b*) == %*y. SOME x. f x* = *y*

Inv :: ′*a set* => (′*a* => ′*b*) => (′*b* => ′*a*)
Inv A f == %*x. SOME y. y* ∈ *A* & *f y* = *x*

21.2 Hilbert's Epsilon-operator

Easier to apply than *someI* if the witness comes from an existential formula

lemma *someI-ex* [*elim?*]: ∃ *x. P x* ==> *P* (*SOME x. P x*)
⟨*proof*⟩

Easier to apply than *someI* because the conclusion has only one occurrence of *P*.

lemma *someI2*: [| *P a*; !!*x. P x* ==> *Q x* |] ==> *Q* (*SOME x. P x*)
⟨*proof*⟩

Easier to apply than *someI2* if the witness comes from an existential formula

lemma *someI2-ex*: [| ∃ *a. P a*; !!*x. P x* ==> *Q x* |] ==> *Q* (*SOME x. P x*)
⟨*proof*⟩

lemma *some-equality* [*intro*]:
[| *P a*; !!*x. P x* ==> *x*=*a* |] ==> (*SOME x. P x*) = *a*
⟨*proof*⟩

lemma *some1-equality*: [| *EX*!*x. P x*; *P a* |] ==> (*SOME x. P x*) = *a*
⟨*proof*⟩

lemma *some-eq-ex*: *P* (*SOME x. P x*) = (∃ *x. P x*)
⟨*proof*⟩

lemma *some-eq-trivial* [*simp*]: (*SOME y. y*=*x*) = *x*
⟨*proof*⟩

lemma *some-sym-eq-trivial* [*simp*]: (*SOME y. x*=*y*) = *x*
⟨*proof*⟩

21.3 Axiom of Choice, Proved Using the Description Operator

Used in *Tools/meson.ML*

lemma *choice*: $\forall x. \exists y. Q\ x\ y ==> \exists f. \forall x. Q\ x\ (f\ x)$
⟨*proof*⟩

lemma *bchoice*: $\forall x{\in}S. \exists y. Q\ x\ y ==> \exists f. \forall x{\in}S. Q\ x\ (f\ x)$
⟨*proof*⟩

21.4 Function Inverse

lemma *inv-id* [*simp*]: $inv\ id = id$
⟨*proof*⟩

A one-to-one function has an inverse.

lemma *inv-f-f* [*simp*]: $inj\ f ==> inv\ f\ (f\ x) = x$
⟨*proof*⟩

lemma *inv-f-eq*: $[|\ inj\ f;\ \ f\ x = y\ |] ==> inv\ f\ y = x$
⟨*proof*⟩

lemma *inj-imp-inv-eq*: $[|\ inj\ f; \forall x.\ f(g\ x) = x\ |] ==> inv\ f = g$
⟨*proof*⟩

But is it useful?

lemma *inj-transfer*:
  assumes *injf*: $inj\ f$ **and** *minor*: $!!y.\ y \in range(f) ==> P(inv\ f\ y)$
  shows $P\ x$
⟨*proof*⟩

lemma *inj-iff*: $(inj\ f) = (inv\ f\ o\ f = id)$
⟨*proof*⟩

lemma *inv-o-cancel*[*simp*]: $inj\ f ==> inv\ f\ o\ f = id$
⟨*proof*⟩

lemma *o-inv-o-cancel*[*simp*]: $inj\ f ==> g\ o\ inv\ f\ o\ f = g$
⟨*proof*⟩

lemma *inv-image-cancel*[*simp*]:
  $inj\ f ==> inv\ f\ `\ f\ `\ S = S$
⟨*proof*⟩

lemma *inj-imp-surj-inv*: $inj\ f ==> surj\ (inv\ f)$
⟨*proof*⟩

lemma *f-inv-f*: $y \in range(f) ==> f(inv\ f\ y) = y$

⟨*proof*⟩

lemma *surj-f-inv-f*: *surj f* ==> *f*(*inv f y*) = *y*
⟨*proof*⟩

lemma *inv-injective*:
  assumes *eq*: *inv f x* = *inv f y*
    and *x*: *x*: *range f*
    and *y*: *y*: *range f*
  shows *x*=*y*
⟨*proof*⟩

lemma *inj-on-inv*: *A* <= *range*(*f*) ==> *inj-on* (*inv f*) *A*
⟨*proof*⟩

lemma *surj-imp-inj-inv*: *surj f* ==> *inj* (*inv f*)
⟨*proof*⟩

lemma *surj-iff*: (*surj f*) = (*f o inv f* = *id*)
⟨*proof*⟩

lemma *surj-imp-inv-eq*: [| *surj f*; ∀ *x*. *g*(*f x*) = *x* |] ==> *inv f* = *g*
⟨*proof*⟩

lemma *bij-imp-bij-inv*: *bij f* ==> *bij* (*inv f*)
⟨*proof*⟩

lemma *inv-equality*: [| !!*x*. *g* (*f x*) = *x*; !!*y*. *f* (*g y*) = *y* |] ==> *inv f* = *g*
⟨*proof*⟩

lemma *inv-inv-eq*: *bij f* ==> *inv* (*inv f*) = *f*
⟨*proof*⟩

lemma *o-inv-distrib*: [| *bij f*; *bij g* |] ==> *inv* (*f o g*) = *inv g o inv f*
⟨*proof*⟩

lemma *image-surj-f-inv-f*: *surj f* ==> *f* ‘ (*inv f* ‘ *A*) = *A*
⟨*proof*⟩

lemma *image-inv-f-f*: *inj f* ==> (*inv f*) ‘ (*f* ‘ *A*) = *A*
⟨*proof*⟩

lemma *inv-image-comp*: *inj f* ==> *inv f* ‘ (*f*‘*X*) = *X*
⟨*proof*⟩

lemma *bij-image-Collect-eq*: *bij f* ==> *f* ‘ *Collect P* = {*y*. *P* (*inv f y*)}

⟨*proof*⟩

lemma *bij-vimage-eq-inv-image*: *bij f ==> f −' A = inv f ' A*
⟨*proof*⟩

21.5 Inverse of a PI-function (restricted domain)

lemma *Inv-f-f*: [| *inj-on f A*; $x \in A$ |] ==> *Inv A f* (*f x*) = *x*
⟨*proof*⟩

lemma *f-Inv-f*: $y \in f`A$ ==> *f* (*Inv A f y*) = *y*
⟨*proof*⟩

lemma *Inv-injective*:
  assumes *eq*: *Inv A f x* = *Inv A f y*
    and *x*: *x*: *f`A*
    and *y*: *y*: *f`A*
  shows *x=y*
⟨*proof*⟩

lemma *inj-on-Inv*: *B* <= *f`A* ==> *inj-on* (*Inv A f*) *B*
⟨*proof*⟩

lemma *Inv-mem*: [| *f ' A* = *B*; $x \in B$ |] ==> *Inv A f x* $\in$ *A*
⟨*proof*⟩

lemma *Inv-f-eq*: [| *inj-on f A*; *f x* = *y*; $x \in A$ |] ==> *Inv A f y* = *x*
⟨*proof*⟩

lemma *Inv-comp*:
  [| *inj-on f* (*g ' A*); *inj-on g A*; $x \in$ *f ' g ' A* |] ==>
  Inv A (*f o g*) *x* = (*Inv A g o Inv* (*g ' A*) *f*) *x*
⟨*proof*⟩

21.6 Other Consequences of Hilbert's Epsilon

Hilbert's Epsilon and the *split* Operator

Looping simprule

lemma *split-paired-Eps*: (*SOME x. P x*) = (*SOME* (*a*,*b*). *P*(*a*,*b*))
⟨*proof*⟩

lemma *Eps-split*: *Eps* (*split P*) = (*SOME xy. P* (*fst xy*) (*snd xy*))
⟨*proof*⟩

lemma *Eps-split-eq* [*simp*]: (@(x',y'). $x = x'$ & $y = y'$) = (*x*,*y*)
⟨*proof*⟩

A relation is wellfounded iff it has no infinite descending chain

lemma *wf-iff-no-infinite-down-chain*:
$wf\ r = (\sim(\exists f.\ \forall i.\ (f(Suc\ i),f\ i) \in r))$
⟨*proof*⟩

A dynamically-scoped fact for TFL

lemma *tfl-some*: $\forall P\ x.\ P\ x\ {-}{-}{>}\ P\ (Eps\ P)$
⟨*proof*⟩

21.7 Least value operator

constdefs
$LeastM :: [{'}a => {'}b{::}ord,\ {'}a => bool] => {'}a$
$LeastM\ m\ P == SOME\ x.\ P\ x\ \&\ (\forall y.\ P\ y\ {-}{-}{>}\ m\ x <= m\ y)$

syntax
$\text{-}LeastM :: [pttrn,\ {'}a => {'}b{::}ord,\ bool] => {'}a \quad (LEAST\ \text{-}\ WRT\ \text{-}.\ \text{-}\ [0,\ 4,\ 10]\ 10)$
translations
$LEAST\ x\ WRT\ m.\ P == LeastM\ m\ (\%x.\ P)$

lemma *LeastMI2*:
$P\ x ==> (!!y.\ P\ y ==> m\ x <= m\ y)$
$==> (!!x.\ P\ x ==> \forall y.\ P\ y\ {-}{-}{>}\ m\ x \leq m\ y ==> Q\ x)$
$==> Q\ (LeastM\ m\ P)$
⟨*proof*⟩

lemma *LeastM-equality*:
$P\ k ==> (!!x.\ P\ x ==> m\ k <= m\ x)$
$==> m\ (LEAST\ x\ WRT\ m.\ P\ x) = (m\ k{::}{'}a{::}order)$
⟨*proof*⟩

lemma *wf-linord-ex-has-least*:
$wf\ r ==> \forall x\ y.\ ((x,y){:}r\hat{\ }{+}) = ((y,x)\sim{:}r\hat{\ }{*}) ==> P\ k$
$==> \exists x.\ P\ x\ \&\ (!y.\ P\ y\ {-}{-}{>}\ (m\ x,m\ y){:}r\hat{\ }{*})$
⟨*proof*⟩

lemma *ex-has-least-nat*:
$P\ k ==> \exists x.\ P\ x\ \&\ (\forall y.\ P\ y\ {-}{-}{>}\ m\ x <= (m\ y{::}nat))$
⟨*proof*⟩

lemma *LeastM-nat-lemma*:
$P\ k ==> P\ (LeastM\ m\ P)\ \&\ (\forall y.\ P\ y\ {-}{-}{>}\ m\ (LeastM\ m\ P) <= (m\ y{::}nat))$
⟨*proof*⟩

lemmas *LeastM-natI* = *LeastM-nat-lemma* [*THEN conjunct1*, *standard*]

lemma *LeastM-nat-le*: $P\ x ==> m\ (LeastM\ m\ P) <= (m\ x{::}nat)$
⟨*proof*⟩

21.8 Greatest value operator

constdefs
  GreatestM :: ['a => 'b::ord, 'a => bool] => 'a
  GreatestM m P == SOME x. P x & (∀ y. P y −−> m y <= m x)

  Greatest :: ('a::ord => bool) => 'a   (**binder** *GREATEST 10*)
  Greatest == GreatestM (%x. x)

syntax
  -*GreatestM* :: [pttrn, 'a=>'b::ord, bool] => 'a
    (*GREATEST - WRT -. -* [0, 4, 10] 10)

translations
  GREATEST x WRT m. P == GreatestM m (%x. P)

lemma *GreatestMI2*:
  P x ==> (!!y. P y ==> m y <= m x)
    ==> (!!x. P x ==> ∀ y. P y −−> m y ≤ m x ==> Q x)
    ==> Q (GreatestM m P)
  ⟨*proof*⟩

lemma *GreatestM-equality*:
 P k ==> (!!x. P x ==> m x <= m k)
    ==> m (GREATEST x WRT m. P x) = (m k::'a::order)
  ⟨*proof*⟩

lemma *Greatest-equality*:
  P (k::'a::order) ==> (!!x. P x ==> x <= k) ==> (GREATEST x. P x) = k
  ⟨*proof*⟩

lemma *ex-has-greatest-nat-lemma*:
  P k ==> ∀ x. P x −−> (∃ y. P y & ~ ((m y::nat) <= m x))
    ==> ∃ y. P y & ~ (m y < m k + n)
  ⟨*proof*⟩

lemma *ex-has-greatest-nat*:
  P k ==> ∀ y. P y −−> m y < b
    ==> ∃ x. P x & (∀ y. P y −−> (m y::nat) <= m x)
  ⟨*proof*⟩

lemma *GreatestM-nat-lemma*:
  P k ==> ∀ y. P y −−> m y < b
    ==> P (GreatestM m P) & (∀ y. P y −−> (m y::nat) <= m (GreatestM m
P))
  ⟨*proof*⟩

lemmas *GreatestM-natI = GreatestM-nat-lemma [THEN conjunct1, standard]*

lemma *GreatestM-nat-le*:

$P\ x ==> \forall y.\ P\ y --> m\ y < b$
$==> (m\ x::nat) <= m\ (GreatestM\ m\ P)$
⟨*proof*⟩

Specialization to *GREATEST*.

lemma *GreatestI*: $P\ (k::nat) ==> \forall y.\ P\ y --> y < b ==> P\ (GREATEST\ x.\ P\ x)$
⟨*proof*⟩

lemma *Greatest-le*:
$P\ x ==> \forall y.\ P\ y --> y < b ==> (x::nat) <= (GREATEST\ x.\ P\ x)$
⟨*proof*⟩

21.9 The Meson proof procedure

21.9.1 Negation Normal Form

de Morgan laws

lemma *meson-not-conjD*: $\tilde{}(P\&Q) ==> \tilde{}P \mid \tilde{}Q$
and *meson-not-disjD*: $\tilde{}(P|Q) ==> \tilde{}P\ \&\ \tilde{}Q$
and *meson-not-notD*: $\tilde{}\tilde{}P ==> P$
and *meson-not-allD*: $!!P.\ \tilde{}(\forall x.\ P(x)) ==> \exists x.\ \tilde{}P(x)$
and *meson-not-exD*: $!!P.\ \tilde{}(\exists x.\ P(x)) ==> \forall x.\ \tilde{}P(x)$
⟨*proof*⟩

Removal of $-->$ and $<->$ (positive and negative occurrences)

lemma *meson-imp-to-disjD*: $P-->Q ==> \tilde{}P \mid Q$
and *meson-not-impD*: $\tilde{}(P-->Q) ==> P\ \&\ \tilde{}Q$
and *meson-iff-to-disjD*: $P=Q ==> (\tilde{}P \mid Q)\ \&\ (\tilde{}Q \mid P)$
and *meson-not-iffD*: $\tilde{}(P=Q) ==> (P \mid Q)\ \&\ (\tilde{}P \mid \tilde{}Q)$
— Much more efficient than $P \wedge \neg Q \vee Q \wedge \neg P$ for computing CNF
and *meson-not-refl-disj-D*: $x\ \tilde{}= x \mid P ==> P$
⟨*proof*⟩

21.9.2 Pulling out the existential quantifiers

Conjunction

lemma *meson-conj-exD1*: $!!P\ Q.\ (\exists x.\ P(x))\ \&\ Q ==> \exists x.\ P(x)\ \&\ Q$
and *meson-conj-exD2*: $!!P\ Q.\ P\ \&\ (\exists x.\ Q(x)) ==> \exists x.\ P\ \&\ Q(x)$
⟨*proof*⟩

Disjunction

lemma *meson-disj-exD*: $!!P\ Q.\ (\exists x.\ P(x)) \mid (\exists x.\ Q(x)) ==> \exists x.\ P(x) \mid Q(x)$
— DO NOT USE with forall-Skolemization: makes fewer schematic variables!!
— With ex-Skolemization, makes fewer Skolem constants
and *meson-disj-exD1*: $!!P\ Q.\ (\exists x.\ P(x)) \mid Q ==> \exists x.\ P(x) \mid Q$
and *meson-disj-exD2*: $!!P\ Q.\ P \mid (\exists x.\ Q(x)) ==> \exists x.\ P \mid Q(x)$
⟨*proof*⟩

21.9.3 Generating clauses for the Meson Proof Procedure

Disjunctions

lemma *meson-disj-assoc*: $(P|Q)|R$ ==> $P|(Q|R)$
and *meson-disj-comm*: $P|Q$ ==> $Q|P$
and *meson-disj-FalseD1*: *False*$|P$ ==> P
and *meson-disj-FalseD2*: $P|$*False* ==> P
⟨*proof*⟩

21.10 Lemmas for Meson, the Model Elimination Procedure

Generation of contrapositives

Inserts negated disjunct after removing the negation; P is a literal. Model elimination requires assuming the negation of every attempted subgoal, hence the negated disjuncts.

lemma *make-neg-rule*: ~$P|Q$ ==> ((~P==>P) ==> Q)
⟨*proof*⟩

Version for Plaisted's "Postive refinement" of the Meson procedure

lemma *make-refined-neg-rule*: ~$P|Q$ ==> (P ==> Q)
⟨*proof*⟩

P should be a literal

lemma *make-pos-rule*: $P|Q$ ==> ((P==>~P) ==> Q)
⟨*proof*⟩

Versions of *make-neg-rule* and *make-pos-rule* that don't insert new assumptions, for ordinary resolution.

lemmas *make-neg-rule'* = *make-refined-neg-rule*

lemma *make-pos-rule'*: [|$P|Q$; ~P|] ==> Q
⟨*proof*⟩

Generation of a goal clause – put away the final literal

lemma *make-neg-goal*: ~P ==> ((~P==>P) ==> *False*)
⟨*proof*⟩

lemma *make-pos-goal*: P ==> ((P==>~P) ==> *False*)
⟨*proof*⟩

21.10.1 Lemmas for Forward Proof

There is a similarity to congruence rules

lemma *conj-forward*: [| P'&Q'; P' ==> P; Q' ==> Q |] ==> P&Q
⟨*proof*⟩

lemma *disj-forward*: $[| P'|Q'; \; P' ==> P; \; Q' ==> Q |] ==> P|Q$
⟨*proof*⟩

lemma *disj-forward2*:
$[| P'|Q'; \; P' ==> P; \; [| Q'; P==>False |] ==> Q |] ==> P|Q$
⟨*proof*⟩

lemma *all-forward*: $[| \forall x. P'(x); \; !!x. P'(x) ==> P(x) |] ==> \forall x. P(x)$
⟨*proof*⟩

lemma *ex-forward*: $[| \exists x. P'(x); \; !!x. P'(x) ==> P(x) |] ==> \exists x. P(x)$
⟨*proof*⟩

Many of these bindings are used by the ATP linkup, and not just by legacy proof scripts.

⟨*ML*⟩

21.11 Meson package

⟨*ML*⟩

21.12 Specification package – Hilbertized version

lemma *exE-some*: $[| Ex\ P \; ; \; c == Eps\ P |] ==> P\ c$
⟨*proof*⟩

⟨*ML*⟩

end

22 Finite-Set: Finite sets

theory *Finite-Set*
imports *Divides*
begin

22.1 Definition and basic properties

inductive *finite* :: $'a\ set => bool$
where
emptyI [*simp*, *intro*!]: *finite* {}
| *insertI* [*simp*, *intro*!]: *finite A* ==> *finite* (*insert a A*)

lemma *ex-new-if-finite*: — does not depend on def of finite at all
assumes ¬ *finite* (*UNIV* :: $'a\ set$) **and** *finite A*
shows $\exists a::'a. \; a \notin A$

⟨*proof*⟩

lemma *finite-induct* [*case-names empty insert*, *induct set*: *finite*]:
finite F ==>
P {} ==> (!!x F. *finite F* ==> $x \notin F$ ==> P F ==> P (*insert x F*)) ==> P F
— Discharging $x \notin F$ entails extra work.
⟨*proof*⟩

lemma *finite-ne-induct*[*case-names singleton insert*, *consumes 2*]:
assumes *fin*: *finite F* **shows** $F \neq \{\} \Longrightarrow$
$[\![\bigwedge x.\ P\{x\};$
$\bigwedge x\ F.\ [\![\mathit{finite}\ F;\ F \neq \{\};\ x \notin F;\ P\ F\ ]\!] \Longrightarrow P\ (\mathit{insert}\ x\ F)\ ]\!]$
$\Longrightarrow P\ F$
⟨*proof*⟩

lemma *finite-subset-induct* [*consumes 2*, *case-names empty insert*]:
assumes *finite F* **and** $F \subseteq A$
and *empty*: P {}
and *insert*: !!a F. *finite F* ==> $a \in A$ ==> $a \notin F$ ==> P F ==> P (*insert a F*)
shows P F
⟨*proof*⟩

Finite sets are the images of initial segments of natural numbers:

lemma *finite-imp-nat-seg-image-inj-on*:
assumes *fin*: *finite A*
shows $\exists$ (n::*nat*) f. $A = f$ ‘ {i. i<n} & *inj-on f* {i. i<n}
⟨*proof*⟩

lemma *nat-seg-image-imp-finite*:
!!f A. $A = f$ ‘ {i::*nat*. i<n} $\Longrightarrow$ *finite A*
⟨*proof*⟩

lemma *finite-conv-nat-seg-image*:
finite A = ($\exists$ (n::*nat*) f. $A = f$ ‘ {i::*nat*. i<n})
⟨*proof*⟩

22.1.1 Finiteness and set theoretic constructions

lemma *finite-UnI*: *finite F* ==> *finite G* ==> *finite* (*F Un G*)
— The union of two finite sets is finite.
⟨*proof*⟩

lemma *finite-subset*: $A \subseteq B$ ==> *finite B* ==> *finite A*
— Every subset of a finite set is finite.
⟨*proof*⟩

lemma *finite-Collect-subset*[*simp*]: *finite A* $\Longrightarrow$ *finite*{$x \in A$. P x}

⟨*proof*⟩

lemma *finite-Un* [*iff*]: *finite* (*F Un G*) = (*finite F* & *finite G*)
⟨*proof*⟩

lemma *finite-Int* [*simp*, *intro*]: *finite F* | *finite G* ==> *finite* (*F Int G*)
— The converse obviously fails.
⟨*proof*⟩

lemma *finite-insert* [*simp*]: *finite* (*insert a A*) = *finite A*
⟨*proof*⟩

lemma *finite-Union*[*simp*, *intro*]:
⟦ *finite A*; !!M. $M \in A \Longrightarrow$ *finite M* ⟧ $\Longrightarrow$ *finite*($\bigcup A$)
⟨*proof*⟩

lemma *finite-empty-induct*:
assumes *finite A*
and *P A*
and !!*a A*. *finite A* ==> *a*:*A* ==> *P A* ==> *P* ($A - \{a\}$)
shows *P* {}
⟨*proof*⟩

lemma *finite-Diff* [*simp*]: *finite B* ==> *finite* ($B - Ba$)
⟨*proof*⟩

lemma *finite-Diff-insert* [*iff*]: *finite* (A − *insert a B*) = *finite* ($A - B$)
⟨*proof*⟩

lemma *finite-Diff-singleton* [*simp*]: *finite* ($A - \{a\}$) = *finite A*
⟨*proof*⟩

Image and Inverse Image over Finite Sets

lemma *finite-imageI*[*simp*]: *finite F* ==> *finite* (*h* ' *F*)
— The image of a finite set is finite.
⟨*proof*⟩

lemma *finite-surj*: *finite A* ==> *B* <= *f* ' *A* ==> *finite B*
⟨*proof*⟩

lemma *finite-range-imageI*:
finite (*range g*) ==> *finite* (*range* (%*x*. *f* (*g x*)))
⟨*proof*⟩

lemma *finite-imageD*: *finite* (*f*'*A*) ==> *inj-on f A* ==> *finite A*
⟨*proof*⟩

lemma *inj-vimage-singleton*: *inj f* ==> *f*−'{*a*} $\subseteq$ {*THE x*. *f x* = *a*}

— The inverse image of a singleton under an injective function is included in a singleton.
⟨*proof*⟩

lemma *finite-vimageI*: $[|finite\ F;\ inj\ h|] ==> finite\ (h\ -`\ F)$
— The inverse image of a finite set under an injective function is finite.
⟨*proof*⟩

The finite UNION of finite sets

lemma *finite-UN-I*: $finite\ A ==> (!!a.\ a{:}A ==> finite\ (B\ a)) ==> finite\ (UN\ a{:}A.\ B\ a)$
⟨*proof*⟩

Strengthen RHS to $(\forall x{\in}A.\ finite\ (B\ x)) \wedge finite\ \{x \in A.\ B\ x \neq \{\}\}$? We'd need to prove $finite\ C \Longrightarrow \forall A\ B.\ UNION\ A\ B \subseteq C \longrightarrow finite\ \{x \in A.\ B\ x \neq \{\}\}$ by induction.

lemma *finite-UN* [*simp*]: $finite\ A ==> finite\ (UNION\ A\ B) = (ALL\ x{:}A.\ finite\ (B\ x))$
⟨*proof*⟩

lemma *finite-Plus*: $[|\ finite\ A;\ finite\ B\ |] ==> finite\ (A <+> B)$
⟨*proof*⟩

Sigma of finite sets

lemma *finite-SigmaI* [*simp*]:
$finite\ A ==> (!!a.\ a{:}A ==> finite\ (B\ a)) ==> finite\ (SIGMA\ a{:}A.\ B\ a)$
⟨*proof*⟩

lemma *finite-cartesian-product*: $[|\ finite\ A;\ finite\ B\ |] ==>$
$finite\ (A <*> B)$
⟨*proof*⟩

lemma *finite-Prod-UNIV*:
$finite\ (UNIV{::}'a\ set) ==> finite\ (UNIV{::}'b\ set) ==> finite\ (UNIV{::}('a * 'b)\ set)$
⟨*proof*⟩

lemma *finite-cartesian-productD1*:
$[|\ finite\ (A <*> B);\ B \neq \{\}\ |] ==> finite\ A$
⟨*proof*⟩

lemma *finite-cartesian-productD2*:
$[|\ finite\ (A <*> B);\ A \neq \{\}\ |] ==> finite\ B$
⟨*proof*⟩

The powerset of a finite set

lemma *finite-Pow-iff* [*iff*]: $finite\ (Pow\ A) = finite\ A$

⟨*proof*⟩

lemma *finite-UnionD*: *finite*$(\bigcup A) \Longrightarrow$ *finite* A
⟨*proof*⟩

lemma *finite-converse* [*iff*]: *finite* $(r\hat{\ }{-1})$ = *finite* r
⟨*proof*⟩

Finiteness of transitive closure (Thanks to Sidi Ehmety)

lemma *finite-Field*: *finite* r ==> *finite* (*Field* r)
— A finite relation has a finite field (= *domain* ∪ *range*.
⟨*proof*⟩

lemma *trancl-subset-Field2*: $r\hat{\ }+$ <= *Field* $r \times$ *Field* r
⟨*proof*⟩

lemma *finite-trancl*: *finite* $(r\hat{\ }+)$ = *finite* r
⟨*proof*⟩

22.2 A fold functional for finite sets

The intended behaviour is *fold* f g z $\{x_1, ..., x_n\} = f\ (g\ x_1)\ (\ldots\ (f\ (g\ x_n)\ z)\ldots)$ if f is associative-commutative. For an application of *fold* se the definitions of sums and products over finite sets.

inductive
foldSet :: ('a => 'a => 'a) => ('b => 'a) => 'a => 'b *set* => 'a => *bool*
for f :: 'a => 'a => 'a
and g :: 'b => 'a
and z :: 'a
where
emptyI [*intro*]: *foldSet* f g z {} z
| *insertI* [*intro*]:
⟦ $x \notin A$; *foldSet* f g z A y ⟧
$\Longrightarrow$ *foldSet* f g z (*insert* x A) (f (g x) y)

inductive-cases *empty-foldSetE* [*elim*!]: *foldSet* f g z {} x

constdefs
fold :: ('a => 'a => 'a) => ('b => 'a) => 'a => 'b *set* => 'a
fold f g z A == *THE* x. *foldSet* f g z A x

A tempting alternative for the definiens is *if finite A then THE x. foldSet f g e A x else e*. It allows the removal of finiteness assumptions from the theorems *fold-commute*, *fold-reindex* and *fold-distrib*. The proofs become ugly, with *rule-format*. It is not worth the effort.

lemma *Diff1-foldSet*:
foldSet f g z $(A - \{x\})$ y ==> x: A ==> *foldSet f g z A* $(f\ (g\ x)\ y)$
⟨*proof*⟩

lemma *foldSet-imp-finite*: *foldSet f g z A x*==> *finite A*
⟨*proof*⟩

lemma *finite-imp-foldSet*: *finite A* ==> *EX x*. *foldSet f g z A x*
⟨*proof*⟩

22.2.1 Commutative monoids

locale *ACf* =
fixes f :: $'a$ => $'a$ => $'a$ (**infixl** · *70*)
assumes *commute*: $x \cdot y = y \cdot x$
and *assoc*: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$
begin

lemma *left-commute*: $x \cdot (y \cdot z) = y \cdot (x \cdot z)$
⟨*proof*⟩

lemmas *AC* = *assoc commute left-commute*

end

locale *ACe* = *ACf* +
fixes e :: $'a$
assumes *ident* [*simp*]: $x \cdot e = x$
begin

lemma *left-ident* [*simp*]: $e \cdot x = x$
⟨*proof*⟩

end

locale *ACIf* = *ACf* +
assumes *idem*: $x \cdot x = x$
begin

lemma *idem2*: $x \cdot (x \cdot y) = x \cdot y$
⟨*proof*⟩

lemmas *ACI* = *AC idem idem2*

end

22.2.2 From *foldSet* to *fold*

lemma *image-less-Suc*: h ‘ $\{i.\ i < Suc\ m\}$ = *insert* $(h\ m)$ $(h$ ‘ $\{i.\ i < m\})$
⟨*proof*⟩

lemma *insert-image-inj-on-eq*:
$[|insert\ (h\ m)\ A = h\ `\ \{i.\ i < Suc\ m\};\ h\ m \notin A;$
$inj\text{-}on\ h\ \{i.\ i < Suc\ m\}|]$
$==> A = h\ `\ \{i.\ i < m\}$
⟨*proof*⟩

lemma *insert-inj-onE*:
assumes *aA*: $insert\ a\ A = h`\{i::nat.\ i<n\}$ **and** *anot*: $a \notin A$
and *inj-on*: $inj\text{-}on\ h\ \{i::nat.\ i<n\}$
shows $\exists\, hm\ m.\ inj\text{-}on\ hm\ \{i::nat.\ i<m\}\ \&\ A = hm\ `\ \{i.\ i<m\}\ \&\ m < n$
⟨*proof*⟩

lemma (**in** *ACf*) *foldSet-determ-aux*:
$!!A\ x\ x'\ h.\ [\![\ A = h`\{i::nat.\ i<n\};\ inj\text{-}on\ h\ \{i.\ i<n\};$
$foldSet\ f\ g\ z\ A\ x;\ foldSet\ f\ g\ z\ A\ x'\ ]\!]$
$\Longrightarrow x' = x$
⟨*proof*⟩

lemma (**in** *ACf*) *foldSet-determ*:
$foldSet\ f\ g\ z\ A\ x ==> foldSet\ f\ g\ z\ A\ y ==> y = x$
⟨*proof*⟩

lemma (**in** *ACf*) *fold-equality*: $foldSet\ f\ g\ z\ A\ y ==> fold\ f\ g\ z\ A = y$
⟨*proof*⟩

The base case for *fold*:

lemma *fold-empty* [*simp*]: $fold\ f\ g\ z\ \{\} = z$
⟨*proof*⟩

lemma (**in** *ACf*) *fold-insert-aux*: $x \notin A ==>$
$(foldSet\ f\ g\ z\ (insert\ x\ A)\ v) =$
$(EX\ y.\ foldSet\ f\ g\ z\ A\ y\ \&\ v = f\ (g\ x)\ y)$
⟨*proof*⟩

The recursion equation for *fold*:

lemma (**in** *ACf*) *fold-insert*[*simp*]:
$finite\ A ==> x \notin A ==> fold\ f\ g\ z\ (insert\ x\ A) = f\ (g\ x)\ (fold\ f\ g\ z\ A)$
⟨*proof*⟩

lemma (**in** *ACf*) *fold-rec*:
assumes *fin*: *finite A* **and** *a*: $a{:}A$
shows $fold\ f\ g\ z\ A = f\ (g\ a)\ (fold\ f\ g\ z\ (A - \{a\}))$
⟨*proof*⟩

A simplified version for idempotent functions:

lemma (**in** *ACIf*) *fold-insert-idem*:
assumes *finA*: *finite A*

shows *fold f g z* (*insert a A*) = *g a* · *fold f g z A*
⟨*proof*⟩

lemma (**in** *ACIf*) *foldI-conv-id*:
finite A ⟹ *fold f g z A* = *fold f id z* (*g* ' *A*)
⟨*proof*⟩

22.2.3 Lemmas about *fold*

lemma (**in** *ACf*) *fold-commute*:
finite A ==> (!!*z*. *f x* (*fold f g z A*) = *fold f g* (*f x z*) *A*)
⟨*proof*⟩

lemma (**in** *ACf*) *fold-nest-Un-Int*:
finite A ==> *finite B*
==> *fold f g* (*fold f g z B*) *A* = *fold f g* (*fold f g z* (*A Int B*)) (*A Un B*)
⟨*proof*⟩

lemma (**in** *ACf*) *fold-nest-Un-disjoint*:
finite A ==> *finite B* ==> *A Int B* = {}
==> *fold f g z* (*A Un B*) = *fold f g* (*fold f g z B*) *A*
⟨*proof*⟩

lemma (**in** *ACf*) *fold-reindex*:
assumes *fin*: *finite A*
shows *inj-on h A* ⟹ *fold f g z* (*h* ' *A*) = *fold f* (*g* ∘ *h*) *z A*
⟨*proof*⟩

lemma (**in** *ACe*) *fold-Un-Int*:
finite A ==> *finite B* ==>
fold f g e A · *fold f g e B* =
fold f g e (*A Un B*) · *fold f g e* (*A Int B*)
⟨*proof*⟩

corollary (**in** *ACe*) *fold-Un-disjoint*:
finite A ==> *finite B* ==> *A Int B* = {} ==>
fold f g e (*A Un B*) = *fold f g e A* · *fold f g e B*
⟨*proof*⟩

lemma (**in** *ACe*) *fold-UN-disjoint*:
⟦ *finite I*; *ALL i*:*I*. *finite* (*A i*);
ALL i:*I*. *ALL j*:*I*. *i* ≠ *j* --> *A i Int A j* = {} ⟧
⟹ *fold f g e* (*UNION I A*) =
fold f (%*i*. *fold f g e* (*A i*)) *e I*
⟨*proof*⟩

Fusion theorem, as described in Graham Hutton's paper, A Tutorial on the Universality and Expressiveness of Fold, JFP 9:4 (355-372), 1999.

lemma (**in** *ACf*) *fold-fusion*:

includes *ACf g*
shows
finite A ==>
(!!*x y*. *h* (*g x y*) = *f x* (*h y*)) ==>
h (*fold g j w A*) = *fold f j* (*h w*) *A*
⟨*proof*⟩

lemma (**in** *ACf*) *fold-cong*:
finite A $\Longrightarrow$ (!!*x*. *x*:*A* ==> *g x* = *h x*) ==> *fold f g z A* = *fold f h z A*
⟨*proof*⟩

lemma (**in** *ACe*) *fold-Sigma*: *finite A* ==> *ALL x*:*A*. *finite* (*B x*) ==>
fold f (%*x*. *fold f* (*g x*) *e* (*B x*)) *e A* =
fold f (*split g*) *e* (*SIGMA x*:*A*. *B x*)
⟨*proof*⟩

lemma (**in** *ACe*) *fold-distrib*: *finite A* $\Longrightarrow$
fold f (%*x*. *f* (*g x*) (*h x*)) *e A* = *f* (*fold f g e A*) (*fold f h e A*)
⟨*proof*⟩

Interpretation of locales – see OrderedGroup.thy

interpretation *AC-add*: *ACe* [*op* + *0*::'*a*::*comm-monoid-add*]
⟨*proof*⟩

interpretation *AC-mult*: *ACe* [*op* ∗ *1*::'*a*::*comm-monoid-mult*]
⟨*proof*⟩

22.3 Generalized summation over a set

constdefs
setsum :: ('*a* => '*b*) => '*a set* => '*b*::*comm-monoid-add*
setsum f A == *if finite A then fold* (*op* +) *f 0 A else 0*

abbreviation
Setsum ($\sum$- [*1000*] *999*) **where**
$\sum$*A* == *setsum* (%*x*. *x*) *A*

Now: lot's of fancy syntax. First, *setsum* ($\lambda x.\ e$) A is written $\sum x{\in}A.\ e$.

syntax
-setsum :: *pttrn* => '*a set* => '*b* => '*b*::*comm-monoid-add* ((*3SUM* -:-. -) [*0*, *51*, *10*] *10*)
syntax (*xsymbols*)
-setsum :: *pttrn* => '*a set* => '*b* => '*b*::*comm-monoid-add* ((*3*$\sum$ -∈-. -) [*0*, *51*, *10*] *10*)
syntax (*HTML* **output**)
-setsum :: *pttrn* => '*a set* => '*b* => '*b*::*comm-monoid-add* ((*3*$\sum$ -∈-. -) [*0*, *51*, *10*] *10*)

translations — Beware of argument permutation!

$SUM\ i{:}A.\ b == setsum\ (\%i.\ b)\ A$
$\sum i{\in}A.\ b == setsum\ (\%i.\ b)\ A$

Instead of $\sum x{\in}\{x.\ P\}.\ e$ we introduce the shorter $\sum x|P.\ e$.

syntax
$\text{-}qsetsum :: pttrn \Rightarrow bool \Rightarrow {'a} \Rightarrow {'a}\ ((3SUM\ \text{-}\ |/\ \text{-}./\ \text{-})\ [0,0,10]\ 10)$
syntax ($xsymbols$)
$\text{-}qsetsum :: pttrn \Rightarrow bool \Rightarrow {'a} \Rightarrow {'a}\ ((3\sum\text{-}\ |\ (\text{-}).\!/\ \text{-})\ [0,0,10]\ 10)$
syntax ($HTML$ **output**)
$\text{-}qsetsum :: pttrn \Rightarrow bool \Rightarrow {'a} \Rightarrow {'a}\ ((3\sum\text{-}\ |\ (\text{-}).\!/\ \text{-})\ [0,0,10]\ 10)$

translations
$SUM\ x|P.\ t => setsum\ (\%x.\ t)\ \{x.\ P\}$
$\sum x|P.\ t => setsum\ (\%x.\ t)\ \{x.\ P\}$

$\langle ML \rangle$

lemma $setsum\text{-}empty$ [$simp$]: $setsum\ f\ \{\} = 0$
$\langle proof \rangle$

lemma $setsum\text{-}insert$ [$simp$]:
$finite\ F ==> a \notin F ==> setsum\ f\ (insert\ a\ F) = f\ a + setsum\ f\ F$
$\langle proof \rangle$

lemma $setsum\text{-}infinite$ [$simp$]: $\sim finite\ A ==> setsum\ f\ A = 0$
$\langle proof \rangle$

lemma $setsum\text{-}reindex$:
$inj\text{-}on\ f\ B ==> setsum\ h\ (f\ `\ B) = setsum\ (h \circ f)\ B$
$\langle proof \rangle$

lemma $setsum\text{-}reindex\text{-}id$:
$inj\text{-}on\ f\ B ==> setsum\ f\ B = setsum\ id\ (f\ `\ B)$
$\langle proof \rangle$

lemma $setsum\text{-}cong$:
$A = B ==> (!!x.\ x{:}B ==> f\ x = g\ x) ==> setsum\ f\ A = setsum\ g\ B$
$\langle proof \rangle$

lemma $strong\text{-}setsum\text{-}cong$[$cong$]:
$A = B ==> (!!x.\ x{:}B =simp=> f\ x = g\ x)$
$==> setsum\ (\%x.\ f\ x)\ A = setsum\ (\%x.\ g\ x)\ B$
$\langle proof \rangle$

lemma $setsum\text{-}cong2$: $[\![\bigwedge x.\ x \in A \Longrightarrow f\ x = g\ x]\!] \Longrightarrow setsum\ f\ A = setsum\ g\ A$
$\langle proof \rangle$

lemma $setsum\text{-}reindex\text{-}cong$:

$[|$*inj-on* f A; $B = f$ ` A; $!!a.\ a{:}A \Longrightarrow g\ a = h\ (f\ a)|]$
$==>$ *setsum* h B = *setsum* g A
⟨*proof*⟩

lemma *setsum-0*[*simp*]: *setsum* (%*i*. *0*) A = *0*
⟨*proof*⟩

lemma *setsum-0'*: *ALL* $a{:}A.\ f\ a = 0$ $==>$ *setsum* f A = *0*
⟨*proof*⟩

lemma *setsum-Un-Int*: *finite* A $==>$ *finite* B $==>$
setsum g (A *Un* B) + *setsum* g (A *Int* B) = *setsum* g A + *setsum* g B
— The reversed orientation looks more natural, but LOOPS as a simprule!
⟨*proof*⟩

lemma *setsum-Un-disjoint*: *finite* A $==>$ *finite* B
$==>$ A *Int* B = {} $==>$ *setsum* g (A *Un* B) = *setsum* g A + *setsum* g B
⟨*proof*⟩

lemma *setsum-UN-disjoint*:
finite I $==>$ (*ALL* $i{:}I$. *finite* ($A\ i$)) $==>$
(*ALL* $i{:}I$. *ALL* $j{:}I$. $i \neq j$ $-->$ $A\ i$ *Int* $A\ j$ = {}) $==>$
setsum f (*UNION* I A) = ($\sum i{\in}I$. *setsum* f ($A\ i$))
⟨*proof*⟩

No need to assume that C is finite. If infinite, the rhs is directly 0, and $\bigcup C$ is also infinite, hence the lhs is also 0.

lemma *setsum-Union-disjoint*:
[| (*ALL* $A{:}C$. *finite* A);
(*ALL* $A{:}C$. *ALL* $B{:}C$. $A \neq B$ $-->$ A *Int* B = {}) |]
$==>$ *setsum* f (*Union* C) = *setsum* (*setsum* f) C
⟨*proof*⟩

lemma *setsum-Sigma*: *finite* A $==>$ *ALL* $x{:}A$. *finite* ($B\ x$) $==>$
($\sum x{\in}A$. ($\sum y{\in}B\ x$. $f\ x\ y$)) = ($\sum (x,y){\in}$(*SIGMA* $x{:}A$. $B\ x$). $f\ x\ y$)
⟨*proof*⟩

Here we can eliminate the finiteness assumptions, by cases.

lemma *setsum-cartesian-product*:
($\sum x{\in}A$. ($\sum y{\in}B$. $f\ x\ y$)) = ($\sum (x,y) \in A$ <*> B. $f\ x\ y$)
⟨*proof*⟩

lemma *setsum-addf*: *setsum* (%x. $f\ x + g\ x$) A = (*setsum* f A + *setsum* g A)
⟨*proof*⟩

22.3.1 Properties in more restricted classes of structures

lemma *setsum-SucD*: $setsum\ f\ A = Suc\ n \ {==>}\ EX\ a{:}A.\ 0 < f\ a$
⟨*proof*⟩

lemma *setsum-eq-0-iff* [*simp*]:
$finite\ F \ {==>}\ (setsum\ f\ F = 0) = (ALL\ a{:}F.\ f\ a = (0{::}nat))$
⟨*proof*⟩

lemma *setsum-Un-nat*: $finite\ A \ {==>}\ finite\ B \ {==>}$
$(setsum\ f\ (A\ Un\ B) :: nat) = setsum\ f\ A + setsum\ f\ B - setsum\ f\ (A\ Int\ B)$
— For the natural numbers, we have subtraction.
⟨*proof*⟩

lemma *setsum-Un*: $finite\ A \ {==>}\ finite\ B \ {==>}$
$(setsum\ f\ (A\ Un\ B) :: 'a :: ab\text{-}group\text{-}add) =$
$setsum\ f\ A + setsum\ f\ B - setsum\ f\ (A\ Int\ B)$
⟨*proof*⟩

lemma *setsum-diff1-nat*: $(setsum\ f\ (A - \{a\}) :: nat) =$
$(if\ a{:}A\ then\ setsum\ f\ A - f\ a\ else\ setsum\ f\ A)$
⟨*proof*⟩

lemma *setsum-diff1*: $finite\ A \Longrightarrow$
$(setsum\ f\ (A - \{a\}) :: ('a{::}ab\text{-}group\text{-}add)) =$
$(if\ a{:}A\ then\ setsum\ f\ A - f\ a\ else\ setsum\ f\ A)$
⟨*proof*⟩

lemma *setsum-diff1'*[*rule-format*]: $finite\ A \Longrightarrow a \in A \longrightarrow (\sum\ x \in A.\ f\ x) = f\ a + (\sum\ x \in (A - \{a\}).\ f\ x)$
⟨*proof*⟩

lemma *setsum-diff-nat*:
assumes $finite\ B$
and $B \subseteq A$
shows $(setsum\ f\ (A - B) :: nat) = (setsum\ f\ A) - (setsum\ f\ B)$
⟨*proof*⟩

lemma *setsum-diff*:
assumes le: $finite\ A\ B \subseteq A$
shows $setsum\ f\ (A - B) = setsum\ f\ A - ((setsum\ f\ B){::}('a{::}ab\text{-}group\text{-}add))$
⟨*proof*⟩

lemma *setsum-mono*:
assumes le: $\bigwedge i.\ i{\in}K \Longrightarrow f\ (i{::}'a) \leq ((g\ i){::}('b{::}\{comm\text{-}monoid\text{-}add,\ pordered\text{-}ab\text{-}semigroup\text{-}add\}))$
shows $(\sum i{\in}K.\ f\ i) \leq (\sum i{\in}K.\ g\ i)$
⟨*proof*⟩

lemma *setsum-strict-mono*:
  fixes f :: $'a \Rightarrow 'b$::{*pordered-cancel-ab-semigroup-add*,*comm-monoid-add*}
  assumes *finite A* $A \neq \{\}$
    and !!x. x:$A \Longrightarrow f\ x < g\ x$
  shows *setsum f A* $<$ *setsum g A*
  ⟨*proof*⟩

lemma *setsum-negf*:
  setsum (%x. $-$ ($f\ x$)::$'a$::*ab-group-add*) $A = -$ *setsum f A*
⟨*proof*⟩

lemma *setsum-subtractf*:
  setsum (%x. (($f\ x$)::$'a$::*ab-group-add*) $-$ $g\ x$) A =
    setsum f A $-$ *setsum g A*
⟨*proof*⟩

lemma *setsum-nonneg*:
  assumes *nn*: $\forall x \in A$. (*0*::$'a$::{*pordered-ab-semigroup-add*,*comm-monoid-add*}) $\leq$
$f\ x$
  shows *0* $\leq$ *setsum f A*
⟨*proof*⟩

lemma *setsum-nonpos*:
  assumes *np*: $\forall x \in A$. $f\ x \leq$ (*0*::$'a$::{*pordered-ab-semigroup-add*,*comm-monoid-add*})
  shows *setsum f A* $\leq$ *0*
⟨*proof*⟩

lemma *setsum-mono2*:
fixes f :: $'a \Rightarrow 'b$ :: {*pordered-ab-semigroup-add-imp-le*,*comm-monoid-add*}
assumes *fin*: *finite B* **and** *sub*: $A \subseteq B$ **and** *nn*: $\bigwedge b.\ b \in B - A \Longrightarrow 0 \leq f\ b$
shows *setsum f A* $\leq$ *setsum f B*
⟨*proof*⟩

lemma *setsum-mono3*: *finite B* ==> *A* <= *B* ==>
    ALL x: *B* $-$ *A*.
      0 <= ((*f x*)::$'a$::{*comm-monoid-add*,*pordered-ab-semigroup-add*}) ==>
        setsum f A <= *setsum f B*
  ⟨*proof*⟩

lemma *setsum-right-distrib*:
  fixes f :: $'a$ => ($'b$::*semiring-0*)
  shows $r *$ *setsum f A* = *setsum* (%n. $r * f\ n$) A
⟨*proof*⟩

lemma *setsum-left-distrib*:
  setsum f A $*$ (r::$'a$::*semiring-0*) = ($\sum n \in A.\ f\ n * r$)
⟨*proof*⟩

lemma *setsum-divide-distrib*:

setsum f A / (r::'a::field) = ($\sum$ n∈A. f n / r)
⟨*proof*⟩

lemma *setsum-abs[iff]*:
fixes *f :: 'a => ('b::pordered-ab-group-add-abs)*
shows *abs (setsum f A) ≤ setsum (%i. abs(f i)) A*
⟨*proof*⟩

lemma *setsum-abs-ge-zero[iff]*:
fixes *f :: 'a => ('b::pordered-ab-group-add-abs)*
shows *0 ≤ setsum (%i. abs(f i)) A*
⟨*proof*⟩

lemma *abs-setsum-abs[simp]*:
fixes *f :: 'a => ('b::pordered-ab-group-add-abs)*
shows *abs ($\sum$ a∈A. abs(f a)) = ($\sum$ a∈A. abs(f a))*
⟨*proof*⟩

Commuting outer and inner summation

lemma *swap-inj-on*:
inj-on (%(i, j). (j, i)) (A × B)
⟨*proof*⟩

lemma *swap-product*:
(%(i, j). (j, i)) ` (A × B) = B × A
⟨*proof*⟩

lemma *setsum-commute*:
($\sum$ i∈A. $\sum$ j∈B. f i j) = ($\sum$ j∈B. $\sum$ i∈A. f i j)
⟨*proof*⟩

lemma *setsum-product*:
fixes *f :: 'a => ('b::semiring-0)*
shows *setsum f A * setsum g B = ($\sum$ i∈A. $\sum$ j∈B. f i * g j)*
⟨*proof*⟩

22.4 Generalized product over a set

constdefs
setprod :: ('a => 'b) => 'a set => 'b::comm-monoid-mult
*setprod f A == if finite A then fold (op *) f 1 A else 1*

abbreviation
Setprod ($\prod$ - *[1000] 999*) **where**
$\prod$ *A == setprod (%x. x) A*

syntax
*-setprod :: pttrn => 'a set => 'b => 'b::comm-monoid-mult ((3PROD -:-. -)
[0, 51, 10] 10)*

syntax (*xsymbols*)
-setprod :: *pttrn* => *'a set* => *'b* => *'b::comm-monoid-mult* ((*3*$\prod$-∈-. -) [*0*, *51*, *10*] *10*)
syntax (*HTML* **output**)
-setprod :: *pttrn* => *'a set* => *'b* => *'b::comm-monoid-mult* ((*3*$\prod$-∈-. -) [*0*, *51*, *10*] *10*)

translations — Beware of argument permutation!
PROD i:A. b == *setprod* (%*i. b*) *A*
$\prod$*i*∈*A. b* == *setprod* (%*i. b*) *A*

Instead of $\prod x{\in}\{x.\ P\}.\ e$ we introduce the shorter $\prod x|P.\ e$.

syntax
-qsetprod :: *pttrn* ⇒ *bool* ⇒ *'a* ⇒ *'a* ((*3PROD* - |/ -./ -) [*0*,*0*,*10*] *10*)
syntax (*xsymbols*)
-qsetprod :: *pttrn* ⇒ *bool* ⇒ *'a* ⇒ *'a* ((*3*$\prod$- | (-)./ -) [*0*,*0*,*10*] *10*)
syntax (*HTML* **output**)
-qsetprod :: *pttrn* ⇒ *bool* ⇒ *'a* ⇒ *'a* ((*3*$\prod$- | (-)./ -) [*0*,*0*,*10*] *10*)

translations
PROD x|P. t => *setprod* (%*x. t*) {*x. P*}
$\prod$*x|P. t* => *setprod* (%*x. t*) {*x. P*}

lemma *setprod-empty* [*simp*]: *setprod f* {} = *1*
⟨*proof*⟩

lemma *setprod-insert* [*simp*]: [| *finite A*; *a* ∉ *A* |] ==>
setprod f (*insert a A*) = *f a* ∗ *setprod f A*
⟨*proof*⟩

lemma *setprod-infinite* [*simp*]: ~ *finite A* ==> *setprod f A* = *1*
⟨*proof*⟩

lemma *setprod-reindex*:
inj-on f B ==> *setprod h* (*f* ‘ *B*) = *setprod* (*h* ∘ *f*) *B*
⟨*proof*⟩

lemma *setprod-reindex-id*: *inj-on f B* ==> *setprod f B* = *setprod id* (*f* ‘ *B*)
⟨*proof*⟩

lemma *setprod-cong*:
A = *B* ==> (!!*x. x*:*B* ==> *f x* = *g x*) ==> *setprod f A* = *setprod g B*
⟨*proof*⟩

lemma *strong-setprod-cong*:
A = *B* ==> (!!*x. x*:*B* =*simp*=> *f x* = *g x*) ==> *setprod f A* = *setprod g B*
⟨*proof*⟩

lemma *setprod-reindex-cong*: *inj-on f A ==>*
B = f ‘ A ==> g = h ∘ f ==> setprod h B = setprod g A
⟨*proof*⟩

lemma *setprod-1*: *setprod (%i. 1) A = 1*
⟨*proof*⟩

lemma *setprod-1'*: *ALL a:F. f a = 1 ==> setprod f F = 1*
⟨*proof*⟩

lemma *setprod-Un-Int*: *finite A ==> finite B*
*==> setprod g (A Un B) * setprod g (A Int B) = setprod g A * setprod g B*
⟨*proof*⟩

lemma *setprod-Un-disjoint*: *finite A ==> finite B*
*==> A Int B = {} ==> setprod g (A Un B) = setprod g A * setprod g B*
⟨*proof*⟩

lemma *setprod-UN-disjoint*:
finite I ==> (ALL i:I. finite (A i)) ==>
(ALL i:I. ALL j:I. i ≠ j --> A i Int A j = {}) ==>
setprod f (UNION I A) = setprod (%i. setprod f (A i)) I
⟨*proof*⟩

lemma *setprod-Union-disjoint*:
[| (ALL A:C. finite A);
(ALL A:C. ALL B:C. A ≠ B --> A Int B = {}) |]
==> setprod f (Union C) = setprod (setprod f) C
⟨*proof*⟩

lemma *setprod-Sigma*: *finite A ==> ALL x:A. finite (B x) ==>*
$(\prod x{\in}A.\ (\prod y{\in}\ B\ x.\ f\ x\ y)) =$
$(\prod (x,y){\in}(SIGMA\ x{:}A.\ B\ x).\ f\ x\ y)$
⟨*proof*⟩

Here we can eliminate the finiteness assumptions, by cases.

lemma *setprod-cartesian-product*:
$(\prod x{\in}A.\ (\prod y{\in}\ B.\ f\ x\ y)) = (\prod (x,y){\in}(A <*> B).\ f\ x\ y)$
⟨*proof*⟩

lemma *setprod-timesf*:
*setprod (%x. f x * g x) A = (setprod f A * setprod g A)*
⟨*proof*⟩

22.4.1 Properties in more restricted classes of structures

lemma *setprod-eq-1-iff* [*simp*]:
finite F ==> (setprod f F = 1) = (ALL a:F. f a = (1::nat))

⟨*proof*⟩

lemma *setprod-zero*:
finite A ==> EX x: A. f x = (0::'a::comm-semiring-1) ==> setprod f A = 0
⟨*proof*⟩

lemma *setprod-nonneg* [*rule-format*]:
(ALL x: A. (0::'a::ordered-idom) ≤ f x) --> 0 ≤ setprod f A
⟨*proof*⟩

lemma *setprod-pos* [*rule-format*]: *(ALL x: A. (0::'a::ordered-idom) < f x)*
--> 0 < setprod f A
⟨*proof*⟩

lemma *setprod-nonzero* [*rule-format*]:
*(ALL x y. (x::'a::comm-semiring-1) * y = 0 --> x = 0 | y = 0) ==>*
finite A ==> (ALL x: A. f x ≠ (0::'a)) --> setprod f A ≠ 0
⟨*proof*⟩

lemma *setprod-zero-eq*:
*(ALL x y. (x::'a::comm-semiring-1) * y = 0 --> x = 0 | y = 0) ==>*
finite A ==> (setprod f A = (0::'a)) = (EX x: A. f x = 0)
⟨*proof*⟩

lemma *setprod-nonzero-field*:
finite A ==> (ALL x: A. f x ≠ (0::'a::idom)) ==> setprod f A ≠ 0
⟨*proof*⟩

lemma *setprod-zero-eq-field*:
finite A ==> (setprod f A = (0::'a::idom)) = (EX x: A. f x = 0)
⟨*proof*⟩

lemma *setprod-Un*: *finite A ==> finite B ==> (ALL x: A Int B. f x ≠ 0) ==>*
(setprod f (A Un B) :: 'a ::{field})
*= setprod f A * setprod f B / setprod f (A Int B)*
⟨*proof*⟩

lemma *setprod-diff1*: *finite A ==> f a ≠ 0 ==>*
(setprod f (A - {a}) :: 'a :: {field}) =
(if a:A then setprod f A / f a else setprod f A)
⟨*proof*⟩

lemma *setprod-inversef*: *finite A ==>*
ALL x: A. f x ≠ (0::'a::{field,division-by-zero}) ==>
setprod (inverse ∘ f) A = inverse (setprod f A)
⟨*proof*⟩

lemma *setprod-dividef*:
[|finite A;

  $\forall x \in A.\ g\ x \neq (0{::}'a{::}\{field,division\text{-}by\text{-}zero\})|]$
  ==> *setprod* (%x. *f x* / *g x*) *A* = *setprod f A* / *setprod g A*
⟨*proof*⟩

22.5 Finite cardinality

This definition, although traditional, is ugly to work with: *card A* == *LEAST n. EX f. A* = $\{f\ i \mid i.\ i < n\}$. But now that we have *setsum* things are easy:

constdefs
card :: *'a set* => *nat*
card A == *setsum* (%x. *1*::*nat*) *A*

lemma *card-empty* [*simp*]: *card* {} = *0*
⟨*proof*⟩

lemma *card-infinite* [*simp*]: ~ *finite A* ==> *card A* = *0*
⟨*proof*⟩

lemma *card-eq-setsum*: *card A* = *setsum* (%x. *1*) *A*
⟨*proof*⟩

lemma *card-insert-disjoint* [*simp*]:
finite A ==> $x \notin A$ ==> *card* (*insert x A*) = *Suc*(*card A*)
⟨*proof*⟩

lemma *card-insert-if*:
finite A ==> *card* (*insert x A*) = (*if x*:*A then card A else Suc*(*card*(*A*)))
⟨*proof*⟩

lemma *card-0-eq* [*simp*,*noatp*]: *finite A* ==> (*card A* = *0*) = (*A* = {})
⟨*proof*⟩

lemma *card-eq-0-iff*: (*card A* = *0*) = (*A* = {} | ~ *finite A*)
⟨*proof*⟩

lemma *card-Suc-Diff1*: *finite A* ==> *x*: *A* ==> *Suc* (*card* (*A* − {*x*})) = *card A*
⟨*proof*⟩

lemma *card-Diff-singleton*:
finite A ==> *x*: *A* ==> *card* (*A* − {*x*}) = *card A* − *1*
⟨*proof*⟩

lemma *card-Diff-singleton-if*:
finite A ==> *card* (*A*−{*x*}) = (*if x* : *A then card A* − *1 else card A*)
⟨*proof*⟩

lemma *card-Diff-insert*[*simp*]:

assumes *finite A* **and** $a{:}A$ **and** a ~: B
shows $card(A - insert\ a\ B) = card(A - B) - 1$
⟨*proof*⟩

lemma *card-insert*: *finite A ==> card (insert x A) = Suc (card (A − {x}))*
⟨*proof*⟩

lemma *card-insert-le*: *finite A ==> card A <= card (insert x A)*
⟨*proof*⟩

lemma *card-mono*: $[\![$ *finite B*; $A \subseteq B$ $]\!] \Longrightarrow$ *card A* $\leq$ *card B*
⟨*proof*⟩

lemma *card-seteq*: *finite B ==> (!!A. A <= B ==> card B <= card A ==> A = B)*
⟨*proof*⟩

lemma *psubset-card-mono*: *finite B ==> A < B ==> card A < card B*
⟨*proof*⟩

lemma *card-Un-Int*: *finite A ==> finite B*
==> card A + card B = card (A Un B) + card (A Int B)
⟨*proof*⟩

lemma *card-Un-disjoint*: *finite A ==> finite B*
==> A Int B = {} ==> card (A Un B) = card A + card B
⟨*proof*⟩

lemma *card-Diff-subset*:
finite B ==> B <= A ==> card (A − B) = card A − card B
⟨*proof*⟩

lemma *card-Diff1-less*: *finite A ==> x: A ==> card (A − {x}) < card A*
⟨*proof*⟩

lemma *card-Diff2-less*:
finite A ==> x: A ==> y: A ==> card (A − {x} − {y}) < card A
⟨*proof*⟩

lemma *card-Diff1-le*: *finite A ==> card (A − {x}) <= card A*
⟨*proof*⟩

lemma *card-psubset*: *finite B ==> A* $\subseteq$ *B ==> card A < card B ==> A < B*
⟨*proof*⟩

lemma *insert-partition*:
$[\![\ x \notin F; \forall c1 \in insert\ x\ F.\ \forall c2 \in insert\ x\ F.\ c1 \neq c2 \longrightarrow c1 \cap c2 = \{\}\ ]\!]$
$\Longrightarrow x \cap \bigcup\ F = \{\}$
⟨*proof*⟩

main cardinality theorem

lemma *card-partition* [*rule-format*]:
$\quad finite\ C ==>$
$\quad\quad finite\ (\bigcup\ C) \ {-}{-}{>}$
$\quad\quad (\forall\, c{\in}C.\ card\ c = k) \ {-}{-}{>}$
$\quad\quad (\forall\, c1 \in C.\ \forall\, c2 \in C.\ c1 \neq c2 \ {-}{-}{>}\ c1 \cap c2 = \{\}) \ {-}{-}{>}$
$\quad\quad k * card(C) = card\ (\bigcup\ C)$
⟨*proof*⟩

The form of a finite set of given cardinality

lemma *card-eq-SucD*:
assumes $card\ A = Suc\ k$
shows $\exists\, b\ B.\ A = insert\ b\ B\ \&\ b \notin B\ \&\ card\ B = k\ \&\ (k{=}0 \longrightarrow B{=}\{\})$
⟨*proof*⟩

lemma *card-Suc-eq*:
$(card\ A = Suc\ k) =$
$(\exists\, b\ B.\ A = insert\ b\ B\ \&\ b \notin B\ \&\ card\ B = k\ \&\ (k{=}0 \longrightarrow B{=}\{\}))$
⟨*proof*⟩

lemma *setsum-constant* [*simp*]: $(\sum x \in A.\ y) = of\text{-}nat(card\ A) * y$
⟨*proof*⟩

lemma *setprod-constant*: $finite\ A ==> (\prod x{\in}\ A.\ (y{::}'a{::}\{recpower,\ comm\text{-}monoid\text{-}mult\}))$
$= y\hat{}(card\ A)$
⟨*proof*⟩

lemma *setsum-bounded*:
assumes *le*: $\bigwedge i.\ i{\in}A \Longrightarrow f\ i \leq (K{::}'a{::}\{semiring\text{-}1,\ pordered\text{-}ab\text{-}semigroup\text{-}add\})$
shows $setsum\ f\ A \leq of\text{-}nat(card\ A) * K$
⟨*proof*⟩

22.5.1 Cardinality of unions

lemma *card-UN-disjoint*:
$\quad finite\ I ==> (ALL\ i{:}I.\ finite\ (A\ i)) ==>$
$\quad\quad (ALL\ i{:}I.\ ALL\ j{:}I.\ i \neq j \ {-}{-}{>}\ A\ i\ Int\ A\ j = \{\}) ==>$
$\quad\quad card\ (UNION\ I\ A) = (\sum i{\in}I.\ card(A\ i))$
⟨*proof*⟩

lemma *card-Union-disjoint*:
$\quad finite\ C ==> (ALL\ A{:}C.\ finite\ A) ==>$
$\quad\quad (ALL\ A{:}C.\ ALL\ B{:}C.\ A \neq B \ {-}{-}{>}\ A\ Int\ B = \{\}) ==>$
$\quad\quad card\ (Union\ C) = setsum\ card\ C$
⟨*proof*⟩

22.5.2 Cardinality of image

The image of a finite set can be expressed using *fold*.

lemma *image-eq-fold*: *finite* A ==> f ' A = *fold* (*op Un*) (%x. {$f\ x$}) {} A
⟨*proof*⟩

lemma *card-image-le*: *finite* A ==> *card* (f ' A) <= *card* A
⟨*proof*⟩

lemma *card-image*: *inj-on* f A ==> *card* (f ' A) = *card* A
⟨*proof*⟩

lemma *endo-inj-surj*: *finite* A ==> f ' $A \subseteq A$ ==> *inj-on* f A ==> f ' A = A
⟨*proof*⟩

lemma *eq-card-imp-inj-on*:
[| *finite* A; *card*(f ' A) = *card* A |] ==> *inj-on* f A
⟨*proof*⟩

lemma *inj-on-iff-eq-card*:
finite A ==> *inj-on* f A = (*card*(f ' A) = *card* A)
⟨*proof*⟩

lemma *card-inj-on-le*:
[|*inj-on* f A; f ' $A \subseteq B$; *finite* B |] ==> *card* $A \leq$ *card* B
⟨*proof*⟩

lemma *card-bij-eq*:
[|*inj-on* f A; f ' $A \subseteq B$; *inj-on* g B; g ' $B \subseteq A$;
finite A; *finite* B |] ==> *card* A = *card* B
⟨*proof*⟩

22.5.3 Cardinality of products

lemma *card-SigmaI* [*simp*]:
⟦ *finite* A; *ALL* a:A. *finite* ($B\ a$) ⟧
$\Longrightarrow$ *card* (*SIGMA* x: A. $B\ x$) = ($\sum a \in A$. *card* ($B\ a$))
⟨*proof*⟩

lemma *card-cartesian-product*: *card* (A <∗> B) = *card*(A) ∗ *card*(B)
⟨*proof*⟩

lemma *card-cartesian-product-singleton*: *card*({x} <∗> A) = *card*(A)
⟨*proof*⟩

22.5.4 Cardinality of the Powerset

lemma *card-Pow*: *finite* A ==> *card* (*Pow* A) = *Suc* (*Suc* 0) ^ *card* A
⟨*proof*⟩

Relates to equivalence classes. Based on a theorem of F. Kammüller.

lemma *dvd-partition*:

finite (*Union C*) ==>
  ALL c : *C*. *k dvd card c* ==>
  (*ALL c1*: *C*. *ALL c2*: *C*. *c1* $\neq$ *c2* --> *c1 Int c2* = {}) ==>
k dvd card (*Union C*)
⟨*proof*⟩

22.5.5 Relating injectivity and surjectivity

lemma *finite-surj-inj*: *finite*(*A*) $\Longrightarrow$ *A* <= *f'A* $\Longrightarrow$ *inj-on f A*
⟨*proof*⟩

lemma *finite-UNIV-surj-inj*: **fixes** *f* :: *'a* $\Rightarrow$ *'a*
shows *finite*(*UNIV*:: *'a set*) $\Longrightarrow$ *surj f* $\Longrightarrow$ *inj f*
⟨*proof*⟩

lemma *finite-UNIV-inj-surj*: **fixes** *f* :: *'a* $\Rightarrow$ *'a*
shows *finite*(*UNIV*:: *'a set*) $\Longrightarrow$ *inj f* $\Longrightarrow$ *surj f*
⟨*proof*⟩

corollary *infinite-UNIV-nat*: ~*finite*(*UNIV*::*nat set*)
⟨*proof*⟩

22.6 A fold functional for non-empty sets

Does not require start value.

inductive
  fold1Set :: (*'a* => *'a* => *'a*) => *'a set* => *'a* => *bool*
  for *f* :: *'a* => *'a* => *'a*
where
  fold1Set-insertI [*intro*]:
  ⟦ *foldSet f id a A x*; *a* $\notin$ *A* ⟧ $\Longrightarrow$ *fold1Set f* (*insert a A*) *x*

constdefs
  fold1 :: (*'a* => *'a* => *'a*) => *'a set* => *'a*
  fold1 f A == *THE x*. *fold1Set f A x*

lemma *fold1Set-nonempty*:
  fold1Set f A x $\Longrightarrow$ *A* $\neq$ {}
  ⟨*proof*⟩

inductive-cases *empty-fold1SetE* [*elim*!]: *fold1Set f* {} *x*

inductive-cases *insert-fold1SetE* [*elim*!]: *fold1Set f* (*insert a X*) *x*

lemma *fold1Set-sing* [*iff*]: (*fold1Set f* {*a*} *b*) = (*a* = *b*)
  ⟨*proof*⟩

lemma *fold1-singleton* [*simp*]: *fold1 f* {*a*} = *a*

⟨*proof*⟩

lemma *finite-nonempty-imp-fold1Set*:
$[\![$ *finite* A; $A \neq \{\}$ $]\!] \Longrightarrow$ *EX* x. *fold1Set f A x*
⟨*proof*⟩

First, some lemmas about *foldSet*.

lemma (**in** *ACf*) *foldSet-insert-swap*:
assumes *fold*: *foldSet f id b A y*
shows $b \notin A \Longrightarrow$ *foldSet f id z* (*insert b A*) $(z \cdot y)$
⟨*proof*⟩

lemma (**in** *ACf*) *foldSet-permute-diff*:
assumes *fold*: *foldSet f id b A x*
shows !!a. $[\![a \in A;\ b \notin A]\!] \Longrightarrow$ *foldSet f id a* (*insert b* $(A - \{a\})$) x
⟨*proof*⟩

lemma (**in** *ACf*) *fold1-eq-fold*:
[|*finite* A; $a \notin A$|] ==> *fold1 f* (*insert a A*) = *fold f id a A*
⟨*proof*⟩

lemma *nonempty-iff*: $(A \neq \{\}) = (\exists\, x\ B.\ A =$ *insert* $x\ B$ & $x \notin B)$
⟨*proof*⟩

lemma (**in** *ACf*) *fold1-insert*:
assumes *nonempty*: $A \neq \{\}$ **and** A: *finite* A $x \notin A$
shows *fold1 f* (*insert x A*) = *f x* (*fold1 f A*)
⟨*proof*⟩

lemma (**in** *ACIf*) *fold1-insert-idem* [*simp*]:
assumes *nonempty*: $A \neq \{\}$ **and** A: *finite* A
shows *fold1 f* (*insert x A*) = *f x* (*fold1 f A*)
⟨*proof*⟩

lemma (**in** *ACIf*) *hom-fold1-commute*:
assumes *hom*: !!$x\ y$. $h(f\ x\ y) = f\ (h\ x)\ (h\ y)$
and N: *finite* N $N \neq \{\}$ **shows** $h($*fold1 f N*$)$ = *fold1 f* (h ‘ N)
⟨*proof*⟩

Now the recursion rules for definitions:

lemma *fold1-singleton-def*: $g =$ *fold1 f* $\Longrightarrow g\ \{a\} = a$
⟨*proof*⟩

lemma (**in** *ACf*) *fold1-insert-def*:
$[\![$ $g =$ *fold1 f*; *finite* A; $x \notin A$; $A \neq \{\}$ $]\!] \Longrightarrow g$ (*insert x A*) $= x \cdot (g\ A)$
⟨*proof*⟩

lemma (**in** *ACIf*) *fold1-insert-idem-def*:
$[\![$ $g =$ *fold1 f*; *finite* A; $A \neq \{\}$ $]\!] \Longrightarrow g$ (*insert x A*) $= x \cdot (g\ A)$

⟨*proof*⟩

22.6.1 Determinancy for *fold1Set*

Not actually used!!

lemma (**in** *ACf*) *foldSet-permute*:
  $[\![foldSet\ f\ id\ b\ (insert\ a\ A)\ x;\ a \notin A;\ b \notin A]\!]$
   $\Longrightarrow foldSet\ f\ id\ a\ (insert\ b\ A)\ x$
⟨*proof*⟩

lemma (**in** *ACf*) *fold1Set-determ*:
  $fold1Set\ f\ A\ x \Longrightarrow fold1Set\ f\ A\ y \Longrightarrow y = x$
⟨*proof*⟩

lemma (**in** *ACf*) *fold1Set-equality*: $fold1Set\ f\ A\ y \Longrightarrow fold1\ f\ A = y$
  ⟨*proof*⟩

declare
  empty-foldSetE [*rule del*] *foldSet.intros* [*rule del*]
  empty-fold1SetE [*rule del*] *insert-fold1SetE* [*rule del*]
  — No more proofs involve these relations.

22.6.2 Semi-Lattices

locale *ACIfSL* = *ord* + *ACIf* +
  assumes *below-def*: $less\text{-}eq\ x\ y \longleftrightarrow x \cdot y = x$
  and *strict-below-def*: $less\ x\ y \longleftrightarrow less\text{-}eq\ x\ y \wedge x \neq y$
begin

notation
  less ((-/ $\prec$ -) [*51*, *51*] *50*)

notation (*xsymbols*)
  less-eq ((-/ $\preceq$ -) [*51*, *51*] *50*)

notation (*HTML* **output**)
  less-eq ((-/ $\preceq$ -) [*51*, *51*] *50*)

lemma *below-refl* [*simp*]: $x \preceq x$
  ⟨*proof*⟩

lemma *below-antisym*:
  assumes *xy*: $x \preceq y$ **and** *yx*: $y \preceq x$
  shows $x = y$
  ⟨*proof*⟩

lemma *below-trans*:
  assumes *xy*: $x \preceq y$ **and** *yz*: $y \preceq z$
  shows $x \preceq z$

⟨*proof*⟩

lemma *below-f-conv* [*simp*,*noatp*]: $x \preceq y \cdot z = (x \preceq y \wedge x \preceq z)$
⟨*proof*⟩

end

interpretation *ACIfSL* < *order*
⟨*proof*⟩

locale *ACIfSLlin* = *ACIfSL* +
assumes *lin*: $x \cdot y \in \{x,y\}$
begin

lemma *above-f-conv*:
$x \cdot y \preceq z = (x \preceq z \vee y \preceq z)$
⟨*proof*⟩

lemma *strict-below-f-conv*[*simp*,*noatp*]: $x \prec y \cdot z = (x \prec y \wedge x \prec z)$
⟨*proof*⟩

lemma *strict-above-f-conv*:
$x \cdot y \prec z = (x \prec z \vee y \prec z)$
⟨*proof*⟩

end

interpretation *ACIfSLlin* < *linorder*
⟨*proof*⟩

22.6.3 Lemmas about *fold1*

lemma (**in** *ACf*) *fold1-Un*:
assumes *A*: *finite* A $A \neq \{\}$
shows *finite* $B \Longrightarrow B \neq \{\} \Longrightarrow A$ *Int* $B = \{\} \Longrightarrow$
fold1 f $(A$ *Un* $B) = f$ (*fold1* f A) (*fold1* f B)
⟨*proof*⟩

lemma (**in** *ACIf*) *fold1-Un2*:
assumes *A*: *finite* A $A \neq \{\}$
shows *finite* $B \Longrightarrow B \neq \{\} \Longrightarrow$
fold1 f $(A$ *Un* $B) = f$ (*fold1* f A) (*fold1* f B)
⟨*proof*⟩

lemma (**in** *ACf*) *fold1-in*:
assumes *A*: *finite* (A) $A \neq \{\}$ **and** *elem*: $\bigwedge x\ y.\ x \cdot y \in \{x,y\}$
shows *fold1* f $A \in A$
⟨*proof*⟩

lemma (**in** *ACIfSL*) *below-fold1-iff*:
assumes *A*: *finite A* $A \neq \{\}$
shows $x \preceq$ *fold1 f A* $= (\forall\, a{\in}A.\ x \preceq a)$
⟨*proof*⟩

lemma (**in** *ACIfSLlin*) *strict-below-fold1-iff*:
finite A $\Longrightarrow A \neq \{\} \Longrightarrow x \prec$ *fold1 f A* $= (\forall\, a{\in}A.\ x \prec a)$
⟨*proof*⟩

lemma (**in** *ACIfSL*) *fold1-belowI*:
assumes *A*: *finite A* $A \neq \{\}$
shows $a \in A \Longrightarrow$ *fold1 f A* $\preceq a$
⟨*proof*⟩

lemma (**in** *ACIfSLlin*) *fold1-below-iff*:
assumes *A*: *finite A* $A \neq \{\}$
shows *fold1 f A* $\preceq x = (\exists\, a{\in}A.\ a \preceq x)$
⟨*proof*⟩

lemma (**in** *ACIfSLlin*) *fold1-strict-below-iff*:
assumes *A*: *finite A* $A \neq \{\}$
shows *fold1 f A* $\prec x = (\exists\, a{\in}A.\ a \prec x)$
⟨*proof*⟩

lemma (**in** *ACIfSLlin*) *fold1-antimono*:
assumes $A \neq \{\}$ **and** $A \subseteq B$ **and** *finite B*
shows *fold1 f B* $\preceq$ *fold1 f A*
⟨*proof*⟩

22.6.4 Fold1 in lattices with *inf* and *sup*

As an application of *fold1* we define infimum and supremum in (not necessarily complete!) lattices over (non-empty) sets by means of *fold1*.

lemma (**in** *lower-semilattice*) *ACf-inf*: *ACf inf*
⟨*proof*⟩

lemma (**in** *upper-semilattice*) *ACf-sup*: *ACf sup*
⟨*proof*⟩

lemma (**in** *lower-semilattice*) *ACIf-inf*: *ACIf inf*
⟨*proof*⟩

lemma (**in** *upper-semilattice*) *ACIf-sup*: *ACIf sup*
⟨*proof*⟩

lemma (**in** *lower-semilattice*) *ACIfSL-inf*: *ACIfSL* (*op* $\leq$) (*op* $<$) *inf*
⟨*proof*⟩

lemma (**in** *upper-semilattice*) *ACIfSL-sup*: *ACIfSL* (%x y. $y \leq x$) (%x y. $y < x$) *sup*
⟨*proof*⟩

context *lattice*
begin

definition
  Inf-fin :: $'a$ *set* $\Rightarrow$ $'a$ ($\bigsqcap_{fin}$- [*900*] *900*)
where
  Inf-fin = *fold1 inf*

definition
  Sup-fin :: $'a$ *set* $\Rightarrow$ $'a$ ($\bigsqcup_{fin}$- [*900*] *900*)
where
  Sup-fin = *fold1 sup*

lemma *Inf-le-Sup* [*simp*]: $\llbracket$ *finite* A; $A \neq \{\}$ $\rrbracket \Longrightarrow \bigsqcap_{fin}A \leq \bigsqcup_{fin}A$
⟨*proof*⟩

lemma *sup-Inf-absorb* [*simp*]:
  $\llbracket$ *finite* A; $A \neq \{\}$; $a \in A$ $\rrbracket \Longrightarrow$ (*sup* a ($\bigsqcap_{fin}A$)) = a
⟨*proof*⟩

lemma *inf-Sup-absorb* [*simp*]:
  $\llbracket$ *finite* A; $A \neq \{\}$; $a \in A$ $\rrbracket \Longrightarrow$ (*inf* a ($\bigsqcup_{fin}A$)) = a
⟨*proof*⟩

end

context *distrib-lattice*
begin

lemma *sup-Inf1-distrib*:
  finite $A \Longrightarrow A \neq \{\} \Longrightarrow$ *sup* x ($\bigsqcap_{fin}A$) = $\bigsqcap_{fin}\{$*sup* x $a|a.$ $a \in A\}$
⟨*proof*⟩

lemma *sup-Inf2-distrib*:
  assumes A: *finite* A $A \neq \{\}$ **and** B: *finite* B $B \neq \{\}$
  shows *sup* ($\bigsqcap_{fin}A$) ($\bigsqcap_{fin}B$) = $\bigsqcap_{fin}\{$*sup* a $b|a$ $b.$ $a \in A \wedge b \in B\}$
⟨*proof*⟩

lemma *inf-Sup1-distrib*:
  finite $A \Longrightarrow A \neq \{\} \Longrightarrow$ *inf* x ($\bigsqcup_{fin}A$) = $\bigsqcup_{fin}\{$*inf* x $a|a.$ $a \in A\}$
⟨*proof*⟩

lemma *inf-Sup2-distrib*:
  assumes A: *finite* A $A \neq \{\}$ **and** B: *finite* B $B \neq \{\}$
  shows *inf* ($\bigsqcup_{fin}A$) ($\bigsqcup_{fin}B$) = $\bigsqcup_{fin}\{$*inf* a $b|a$ $b.$ $a \in A \wedge b \in B\}$

⟨*proof*⟩

end

context *complete-lattice*
begin

Coincidence on finite sets in complete lattices:

lemma *Inf-fin-Inf*:
$finite\ A \Longrightarrow A \neq \{\} \Longrightarrow \bigsqcap_{fin} A = Inf\ A$
⟨*proof*⟩

lemma *Sup-fin-Sup*:
$finite\ A \Longrightarrow A \neq \{\} \Longrightarrow \bigsqcup_{fin} A = Sup\ A$
⟨*proof*⟩

end

22.6.5 Fold1 in linear orders with *min* and *max*

As an application of *fold1* we define minimum and maximum in (not necessarily complete!) linear orders over (non-empty) sets by means of *fold1*.

context *linorder*
begin

definition
$Min :: 'a\ set \Rightarrow 'a$
where
$Min = fold1\ min$

definition
$Max :: 'a\ set \Rightarrow 'a$
where
$Max = fold1\ max$

end context *linorder* **begin**

recall: *min* and *max* behave like *inf* and *sup*

lemma *ACIf-min*: *ACIf min*
⟨*proof*⟩

lemma *ACf-min*: *ACf min*
⟨*proof*⟩

lemma *ACIfSL-min*: $ACIfSL\ (op \leq)\ (op <)\ min$
⟨*proof*⟩

lemma *ACIfSLlin-min*: $ACIfSLlin\ (op \leq)\ (op <)\ min$

⟨*proof*⟩

lemma *ACIf-max*: *ACIf max*
⟨*proof*⟩

lemma *ACf-max*: *ACf max*
⟨*proof*⟩

lemma *ACIfSL-max*: *ACIfSL* $(\lambda x\ y.\ y \leq x)$ $(\lambda x\ y.\ y < x)$ *max*
⟨*proof*⟩

lemma *ACIfSLlin-max*: *ACIfSLlin* $(\lambda x\ y.\ y \leq x)$ $(\lambda x\ y.\ y < x)$ *max*
⟨*proof*⟩

lemmas *Min-singleton* [*simp*] = *fold1-singleton-def* [*OF Min-def*]
lemmas *Max-singleton* [*simp*] = *fold1-singleton-def* [*OF Max-def*]
lemmas *Min-insert* [*simp*] = *ACIf.fold1-insert-idem-def* [*OF ACIf-min Min-def*]
lemmas *Max-insert* [*simp*] = *ACIf.fold1-insert-idem-def* [*OF ACIf-max Max-def*]

lemma *Min-in* [*simp*]:
shows *finite* $A \Longrightarrow A \neq \{\} \Longrightarrow$ *Min* $A \in A$
⟨*proof*⟩

lemma *Max-in* [*simp*]:
shows *finite* $A \Longrightarrow A \neq \{\} \Longrightarrow$ *Max* $A \in A$
⟨*proof*⟩

lemma *Min-antimono*: $[\![\ M \subseteq N;\ M \neq \{\};$ *finite* $N\ ]\!] \Longrightarrow$ *Min* $N \leq$ *Min* M
⟨*proof*⟩

lemma *Max-mono*: $[\![\ M \subseteq N;\ M \neq \{\};$ *finite* $N\ ]\!] \Longrightarrow$ *Max* $M \leq$ *Max* N
⟨*proof*⟩

lemma *Min-le* [*simp*]: $[\![$ *finite* $A;\ A \neq \{\};\ x \in A\ ]\!] \Longrightarrow$ *Min* $A \leq x$
⟨*proof*⟩

lemma *Max-ge* [*simp*]: $[\![$ *finite* $A;\ A \neq \{\};\ x \in A\ ]\!] \Longrightarrow x \leq$ *Max* A
⟨*proof*⟩

lemma *Min-ge-iff* [*simp*,*noatp*]:
$[\![$ *finite* $A;\ A \neq \{\}\ ]\!] \Longrightarrow x \leq$ *Min* $A \longleftrightarrow (\forall\, a{\in}A.\ x \leq a)$
⟨*proof*⟩

lemma *Max-le-iff* [*simp*,*noatp*]:
$[\![$ *finite* $A;\ A \neq \{\}\ ]\!] \Longrightarrow$ *Max* $A \leq x \longleftrightarrow (\forall\, a{\in}A.\ a \leq x)$
⟨*proof*⟩

lemma *Min-gr-iff* [*simp*,*noatp*]:
$[\![$ *finite* $A;\ A \neq \{\}\ ]\!] \Longrightarrow x <$ *Min* $A \longleftrightarrow (\forall\, a{\in}A.\ x < a)$

$\langle proof \rangle$

lemma *Max-less-iff* [*simp*,*noatp*]:
$[\![\textit{finite } A;\ A \neq \{\}\ ]\!] \Longrightarrow \textit{Max } A < x \longleftrightarrow (\forall\, a{\in}A.\ a < x)$
$\langle proof \rangle$

lemma *Min-le-iff* [*noatp*]:
$[\![\textit{finite } A;\ A \neq \{\}\ ]\!] \Longrightarrow \textit{Min } A \leq x \longleftrightarrow (\exists\, a{\in}A.\ a \leq x)$
$\langle proof \rangle$

lemma *Max-ge-iff* [*noatp*]:
$[\![\textit{finite } A;\ A \neq \{\}\ ]\!] \Longrightarrow x \leq \textit{Max } A \longleftrightarrow (\exists\, a{\in}A.\ x \leq a)$
$\langle proof \rangle$

lemma *Min-less-iff* [*noatp*]:
$[\![\textit{finite } A;\ A \neq \{\}\ ]\!] \Longrightarrow \textit{Min } A < x \longleftrightarrow (\exists\, a{\in}A.\ a < x)$
$\langle proof \rangle$

lemma *Max-gr-iff* [*noatp*]:
$[\![\textit{finite } A;\ A \neq \{\}\ ]\!] \Longrightarrow x < \textit{Max } A \longleftrightarrow (\exists\, a{\in}A.\ x < a)$
$\langle proof \rangle$

lemma *Min-Un*: $[\![\textit{finite } A;\ A \neq \{\};\ \textit{finite } B;\ B \neq \{\}]\!]$
$\Longrightarrow \textit{Min } (A \cup B) = \textit{min } (\textit{Min } A)\ (\textit{Min } B)$
$\langle proof \rangle$

lemma *Max-Un*: $[\![\textit{finite } A;\ A \neq \{\};\ \textit{finite } B;\ B \neq \{\}]\!]$
$\Longrightarrow \textit{Max } (A \cup B) = \textit{max } (\textit{Max } A)\ (\textit{Max } B)$
$\langle proof \rangle$

lemma *hom-Min-commute*:
$(\bigwedge x\ y.\ h\ (\textit{min } x\ y) = \textit{min } (h\ x)\ (h\ y))$
$\Longrightarrow \textit{finite } N \Longrightarrow N \neq \{\} \Longrightarrow h\ (\textit{Min } N) = \textit{Min } (h\ \text{`}\ N)$
$\langle proof \rangle$

lemma *hom-Max-commute*:
$(\bigwedge x\ y.\ h\ (\textit{max } x\ y) = \textit{max } (h\ x)\ (h\ y))$
$\Longrightarrow \textit{finite } N \Longrightarrow N \neq \{\} \Longrightarrow h\ (\textit{Max } N) = \textit{Max } (h\ \text{`}\ N)$
$\langle proof \rangle$

end

context *ordered-ab-semigroup-add*
begin

lemma *add-Min-commute*:
fixes k
assumes *finite* N **and** $N \neq \{\}$
shows $k + \textit{Min } N = \textit{Min } \{k + m \mid m.\ m \in N\}$

⟨*proof*⟩

lemma *add-Max-commute*:
  fixes k
  assumes *finite* N **and** $N \neq \{\}$
  shows $k + Max\ N = Max\ \{k + m \mid m.\ m \in N\}$
⟨*proof*⟩

end

22.7 Class *finite* and code generation

lemma *finite-code* [*code func*]:
  finite $\{\} \longleftrightarrow True$
  finite (*insert a A*) $\longleftrightarrow$ *finite A*
  ⟨*proof*⟩

lemma *card-code* [*code func*]:
  card $\{\} = 0$
  card (*insert a A*) =
    (*if finite A then Suc* (*card* $(A - \{a\})$) *else card* (*insert a A*))
  ⟨*proof*⟩

⟨*ML*⟩
class *finite* (**attach** *UNIV*) = *type* +
  fixes *itself* :: *'a itself*
  assumes *finite-UNIV*: *finite* (*UNIV* :: *'a set*)
⟨*ML*⟩
hide *const finite*

lemma *finite* [*simp*]: *finite* (*A* :: *'a::finite set*)
  ⟨*proof*⟩

lemma *univ-unit* [*noatp*]:
  UNIV = {()} ⟨*proof*⟩

instance *unit* :: *finite*
  Finite-Set.itself $\equiv$ *TYPE*(*unit*)
⟨*proof*⟩

lemmas [*code func*] = *univ-unit*

lemma *univ-bool* [*noatp*]:
  UNIV = {*False*, *True*} ⟨*proof*⟩

instance *bool* :: *finite*
  itself $\equiv$ *TYPE*(*bool*)
⟨*proof*⟩

lemmas [*code func*] = *univ-bool*

instance $*$ :: (*finite*, *finite*) *finite*
  itself $\equiv$ *TYPE*('*a*::*finite*)
⟨*proof*⟩

lemma *univ-prod* [*noatp*, *code func*]:
  $UNIV = (UNIV :: 'a::finite\ set) \times (UNIV :: 'b::finite\ set)$
  ⟨*proof*⟩

instance $+$ :: (*finite*, *finite*) *finite*
  itself $\equiv$ *TYPE*('*a*::*finite* + '*b*::*finite*)
⟨*proof*⟩

lemma *univ-sum* [*noatp*, *code func*]:
  $UNIV = (UNIV :: 'a::finite\ set) <+> (UNIV :: 'b::finite\ set)$
  ⟨*proof*⟩

instance *set* :: (*finite*) *finite*
  itself $\equiv$ *TYPE*('*a*::*finite set*)
⟨*proof*⟩

lemma *univ-set* [*noatp*, *code func*]:
  $UNIV = Pow\ (UNIV :: 'a::finite\ set)$ ⟨*proof*⟩

lemma *inj-graph*: $inj\ (\%f.\ \{(x, y).\ y = f\ x\})$
  ⟨*proof*⟩

instance *fun* :: (*finite*, *finite*) *finite*
  itself $\equiv$ *TYPE*('*a*::*finite* $\Rightarrow$ '*b*::*finite*)
⟨*proof*⟩

hide (**open**) *const itself*

22.8 Equality and order on functions

instance *fun* :: (*finite*, *eq*) *eq* ⟨*proof*⟩

lemma *eq-fun* [*code func*]:
  fixes $f\ g :: 'a::finite \Rightarrow 'b::eq$
  shows $f = g \longleftrightarrow (\forall x \in UNIV.\ f\ x = g\ x)$
  ⟨*proof*⟩

lemma *order-fun* [*code func*]:
  fixes $f\ g :: 'a::finite \Rightarrow 'b::order$
  shows $f \le g \longleftrightarrow (\forall x \in UNIV.\ f\ x \le g\ x)$
    and $f < g \longleftrightarrow f \le g \wedge (\exists x \in UNIV.\ f\ x \neq g\ x)$
  ⟨*proof*⟩

end

23 Datatype: Analogues of the Cartesian Product and Disjoint Sum for Datatypes

theory *Datatype*
imports *Finite-Set*
uses *Tools/datatype-codegen.ML*
begin

typedef (*Node*)
  $('a,'b)\ node = \{p.\ EX\ f\ x\ k.\ p = (f{::}nat{=}{>}'b{+}nat,\ x{::}'a{+}nat)\ \&\ f\ k = Inr\ 0\}$
    — it is a subtype of $(nat{=}{>}'b{+}nat) * ('a{+}nat)$
  $\langle proof \rangle$

Datatypes will be represented by sets of type *node*

types $'a\ item\ = ('a,\ unit)\ node\ set$
    $('a,\ 'b)\ dtree = ('a,\ 'b)\ node\ set$

consts
  $apfst\ :: ['a{=}{>}'c,\ 'a{*}'b] => 'c{*}'b$
  $Push\ :: [('b + nat),\ nat => ('b + nat)] => (nat => ('b + nat))$

  $Push\text{-}Node :: [('b + nat),\ ('a,\ 'b)\ node] => ('a,\ 'b)\ node$
  $ndepth\ :: ('a,\ 'b)\ node => nat$

  $Atom\ :: ('a + nat) => ('a,\ 'b)\ dtree$
  $Leaf\ :: 'a => ('a,\ 'b)\ dtree$
  $Numb\ :: nat => ('a,\ 'b)\ dtree$
  $Scons\ :: [('a,\ 'b)\ dtree,\ ('a,\ 'b)\ dtree] => ('a,\ 'b)\ dtree$
  $In0\ :: ('a,\ 'b)\ dtree => ('a,\ 'b)\ dtree$
  $In1\ :: ('a,\ 'b)\ dtree => ('a,\ 'b)\ dtree$
  $Lim\ :: ('b => ('a,\ 'b)\ dtree) => ('a,\ 'b)\ dtree$

  $ntrunc\ :: [nat,\ ('a,\ 'b)\ dtree] => ('a,\ 'b)\ dtree$

  $uprod\ :: [('a,\ 'b)\ dtree\ set,\ ('a,\ 'b)\ dtree\ set]=> ('a,\ 'b)\ dtree\ set$
  $usum\ :: [('a,\ 'b)\ dtree\ set,\ ('a,\ 'b)\ dtree\ set]=> ('a,\ 'b)\ dtree\ set$

  $Split\ :: [[('a,\ 'b)\ dtree,\ ('a,\ 'b)\ dtree]{=}{>}'c,\ ('a,\ 'b)\ dtree] => 'c$
  $Case\ :: [[('a,\ 'b)\ dtree]{=}{>}'c,\ [('a,\ 'b)\ dtree]{=}{>}'c,\ ('a,\ 'b)\ dtree] => 'c$

  $dprod\ :: [(('a,\ 'b)\ dtree * ('a,\ 'b)\ dtree)set,\ (('a,\ 'b)\ dtree * ('a,\ 'b)\ dtree)set]$
    $=> (('a,\ 'b)\ dtree * ('a,\ 'b)\ dtree)set$
  $dsum\ :: [(('a,\ 'b)\ dtree * ('a,\ 'b)\ dtree)set,\ (('a,\ 'b)\ dtree * ('a,\ 'b)\ dtree)set]$
    $=> (('a,\ 'b)\ dtree * ('a,\ 'b)\ dtree)set$

defs

Push-Node-def: *Push-Node == (%n x. Abs-Node (apfst (Push n) (Rep-Node x)))*

apfst-def: *apfst == (%f (x,y). (f(x),y))*
Push-def: *Push == (%b h. nat-case b h)*

Atom-def: *Atom == (%x. {Abs-Node((%k. Inr 0, x))})*
Scons-def: *Scons M N == (Push-Node (Inr 1) ‘ M) Un (Push-Node (Inr (Suc 1)) ‘ N)*

Leaf-def: *Leaf == Atom o Inl*
Numb-def: *Numb == Atom o Inr*

In0-def: *In0(M) == Scons (Numb 0) M*
In1-def: *In1(M) == Scons (Numb 1) M*

Lim-def: *Lim f == Union {z. ? x. z = Push-Node (Inl x) ‘ (f x)}*

ndepth-def: *ndepth(n) == (%(f,x). LEAST k. f k = Inr 0) (Rep-Node n)*
ntrunc-def: *ntrunc k N == {n. n:N & ndepth(n)<k}*

uprod-def: *uprod A B == UN x:A. UN y:B. { Scons x y }*
usum-def: *usum A B == In0‘A Un In1‘B*

Split-def: *Split c M == THE u. EX x y. M = Scons x y & u = c x y*

Case-def: *Case c d M == THE u. (EX x . M = In0(x) & u = c(x))*
| (EX y . M = In1(y) & u = d(y))

dprod-def: *dprod r s == UN (x,x′):r. UN (y,y′):s. {(Scons x y, Scons x′ y′)}*

dsum-def: *dsum r s == (UN (x,x′):r. {(In0(x),In0(x′))}) Un*
(UN (y,y′):s. {(In1(y),In1(y′))})

lemma *apfst-conv* [*simp*, *code*]: *apfst f* (*a*, *b*) = (*f a*, *b*)
⟨*proof*⟩

lemma *apfst-convE*:
[| *q* = *apfst f p*; !!*x y*. [| *p* = (*x*,*y*); *q* = (*f*(*x*),*y*) |] ==> *R*
|] ==> *R*
⟨*proof*⟩

lemma *Push-inject1*: *Push i f* = *Push j g* ==> *i*=*j*
⟨*proof*⟩

lemma *Push-inject2*: *Push i f* = *Push j g* ==> *f*=*g*
⟨*proof*⟩

lemma *Push-inject*:
[| *Push i f* =*Push j g*; [| *i*=*j*; *f*=*g* |] ==> *P* |] ==> *P*
⟨*proof*⟩

lemma *Push-neq-K0*: *Push* (*Inr* (*Suc k*)) *f* = (%*z*. *Inr 0*) ==> *P*
⟨*proof*⟩

lemmas *Abs-Node-inj* = *Abs-Node-inject* [*THEN* [*2*] *rev-iffD1*, *standard*]

lemma *Node-K0-I*: (%*k*. *Inr 0*, *a*) : *Node*
⟨*proof*⟩

lemma *Node-Push-I*: *p*: *Node* ==> *apfst* (*Push i*) *p* : *Node*
⟨*proof*⟩

23.1 Freeness: Distinctness of Constructors

lemma *Scons-not-Atom* [*iff*]: *Scons M N* ≠ *Atom*(*a*)
⟨*proof*⟩

lemmas *Atom-not-Scons* [*iff*] = *Scons-not-Atom* [*THEN not-sym*, *standard*]

lemma *inj-Atom*: *inj*(*Atom*)
⟨*proof*⟩
lemmas *Atom-inject* = *inj-Atom* [*THEN injD*, *standard*]

lemma *Atom-Atom-eq* [*iff*]: (*Atom*(*a*)=*Atom*(*b*)) = (*a*=*b*)
⟨*proof*⟩

lemma *inj-Leaf*: *inj*(*Leaf*)
⟨*proof*⟩

lemmas *Leaf-inject* [*dest*!] = *inj-Leaf* [*THEN injD*, *standard*]

lemma *inj-Numb*: *inj*(*Numb*)
⟨*proof*⟩

lemmas *Numb-inject* [*dest*!] = *inj-Numb* [*THEN injD*, *standard*]

lemma *Push-Node-inject*:
  [| *Push-Node i m* =*Push-Node j n*; [| *i*=*j*; *m*=*n* |] ==> *P*
  |] ==> *P*
⟨*proof*⟩

lemma *Scons-inject-lemma1*: *Scons M N* <= *Scons M′ N′* ==> *M*<=*M′*
⟨*proof*⟩

lemma *Scons-inject-lemma2*: *Scons M N* <= *Scons M′ N′* ==> *N*<=*N′*
⟨*proof*⟩

lemma *Scons-inject1*: *Scons M N* = *Scons M′ N′* ==> *M*=*M′*
⟨*proof*⟩

lemma *Scons-inject2*: *Scons M N* = *Scons M′ N′* ==> *N*=*N′*
⟨*proof*⟩

lemma *Scons-inject*:
  [| *Scons M N* = *Scons M′ N′*; [| *M*=*M′*; *N*=*N′* |] ==> *P* |] ==> *P*
⟨*proof*⟩

lemma *Scons-Scons-eq* [*iff*]: (*Scons M N* = *Scons M′ N′*) = (*M*=*M′* & *N*=*N′*)
⟨*proof*⟩

lemma *Scons-not-Leaf* [*iff*]: *Scons M N* $\neq$ *Leaf*(*a*)
⟨*proof*⟩

lemmas *Leaf-not-Scons* [*iff*] = *Scons-not-Leaf* [*THEN not-sym*, *standard*]

lemma *Scons-not-Numb* [*iff*]: *Scons M N* $\neq$ *Numb*(*k*)
⟨*proof*⟩

lemmas *Numb-not-Scons* [*iff*] = *Scons-not-Numb* [*THEN not-sym*, *standard*]

lemma *Leaf-not-Numb* [*iff*]: *Leaf*(*a*) $\neq$ *Numb*(*k*)
⟨*proof*⟩

lemmas *Numb-not-Leaf* [*iff*] = *Leaf-not-Numb* [*THEN not-sym*, *standard*]

lemma *ndepth-K0*: *ndepth* (*Abs-Node*(%*k*. *Inr 0*, *x*)) = *0*
⟨*proof*⟩

lemma *ndepth-Push-Node-aux*:
nat-case (*Inr* (*Suc i*)) *f k* = *Inr 0* --> *Suc*(*LEAST x*. *f x* = *Inr 0*) <= *k*
⟨*proof*⟩

lemma *ndepth-Push-Node*:
ndepth (*Push-Node* (*Inr* (*Suc i*)) *n*) = *Suc*(*ndepth*(*n*))
⟨*proof*⟩

lemma *ntrunc-0* [*simp*]: *ntrunc 0 M* = {}
⟨*proof*⟩

lemma *ntrunc-Atom* [*simp*]: *ntrunc* (*Suc k*) (*Atom a*) = *Atom*(*a*)
⟨*proof*⟩

lemma *ntrunc-Leaf* [*simp*]: *ntrunc* (*Suc k*) (*Leaf a*) = *Leaf*(*a*)

⟨proof⟩

lemma *ntrunc-Numb* [*simp*]: *ntrunc* (*Suc k*) (*Numb i*) = *Numb*(*i*)
⟨proof⟩

lemma *ntrunc-Scons* [*simp*]:
    ntrunc (*Suc k*) (*Scons M N*) = *Scons* (*ntrunc k M*) (*ntrunc k N*)
⟨proof⟩

lemma *ntrunc-one-In0* [*simp*]: *ntrunc* (*Suc 0*) (*In0 M*) = {}
⟨proof⟩

lemma *ntrunc-In0* [*simp*]: *ntrunc* (*Suc*(*Suc k*)) (*In0 M*) = *In0* (*ntrunc* (*Suc k*) *M*)
⟨proof⟩

lemma *ntrunc-one-In1* [*simp*]: *ntrunc* (*Suc 0*) (*In1 M*) = {}
⟨proof⟩

lemma *ntrunc-In1* [*simp*]: *ntrunc* (*Suc*(*Suc k*)) (*In1 M*) = *In1* (*ntrunc* (*Suc k*) *M*)
⟨proof⟩

23.2 Set Constructions

lemma *uprodI* [*intro*!]: [| *M*:*A*; *N*:*B* |] ==> *Scons M N* : *uprod A B*
⟨proof⟩

lemma *uprodE* [*elim*!]:
    [| *c* : *uprod A B*;
        !!*x y*. [| *x*:*A*; *y*:*B*; *c* = *Scons x y* |] ==> *P*
     |] ==> *P*
⟨proof⟩

lemma *uprodE2*: [| *Scons M N* : *uprod A B*; [| *M*:*A*; *N*:*B* |] ==> *P* |] ==> *P*
⟨proof⟩

lemma *usum-In0I* [*intro*]: *M*:*A* ==> *In0*(*M*) : *usum A B*
⟨proof⟩

lemma *usum-In1I* [*intro*]: *N*:*B* ==> *In1*(*N*) : *usum A B*
⟨*proof*⟩

lemma *usumE* [*elim*!]:
  [| *u* : *usum A B*;
    !!*x*. [| *x*:*A*; *u*=*In0*(*x*) |] ==> *P*;
    !!*y*. [| *y*:*B*; *u*=*In1*(*y*) |] ==> *P*
  |] ==> *P*
⟨*proof*⟩

lemma *In0-not-In1* [*iff*]: *In0*(*M*) ≠ *In1*(*N*)
⟨*proof*⟩

lemmas *In1-not-In0* [*iff*] = *In0-not-In1* [*THEN not-sym*, *standard*]

lemma *In0-inject*: *In0*(*M*) = *In0*(*N*) ==> *M*=*N*
⟨*proof*⟩

lemma *In1-inject*: *In1*(*M*) = *In1*(*N*) ==> *M*=*N*
⟨*proof*⟩

lemma *In0-eq* [*iff*]: (*In0 M* = *In0 N*) = (*M*=*N*)
⟨*proof*⟩

lemma *In1-eq* [*iff*]: (*In1 M* = *In1 N*) = (*M*=*N*)
⟨*proof*⟩

lemma *inj-In0*: *inj In0*
⟨*proof*⟩

lemma *inj-In1*: *inj In1*
⟨*proof*⟩

lemma *Lim-inject*: *Lim f* = *Lim g* ==> *f* = *g*
⟨*proof*⟩

lemma *ntrunc-subsetI*: *ntrunc k M* <= *M*
⟨*proof*⟩

lemma *ntrunc-subsetD*: (!!k. *ntrunc* k M <= N) ==> M<=N
⟨*proof*⟩

lemma *ntrunc-equality*: (!!k. *ntrunc* k M = *ntrunc* k N) ==> M=N
⟨*proof*⟩

lemma *ntrunc-o-equality*:
[| !!k. (*ntrunc*(k) *o* *h1*) = (*ntrunc*(k) *o* *h2*) |] ==> *h1*=*h2*
⟨*proof*⟩

lemma *uprod-mono*: [| A<=A'; B<=B' |] ==> *uprod* A B <= *uprod* A' B'
⟨*proof*⟩

lemma *usum-mono*: [| A<=A'; B<=B' |] ==> *usum* A B <= *usum* A' B'
⟨*proof*⟩

lemma *Scons-mono*: [| M<=M'; N<=N' |] ==> *Scons* M N <= *Scons* M' N'
⟨*proof*⟩

lemma *In0-mono*: M<=N ==> *In0*(M) <= *In0*(N)
⟨*proof*⟩

lemma *In1-mono*: M<=N ==> *In1*(M) <= *In1*(N)
⟨*proof*⟩

lemma *Split* [*simp*]: *Split* c (*Scons* M N) = c M N
⟨*proof*⟩

lemma *Case-In0* [*simp*]: *Case* c d (*In0* M) = c(M)
⟨*proof*⟩

lemma *Case-In1* [*simp*]: *Case* c d (*In1* N) = d(N)
⟨*proof*⟩

lemma *ntrunc-UN1*: *ntrunc* k (*UN* x. f(x)) = (*UN* x. *ntrunc* k (f x))
⟨*proof*⟩

lemma *Scons-UN1-x*: *Scons* (*UN* x. f x) M = (*UN* x. *Scons* (f x) M)

⟨*proof*⟩

lemma *Scons-UN1-y*: $Scons\ M\ (UN\ x.\ f\ x) = (UN\ x.\ Scons\ M\ (f\ x))$
⟨*proof*⟩

lemma *In0-UN1*: $In0(UN\ x.\ f(x)) = (UN\ x.\ In0(f(x)))$
⟨*proof*⟩

lemma *In1-UN1*: $In1(UN\ x.\ f(x)) = (UN\ x.\ In1(f(x)))$
⟨*proof*⟩

lemma *dprodI* [*intro*!]:
  [| (M,M'):r; (N,N'):s |] ==> $(Scons\ M\ N,\ Scons\ M'\ N')$: *dprod r s*
⟨*proof*⟩

lemma *dprodE* [*elim*!]:
  [| c : *dprod r s*;
    !!$x\ y\ x'\ y'$. [| (x,x') : r; (y,y') : s;
              $c = (Scons\ x\ y,\ Scons\ x'\ y')$ |] ==> P
  |] ==> P
⟨*proof*⟩

lemma *dsum-In0I* [*intro*]: (M,M'):r ==> $(In0(M),\ In0(M'))$: *dsum r s*
⟨*proof*⟩

lemma *dsum-In1I* [*intro*]: (N,N'):s ==> $(In1(N),\ In1(N'))$: *dsum r s*
⟨*proof*⟩

lemma *dsumE* [*elim*!]:
  [| w : *dsum r s*;
    !!$x\ x'$. [| (x,x') : r; $w = (In0(x),\ In0(x'))$ |] ==> P;
    !!$y\ y'$. [| (y,y') : s; $w = (In1(y),\ In1(y'))$ |] ==> P
  |] ==> P
⟨*proof*⟩

lemma *dprod-mono*: [| r<=r'; s<=s' |] ==> *dprod r s* <= *dprod* r' s'
⟨*proof*⟩

lemma *dsum-mono*: [| r<=r'; s<=s' |] ==> *dsum r s* <= *dsum* r' s'

⟨*proof*⟩

lemma *dprod-Sigma*: (*dprod* (*A* <*> *B*) (*C* <*> *D*)) <= (*uprod A C*) <*> (*uprod B D*)
⟨*proof*⟩

lemmas *dprod-subset-Sigma* = *subset-trans* [*OF dprod-mono dprod-Sigma*, *standard*]

lemma *dprod-subset-Sigma2*:
(*dprod* (*Sigma A B*) (*Sigma C D*)) <=
Sigma (*uprod A C*) (*Split* (%*x y*. *uprod* (*B x*) (*D y*)))
⟨*proof*⟩

lemma *dsum-Sigma*: (*dsum* (*A* <*> *B*) (*C* <*> *D*)) <= (*usum A C*) <*> (*usum B D*)
⟨*proof*⟩

lemmas *dsum-subset-Sigma* = *subset-trans* [*OF dsum-mono dsum-Sigma*, *standard*]

lemma *Domain-dprod* [*simp*]: *Domain* (*dprod r s*) = *uprod* (*Domain r*) (*Domain s*)
⟨*proof*⟩

lemma *Domain-dsum* [*simp*]: *Domain* (*dsum r s*) = *usum* (*Domain r*) (*Domain s*)
⟨*proof*⟩

hides popular names

hide (**open**) *type node item*
hide (**open**) *const Push Node Atom Leaf Numb Lim Split Case*

24 Datatypes

24.1 Representing sums

rep-datatype *sum*
distinct *Inl-not-Inr Inr-not-Inl*
inject *Inl-eq Inr-eq*
induction *sum-induct*

lemma *size-sum* [*code func*]:
size $(x :: 'a + 'b) = 0$ ⟨*proof*⟩

lemma *sum-case-KK*[*simp*]: *sum-case* (%*x*. *a*) (%*x*. *a*) = (%*x*. *a*)
⟨*proof*⟩

lemma *surjective-sum*: *sum-case* (%*x*::'*a*. *f* (*Inl x*)) (%*y*::'*b*. *f* (*Inr y*)) *s* = *f*(*s*)
⟨*proof*⟩

lemma *sum-case-weak-cong*: *s* = *t* ==> *sum-case f g s* = *sum-case f g t*
— Prevents simplification of f and g: much faster.
⟨*proof*⟩

lemma *sum-case-inject*:
sum-case f1 f2 = *sum-case g1 g2* ==> (*f1* = *g1* ==> *f2* = *g2* ==> *P*) ==> *P*
⟨*proof*⟩

constdefs
Suml :: ('*a* => '*c*) => '*a* + '*b* => '*c*
Suml == (%*f*. *sum-case f arbitrary*)

Sumr :: ('*b* => '*c*) => '*a* + '*b* => '*c*
Sumr == *sum-case arbitrary*

lemma *Suml-inject*: *Suml f* = *Suml g* ==> *f* = *g*
⟨*proof*⟩

lemma *Sumr-inject*: *Sumr f* = *Sumr g* ==> *f* = *g*
⟨*proof*⟩

hide (**open**) *const Suml Sumr*

24.2 The option datatype

datatype '*a option* = *None* | *Some* '*a*

lemma *not-None-eq* [*iff*]: (*x* ~= *None*) = (*EX y*. *x* = *Some y*)
⟨*proof*⟩

lemma *not-Some-eq* [*iff*]: (*ALL y*. *x* ~= *Some y*) = (*x* = *None*)
⟨*proof*⟩

Although it may appear that both of these equalities are helpful only when applied to assumptions, in practice it seems better to give them the uniform iff attribute.

lemma *option-caseE*:
assumes *c*: (*case x of None* => *P* | *Some y* => *Q y*)
obtains

  (*None*) $x = None$ **and** P
  | (*Some*) y **where** $x = Some\ y$ **and** $Q\ y$
  ⟨*proof*⟩

lemma *insert-None-conv-UNIV*: *insert None* (*range Some*) = *UNIV*
  ⟨*proof*⟩

instance *option* :: (*finite*) *finite*
⟨*proof*⟩

lemma *univ-option* [*noatp*, *code func*]:
  UNIV = *insert* (*None* :: ′*a*::*finite option*) (*image Some UNIV*)
  ⟨*proof*⟩

24.2.1 Operations

consts
  the :: ′*a option* => ′*a*
primrec
  the (*Some x*) = *x*

consts
  o2s :: ′*a option* => ′*a set*
primrec
  o2s None = {}
  o2s (*Some x*) = {*x*}

lemma *ospec* [*dest*]: (*ALL x*:*o2s A*. *P x*) ==> *A* = *Some x* ==> *P x*
  ⟨*proof*⟩

⟨*ML*⟩

lemma *elem-o2s* [*iff*]: (*x* : *o2s xo*) = (*xo* = *Some x*)
  ⟨*proof*⟩

lemma *o2s-empty-eq* [*simp*]: (*o2s xo* = {}) = (*xo* = *None*)
  ⟨*proof*⟩

constdefs
  option-map :: (′*a* => ′*b*) => (′*a option* => ′*b option*)
  option-map == %*f y*. *case y of None* => *None* | *Some x* => *Some* (*f x*)

lemmas [*code func del*] = *option-map-def*

lemma *option-map-None* [*simp*, *code*]: *option-map f None* = *None*
  ⟨*proof*⟩

lemma *option-map-Some* [*simp*, *code*]: *option-map f* (*Some x*) = *Some* (*f x*)
  ⟨*proof*⟩

lemma *option-map-is-None* [*iff*]:
  (*option-map f opt* = *None*) = (*opt* = *None*)
⟨*proof*⟩

lemma *option-map-eq-Some* [*iff*]:
  (*option-map f xo* = *Some y*) = (*EX z*. *xo* = *Some z* & *f z* = *y*)
⟨*proof*⟩

lemma *option-map-comp*:
  option-map f (*option-map g opt*) = *option-map* (*f o g*) *opt*
⟨*proof*⟩

lemma *option-map-o-sum-case* [*simp*]:
  option-map f o sum-case g h = *sum-case* (*option-map f o g*) (*option-map f o h*)
⟨*proof*⟩

24.2.2 Code generator setup

⟨*ML*⟩

definition
  is-none :: *'a option* ⇒ *bool* **where**
  is-none-none [*code post*, *symmetric*, *code inline*]: *is-none x* ⟷ *x* = *None*

lemma *is-none-code* [*code*]:
  shows *is-none None* ⟷ *True*
  and *is-none* (*Some x*) ⟷ *False*
⟨*proof*⟩

hide (**open**) *const is-none*

code-type *option*
  (*SML* - *option*)
  (*OCaml* - *option*)
  (*Haskell Maybe* -)

code-const *None* **and** *Some*
  (*SML NONE* **and** *SOME*)
  (*OCaml None* **and** *Some* -)
  (*Haskell Nothing* **and** *Just*)

code-instance *option* :: *eq*
  (*Haskell* −)

code-const *op* = :: *'a*::*eq option* ⇒ *'a option* ⇒ *bool*
  (*Haskell* **infixl** *4* ==)

code-reserved *SML*

option NONE SOME

code-reserved *OCaml*
option None Some

code-modulename *SML*
Datatype Nat

code-modulename *OCaml*
Datatype Nat

code-modulename *Haskell*
Datatype Nat

end

25 Equiv-Relations: Equivalence Relations in Higher-Order Set Theory

theory *Equiv-Relations*
imports *Finite-Set Relation*
begin

25.1 Equivalence relations

locale *equiv* =
fixes *A* **and** *r*
assumes *refl*: *refl A r*
and *sym*: *sym r*
and *trans*: *trans r*

Suppes, Theorem 70: r is an equiv relation iff $r^{-1}\ O\ r = r$.

First half: *equiv A r ==>* $r^{-1}\ O\ r = r$.

lemma *sym-trans-comp-subset*:
sym r ==> trans r ==> $r^{-1}\ O\ r \subseteq r$
⟨*proof*⟩

lemma *refl-comp-subset*: *refl A r ==>* $r \subseteq r^{-1}\ O\ r$
⟨*proof*⟩

lemma *equiv-comp-eq*: *equiv A r ==>* $r^{-1}\ O\ r = r$
⟨*proof*⟩

Second half.

lemma *comp-equivI*:
$r^{-1}\ O\ r = r$ *==> Domain r = A ==> equiv A r*
⟨*proof*⟩

25.2 Equivalence classes

lemma *equiv-class-subset*:
$equiv\ A\ r \Longrightarrow (a,\ b) \in r \Longrightarrow r``\{a\} \subseteq r``\{b\}$
— lemma for the next result
⟨*proof*⟩

theorem *equiv-class-eq*: $equiv\ A\ r \Longrightarrow (a,\ b) \in r \Longrightarrow r``\{a\} = r``\{b\}$
⟨*proof*⟩

lemma *equiv-class-self*: $equiv\ A\ r \Longrightarrow a \in A \Longrightarrow a \in r``\{a\}$
⟨*proof*⟩

lemma *subset-equiv-class*:
$equiv\ A\ r \Longrightarrow r``\{b\} \subseteq r``\{a\} \Longrightarrow b \in A \Longrightarrow (a,b) \in r$
— lemma for the next result
⟨*proof*⟩

lemma *eq-equiv-class*:
$r``\{a\} = r``\{b\} \Longrightarrow equiv\ A\ r \Longrightarrow b \in A \Longrightarrow (a,\ b) \in r$
⟨*proof*⟩

lemma *equiv-class-nondisjoint*:
$equiv\ A\ r \Longrightarrow x \in (r``\{a\} \cap r``\{b\}) \Longrightarrow (a,\ b) \in r$
⟨*proof*⟩

lemma *equiv-type*: $equiv\ A\ r \Longrightarrow r \subseteq A \times A$
⟨*proof*⟩

theorem *equiv-class-eq-iff*:
$equiv\ A\ r \Longrightarrow ((x,\ y) \in r) = (r``\{x\} = r``\{y\}\ \&\ x \in A\ \&\ y \in A)$
⟨*proof*⟩

theorem *eq-equiv-class-iff*:
$equiv\ A\ r \Longrightarrow x \in A \Longrightarrow y \in A \Longrightarrow (r``\{x\} = r``\{y\}) = ((x,\ y) \in r)$
⟨*proof*⟩

25.3 Quotients

constdefs
$quotient :: ['a\ set,\ ('a*'a)\ set] => 'a\ set\ set$ (**infixl** $'//'\ 90$)
$A//r == \bigcup x \in A.\ \{r``\{x\}\}$ — set of equiv classes

lemma *quotientI*: $x \in A \Longrightarrow r``\{x\} \in A//r$
⟨*proof*⟩

lemma *quotientE*:
$X \in A//r \Longrightarrow (!!x.\ X = r``\{x\} \Longrightarrow x \in A \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

lemma *Union-quotient*: *equiv A r ==> Union (A//r) = A*
⟨*proof*⟩

lemma *quotient-disj*:
equiv A r ==> X $\in$ *A//r ==> Y* $\in$ *A//r ==> X = Y | (X* $\cap$ *Y = {})*
⟨*proof*⟩

lemma *quotient-eqI*:
[|equiv A r; X $\in$ *A//r; Y* $\in$ *A//r; x* $\in$ *X; y* $\in$ *Y; (x,y)* $\in$ *r|] ==> X = Y*
⟨*proof*⟩

lemma *quotient-eq-iff*:
[|equiv A r; X $\in$ *A//r; Y* $\in$ *A//r; x* $\in$ *X; y* $\in$ *Y|] ==> (X = Y) = ((x,y)* $\in$ *r)*
⟨*proof*⟩

lemma *eq-equiv-class-iff2*:
$\llbracket$ *equiv A r; x* $\in$ *A; y* $\in$ *A* $\rrbracket \Longrightarrow$ *({x}//r = {y}//r) = ((x,y) : r)*
⟨*proof*⟩

lemma *quotient-empty* [*simp*]: *{}//r = {}*
⟨*proof*⟩

lemma *quotient-is-empty* [*iff*]: *(A//r = {}) = (A = {})*
⟨*proof*⟩

lemma *quotient-is-empty2* [*iff*]: *({} = A//r) = (A = {})*
⟨*proof*⟩

lemma *singleton-quotient*: *{x}//r = {r `` {x}}*
⟨*proof*⟩

lemma *quotient-diff1*:
$\llbracket$ *inj-on (%a. {a}//r) A; a* $\in$ *A* $\rrbracket \Longrightarrow$ *(A − {a})//r = A//r − {a}//r*
⟨*proof*⟩

25.4 Defining unary operations upon equivalence classes

A congruence-preserving function

locale *congruent =*
fixes *r* **and** *f*
assumes *congruent*: *(y,z)* $\in$ *r ==> f y = f z*

abbreviation
*RESPECTS :: ('a => 'b) => ('a * 'a) set => bool*
(**infixr** *respects 80*) **where**
f respects r == congruent r f

lemma *UN-constant-eq*: $a \in A ==> \forall y \in A.\ f\ y = c ==> (\bigcup y \in A.\ f(y))=c$
— lemma required to prove *UN-equiv-class*
⟨*proof*⟩

lemma *UN-equiv-class*:
$equiv\ A\ r ==> f\ respects\ r ==> a \in A$
$==> (\bigcup x \in r``\{a\}.\ f\ x) = f\ a$
— Conversion rule
⟨*proof*⟩

lemma *UN-equiv-class-type*:
$equiv\ A\ r ==> f\ respects\ r ==> X \in A//r ==>$
$(!!x.\ x \in A ==> f\ x \in B) ==> (\bigcup x \in X.\ f\ x) \in B$
⟨*proof*⟩

Sufficient conditions for injectiveness. Could weaken premises! major premise could be an inclusion; bcong could be $!!y.\ y \in A ==> f\ y \in B$.

lemma *UN-equiv-class-inject*:
$equiv\ A\ r ==> f\ respects\ r ==>$
$(\bigcup x \in X.\ f\ x) = (\bigcup y \in Y.\ f\ y) ==> X \in A//r ==> Y \in A//r$
$==> (!!x\ y.\ x \in A ==> y \in A ==> f\ x = f\ y ==> (x,\ y) \in r)$
$==> X = Y$
⟨*proof*⟩

25.5 Defining binary operations upon equivalence classes

A congruence-preserving function of two arguments

locale *congruent2* =
fixes *r1* **and** *r2* **and** *f*
assumes *congruent2*:
$(y1,z1) \in r1 ==> (y2,z2) \in r2 ==> f\ y1\ y2 = f\ z1\ z2$

Abbreviation for the common case where the relations are identical

abbreviation
RESPECTS2:: $['a => 'a => 'b,\ ('a * 'a)\ set] => bool$
(**infixr** *respects2 80*) **where**
$f\ respects2\ r == congruent2\ r\ r\ f$

lemma *congruent2-implies-congruent*:
$equiv\ A\ r1 ==> congruent2\ r1\ r2\ f ==> a \in A ==> congruent\ r2\ (f\ a)$
⟨*proof*⟩

lemma *congruent2-implies-congruent-UN*:
$equiv\ A1\ r1 ==> equiv\ A2\ r2 ==> congruent2\ r1\ r2\ f ==> a \in A2 ==>$
$congruent\ r1\ (\lambda x1.\ \bigcup x2 \in r2``\{a\}.\ f\ x1\ x2)$

⟨*proof*⟩

lemma *UN-equiv-class2*:
equiv A1 r1 ==> equiv A2 r2 ==> congruent2 r1 r2 f ==> a1 ∈ *A1 ==> a2* ∈ *A2*
$==> (\bigcup x1 \in r1``\{a1\}. \bigcup x2 \in r2``\{a2\}. f\ x1\ x2) = f\ a1\ a2$
⟨*proof*⟩

lemma *UN-equiv-class-type2*:
equiv A1 r1 ==> equiv A2 r2 ==> congruent2 r1 r2 f
$==> X1 \in A1//r1 ==> X2 \in A2//r2$
$==> (!!x1\ x2.\ x1 \in A1 ==> x2 \in A2 ==> f\ x1\ x2 \in B)$
$==> (\bigcup x1 \in X1. \bigcup x2 \in X2. f\ x1\ x2) \in B$
⟨*proof*⟩

lemma *UN-UN-split-split-eq*:
$(\bigcup (x1, x2) \in X. \bigcup (y1, y2) \in Y. A\ x1\ x2\ y1\ y2) =$
$(\bigcup x \in X. \bigcup y \in Y. (\lambda(x1, x2). (\lambda(y1, y2). A\ x1\ x2\ y1\ y2)\ y)\ x)$
— Allows a natural expression of binary operators,
— without explicit calls to *split*
⟨*proof*⟩

lemma *congruent2I*:
equiv A1 r1 ==> equiv A2 r2
$==> (!!y\ z\ w.\ w \in A2 ==> (y,z) \in r1 ==> f\ y\ w = f\ z\ w)$
$==> (!!y\ z\ w.\ w \in A1 ==> (y,z) \in r2 ==> f\ w\ y = f\ w\ z)$
==> congruent2 r1 r2 f
— Suggested by John Harrison – the two subproofs may be
— *much* simpler than the direct proof.
⟨*proof*⟩

lemma *congruent2-commuteI*:
assumes *equivA*: *equiv A r*
and *commute*: $!!y\ z.\ y \in A ==> z \in A ==> f\ y\ z = f\ z\ y$
and *congt*: $!!y\ z\ w.\ w \in A ==> (y,z) \in r ==> f\ w\ y = f\ w\ z$
shows *f respects2 r*
⟨*proof*⟩

25.6 Quotients and finiteness

Suggested by Florian Kammüller

lemma *finite-quotient*: *finite A ==> r* ⊆ *A* × *A ==> finite* $(A//r)$
— recall *equiv ?A ?r* $\Longrightarrow$ *?r* ⊆ *?A* × *?A*
⟨*proof*⟩

lemma *finite-equiv-class*:
finite A ==> r ⊆ *A* × *A ==> X* ∈ *A//r ==> finite X*
⟨*proof*⟩

lemma *equiv-imp-dvd-card*:
$finite\ A\ {=}{=}{>}\ equiv\ A\ r\ {=}{=}{>}\ \forall X \in A//r.\ k\ dvd\ card\ X$
$\quad {=}{=}{>}\ k\ dvd\ card\ A$
$\langle proof \rangle$

lemma *card-quotient-disjoint*:
$[\![\ finite\ A;\ inj\text{-}on\ (\lambda x.\ \{x\}\ //\ r)\ A\ ]\!] \Longrightarrow card(A//r) = card\ A$
$\langle proof \rangle$

end

26 IntDef: The Integers as Equivalence Classes over Pairs of Natural Numbers

theory *IntDef*
imports *Equiv-Relations Nat*
begin

the equivalence relation underlying the integers

definition
$intrel :: ((nat \times nat) \times (nat \times nat))\ set$
where
$intrel = \{((x,\ y),\ (u,\ v))\ |\ x\ y\ u\ v.\ x + v = u + y\ \}$

typedef *(Integ)*
$int = UNIV//intrel$
$\langle proof \rangle$

instance *int* :: *zero*
$Zero\text{-}int\text{-}def$: $0 \equiv Abs\text{-}Integ\ (intrel\ ``\ \{(0,\ 0)\})\ \langle proof \rangle$

instance *int* :: *one*
$One\text{-}int\text{-}def$: $1 \equiv Abs\text{-}Integ\ (intrel\ ``\ \{(1,\ 0)\})\ \langle proof \rangle$

instance *int* :: *plus*
$add\text{-}int\text{-}def$: $z + w \equiv Abs\text{-}Integ$
$\quad (\bigcup (x,\ y) \in Rep\text{-}Integ\ z.\ \bigcup (u,\ v) \in Rep\text{-}Integ\ w.$
$\quad\quad intrel\ ``\ \{(x + u,\ y + v)\})\ \langle proof \rangle$

instance *int* :: *minus*
$minus\text{-}int\text{-}def$:
$\quad - z \equiv Abs\text{-}Integ\ (\bigcup (x,\ y) \in Rep\text{-}Integ\ z.\ intrel\ ``\ \{(y,\ x)\})$
$diff\text{-}int\text{-}def$: $z - w \equiv z + (-w)\ \langle proof \rangle$

instance *int* :: *times*
$mult\text{-}int\text{-}def$: $z * w \equiv Abs\text{-}Integ$
$\quad (\bigcup (x,\ y) \in Rep\text{-}Integ\ z.\ \bigcup (u,v\ ) \in Rep\text{-}Integ\ w.$

intrel `` $\{(x*u + y*v,\ x*v + y*u)\}$) ⟨*proof*⟩

instance *int* :: *ord*
le-int-def:
$z \le w \equiv \exists x\ y\ u\ v.\ x+v \le u+y \wedge (x,\ y) \in$ *Rep-Integ* $z \wedge (u,\ v) \in$ *Rep-Integ* w
less-int-def: $z < w \equiv z \le w \wedge z \ne w$ ⟨*proof*⟩

lemmas [*code func del*] = *Zero-int-def One-int-def add-int-def minus-int-def mult-int-def le-int-def less-int-def*

26.1 Construction of the Integers

lemma *intrel-iff* [*simp*]: $(((x,y),(u,v)) \in$ *intrel*$) = (x+v = u+y)$
⟨*proof*⟩

lemma *equiv-intrel*: *equiv UNIV intrel*
⟨*proof*⟩

Reduces equality of equivalence classes to the *intrel* relation: (*intrel* `` $\{x\}$ = *intrel* `` $\{y\}$) = $((x,\ y) \in$ *intrel*)

lemmas *equiv-intrel-iff* [*simp*] = *eq-equiv-class-iff* [*OF equiv-intrel UNIV-I UNIV-I*]

All equivalence classes belong to set of representatives

lemma [*simp*]: *intrel*``$\{(x,y)\} \in$ *Integ*
⟨*proof*⟩

Reduces equality on abstractions to equality on representatives: $[\![x \in$ *Integ*; $y \in$ *Integ*$]\!] \Longrightarrow$ (*Abs-Integ* x = *Abs-Integ* y) = $(x = y)$

declare *Abs-Integ-inject* [*simp*,*noatp*] *Abs-Integ-inverse* [*simp*,*noatp*]

Case analysis on the representation of an integer as an equivalence class of pairs of naturals.

lemma *eq-Abs-Integ* [*case-names Abs-Integ*, *cases type*: *int*]:
(!!x y. z = *Abs-Integ*(*intrel*``$\{(x,y)\}$) ==> P) ==> P
⟨*proof*⟩

26.2 Arithmetic Operations

lemma *minus*: − *Abs-Integ*(*intrel*``$\{(x,y)\}$) = *Abs-Integ*(*intrel* `` $\{(y,x)\}$)
⟨*proof*⟩

lemma *add*:
Abs-Integ (*intrel*``$\{(x,y)\}$) + *Abs-Integ* (*intrel*``$\{(u,v)\}$) =
Abs-Integ (*intrel*``$\{(x+u,\ y+v)\}$)
⟨*proof*⟩

Congruence property for multiplication

lemma *mult-congruent2*:

*(%p1 p2. (%(x,y). (%(u,v). intrel``{(x*u + y*v, x*v + y*u)}) p2) p1)*
respects2 intrel
⟨*proof*⟩

lemma *mult*:
Abs-Integ((intrel``{(x,y)})) \* Abs-Integ((intrel``{(u,v)})) =
*Abs-Integ(intrel `` {(x*u + y*v, x*v + y*u)})*
⟨*proof*⟩

The integers form a *comm-ring-1*

instance *int* :: *comm-ring-1*
⟨*proof*⟩

lemma *int-def*: *of-nat m = Abs-Integ (intrel `` {(m, 0)})*
⟨*proof*⟩

26.3 The ≤ Ordering

lemma *le*:
(Abs-Integ(intrel``{(x,y)}) ≤ Abs-Integ(intrel``{(u,v)})) = (x+v ≤ u+y)
⟨*proof*⟩

lemma *less*:
(Abs-Integ(intrel``{(x,y)}) < Abs-Integ(intrel``{(u,v)})) = (x+v < u+y)
⟨*proof*⟩

instance *int* :: *linorder*
⟨*proof*⟩

instance *int* :: *pordered-cancel-ab-semigroup-add*
⟨*proof*⟩

Strict Monotonicity of Multiplication

strict, in 1st argument; proof is by induction on k¿0

lemma *zmult-zless-mono2-lemma*:
*(i::int)<j ==> 0<k ==> of-nat k * i < of-nat k * j*
⟨*proof*⟩

lemma *zero-le-imp-eq-int*: *(0::int) ≤ k ==> ∃ n. k = of-nat n*
⟨*proof*⟩

lemma *zero-less-imp-eq-int*: *(0::int) < k ==> ∃ n>0. k = of-nat n*
⟨*proof*⟩

lemma *zmult-zless-mono2*: *[| i<j; (0::int) < k |] ==> k*i < k*j*
⟨*proof*⟩

instance *int* :: *abs*

zabs-def: $|i{::}int| \equiv$ *if* $i < 0$ *then* $-\ i$ *else* i ⟨*proof*⟩
instance *int* :: *sgn*
zsgn-def: $sgn(i{::}int) \equiv$ (*if* $i{=}0$ *then* 0 *else if* $0{<}i$ *then* 1 *else* $-\ 1$) ⟨*proof*⟩

instance *int* :: *distrib-lattice*
inf $\equiv$ *min*
sup $\equiv$ *max*
⟨*proof*⟩

The integers form an ordered integral domain

instance *int* :: *ordered-idom*
⟨*proof*⟩

lemma *zless-imp-add1-zle*: $w{<}z$ ==> $w + (1{::}int) \leq z$
⟨*proof*⟩

26.4 Magnitude of an Integer, as a Natural Number: *nat*

definition
nat :: *int* $\Rightarrow$ *nat*
where
[*code func del*]: *nat* z = *contents* ($\bigcup (x,\ y) \in$ *Rep-Integ* $z.\ \{x{-}y\}$)

lemma *nat*: *nat* (*Abs-Integ* (*intrel*``$\{(x,y)\}$)) $= x{-}y$
⟨*proof*⟩

lemma *nat-int* [*simp*]: *nat* (*of-nat* n) $= n$
⟨*proof*⟩

lemma *nat-zero* [*simp*]: *nat* $0 = 0$
⟨*proof*⟩

lemma *int-nat-eq* [*simp*]: *of-nat* (*nat* z) = (*if* $0 \leq z$ *then* z *else* 0)
⟨*proof*⟩

corollary *nat-0-le*: $0 \leq z$ ==> *of-nat* (*nat* z) $= z$
⟨*proof*⟩

lemma *nat-le-0* [*simp*]: $z \leq 0$ ==> *nat* $z = 0$
⟨*proof*⟩

lemma *nat-le-eq-zle*: $0 < w \mid 0 \leq z$ ==> (*nat* $w \leq$ *nat* z) = ($w{\leq}z$)
⟨*proof*⟩

An alternative condition is $(0{::}'a) \leq w$

corollary *nat-mono-iff*: $0 < z$ ==> (*nat* $w <$ *nat* z) = ($w < z$)
⟨*proof*⟩

corollary *nat-less-eq-zless*: $0 \leq w$ ==> (*nat* $w <$ *nat* z) = ($w{<}z$)

⟨*proof*⟩

lemma *zless-nat-conj* [*simp*]: $(nat\ w < nat\ z) = (0 < z\ \&\ w < z)$
⟨*proof*⟩

lemma *nonneg-eq-int*:
  fixes $z :: int$
  assumes $0 \le z$ **and** $\bigwedge m.\ z = of\text{-}nat\ m \Longrightarrow P$
  shows P
  ⟨*proof*⟩

lemma *nat-eq-iff*: $(nat\ w = m) = (if\ 0 \le w\ then\ w = of\text{-}nat\ m\ else\ m{=}0)$
⟨*proof*⟩

corollary *nat-eq-iff2*: $(m = nat\ w) = (if\ 0 \le w\ then\ w = of\text{-}nat\ m\ else\ m{=}0)$
⟨*proof*⟩

lemma *nat-less-iff*: $0 \le w ==> (nat\ w < m) = (w < of\text{-}nat\ m)$
⟨*proof*⟩

lemma *int-eq-iff*: $(of\text{-}nat\ m = z) = (m = nat\ z\ \&\ 0 \le z)$
⟨*proof*⟩

lemma *zero-less-nat-eq* [*simp*]: $(0 < nat\ z) = (0 < z)$
⟨*proof*⟩

lemma *nat-add-distrib*:
    $[|\ (0::int) \le z;\ \ 0 \le z'\ |] ==> nat\ (z{+}z') = nat\ z + nat\ z'$
⟨*proof*⟩

lemma *nat-diff-distrib*:
    $[|\ (0::int) \le z';\ \ z' \le z\ |] ==> nat\ (z{-}z') = nat\ z - nat\ z'$
⟨*proof*⟩

lemma *nat-zminus-int* [*simp*]: $nat\ (-\ (of\text{-}nat\ n)) = 0$
⟨*proof*⟩

lemma *zless-nat-eq-int-zless*: $(m < nat\ z) = (of\text{-}nat\ m < z)$
⟨*proof*⟩

26.5 Lemmas about the Function *of-nat* and Orderings

lemma *negative-zless-0*: $-\ (of\text{-}nat\ (Suc\ n)) < (0 :: int)$
⟨*proof*⟩

lemma *negative-zless* [*iff*]: $-\ (of\text{-}nat\ (Suc\ n)) < (of\text{-}nat\ m :: int)$
⟨*proof*⟩

lemma *negative-zle-0*: $-\ of\text{-}nat\ n \le (0 :: int)$

⟨*proof*⟩

lemma *negative-zle* [*iff*]: $-$ *of-nat* $n \leq$ (*of-nat* m :: *int*)
⟨*proof*⟩

lemma *not-zle-0-negative* [*simp*]: ~ ($0 \leq -$ (*of-nat* (*Suc* n) :: *int*))
⟨*proof*⟩

lemma *int-zle-neg*: ((*of-nat* n :: *int*) $\leq -$ *of-nat* m) = ($n = 0$ & $m = 0$)
⟨*proof*⟩

lemma *not-int-zless-negative* [*simp*]: ~ ((*of-nat* n :: *int*) $< -$ *of-nat* m)
⟨*proof*⟩

lemma *negative-eq-positive* [*simp*]: (($-$ *of-nat* n :: *int*) = *of-nat* m) = ($n = 0$ & $m = 0$)
⟨*proof*⟩

lemma *zle-iff-zadd*: (w::*int*) $\leq z \longleftrightarrow (\exists n.\ z = w +$ *of-nat* n)
⟨*proof*⟩

lemma *zadd-int-left*: *of-nat* m + (*of-nat* $n + z$) = *of-nat* ($m + n$) + (z::*int*)
⟨*proof*⟩

lemma *int-Suc0-eq-1*: *of-nat* (*Suc* 0) = (1::*int*)
⟨*proof*⟩

This version is proved for all ordered rings, not just integers! It is proved here because attribute *arith-split* is not available in theory *Ring-and-Field*. But is it really better than just rewriting with *abs-if*?

lemma *abs-split* [*arith-split*,*noatp*]:
  P(*abs*(a::$'a$::*ordered-idom*)) = (($0 \leq a$ --> $P\ a$) & ($a < 0$ --> $P(-a)$))
⟨*proof*⟩

26.6 Constants *neg* and *iszero*

definition
  neg :: $'a$::*ordered-idom* $\Rightarrow$ *bool*
where
  neg $Z \longleftrightarrow Z < 0$

definition
  iszero :: $'a$::*semiring-1* $\Rightarrow$ *bool*
where
  iszero $z \longleftrightarrow z = 0$

lemma *not-neg-int* [*simp*]: ~ *neg* (*of-nat* n)
⟨*proof*⟩

lemma *neg-zminus-int* [*simp*]: *neg* (− (*of-nat* (*Suc n*)))
⟨*proof*⟩

lemmas *neg-eq-less-0* = *neg-def*

lemma *not-neg-eq-ge-0*: (~*neg x*) = (*0* ≤ *x*)
⟨*proof*⟩

To simplify inequalities when Numeral1 can get simplified to 1

lemma *not-neg-0*: ~ *neg 0*
⟨*proof*⟩

lemma *not-neg-1*: ~ *neg 1*
⟨*proof*⟩

lemma *iszero-0*: *iszero 0*
⟨*proof*⟩

lemma *not-iszero-1*: ~ *iszero 1*
⟨*proof*⟩

lemma *neg-nat*: *neg z* ==> *nat z* = *0*
⟨*proof*⟩

lemma *not-neg-nat*: ~ *neg z* ==> *of-nat* (*nat z*) = *z*
⟨*proof*⟩

26.7 Embedding of the Integers into any *ring-1*: *of-int*

context *ring-1*
begin

term *of-nat*

definition
  of-int :: *int* ⇒ *'a*
where
  of-int z = *contents* ($\bigcup$(*i*, *j*) ∈ *Rep-Integ z*. { *of-nat i* − *of-nat j* })
lemmas [*code func del*] = *of-int-def*

lemma *of-int*: *of-int* (*Abs-Integ* (*intrel* ‘‘ {(*i*,*j*)})) = *of-nat i* − *of-nat j*
⟨*proof*⟩

lemma *of-int-0* [*simp*]: *of-int 0* = *0*
⟨*proof*⟩

lemma *of-int-1* [*simp*]: *of-int 1* = *1*
⟨*proof*⟩

lemma *of-int-add* [*simp*]: *of-int* $(w+z)$ = *of-int* w + *of-int* z
⟨*proof*⟩

lemma *of-int-minus* [*simp*]: *of-int* $(-z) = -$ (*of-int* z)
⟨*proof*⟩

lemma *of-int-mult* [*simp*]: *of-int* $(w*z)$ = *of-int* w $*$ *of-int* z
⟨*proof*⟩

Collapse nested embeddings

lemma *of-int-of-nat-eq* [*simp*]: *of-int* (*Nat.of-nat* n) = *of-nat* n
⟨*proof*⟩

end

lemma *of-int-diff* [*simp*]: *of-int* $(w-z)$ = *of-int* w $-$ *of-int* z
⟨*proof*⟩

lemma *of-int-le-iff* [*simp*]:
(*of-int* $w \leq$ (*of-int* z::'a::*ordered-idom*)) = $(w \leq z)$
⟨*proof*⟩

Special cases where either operand is zero

lemmas *of-int-0-le-iff* [*simp*] = *of-int-le-iff* [*of 0*, *simplified*]
lemmas *of-int-le-0-iff* [*simp*] = *of-int-le-iff* [*of - 0*, *simplified*]

lemma *of-int-less-iff* [*simp*]:
(*of-int* $w <$ (*of-int* z::'a::*ordered-idom*)) = $(w < z)$
⟨*proof*⟩

Special cases where either operand is zero

lemmas *of-int-0-less-iff* [*simp*] = *of-int-less-iff* [*of 0*, *simplified*]
lemmas *of-int-less-0-iff* [*simp*] = *of-int-less-iff* [*of - 0*, *simplified*]

Class for unital rings with characteristic zero. Includes non-ordered rings like the complex numbers.

class *ring-char-0* = *ring-1* + *semiring-char-0*
begin

lemma *of-int-eq-iff* [*simp*]:
of-int w = *of-int* $z \longleftrightarrow w = z$
⟨*proof*⟩

Special cases where either operand is zero

lemmas *of-int-0-eq-iff* [*simp*] = *of-int-eq-iff* [*of 0*, *simplified*]
lemmas *of-int-eq-0-iff* [*simp*] = *of-int-eq-iff* [*of - 0*, *simplified*]

end

Every *ordered-idom* has characteristic zero.

instance *ordered-idom* $\subseteq$ *ring-char-0* $\langle proof \rangle$

lemma *of-int-eq-id* [*simp*]: *of-int* = *id*
$\langle proof \rangle$

context *ring-1*
begin

lemma *of-nat-nat*: $0 \leq z \Longrightarrow$ *of-nat* (*nat* z) = *of-int* z
$\langle proof \rangle$

end

26.8 The Set of Integers

context *ring-1*
begin

definition
Ints :: *'a set*
where
Ints = *range of-int*

end

notation (*xsymbols*)
Ints ($\mathbb{Z}$)

context *ring-1*
begin

lemma *Ints-0* [*simp*]: $0 \in \mathbb{Z}$
$\langle proof \rangle$

lemma *Ints-1* [*simp*]: $1 \in \mathbb{Z}$
$\langle proof \rangle$

lemma *Ints-add* [*simp*]: $a \in \mathbb{Z} \Longrightarrow b \in \mathbb{Z} \Longrightarrow a + b \in \mathbb{Z}$
$\langle proof \rangle$

lemma *Ints-minus* [*simp*]: $a \in \mathbb{Z} \Longrightarrow -a \in \mathbb{Z}$
$\langle proof \rangle$

lemma *Ints-mult* [*simp*]: $a \in \mathbb{Z} \Longrightarrow b \in \mathbb{Z} \Longrightarrow a * b \in \mathbb{Z}$
$\langle proof \rangle$

lemma *Ints-cases* [*cases set*: *Ints*]:
assumes $q \in \mathbb{Z}$

obtains $(of\text{-}int)\ z$ **where** $q = of\text{-}int\ z$
$\langle proof \rangle$

lemma *Ints-induct* [*case-names of-int*, *induct set*: *Ints*]:
$q \in \mathbb{Z} \Longrightarrow (\bigwedge z.\ P\ (of\text{-}int\ z)) \Longrightarrow P\ q$
$\langle proof \rangle$

end

lemma *Ints-diff* [*simp*]: $a \in \mathbb{Z} \Longrightarrow b \in \mathbb{Z} \Longrightarrow a - b \in \mathbb{Z}$
$\langle proof \rangle$

26.9 *setsum* and *setprod*

By Jeremy Avigad

lemma *of-nat-setsum*: $of\text{-}nat\ (setsum\ f\ A) = (\sum x{\in}A.\ of\text{-}nat(f\ x))$
$\langle proof \rangle$

lemma *of-int-setsum*: $of\text{-}int\ (setsum\ f\ A) = (\sum x{\in}A.\ of\text{-}int(f\ x))$
$\langle proof \rangle$

lemma *of-nat-setprod*: $of\text{-}nat\ (setprod\ f\ A) = (\prod x{\in}A.\ of\text{-}nat(f\ x))$
$\langle proof \rangle$

lemma *of-int-setprod*: $of\text{-}int\ (setprod\ f\ A) = (\prod x{\in}A.\ of\text{-}int(f\ x))$
$\langle proof \rangle$

lemma *setprod-nonzero-nat*:
$finite\ A ==> (\forall x \in A.\ f\ x \neq (0::nat)) ==> setprod\ f\ A \neq 0$
$\langle proof \rangle$

lemma *setprod-zero-eq-nat*:
$finite\ A ==> (setprod\ f\ A = (0::nat)) = (\exists x \in A.\ f\ x = 0)$
$\langle proof \rangle$

lemma *setprod-nonzero-int*:
$finite\ A ==> (\forall x \in A.\ f\ x \neq (0::int)) ==> setprod\ f\ A \neq 0$
$\langle proof \rangle$

lemma *setprod-zero-eq-int*:
$finite\ A ==> (setprod\ f\ A = (0::int)) = (\exists x \in A.\ f\ x = 0)$
$\langle proof \rangle$

lemmas *int-setsum* = *of-nat-setsum* [**where** *'a=int*]
lemmas *int-setprod* = *of-nat-setprod* [**where** *'a=int*]

26.10 Further properties

Now we replace the case analysis rule by a more conventional one: whether an integer is negative or not.

lemma *zless-iff-Suc-zadd*:
$(w :: int) < z \longleftrightarrow (\exists\, n.\ z = w + \textit{of-nat}\ (\textit{Suc}\ n))$
⟨*proof*⟩

lemma *negD*: $(x :: int) < 0 \Longrightarrow \exists\, n.\ x = -\ (\textit{of-nat}\ (\textit{Suc}\ n))$
⟨*proof*⟩

theorem *int-cases* [*cases type*: *int*, *case-names nonneg neg*]:
[|!! *n*. (*z* :: *int*) = *of-nat n* ==> *P*; !! *n*. *z* = − (*of-nat* (*Suc n*)) ==> *P* |]
==> *P*
⟨*proof*⟩

theorem *int-induct* [*induct type*: *int*, *case-names nonneg neg*]:
[|!! *n*. *P* (*of-nat n* :: *int*); !!*n*. *P* (− (*of-nat* (*Suc n*))) |] ==> *P z*
⟨*proof*⟩

Contributed by Brian Huffman

theorem *int-diff-cases*:
obtains (*diff*) *m n* **where** (*z*::*int*) = *of-nat m* − *of-nat n*
⟨*proof*⟩

26.11 Legacy theorems

lemmas *zminus-zminus* = *minus-minus* [*of z*::*int*, *standard*]
lemmas *zminus-0* = *minus-zero* [**where** ′*a*=*int*]
lemmas *zminus-zadd-distrib* = *minus-add-distrib* [*of z*::*int w*, *standard*]
lemmas *zadd-commute* = *add-commute* [*of z*::*int w*, *standard*]
lemmas *zadd-assoc* = *add-assoc* [*of z1*::*int z2 z3*, *standard*]
lemmas *zadd-left-commute* = *add-left-commute* [*of x*::*int y z*, *standard*]
lemmas *zadd-ac* = *zadd-assoc zadd-commute zadd-left-commute*
lemmas *zmult-ac* = *OrderedGroup.mult-ac*
lemmas *zadd-0* = *OrderedGroup.add-0-left* [*of z*::*int*, *standard*]
lemmas *zadd-0-right* = *OrderedGroup.add-0-left* [*of z*::*int*, *standard*]
lemmas *zadd-zminus-inverse2* = *left-minus* [*of z*::*int*, *standard*]
lemmas *zmult-zminus* = *mult-minus-left* [*of z*::*int w*, *standard*]
lemmas *zmult-commute* = *mult-commute* [*of z*::*int w*, *standard*]
lemmas *zmult-assoc* = *mult-assoc* [*of z1*::*int z2 z3*, *standard*]
lemmas *zadd-zmult-distrib* = *left-distrib* [*of z1*::*int z2 w*, *standard*]
lemmas *zadd-zmult-distrib2* = *right-distrib* [*of w*::*int z1 z2*, *standard*]
lemmas *zdiff-zmult-distrib* = *left-diff-distrib* [*of z1*::*int z2 w*, *standard*]
lemmas *zdiff-zmult-distrib2* = *right-diff-distrib* [*of w*::*int z1 z2*, *standard*]

lemmas *int-distrib* =
zadd-zmult-distrib zadd-zmult-distrib2
zdiff-zmult-distrib zdiff-zmult-distrib2

lemmas *zmult-1* = *mult-1-left* [*of z::int*, *standard*]
lemmas *zmult-1-right* = *mult-1-right* [*of z::int*, *standard*]

lemmas *zle-refl* = *order-refl* [*of w::int*, *standard*]
lemmas *zle-trans* = *order-trans* [**where** *'a=int* **and** *x=i* **and** *y=j* **and** *z=k*, *standard*]
lemmas *zle-anti-sym* = *order-antisym* [*of z::int w*, *standard*]
lemmas *zle-linear* = *linorder-linear* [*of z::int w*, *standard*]
lemmas *zless-linear* = *linorder-less-linear* [**where** *'a* = *int*]

lemmas *zadd-left-mono* = *add-left-mono* [*of i::int j k*, *standard*]
lemmas *zadd-strict-right-mono* = *add-strict-right-mono* [*of i::int j k*, *standard*]
lemmas *zadd-zless-mono* = *add-less-le-mono* [*of w'::int w z' z*, *standard*]

lemmas *int-0-less-1* = *zero-less-one* [**where** *'a=int*]
lemmas *int-0-neq-1* = *zero-neq-one* [**where** *'a=int*]

lemmas *inj-int* = *inj-of-nat* [**where** *'a=int*]
lemmas *int-int-eq* = *of-nat-eq-iff* [**where** *'a=int*]
lemmas *zadd-int* = *of-nat-add* [**where** *'a=int*, *symmetric*]
lemmas *int-mult* = *of-nat-mult* [**where** *'a=int*]
lemmas *zmult-int* = *of-nat-mult* [**where** *'a=int*, *symmetric*]
lemmas *int-eq-0-conv* = *of-nat-eq-0-iff* [**where** *'a=int* **and** *m=n*, *standard*]
lemmas *zless-int* = *of-nat-less-iff* [**where** *'a=int*]
lemmas *int-less-0-conv* = *of-nat-less-0-iff* [**where** *'a=int* **and** *m=k*, *standard*]
lemmas *zero-less-int-conv* = *of-nat-0-less-iff* [**where** *'a=int*]
lemmas *zle-int* = *of-nat-le-iff* [**where** *'a=int*]
lemmas *zero-zle-int* = *of-nat-0-le-iff* [**where** *'a=int*]
lemmas *int-le-0-conv* = *of-nat-le-0-iff* [**where** *'a=int* **and** *m=n*, *standard*]
lemmas *int-0* = *of-nat-0* [**where** *'a=int*]
lemmas *int-1* = *of-nat-1* [**where** *'a=int*]
lemmas *int-Suc* = *of-nat-Suc* [**where** *'a=int*]
lemmas *abs-int-eq* = *abs-of-nat* [**where** *'a=int* **and** *n=m*, *standard*]
lemmas *of-int-int-eq* = *of-int-of-nat-eq* [**where** *'a=int*]
lemmas *zdiff-int* = *of-nat-diff* [**where** *'a=int*, *symmetric*]
lemmas *zless-le* = *less-int-def* [*THEN meta-eq-to-obj-eq*]
lemmas *int-eq-of-nat* = *TrueI*

abbreviation
  int :: *nat* ⇒ *int*
where
  int ≡ *of-nat*

end

27 Numeral: Arithmetic on Binary Integers

theory *Numeral*
imports *Datatype IntDef*
uses
 (*Tools/numeral.ML*)
 (*Tools/numeral-syntax.ML*)
begin

27.1 Binary representation

This formalization defines binary arithmetic in terms of the integers rather than using a datatype. This avoids multiple representations (leading zeroes, etc.) See *ZF/Tools/twos−compl.ML*, function *int-of-binary*, for the numerical interpretation.

The representation expects that $(m \ mod \ 2)$ is 0 or 1, even if m is negative; For instance, $-5 \ div \ 2 = -3$ and $-5 \ mod \ 2 = 1$; thus $-5 = (-3)*2 + 1$.

datatype *bit* = *B0* | *B1*

Type *bit* avoids the use of type *bool*, which would make all of the rewrite rules higher-order.

definition
 Pls :: *int* **where**
 [*code func del*]: *Pls* = *0*

definition
 Min :: *int* **where**
 [*code func del*]: *Min* = − *1*

definition
 Bit :: *int* ⇒ *bit* ⇒ *int* (**infixl** *BIT 90*) **where**
 [*code func del*]: *k BIT b* = (*case b of B0* ⇒ *0* | *B1* ⇒ *1*) + *k* + *k*

class *number* = *type* + — for numeric types: nat, int, real, ...
 fixes *number-of* :: *int* ⇒ *'a*

⟨*ML*⟩

syntax
 -Numeral :: *num-const* ⇒ *'a* (-)

⟨*ML*⟩

abbreviation
 Numeral0 ≡ *number-of Pls*

abbreviation
 Numeral1 ≡ *number-of* (*Pls BIT B1*)

lemma *Let-number-of* [*simp*]: *Let* (*number-of v*) *f* = *f* (*number-of v*)
— Unfold all *let*s involving constants
⟨*proof*⟩

definition
succ :: *int* ⇒ *int* **where**
[*code func del*]: *succ k* = *k* + *1*

definition
pred :: *int* ⇒ *int* **where**
[*code func del*]: *pred k* = *k* − *1*

lemmas
max-number-of [*simp*] = *max-def*
[*of number-of u number-of v*, *standard*, *simp*]
and
min-number-of [*simp*] = *min-def*
[*of number-of u number-of v*, *standard*, *simp*]
— unfolding *minx* and *max* on numerals

lemmas *numeral-simps* =
succ-def pred-def Pls-def Min-def Bit-def

Removal of leading zeroes

lemma *Pls-0-eq* [*simp*, *code post*]:
Pls BIT B0 = *Pls*
⟨*proof*⟩

lemma *Min-1-eq* [*simp*, *code post*]:
Min BIT B1 = *Min*
⟨*proof*⟩

27.2 The Functions *succ*, *pred* and *uminus*

lemma *succ-Pls* [*simp*]:
succ Pls = *Pls BIT B1*
⟨*proof*⟩

lemma *succ-Min* [*simp*]:
succ Min = *Pls*
⟨*proof*⟩

lemma *succ-1* [*simp*]:
succ (*k BIT B1*) = *succ k BIT B0*
⟨*proof*⟩

lemma *succ-0* [*simp*]:
succ (*k BIT B0*) = *k BIT B1*

⟨*proof*⟩

lemma *pred-Pls* [*simp*]:
pred Pls = *Min*
⟨*proof*⟩

lemma *pred-Min* [*simp*]:
pred Min = *Min BIT B0*
⟨*proof*⟩

lemma *pred-1* [*simp*]:
pred (*k BIT B1*) = *k BIT B0*
⟨*proof*⟩

lemma *pred-0* [*simp*]:
pred (*k BIT B0*) = *pred k BIT B1*
⟨*proof*⟩

lemma *minus-Pls* [*simp*]:
− *Pls* = *Pls*
⟨*proof*⟩

lemma *minus-Min* [*simp*]:
− *Min* = *Pls BIT B1*
⟨*proof*⟩

lemma *minus-1* [*simp*]:
− (*k BIT B1*) = *pred* (− *k*) *BIT B1*
⟨*proof*⟩

lemma *minus-0* [*simp*]:
− (*k BIT B0*) = (− *k*) *BIT B0*
⟨*proof*⟩

27.3 Binary Addition and Multiplication: *op* + and *op* ∗

lemma *add-Pls* [*simp*]:
Pls + *k* = *k*
⟨*proof*⟩

lemma *add-Min* [*simp*]:
Min + *k* = *pred k*
⟨*proof*⟩

lemma *add-BIT-11* [*simp*]:
(*k BIT B1*) + (*l BIT B1*) = (*k* + *succ l*) *BIT B0*
⟨*proof*⟩

lemma *add-BIT-10* [*simp*]:

$(k\ BIT\ B1) + (l\ BIT\ B0) = (k + l)\ BIT\ B1$
⟨*proof*⟩

lemma *add-BIT-0* [*simp*]:
$(k\ BIT\ B0) + (l\ BIT\ b) = (k + l)\ BIT\ b$
⟨*proof*⟩

lemma *add-Pls-right* [*simp*]:
$k + Pls = k$
⟨*proof*⟩

lemma *add-Min-right* [*simp*]:
$k + Min = pred\ k$
⟨*proof*⟩

lemma *mult-Pls* [*simp*]:
$Pls * w = Pls$
⟨*proof*⟩

lemma *mult-Min* [*simp*]:
$Min * k = - k$
⟨*proof*⟩

lemma *mult-num1* [*simp*]:
$(k\ BIT\ B1) * l = ((k * l)\ BIT\ B0) + l$
⟨*proof*⟩

lemma *mult-num0* [*simp*]:
$(k\ BIT\ B0) * l = (k * l)\ BIT\ B0$
⟨*proof*⟩

27.4 Converting Numerals to Rings: *number-of*

axclass *number-ring* ⊆ *number*, *comm-ring-1*
number-of-eq: *number-of* k = *of-int* k

self-embedding of the integers

instance *int* :: *number-ring*
int-number-of-def: *number-of* $w \equiv$ *of-int* w
⟨*proof*⟩

lemmas [*code func del*] = *int-number-of-def*

lemma *number-of-is-id*:
number-of $(k::int) = k$
⟨*proof*⟩

lemma *number-of-succ*:
number-of $(succ\ k) = (1 +$ *number-of* k ::'a::*number-ring*)

⟨*proof*⟩

lemma *number-of-pred*:
number-of (*pred w*) = (− *1* + *number-of w* ::′*a*::*number-ring*)
⟨*proof*⟩

lemma *number-of-minus*:
number-of (*uminus w*) = (− (*number-of w*)::′*a*::*number-ring*)
⟨*proof*⟩

lemma *number-of-add*:
number-of (*v* + *w*) = (*number-of v* + *number-of w*::′*a*::*number-ring*)
⟨*proof*⟩

lemma *number-of-mult*:
number-of (*v* ∗ *w*) = (*number-of v* ∗ *number-of w*::′*a*::*number-ring*)
⟨*proof*⟩

The correctness of shifting. But it doesn't seem to give a measurable speed-up.

lemma *double-number-of-BIT*:
(*1* + *1*) ∗ *number-of w* = (*number-of* (*w BIT B0*) ::′*a*::*number-ring*)
⟨*proof*⟩

Converting numerals 0 and 1 to their abstract versions.

lemma *numeral-0-eq-0* [*simp*]:
Numeral0 = (*0*::′*a*::*number-ring*)
⟨*proof*⟩

lemma *numeral-1-eq-1* [*simp*]:
Numeral1 = (*1*::′*a*::*number-ring*)
⟨*proof*⟩

Special-case simplification for small constants.

Unary minus for the abstract constant 1. Cannot be inserted as a simprule until later: it is *number-of-Min* re-oriented!

lemma *numeral-m1-eq-minus-1*:
(−*1*::′*a*::*number-ring*) = − *1*
⟨*proof*⟩

lemma *mult-minus1* [*simp*]:
−*1* ∗ *z* = −(*z*::′*a*::*number-ring*)
⟨*proof*⟩

lemma *mult-minus1-right* [*simp*]:
z ∗ −*1* = −(*z*::′*a*::*number-ring*)
⟨*proof*⟩

lemma *minus-number-of-mult* [*simp*]:
$-$ (*number-of w*) $*$ z = *number-of* (*uminus w*) $*$ (z::$'a$::*number-ring*)
⟨*proof*⟩

Subtraction

lemma *diff-number-of-eq*:
number-of v $-$ *number-of w* =
(*number-of* (v + *uminus w*)::$'a$::*number-ring*)
⟨*proof*⟩

lemma *number-of-Pls*:
number-of Pls = (0::$'a$::*number-ring*)
⟨*proof*⟩

lemma *number-of-Min*:
number-of Min = ($-$ 1::$'a$::*number-ring*)
⟨*proof*⟩

lemma *number-of-BIT*:
number-of(*w BIT x*) = (*case x of B0 => 0 | B1 =>* (1::$'a$::*number-ring*))
+ (*number-of w*) + (*number-of w*)
⟨*proof*⟩

27.5 Equality of Binary Numbers

First version by Norbert Voelker

lemma *eq-number-of-eq*:
((*number-of x*::$'a$::*number-ring*) = *number-of y*) =
iszero (*number-of* (x + *uminus y*) :: $'a$)
⟨*proof*⟩

lemma *iszero-number-of-Pls*:
iszero ((*number-of Pls*)::$'a$::*number-ring*)
⟨*proof*⟩

lemma *nonzero-number-of-Min*:
~ *iszero* ((*number-of Min*)::$'a$::*number-ring*)
⟨*proof*⟩

27.6 Comparisons, for Ordered Rings

lemmas *double-eq-0-iff* = *double-zero*

lemma *le-imp-0-less*:
assumes *le*: $0 \leq z$
shows (0::*int*) $< 1 + z$
⟨*proof*⟩

lemma *odd-nonzero*:
$1 + z + z \neq (0::int)$
⟨*proof*⟩

The premise involving $\mathbb{Z}$ prevents $a = (1::'a) / (2::'a)$.

lemma *Ints-double-eq-0-iff*:
assumes *in-Ints*: $a \in Ints$
shows $(a + a = 0) = (a = (0::'a::ring\text{-}char\text{-}0))$
⟨*proof*⟩

lemma *Ints-odd-nonzero*:
assumes *in-Ints*: $a \in Ints$
shows $1 + a + a \neq (0::'a::ring\text{-}char\text{-}0)$
⟨*proof*⟩

lemma *Ints-number-of*:
$(number\text{-}of\ w :: 'a::number\text{-}ring) \in Ints$
⟨*proof*⟩

lemma *iszero-number-of-BIT*:
$iszero\ (number\text{-}of\ (w\ BIT\ x)::'a) =$
$(x = B0 \wedge iszero\ (number\text{-}of\ w::'a::\{ring\text{-}char\text{-}0,number\text{-}ring\}))$
⟨*proof*⟩

lemma *iszero-number-of-0*:
$iszero\ (number\text{-}of\ (w\ BIT\ B0) :: 'a::\{ring\text{-}char\text{-}0,number\text{-}ring\}) =$
$iszero\ (number\text{-}of\ w :: 'a)$
⟨*proof*⟩

lemma *iszero-number-of-1*:
$\sim iszero\ (number\text{-}of\ (w\ BIT\ B1)::'a::\{ring\text{-}char\text{-}0,number\text{-}ring\})$
⟨*proof*⟩

27.7 The Less-Than Relation

lemma *less-number-of-eq-neg*:
$((number\text{-}of\ x::'a::\{ordered\text{-}idom,number\text{-}ring\}) < number\text{-}of\ y)$
$= neg\ (number\text{-}of\ (x + uminus\ y) :: 'a)$
⟨*proof*⟩

If *Numeral0* is rewritten to 0 then this rule can't be applied: *Numeral0* IS *Numeral0*

lemma *not-neg-number-of-Pls*:
$\sim neg\ (number\text{-}of\ Pls ::'a::\{ordered\text{-}idom,number\text{-}ring\})$
⟨*proof*⟩

lemma *neg-number-of-Min*:
$neg\ (number\text{-}of\ Min ::'a::\{ordered\text{-}idom,number\text{-}ring\})$
⟨*proof*⟩

lemma *double-less-0-iff*:
$(a + a < 0) = (a < (0::'a::ordered\text{-}idom))$
⟨*proof*⟩

lemma *odd-less-0*:
$(1 + z + z < 0) = (z < (0::int))$
⟨*proof*⟩

The premise involving $\mathbb{Z}$ prevents $a = (1::'a) / (2::'a)$.

lemma *Ints-odd-less-0*:
assumes *in-Ints*: $a \in Ints$
shows $(1 + a + a < 0) = (a < (0::'a::ordered\text{-}idom))$
⟨*proof*⟩

lemma *neg-number-of-BIT*:
neg (*number-of* (*w BIT x*)::'*a*) =
neg (*number-of w* :: '*a*::{*ordered-idom*,*number-ring*})
⟨*proof*⟩

Less-Than or Equals

Reduces $a \leq b$ to $\neg\ b < a$ for ALL numerals.

lemmas *le-number-of-eq-not-less* =
linorder-not-less [*of number-of w number-of v*, *symmetric*,
standard]

lemma *le-number-of-eq*:
((*number-of x*::'*a*::{*ordered-idom*,*number-ring*}) ≤ *number-of y*)
= (~ (*neg* (*number-of* (*y* + *uminus x*) :: '*a*)))
⟨*proof*⟩

Absolute value (*abs*)

lemma *abs-number-of*:
abs(*number-of x*::'*a*::{*ordered-idom*,*number-ring*}) =
(*if number-of x* < (*0*::'*a*) *then* −*number-of x else number-of x*)
⟨*proof*⟩

Re-orientation of the equation nnn=x

lemma *number-of-reorient*:
(*number-of w* = *x*) = (*x* = *number-of w*)
⟨*proof*⟩

27.8 Simplification of arithmetic operations on integer constants.

lemmas *arith-extra-simps* [*standard*, *simp*] =
number-of-add [*symmetric*]
number-of-minus [*symmetric*] *numeral-m1-eq-minus-1* [*symmetric*]

number-of-mult [*symmetric*]
diff-number-of-eq abs-number-of

For making a minimal simpset, one must include these default simprules. Also include *simp-thms*.

lemmas *arith-simps* =
bit.distinct
Pls-0-eq Min-1-eq
pred-Pls pred-Min pred-1 pred-0
succ-Pls succ-Min succ-1 succ-0
add-Pls add-Min add-BIT-0 add-BIT-10 add-BIT-11
minus-Pls minus-Min minus-1 minus-0
mult-Pls mult-Min mult-num1 mult-num0
add-Pls-right add-Min-right
abs-zero abs-one arith-extra-simps

Simplification of relational operations

lemmas *rel-simps* [*simp*] =
eq-number-of-eq iszero-0 nonzero-number-of-Min
iszero-number-of-0 iszero-number-of-1
less-number-of-eq-neg
not-neg-number-of-Pls not-neg-0 not-neg-1 not-iszero-1
neg-number-of-Min neg-number-of-BIT
le-number-of-eq

27.9 Simplification of arithmetic when nested to the right.

lemma *add-number-of-left* [*simp*]:
$number\text{-}of\ v + (number\text{-}of\ w + z) =$
$(number\text{-}of(v + w) + z::'a::number\text{-}ring)$
⟨*proof*⟩

lemma *mult-number-of-left* [*simp*]:
$number\text{-}of\ v * (number\text{-}of\ w * z) =$
$(number\text{-}of(v * w) * z::'a::number\text{-}ring)$
⟨*proof*⟩

lemma *add-number-of-diff1*:
$number\text{-}of\ v + (number\text{-}of\ w - c) =$
$number\text{-}of(v + w) - (c::'a::number\text{-}ring)$
⟨*proof*⟩

lemma *add-number-of-diff2* [*simp*]:
$number\text{-}of\ v + (c - number\text{-}of\ w) =$
$number\text{-}of\ (v + uminus\ w) + (c::'a::number\text{-}ring)$
⟨*proof*⟩

27.10 Configuration of the code generator

instance *int* :: *eq* ⟨*proof*⟩

code-datatype *Pls Min Bit number-of* :: *int* ⇒ *int*

definition
int-aux :: *nat* ⇒ *int* ⇒ *int* **where**
int-aux n i = *int n* + *i*

lemma [*code*]:
int-aux 0 i = *i*
int-aux (*Suc n*) *i* = *int-aux n* (*i* + *1*) — tail recursive
⟨*proof*⟩

lemma [*code*, *code unfold*, *code inline del*]:
int n = *int-aux n 0*
⟨*proof*⟩

definition
nat-aux :: *int* ⇒ *nat* ⇒ *nat* **where**
nat-aux i n = *nat i* + *n*

lemma [*code*]:
nat-aux i n = (*if i* ≤ *0 then n else nat-aux* (*i* − *1*) (*Suc n*)) — tail recursive
⟨*proof*⟩

lemma [*code*]: *nat i* = *nat-aux i 0*
⟨*proof*⟩

lemma *zero-is-num-zero* [*code func*, *code inline*, *symmetric*, *code post*]:
(*0*::*int*) = *Numeral0*
⟨*proof*⟩

lemma *one-is-num-one* [*code func*, *code inline*, *symmetric*, *code post*]:
(*1*::*int*) = *Numeral1*
⟨*proof*⟩

code-modulename *SML*
IntDef Integer

code-modulename *OCaml*
IntDef Integer

code-modulename *Haskell*
IntDef Integer

code-modulename *SML*
Numeral Integer

code-modulename *OCaml*
Numeral Integer

code-modulename *Haskell*
Numeral Integer

types-code
int (*int*)
attach (*term-of*) ⟪
val term-of-int = *HOLogic.mk-number HOLogic.intT*;
⟫
attach (*test*) ⟪
fun gen-int i = *one-of* [~*1*, *1*] ∗ *random-range 0 i*;
⟫

⟨*ML*⟩

consts-code
number-of :: *int* ⇒ *int* ((-))
0 :: *int* (*0*)
1 :: *int* (*1*)
uminus :: *int* => *int* (~)
op + :: *int* => *int* => *int* ((- +/ -))
op ∗ :: *int* => *int* => *int* ((- ∗/ -))
op ≤ :: *int* => *int* => *bool* ((- <=/ -))
op < :: *int* => *int* => *bool* ((- </ -))

quickcheck-params [*default-type* = *int*]

hide (**open**) *const Pls Min B0 B1 succ pred*

end

28 Wellfounded-Relations: Well-founded Relations

theory *Wellfounded-Relations*
imports *Finite-Set*
begin

Derived WF relations such as inverse image, lexicographic product and measure. The simple relational product, in which (x', y') precedes (x, y) if $x' < x$ and $y' < y$, is a subset of the lexicographic product, and therefore does not need to be defined separately.

constdefs
less-than :: (*nat*∗*nat*)*set*
less-than == *pred-nat*^+

measure :: $('a => nat) => ('a * 'a)set$
measure == *inv-image less-than*

lex-prod :: $[('a*'a)set, ('b*'b)set] => (('a*'b)*('a*'b))set$
(**infixr** $<*lex*>$ *80*)
$ra <*lex*> rb == \{((a,b),(a',b')).\ (a,a') : ra \mid a=a' \ \&\ (b,b') : rb\}$

finite-psubset :: $('a\ set * 'a\ set)\ set$
— finite proper subset
finite-psubset == $\{(A,B).\ A < B \ \&\ finite\ B\}$

same-fst :: $('a => bool) => ('a => ('b * 'b)set) => (('a*'b)*('a*'b))set$
same-fst $P\ R == \{((x',y'),(x,y))\ .\ x'=x \ \&\ P\ x \ \&\ (y',y) : R\ x\}$
— For *rec-def* declarations where the first n parameters stay unchanged in the recursive call. See *Library/While-Combinator.thy* for an application.

28.1 Measure Functions make Wellfounded Relations

28.1.1 'Less than' on the natural numbers

lemma *wf-less-than* [*iff*]: *wf less-than*
⟨*proof*⟩

lemma *trans-less-than* [*iff*]: *trans less-than*
⟨*proof*⟩

lemma *less-than-iff* [*iff*]: $((x,y)\text{: } less\text{-}than) = (x<y)$
⟨*proof*⟩

lemma *full-nat-induct*:
assumes *ih*: $(!!n.\ (ALL\ m.\ Suc\ m <= n --> P\ m) ==> P\ n)$
shows $P\ n$
⟨*proof*⟩

28.1.2 The Inverse Image into a Wellfounded Relation is Wellfounded.

lemma *wf-inv-image* [*simp*,*intro*!]: $wf(r) ==> wf(inv\text{-}image\ r\ (f::'a=>'b))$
⟨*proof*⟩

lemma *in-inv-image*[*simp*]: $((x,y) : inv\text{-}image\ r\ f) = ((f\ x,\ f\ y) : r)$
⟨*proof*⟩

28.1.3 Finally, All Measures are Wellfounded.

lemma *in-measure*[*simp*]: $((x,y) : measure\ f) = (f\ x < f\ y)$
⟨*proof*⟩

lemma *wf-measure* [*iff*]: *wf* (*measure f*)
⟨*proof*⟩

lemma *measure-induct-rule* [*case-names less*]:
  fixes f :: $'a \Rightarrow nat$
  assumes *step*: $\bigwedge x.\ (\bigwedge y.\ f\ y < f\ x \Longrightarrow P\ y) \Longrightarrow P\ x$
  shows $P\ a$
⟨*proof*⟩

lemma *measure-induct*:
  fixes f :: $'a \Rightarrow nat$
  shows $(\bigwedge x.\ \forall y.\ f\ y < f\ x \longrightarrow P\ y \Longrightarrow P\ x) \Longrightarrow P\ a$
  ⟨*proof*⟩

lemma (**in** *linorder*)
  finite-linorder-induct[*consumes 1*, *case-names empty insert*]:
 $finite\ A \Longrightarrow P\ \{\} \Longrightarrow$
  $(!!A\ b.\ finite\ A \Longrightarrow ALL\ a{:}A.\ a < b \Longrightarrow P\ A \Longrightarrow P(insert\ b\ A))$
  $\Longrightarrow P\ A$
⟨*proof*⟩

28.2 Other Ways of Constructing Wellfounded Relations

Wellfoundedness of lexicographic combinations

lemma *wf-lex-prod* [*intro*!]: $[|\ wf(ra);\ wf(rb)\ |] ==> wf(ra <*lex*> rb)$
⟨*proof*⟩

lemma *in-lex-prod*[*simp*]:
  $(((a,b),(a',b')){:}\ r <*lex*> s) = ((a,a'){:}\ r \vee (a = a' \wedge (b,\ b') : s))$
  ⟨*proof*⟩

lexicographic combinations with measure functions

definition
  mlex-prod :: $('a \Rightarrow nat) \Rightarrow ('a \times 'a)\ set \Rightarrow ('a \times 'a)\ set$ (**infixr** $<*mlex*>$ *80*)
where
  $f <*mlex*> R = inv\text{-}image\ (less\text{-}than <*lex*> R)\ (\%x.\ (f\ x,\ x))$

lemma *wf-mlex*: $wf\ R \Longrightarrow wf\ (f <*mlex*> R)$
⟨*proof*⟩

lemma *mlex-less*: $f\ x < f\ y \Longrightarrow (x,\ y) \in f <*mlex*> R$
⟨*proof*⟩

lemma *mlex-leq*: $f\ x \le f\ y \Longrightarrow (x,\ y) \in R \Longrightarrow (x,\ y) \in f <*mlex*> R$
⟨*proof*⟩

Transitivity of WF combinators.

lemma *trans-lex-prod* [*intro*!]:
    $[|\ trans\ R1;\ trans\ R2\ |] ==> trans\ (R1 <*lex*> R2)$
⟨*proof*⟩

28.2.1 Wellfoundedness of proper subset on finite sets.

lemma *wf-finite-psubset*: *wf* (*finite-psubset*)
⟨*proof*⟩

lemma *trans-finite-psubset*: *trans finite-psubset*
⟨*proof*⟩

28.2.2 Wellfoundedness of finite acyclic relations

This proof belongs in this theory because it needs Finite.

lemma *finite-acyclic-wf* [*rule-format*]: *finite r ==> acyclic r --> wf r*
⟨*proof*⟩

lemma *finite-acyclic-wf-converse*: [|*finite r*; *acyclic r*|] ==> *wf* (*r*^-*1*)
⟨*proof*⟩

lemma *wf-iff-acyclic-if-finite*: *finite r ==> wf r = acyclic r*
⟨*proof*⟩

28.2.3 Wellfoundedness of *same-fst*

lemma *same-fstI* [*intro*!]:
[| *P x*; (*y'*,*y*) : *R x* |] ==> ((*x*,*y'*),(*x*,*y*)) : *same-fst P R*
⟨*proof*⟩

lemma *wf-same-fst*:
assumes *prem*: (!!*x*. *P x* ==> *wf*(*R x*))
shows *wf*(*same-fst P R*)
⟨*proof*⟩

28.3 Weakly decreasing sequences (w.r.t. some well-founded order) stabilize.

This material does not appear to be used any longer.

lemma *lemma1*: [| *ALL i*. (*f* (*Suc i*), *f i*) : *r*^* |] ==> (*f* (*i*+*k*), *f i*) : *r*^*
⟨*proof*⟩

lemma *lemma2*: [| *ALL i*. (*f* (*Suc i*), *f i*) : *r*^*; *wf* (*r*^+) |]
==> *ALL m*. *f m* = *x* --> (*EX i*. *ALL k*. *f* (*m*+*i*+*k*) = *f* (*m*+*i*))
⟨*proof*⟩

lemma *wf-weak-decr-stable*: [| *ALL i*. (*f* (*Suc i*), *f i*) : *r*^*; *wf* (*r*^+) |]
==> *EX i*. *ALL k*. *f* (*i*+*k*) = *f i*
⟨*proof*⟩

lemma *weak-decr-stable*:
ALL i. *f* (*Suc i*) <= ((*f i*)::*nat*) ==> *EX i*. *ALL k*. *f* (*i*+*k*) = *f i*

⟨*proof*⟩

⟨*ML*⟩

end

29 IntArith: Integer arithmetic

theory *IntArith*
imports *Numeral Wellfounded-Relations*
uses
  ~~/src/Provers/Arith/assoc-fold.ML
  ~~/src/Provers/Arith/cancel-numerals.ML
  ~~/src/Provers/Arith/combine-numerals.ML
  (*int-arith1.ML*)
begin

29.1 Inequality Reasoning for the Arithmetic Simproc

lemma *add-numeral-0*: *Numeral0* + *a* = (*a*::'*a*::*number-ring*)
⟨*proof*⟩

lemma *add-numeral-0-right*: *a* + *Numeral0* = (*a*::'*a*::*number-ring*)
⟨*proof*⟩

lemma *mult-numeral-1*: *Numeral1* ∗ *a* = (*a*::'*a*::*number-ring*)
⟨*proof*⟩

lemma *mult-numeral-1-right*: *a* ∗ *Numeral1* = (*a*::'*a*::*number-ring*)
⟨*proof*⟩

lemma *divide-numeral-1*: *a* / *Numeral1* = (*a*::'*a*::{*number-ring*,*field*})
⟨*proof*⟩

lemma *inverse-numeral-1*:
  inverse Numeral1 = (*Numeral1*::'*a*::{*number-ring*,*field*})
⟨*proof*⟩

Theorem lists for the cancellation simprocs. The use of binary numerals for 0 and 1 reduces the number of special cases.

lemmas *add-0s* = *add-numeral-0 add-numeral-0-right*
lemmas *mult-1s* = *mult-numeral-1 mult-numeral-1-right*
    mult-minus1 mult-minus1-right

29.2 Special Arithmetic Rules for Abstract 0 and 1

Arithmetic computations are defined for binary literals, which leaves 0 and 1 as special cases. Addition already has rules for 0, but not 1. Multiplication and unary minus already have rules for both 0 and 1.

lemma *binop-eq*: $[|f\ x\ y = g\ x\ y;\ x = x';\ y = y'|] ==> f\ x'\ y' = g\ x'\ y'$
⟨*proof*⟩

lemmas *add-number-of-eq* = *number-of-add* [*symmetric*]

Allow 1 on either or both sides

lemma *one-add-one-is-two*: $1 + 1 = (2::'a::number\text{-}ring)$
⟨*proof*⟩

lemmas *add-special* =
one-add-one-is-two
binop-eq [*of op* +, *OF add-number-of-eq numeral-1-eq-1 refl*, *standard*]
binop-eq [*of op* +, *OF add-number-of-eq refl numeral-1-eq-1*, *standard*]

Allow 1 on either or both sides (1-1 already simplifies to 0)

lemmas *diff-special* =
binop-eq [*of op* −, *OF diff-number-of-eq numeral-1-eq-1 refl*, *standard*]
binop-eq [*of op* −, *OF diff-number-of-eq refl numeral-1-eq-1*, *standard*]

Allow 0 or 1 on either side with a binary numeral on the other

lemmas *eq-special* =
binop-eq [*of op* =, *OF eq-number-of-eq numeral-0-eq-0 refl*, *standard*]
binop-eq [*of op* =, *OF eq-number-of-eq numeral-1-eq-1 refl*, *standard*]
binop-eq [*of op* =, *OF eq-number-of-eq refl numeral-0-eq-0*, *standard*]
binop-eq [*of op* =, *OF eq-number-of-eq refl numeral-1-eq-1*, *standard*]

Allow 0 or 1 on either side with a binary numeral on the other

lemmas *less-special* =
binop-eq [*of op* <, *OF less-number-of-eq-neg numeral-0-eq-0 refl*, *standard*]
binop-eq [*of op* <, *OF less-number-of-eq-neg numeral-1-eq-1 refl*, *standard*]
binop-eq [*of op* <, *OF less-number-of-eq-neg refl numeral-0-eq-0*, *standard*]
binop-eq [*of op* <, *OF less-number-of-eq-neg refl numeral-1-eq-1*, *standard*]

Allow 0 or 1 on either side with a binary numeral on the other

lemmas *le-special* =
binop-eq [*of op* ≤, *OF le-number-of-eq numeral-0-eq-0 refl*, *standard*]
binop-eq [*of op* ≤, *OF le-number-of-eq numeral-1-eq-1 refl*, *standard*]
binop-eq [*of op* ≤, *OF le-number-of-eq refl numeral-0-eq-0*, *standard*]
binop-eq [*of op* ≤, *OF le-number-of-eq refl numeral-1-eq-1*, *standard*]

lemmas *arith-special*[*simp*] =

add-special diff-special eq-special less-special le-special

lemma *min-max-01*: *min* (*0*::*int*) *1* = *0* & *min* (*1*::*int*) *0* = *0* &
max (*0*::*int*) *1* = *1* & *max* (*1*::*int*) *0* = *1*
⟨*proof*⟩

lemmas *min-max-special*[*simp*] =
min-max-01
max-def [*of 0*::*int number-of v*, *standard*, *simp*]
min-def [*of 0*::*int number-of v*, *standard*, *simp*]
max-def [*of number-of u 0*::*int*, *standard*, *simp*]
min-def [*of number-of u 0*::*int*, *standard*, *simp*]
max-def [*of 1*::*int number-of v*, *standard*, *simp*]
min-def [*of 1*::*int number-of v*, *standard*, *simp*]
max-def [*of number-of u 1*::*int*, *standard*, *simp*]
min-def [*of number-of u 1*::*int*, *standard*, *simp*]

⟨*ML*⟩

29.3 Lemmas About Small Numerals

lemma *of-int-m1* [*simp*]: *of-int* −*1* = (−*1* :: ′*a* :: *number-ring*)
⟨*proof*⟩

lemma *abs-minus-one* [*simp*]: *abs* (−*1*) = (*1*::′*a*::{*ordered-idom*,*number-ring*})
⟨*proof*⟩

lemma *abs-power-minus-one* [*simp*]:
abs(−*1* ˆ *n*) = (*1*::′*a*::{*ordered-idom*,*number-ring*,*recpower*})
⟨*proof*⟩

lemma *of-int-number-of-eq*:
of-int (*number-of v*) = (*number-of v* :: ′*a* :: *number-ring*)
⟨*proof*⟩

Lemmas for specialist use, NOT as default simprules

lemma *mult-2*: *2* ∗ *z* = (*z*+*z*::′*a*::*number-ring*)
⟨*proof*⟩

lemma *mult-2-right*: *z* ∗ *2* = (*z*+*z*::′*a*::*number-ring*)
⟨*proof*⟩

29.4 More Inequality Reasoning

lemma *zless-add1-eq*: (*w* < *z* + (*1*::*int*)) = (*w*<*z* | *w*=*z*)
⟨*proof*⟩

lemma *add1-zle-eq*: (*w* + (*1*::*int*) ≤ *z*) = (*w*<*z*)

⟨*proof*⟩

lemma *zle-diff1-eq* [*simp*]: $(w \leq z - (1{::}int)) = (w<z)$
⟨*proof*⟩

lemma *zle-add1-eq-le* [*simp*]: $(w < z + (1{::}int)) = (w\leq z)$
⟨*proof*⟩

lemma *int-one-le-iff-zero-less*: $((1{::}int) \leq z) = (0 < z)$
⟨*proof*⟩

29.5 The Functions *nat* and *int*

Simplify the terms *int 0*, *int* (*Suc 0*) and $w + - z$

declare *Zero-int-def* [*symmetric*, *simp*]
declare *One-int-def* [*symmetric*, *simp*]

lemmas *diff-int-def-symmetric* = *diff-int-def* [*symmetric*, *simp*]

lemma *nat-0*: $nat\ 0 = 0$
⟨*proof*⟩

lemma *nat-1*: $nat\ 1 = Suc\ 0$
⟨*proof*⟩

lemma *nat-2*: $nat\ 2 = Suc\ (Suc\ 0)$
⟨*proof*⟩

lemma *one-less-nat-eq* [*simp*]: $(Suc\ 0 < nat\ z) = (1 < z)$
⟨*proof*⟩

This simplifies expressions of the form $int\ n = z$ where z is an integer literal.

lemmas *int-eq-iff-number-of* [*simp*] = *int-eq-iff* [*of - number-of v*, *standard*]

lemma *split-nat* [*arith-split*]:
  $P(nat(i{::}int)) = ((\forall n.\ i = int\ n \longrightarrow P\ n)\ \&\ (i < 0 \longrightarrow P\ 0))$
  (**is** $?P = (?L\ \&\ ?R)$)
⟨*proof*⟩

context *ring-1*
begin

lemma *of-int-of-nat*:
  $of\text{-}int\ k = (if\ k < 0\ then\ -\ of\text{-}nat\ (nat\ (-\ k))\ else\ of\text{-}nat\ (nat\ k))$
⟨*proof*⟩

end

lemma *nat-mult-distrib*: $(0{::}int) \leq z \Longrightarrow nat\ (z*z') = nat\ z * nat\ z'$

⟨*proof*⟩

lemma *nat-mult-distrib-neg*: $z \le (0::int) ==> nat(z*z') = nat(-z) * nat(-z')$
⟨*proof*⟩

lemma *nat-abs-mult-distrib*: $nat\ (abs\ (w * z)) = nat\ (abs\ w) * nat\ (abs\ z)$
⟨*proof*⟩

29.6 Induction principles for int

Well-founded segments of the integers

definition
  int-ge-less-than :: *int* => (*int* * *int*) *set*
where
  int-ge-less-than $d = \{(z',z).\ d \le z'\ \&\ z' < z\}$

theorem *wf-int-ge-less-than*: *wf* (*int-ge-less-than d*)
⟨*proof*⟩

This variant looks odd, but is typical of the relations suggested by Rank-Finder.

definition
  int-ge-less-than2 :: *int* => (*int* * *int*) *set*
where
  int-ge-less-than2 $d = \{(z',z).\ d \le z\ \&\ z' < z\}$

theorem *wf-int-ge-less-than2*: *wf* (*int-ge-less-than2 d*)
⟨*proof*⟩

theorem *int-ge-induct* [*case-names base step*, *induct set*:*int*]:
  fixes *i* :: *int*
  assumes *ge*: $k \le i$ **and**
    base: $P\ k$ **and**
    step: $\bigwedge i.\ k \le i \Longrightarrow P\ i \Longrightarrow P\ (i + 1)$
  shows $P\ i$
⟨*proof*⟩

theorem *int-gr-induct*[*case-names base step*,*induct set*:*int*]:
  assumes *gr*: $k < (i::int)$ **and**
    base: $P(k+1)$ **and**
    step: $\bigwedge i.\ [\![k < i;\ P\ i]\!] \Longrightarrow P(i+1)$
  shows $P\ i$
⟨*proof*⟩

theorem *int-le-induct*[*consumes 1*,*case-names base step*]:
  assumes *le*: $i \le (k::int)$ **and**

base: $P(k)$ **and**
step: $\bigwedge i.\ [\![i \le k;\ P\ i]\!] \Longrightarrow P(i - 1)$
shows $P\ i$
⟨*proof*⟩

theorem *int-less-induct* [*consumes 1*,*case-names base step*]:
assumes *less*: $(i::int) < k$ **and**
base: $P(k - 1)$ **and**
step: $\bigwedge i.\ [\![i < k;\ P\ i]\!] \Longrightarrow P(i - 1)$
shows $P\ i$
⟨*proof*⟩

29.7 Intermediate value theorems

lemma *int-val-lemma*:
$(\forall\, i{<}n::nat.\ abs(f(i{+}1) - f\ i) \le 1) \dashrightarrow$
$f\ 0 \le k \dashrightarrow k \le f\ n \dashrightarrow (\exists\, i \le n.\ f\ i = (k::int))$
⟨*proof*⟩

lemmas *nat0-intermed-int-val* = *int-val-lemma* [*rule-format* (*no-asm*)]

lemma *nat-intermed-int-val*:
$[|\ \forall\, i.\ m \le i\ \&\ i < n \dashrightarrow abs(f(i + 1::nat) - f\ i) \le 1;\ m < n;$
$f\ m \le k;\ k \le f\ n\ |] ==> ?\ i.\ m \le i\ \&\ i \le n\ \&\ f\ i = (k::int)$
⟨*proof*⟩

29.8 Products and 1, by T. M. Rasmussen

lemma *zabs-less-one-iff* [*simp*]: $(|z| < 1) = (z = (0::int))$
⟨*proof*⟩

lemma *abs-zmult-eq-1*: $(|m * n| = 1) ==> |m| = (1::int)$
⟨*proof*⟩

lemma *pos-zmult-eq-1-iff-lemma*: $(m * n = 1) ==> m = (1::int)\ |\ m = -1$
⟨*proof*⟩

lemma *pos-zmult-eq-1-iff*: $0 < (m::int) ==> (m * n = 1) = (m = 1\ \&\ n = 1)$
⟨*proof*⟩

lemma *zmult-eq-1-iff*: $(m{*}n = (1::int)) = ((m = 1\ \&\ n = 1)\ |\ (m = -1\ \&\ n = -1))$
⟨*proof*⟩

lemma *infinite-UNIV-int*: $\sim finite(UNIV::int\ set)$
⟨*proof*⟩

29.9 Legacy ML bindings

⟨*ML*⟩

end

30 Accessible-Part: The accessible part of a relation

theory *Accessible-Part*
imports *Wellfounded-Recursion*
begin

30.1 Inductive definition

Inductive definition of the accessible part *acc r* of a relation; see also [**?**].

inductive-set
  acc :: (*'a* ∗ *'a*) *set* => *'a set*
  for *r* :: (*'a* ∗ *'a*) *set*
  where
    accI: (!!*y*. (*y*, *x*) : *r* ==> *y* : *acc r*) ==> *x* : *acc r*

abbreviation
  termip :: (*'a* => *'a* => *bool*) => *'a* => *bool* **where**
  termip r == *accp* (r^{-1-1})

abbreviation
  termi :: (*'a* ∗ *'a*) *set* => *'a set* **where**
  termi r == *acc* (r^{-1})

lemmas *accpI* = *accp.accI*

30.2 Induction rules

theorem *accp-induct*:
  assumes *major*: *accp r a*
  assumes *hyp*: !!*x*. *accp r x* ==> ∀ *y*. *r y x* −−> *P y* ==> *P x*
  shows *P a*
  ⟨*proof*⟩

theorems *accp-induct-rule* = *accp-induct* [*rule-format*, *induct set*: *accp*]

theorem *accp-downward*: *accp r b* ==> *r a b* ==> *accp r a*
  ⟨*proof*⟩

lemma *not-accp-down*:
  assumes *na*: ¬ *accp R x*

obtains z **where** $R\ z\ x$ **and** $\neg$ $accp\ R\ z$
⟨*proof*⟩

lemma *accp-downwards-aux*: $r^{**}\ b\ a$ ==> $accp\ r\ a$ --> $accp\ r\ b$
⟨*proof*⟩

theorem *accp-downwards*: $accp\ r\ a$ ==> $r^{**}\ b\ a$ ==> $accp\ r\ b$
⟨*proof*⟩

theorem *accp-wfPI*: $\forall\, x.\ accp\ r\ x$ ==> $wfP\ r$
⟨*proof*⟩

theorem *accp-wfPD*: $wfP\ r$ ==> $accp\ r\ x$
⟨*proof*⟩

theorem *wfP-accp-iff*: $wfP\ r = (\forall\, x.\ accp\ r\ x)$
⟨*proof*⟩

Smaller relations have bigger accessible parts:

lemma *accp-subset*:
assumes *sub*: $R1 \le R2$
shows $accp\ R2 \le accp\ R1$
⟨*proof*⟩

This is a generalized induction theorem that works on subsets of the accessible part.

lemma *accp-subset-induct*:
assumes *subset*: $D \le accp\ R$
and *dcl*: $\bigwedge x\ z.\ [\![D\ x;\ R\ z\ x]\!] \Longrightarrow D\ z$
and $D\ x$
and *istep*: $\bigwedge x.\ [\![D\ x;\ (\bigwedge z.\ R\ z\ x \Longrightarrow P\ z)]\!] \Longrightarrow P\ x$
shows $P\ x$
⟨*proof*⟩

Set versions of the above theorems

lemmas *acc-induct* = *accp-induct* [*to-set*]

lemmas *acc-induct-rule* = *acc-induct* [*rule-format*, *induct set*: *acc*]

lemmas *acc-downward* = *accp-downward* [*to-set*]

lemmas *not-acc-down* = *not-accp-down* [*to-set*]

lemmas *acc-downwards-aux* = *accp-downwards-aux* [*to-set*]

lemmas *acc-downwards* = *accp-downwards* [*to-set*]

lemmas *acc-wfI* = *accp-wfPI* [*to-set*]

lemmas *acc-wfD* = *accp-wfPD* [*to-set*]

lemmas *wf-acc-iff* = *wfP-accp-iff* [*to-set*]

lemmas *acc-subset* = *accp-subset* [*to-set*]

lemmas *acc-subset-induct* = *accp-subset-induct* [*to-set*]

end

31 FunDef: General recursive function definitions

theory *FunDef*
imports *Accessible-Part*
uses
(*Tools/function-package/fundef-lib.ML*)
(*Tools/function-package/fundef-common.ML*)
(*Tools/function-package/inductive-wrap.ML*)
(*Tools/function-package/context-tree.ML*)
(*Tools/function-package/fundef-core.ML*)
(*Tools/function-package/mutual.ML*)
(*Tools/function-package/pattern-split.ML*)
(*Tools/function-package/fundef-package.ML*)
(*Tools/function-package/auto-term.ML*)
begin

Definitions with default value.

definition
THE-default :: $'a \Rightarrow ('a \Rightarrow bool) \Rightarrow 'a$ **where**
THE-default $d\ P = (if\ (\exists!x.\ P\ x)\ then\ (THE\ x.\ P\ x)\ else\ d)$

lemma *THE-defaultI'*: $\exists!x.\ P\ x \Longrightarrow P\ (THE\text{-}default\ d\ P)$
⟨*proof*⟩

lemma *THE-default1-equality*:
$[\![\exists!x.\ P\ x;\ P\ a]\!] \Longrightarrow THE\text{-}default\ d\ P = a$
⟨*proof*⟩

lemma *THE-default-none*:
$\neg(\exists!x.\ P\ x) \Longrightarrow THE\text{-}default\ d\ P = d$
⟨*proof*⟩

lemma *fundef-ex1-existence*:
assumes *f-def*: $f == (\lambda x::'a.\ THE\text{-}default\ (d\ x)\ (\lambda y.\ G\ x\ y))$
assumes *ex1*: $\exists!y.\ G\ x\ y$
shows $G\ x\ (f\ x)$
⟨*proof*⟩

lemma *fundef-ex1-uniqueness*:
  assumes *f-def*: $f == (\lambda x{::}'a.\ \mathit{THE\text{-}default}\ (d\ x)\ (\lambda y.\ G\ x\ y))$
  assumes *ex1*: $\exists!y.\ G\ x\ y$
  assumes *elm*: $G\ x\ (h\ x)$
  shows $h\ x = f\ x$
  ⟨*proof*⟩

lemma *fundef-ex1-iff*:
  assumes *f-def*: $f == (\lambda x{::}'a.\ \mathit{THE\text{-}default}\ (d\ x)\ (\lambda y.\ G\ x\ y))$
  assumes *ex1*: $\exists!y.\ G\ x\ y$
  shows $(G\ x\ y) = (f\ x = y)$
  ⟨*proof*⟩

lemma *fundef-default-value*:
  assumes *f-def*: $f == (\lambda x{::}'a.\ \mathit{THE\text{-}default}\ (d\ x)\ (\lambda y.\ G\ x\ y))$
  assumes *graph*: $\bigwedge x\ y.\ G\ x\ y \Longrightarrow D\ x$
  assumes $\neg\ D\ x$
  shows $f\ x = d\ x$
⟨*proof*⟩

definition *in-rel-def*[*simp*]:
  $\mathit{in\text{-}rel}\ R\ x\ y == (x,\ y) \in R$

lemma *wf-in-rel*:
  $\mathit{wf}\ R \Longrightarrow \mathit{wfP}\ (\mathit{in\text{-}rel}\ R)$
  ⟨*proof*⟩

⟨*ML*⟩

lemma *let-cong* [*fundef-cong*]:
  $M = N \Longrightarrow (\bigwedge x.\ x = N \Longrightarrow f\ x = g\ x) \Longrightarrow \mathit{Let}\ M\ f = \mathit{Let}\ N\ g$
  ⟨*proof*⟩

lemmas [*fundef-cong*] =
  if-cong image-cong INT-cong UN-cong
  bex-cong ball-cong imp-cong

lemma *split-cong* [*fundef-cong*]:
  $(\bigwedge x\ y.\ (x,\ y) = q \Longrightarrow f\ x\ y = g\ x\ y) \Longrightarrow p = q$
    $\Longrightarrow \mathit{split}\ f\ p = \mathit{split}\ g\ q$
  ⟨*proof*⟩

lemma *comp-cong* [*fundef-cong*]:
  $f\ (g\ x) = f'\ (g'\ x') \Longrightarrow (f\ o\ g)\ x = (f'\ o\ g')\ x'$
  ⟨*proof*⟩

end

32 IntDiv: The Division Operators div and mod; the Divides Relation dvd

theory *IntDiv*
imports *IntArith Divides FunDef*
begin

constdefs
$quorem :: (int{*}int) * (int{*}int) => bool$
— definition of quotient and remainder
$[code\ func]$: $quorem == \%((a,b), (q,r)).$
$a = b{*}q + r\ \&$
$(if\ 0 < b\ then\ 0 \leq r\ \&\ r<b\ else\ b<r\ \&\ r \leq 0)$

$adjust :: [int, int{*}int] => int{*}int$
— for the division algorithm
$[code\ func]$: $adjust\ b == \%(q,r).\ if\ 0 \leq r-b\ then\ (2{*}q + 1, r-b)$
$else\ (2{*}q, r)$

algorithm for the case $a \geq 0$, $b>0$

function
$posDivAlg :: int \Rightarrow int \Rightarrow int \times int$
where
$posDivAlg\ a\ b =$
$(if\ (a<b \mid b \leq 0)\ then\ (0,a)$
$else\ adjust\ b\ (posDivAlg\ a\ (2{*}b)))$
$\langle proof \rangle$
termination $\langle proof \rangle$

algorithm for the case $a<0$, $b>0$

function
$negDivAlg :: int \Rightarrow int \Rightarrow int \times int$
where
$negDivAlg\ a\ b\ =$
$(if\ (0 \leq a+b \mid b \leq 0)\ then\ (-1,a+b)$
$else\ adjust\ b\ (negDivAlg\ a\ (2{*}b)))$
$\langle proof \rangle$
termination $\langle proof \rangle$

algorithm for the general case $b \neq (0::'a)$

constdefs
$negateSnd :: int{*}int => int{*}int$
$[code\ func]$: $negateSnd == \%(q,r).\ (q,-r)$

definition
$divAlg :: int \times int \Rightarrow int \times int$

— The full division algorithm considers all possible signs for a, b including the special case $a=0,\ b<0$ because *negDivAlg* requires $a < (0::'a)$.

where

$divAlg = (\lambda(a,\ b).\ (if\ 0\le a\ then$
$\quad if\ 0\le b\ then\ posDivAlg\ a\ b$
$\quad else\ if\ a=0\ then\ (0,\ 0)$
$\qquad else\ negateSnd\ (negDivAlg\ (-a)\ (-b))$
$else$
$\quad if\ 0<b\ then\ negDivAlg\ a\ b$
$\quad else\ negateSnd\ (posDivAlg\ (-a)\ (-b))))$

instance *int* :: *Divides.div*
div-def: $a\ div\ b == fst\ (divAlg\ (a,\ b))$
mod-def: $a\ mod\ b == snd\ (divAlg\ (a,\ b))$ ⟨*proof*⟩

lemma *divAlg-mod-div*:
$divAlg\ (p,\ q) = (p\ div\ q,\ p\ mod\ q)$
⟨*proof*⟩

Here is the division algorithm in ML:

```
fun posDivAlg (a,b) =
  if a<b then (0,a)
  else let val (q,r) = posDivAlg(a, 2*b)
           in  if 0\<le>r-b then (2*q+1, r-b) else (2*q, r)
       end

fun negDivAlg (a,b) =
  if 0\<le>a+b then (~1,a+b)
  else let val (q,r) = negDivAlg(a, 2*b)
           in  if 0\<le>r-b then (2*q+1, r-b) else (2*q, r)
       end;

fun negateSnd (q,r:int) = (q,~r);

fun divAlg (a,b) = if 0\<le>a then
                      if b>0 then posDivAlg (a,b)
                       else if a=0 then (0,0)
                            else negateSnd (negDivAlg (~a,~b))
                   else
                      if 0<b then negDivAlg (a,b)
                      else        negateSnd (posDivAlg (~a,~b));
```

32.1 Uniqueness and Monotonicity of Quotients and Remainders

lemma *unique-quotient-lemma*:
$[| \ b*q' + r' \ \leq b*q + r; \ \ 0 \leq r'; \ \ r' < b; \ \ r < b \ |]$
$==> q' \leq (q::int)$
⟨*proof*⟩

lemma *unique-quotient-lemma-neg*:
$[| \ b*q' + r' \leq b*q + r; \ \ r \leq 0; \ \ b < r; \ \ b < r' \ |]$
$==> q \leq (q'::int)$
⟨*proof*⟩

lemma *unique-quotient*:
$[| \ quorem \ ((a,b), \ (q,r)); \ \ quorem \ ((a,b), \ (q',r')); \ \ b \neq 0 \ |]$
$==> q = q'$
⟨*proof*⟩

lemma *unique-remainder*:
$[| \ quorem \ ((a,b), \ (q,r)); \ \ quorem \ ((a,b), \ (q',r')); \ \ b \neq 0 \ |]$
$==> r = r'$
⟨*proof*⟩

32.2 Correctness of *posDivAlg*, the Algorithm for Non-Negative Dividends

And positive divisors

lemma *adjust-eq* [*simp*]:
$adjust \ b \ (q,r) =$
$(let \ diff = r-b \ in$
$if \ 0 \leq diff \ then \ (2*q + 1, \ diff)$
$else \ (2*q, \ r))$
⟨*proof*⟩

declare *posDivAlg.simps* [*simp del*]

use with a simproc to avoid repeatedly proving the premise

lemma *posDivAlg-eqn*:
$0 < b ==>$
$posDivAlg \ a \ b = (if \ a<b \ then \ (0,a) \ else \ adjust \ b \ (posDivAlg \ a \ (2*b)))$
⟨*proof*⟩

Correctness of *posDivAlg*: it computes quotients correctly

theorem *posDivAlg-correct*:
assumes $0 \leq a$ **and** $0 < b$
shows $quorem \ ((a, \ b), \ posDivAlg \ a \ b)$
⟨*proof*⟩

32.3 Correctness of *negDivAlg*, the Algorithm for Negative Dividends

And positive divisors

declare *negDivAlg.simps* [*simp del*]

use with a simproc to avoid repeatedly proving the premise

lemma *negDivAlg-eqn*:
  $0 < b ==>$
   $negDivAlg\ a\ b =$
    $(if\ 0 \leq a+b\ then\ (-1, a+b)\ else\ adjust\ b\ (negDivAlg\ a\ (2*b)))$
⟨*proof*⟩

lemma *negDivAlg-correct*:
 assumes $a < 0$ **and** $b > 0$
 shows $quorem\ ((a,\ b),\ negDivAlg\ a\ b)$
⟨*proof*⟩

32.4 Existence Shown by Proving the Division Algorithm to be Correct

lemma *quorem-0*: $b \neq 0 ==> quorem\ ((0,b),\ (0,0))$
⟨*proof*⟩

lemma *posDivAlg-0* [*simp*]: $posDivAlg\ 0\ b = (0,\ 0)$
⟨*proof*⟩

lemma *negDivAlg-minus1* [*simp*]: $negDivAlg\ -1\ b = (-1,\ b - 1)$
⟨*proof*⟩

lemma *negateSnd-eq* [*simp*]: $negateSnd(q,r) = (q,-r)$
⟨*proof*⟩

lemma *quorem-neg*: $quorem\ ((-a,-b),\ qr) ==> quorem\ ((a,b),\ negateSnd\ qr)$
⟨*proof*⟩

lemma *divAlg-correct*: $b \neq 0 ==> quorem\ ((a,b),\ divAlg\ (a,\ b))$
⟨*proof*⟩

Arbitrary definitions for division by zero. Useful to simplify certain equations.

lemma *DIVISION-BY-ZERO* [*simp*]: $a\ div\ (0::int) = 0\ \&\ a\ mod\ (0::int) = a$
⟨*proof*⟩

Basic laws about division and remainder

lemma *zmod-zdiv-equality*: $(a::int) = b * (a\ div\ b) + (a\ mod\ b)$
⟨*proof*⟩

lemma *zdiv-zmod-equality*: $(b * (a\ div\ b) + (a\ mod\ b)) + k = (a::int)+k$
⟨*proof*⟩

lemma *zdiv-zmod-equality2*: $((a\ div\ b) * b + (a\ mod\ b)) + k = (a::int)+k$
⟨*proof*⟩

Tool setup

⟨*ML*⟩

lemma *pos-mod-conj* : $(0::int) < b ==> 0 \leq a\ mod\ b\ \&\ a\ mod\ b < b$
⟨*proof*⟩

lemmas *pos-mod-sign* [*simp*] = *pos-mod-conj* [*THEN conjunct1*, *standard*]
  and *pos-mod-bound* [*simp*] = *pos-mod-conj* [*THEN conjunct2*, *standard*]

lemma *neg-mod-conj* : $b < (0::int) ==> a\ mod\ b \leq 0\ \&\ b < a\ mod\ b$
⟨*proof*⟩

lemmas *neg-mod-sign* [*simp*] = *neg-mod-conj* [*THEN conjunct1*, *standard*]
  and *neg-mod-bound* [*simp*] = *neg-mod-conj* [*THEN conjunct2*, *standard*]

32.5 General Properties of div and mod

lemma *quorem-div-mod*: $b \neq 0 ==> quorem\ ((a,\ b),\ (a\ div\ b,\ a\ mod\ b))$
⟨*proof*⟩

lemma *quorem-div*: $[|\ quorem((a,b),(q,r));\ \ b \neq 0\ |] ==> a\ div\ b = q$
⟨*proof*⟩

lemma *quorem-mod*: $[|\ quorem((a,b),(q,r));\ \ b \neq 0\ |] ==> a\ mod\ b = r$
⟨*proof*⟩

lemma *div-pos-pos-trivial*: $[|\ (0::int) \leq a;\ \ a < b\ |] ==> a\ div\ b = 0$
⟨*proof*⟩

lemma *div-neg-neg-trivial*: $[|\ a \leq (0::int);\ \ b < a\ |] ==> a\ div\ b = 0$
⟨*proof*⟩

lemma *div-pos-neg-trivial*: $[|\ (0::int) < a;\ \ a+b \leq 0\ |] ==> a\ div\ b = -1$
⟨*proof*⟩

lemma *mod-pos-pos-trivial*: $[|\ (0::int) \leq a;\ \ a < b\ |] ==> a\ mod\ b = a$
⟨*proof*⟩

lemma *mod-neg-neg-trivial*: $[|\ a \leq (0::int);\ \ b < a\ |] ==> a\ mod\ b = a$
⟨*proof*⟩

lemma *mod-pos-neg-trivial*: [| (0::int) < a; a+b ≤ 0 |] ==> a mod b = a+b
⟨proof⟩

There is no *mod-neg-pos-trivial*.

lemma *zdiv-zminus-zminus* [simp]: (−a) div (−b) = a div (b::int)
⟨proof⟩

lemma *zmod-zminus-zminus* [simp]: (−a) mod (−b) = − (a mod (b::int))
⟨proof⟩

32.6 Laws for div and mod with Unary Minus

lemma *zminus1-lemma*:
    quorem((a,b),(q,r))
     ==> quorem ((−a,b), (if r=0 then −q else −q − 1),
                      (if r=0 then 0 else b−r))
⟨proof⟩

lemma *zdiv-zminus1-eq-if*:
    b ≠ (0::int)
     ==> (−a) div b =
        (if a mod b = 0 then − (a div b) else − (a div b) − 1)
⟨proof⟩

lemma *zmod-zminus1-eq-if*:
    (−a::int) mod b = (if a mod b = 0 then 0 else b − (a mod b))
⟨proof⟩

lemma *zdiv-zminus2*: a div (−b) = (−a::int) div b
⟨proof⟩

lemma *zmod-zminus2*: a mod (−b) = − ((−a::int) mod b)
⟨proof⟩

lemma *zdiv-zminus2-eq-if*:
    b ≠ (0::int)
     ==> a div (−b) =
        (if a mod b = 0 then − (a div b) else − (a div b) − 1)
⟨proof⟩

lemma *zmod-zminus2-eq-if*:
    a mod (−b::int) = (if a mod b = 0 then 0 else (a mod b) − b)
⟨proof⟩

32.7 Division of a Number by Itself

lemma *self-quotient-aux1*: [| (0::int) < a; a = r + a*q; r < a |] ==> 1 ≤ q

⟨*proof*⟩

lemma *self-quotient-aux2*: $[| (0::int) < a; a = r + a*q; 0 \leq r |] ==> q \leq 1$
⟨*proof*⟩

lemma *self-quotient*: $[| quorem((a,a),(q,r)); a \neq (0::int) |] ==> q = 1$
⟨*proof*⟩

lemma *self-remainder*: $[| quorem((a,a),(q,r)); a \neq (0::int) |] ==> r = 0$
⟨*proof*⟩

lemma *zdiv-self* [*simp*]: $a \neq 0 ==> a\ div\ a = (1::int)$
⟨*proof*⟩

lemma *zmod-self* [*simp*]: $a\ mod\ a = (0::int)$
⟨*proof*⟩

32.8 Computation of Division and Remainder

lemma *zdiv-zero* [*simp*]: $(0::int)\ div\ b = 0$
⟨*proof*⟩

lemma *div-eq-minus1*: $(0::int) < b ==> -1\ div\ b = -1$
⟨*proof*⟩

lemma *zmod-zero* [*simp*]: $(0::int)\ mod\ b = 0$
⟨*proof*⟩

lemma *zdiv-minus1*: $(0::int) < b ==> -1\ div\ b = -1$
⟨*proof*⟩

lemma *zmod-minus1*: $(0::int) < b ==> -1\ mod\ b = b - 1$
⟨*proof*⟩

a positive, b positive

lemma *div-pos-pos*: $[| 0 < a; 0 \leq b |] ==> a\ div\ b = fst\ (posDivAlg\ a\ b)$
⟨*proof*⟩

lemma *mod-pos-pos*: $[| 0 < a; 0 \leq b |] ==> a\ mod\ b = snd\ (posDivAlg\ a\ b)$
⟨*proof*⟩

a negative, b positive

lemma *div-neg-pos*: $[| a < 0; 0 < b |] ==> a\ div\ b = fst\ (negDivAlg\ a\ b)$
⟨*proof*⟩

lemma *mod-neg-pos*: $[| a < 0; 0 < b |] ==> a\ mod\ b = snd\ (negDivAlg\ a\ b)$
⟨*proof*⟩

a positive, b negative

lemma *div-pos-neg*:
$[|\ 0 < a;\ \ b < 0\ |] ==> a\ div\ b = fst\ (negateSnd\ (negDivAlg\ (-a)\ (-b)))$
⟨*proof*⟩

lemma *mod-pos-neg*:
$[|\ 0 < a;\ \ b < 0\ |] ==> a\ mod\ b = snd\ (negateSnd\ (negDivAlg\ (-a)\ (-b)))$
⟨*proof*⟩

a negative, b negative

lemma *div-neg-neg*:
$[|\ a < 0;\ \ b \le 0\ |] ==> a\ div\ b = fst\ (negateSnd\ (posDivAlg\ (-a)\ (-b)))$
⟨*proof*⟩

lemma *mod-neg-neg*:
$[|\ a < 0;\ \ b \le 0\ |] ==> a\ mod\ b = snd\ (negateSnd\ (posDivAlg\ (-a)\ (-b)))$
⟨*proof*⟩

Simplify expresions in which div and mod combine numerical constants

lemma *quoremI*:
$[\![a == b * q + r;\ if\ 0 < b\ then\ 0 \le r \wedge r < b\ else\ b < r \wedge r \le 0]\!]$
$\Longrightarrow quorem\ ((a,\ b),\ (q,\ r))$
⟨*proof*⟩

lemmas *quorem-div-eq* = *quoremI* [*THEN quorem-div*, *THEN eq-reflection*]
lemmas *quorem-mod-eq* = *quoremI* [*THEN quorem-mod*, *THEN eq-reflection*]
lemmas *arithmetic-simps* =
arith-simps
add-special
OrderedGroup.add-0-left
OrderedGroup.add-0-right
mult-zero-left
mult-zero-right
mult-1-left
mult-1-right

⟨*ML*⟩

lemmas *div-pos-pos-number-of* =
div-pos-pos [*of number-of v number-of w*, *standard*]

lemmas *div-neg-pos-number-of* =
div-neg-pos [*of number-of v number-of w*, *standard*]

lemmas *div-pos-neg-number-of* =
div-pos-neg [*of number-of v number-of w*, *standard*]

lemmas *div-neg-neg-number-of* =
div-neg-neg [*of number-of v number-of w*, *standard*]

lemmas *mod-pos-pos-number-of* =
mod-pos-pos [*of number-of v number-of w*, *standard*]

lemmas *mod-neg-pos-number-of* =
mod-neg-pos [*of number-of v number-of w*, *standard*]

lemmas *mod-pos-neg-number-of* =
mod-pos-neg [*of number-of v number-of w*, *standard*]

lemmas *mod-neg-neg-number-of* =
mod-neg-neg [*of number-of v number-of w*, *standard*]

lemmas *posDivAlg-eqn-number-of* [*simp*] =
posDivAlg-eqn [*of number-of v number-of w*, *standard*]

lemmas *negDivAlg-eqn-number-of* [*simp*] =
negDivAlg-eqn [*of number-of v number-of w*, *standard*]

Special-case simplification

lemma *zmod-1* [*simp*]: *a mod* (*1*::*int*) = *0*
⟨*proof*⟩

lemma *zdiv-1* [*simp*]: *a div* (*1*::*int*) = *a*
⟨*proof*⟩

lemma *zmod-minus1-right* [*simp*]: *a mod* (−*1*::*int*) = *0*
⟨*proof*⟩

lemma *zdiv-minus1-right* [*simp*]: *a div* (−*1*::*int*) = −*a*
⟨*proof*⟩

lemmas *div-pos-pos-1-number-of* [*simp*] =
div-pos-pos [*OF int-0-less-1*, *of number-of w*, *standard*]

lemmas *div-pos-neg-1-number-of* [*simp*] =
div-pos-neg [*OF int-0-less-1*, *of number-of w*, *standard*]

lemmas *mod-pos-pos-1-number-of* [*simp*] =
mod-pos-pos [*OF int-0-less-1*, *of number-of w*, *standard*]

lemmas *mod-pos-neg-1-number-of* [*simp*] =
mod-pos-neg [*OF int-0-less-1*, *of number-of w*, *standard*]

lemmas *posDivAlg-eqn-1-number-of* [*simp*] =
posDivAlg-eqn [*of* **concl**: *1 number-of w*, *standard*]

lemmas *negDivAlg-eqn-1-number-of* [*simp*] =
negDivAlg-eqn [*of* **concl**: *1 number-of w*, *standard*]

32.9 Monotonicity in the First Argument (Dividend)

lemma *zdiv-mono1*: $[| \; a \le a'; \; 0 < (b{::}int) \; |] ==> a \; div \; b \le a' \; div \; b$
⟨*proof*⟩

lemma *zdiv-mono1-neg*: $[| \; a \le a'; \; (b{::}int) < 0 \; |] ==> a' \; div \; b \le a \; div \; b$
⟨*proof*⟩

32.10 Monotonicity in the Second Argument (Divisor)

lemma *q-pos-lemma*:
$[| \; 0 \le b'{*}q' + r'; \; r' < b'; \; 0 < b' \; |] ==> 0 \le (q'{::}int)$
⟨*proof*⟩

lemma *zdiv-mono2-lemma*:
$[| \; b{*}q + r = b'{*}q' + r'; \; 0 \le b'{*}q' + r';$
$r' < b'; \; 0 \le r; \; 0 < b'; \; b' \le b \; |]$
$==> q \le (q'{::}int)$
⟨*proof*⟩

lemma *zdiv-mono2*:
$[| \; (0{::}int) \le a; \; 0 < b'; \; b' \le b \; |] ==> a \; div \; b \le a \; div \; b'$
⟨*proof*⟩

lemma *q-neg-lemma*:
$[| \; b'{*}q' + r' < 0; \; 0 \le r'; \; 0 < b' \; |] ==> q' \le (0{::}int)$
⟨*proof*⟩

lemma *zdiv-mono2-neg-lemma*:
$[| \; b{*}q + r = b'{*}q' + r'; \; b'{*}q' + r' < 0;$
$r < b; \; 0 \le r'; \; 0 < b'; \; b' \le b \; |]$
$==> q' \le (q{::}int)$
⟨*proof*⟩

lemma *zdiv-mono2-neg*:
$[| \; a < (0{::}int); \; 0 < b'; \; b' \le b \; |] ==> a \; div \; b' \le a \; div \; b$
⟨*proof*⟩

32.11 More Algebraic Laws for div and mod

proving (a*b) div c = a * (b div c) + a * (b mod c)

lemma *zmult1-lemma*:
  [| *quorem*((*b*,*c*),(*q*,*r*)); $c \neq 0$ |]
  ==> *quorem* ((*a*∗*b*, *c*), (*a*∗*q* + *a*∗*r div c*, *a*∗*r mod c*))
⟨*proof*⟩

lemma *zdiv-zmult1-eq*: (*a*∗*b*) *div c* = *a*∗(*b div c*) + *a*∗(*b mod c*) *div* (*c*::*int*)
⟨*proof*⟩

lemma *zmod-zmult1-eq*: (*a*∗*b*) *mod c* = *a*∗(*b mod c*) *mod* (*c*::*int*)
⟨*proof*⟩

lemma *zmod-zmult1-eq'*: (*a*∗*b*) *mod* (*c*::*int*) = ((*a mod c*) ∗ *b*) *mod c*
⟨*proof*⟩

lemma *zmod-zmult-distrib*: (*a*∗*b*) *mod* (*c*::*int*) = ((*a mod c*) ∗ (*b mod c*)) *mod c*
⟨*proof*⟩

lemma *zdiv-zmult-self1* [*simp*]: $b \neq$ (*0*::*int*) ==> (*a*∗*b*) *div b* = *a*
⟨*proof*⟩

lemma *zdiv-zmult-self2* [*simp*]: $b \neq$ (*0*::*int*) ==> (*b*∗*a*) *div b* = *a*
⟨*proof*⟩

lemma *zmod-zmult-self1* [*simp*]: (*a*∗*b*) *mod b* = (*0*::*int*)
⟨*proof*⟩

lemma *zmod-zmult-self2* [*simp*]: (*b*∗*a*) *mod b* = (*0*::*int*)
⟨*proof*⟩

lemma *zmod-eq-0-iff*: (*m mod d* = *0*) = (*EX q*::*int*. *m* = *d*∗*q*)
⟨*proof*⟩

lemmas *zmod-eq-0D* [*dest!*] = *zmod-eq-0-iff* [*THEN iffD1*]

proving (a+b) div c = a div c + b div c + ((a mod c + b mod c) div c)

lemma *zadd1-lemma*:
  [| *quorem*((*a*,*c*),(*aq*,*ar*)); *quorem*((*b*,*c*),(*bq*,*br*)); $c \neq 0$ |]
  ==> *quorem* ((*a*+*b*, *c*), (*aq* + *bq* + (*ar*+*br*) *div c*, (*ar*+*br*) *mod c*))
⟨*proof*⟩

lemma *zdiv-zadd1-eq*:
  (*a*+*b*) *div* (*c*::*int*) = *a div c* + *b div c* + ((*a mod c* + *b mod c*) *div c*)
⟨*proof*⟩

lemma *zmod-zadd1-eq*: (*a*+*b*) *mod* (*c*::*int*) = (*a mod c* + *b mod c*) *mod c*
⟨*proof*⟩

lemma *mod-div-trivial* [*simp*]: (*a mod b*) *div b* = (*0*::*int*)

⟨proof⟩

lemma *mod-mod-trivial* [*simp*]: $(a\ mod\ b)\ mod\ b = a\ mod\ (b::int)$
⟨proof⟩

lemma *zmod-zadd-left-eq*: $(a+b)\ mod\ (c::int) = ((a\ mod\ c) + b)\ mod\ c$
⟨proof⟩

lemma *zmod-zadd-right-eq*: $(a+b)\ mod\ (c::int) = (a + (b\ mod\ c))\ mod\ c$
⟨proof⟩

lemma *zdiv-zadd-self1*[*simp*]: $a \neq (0::int) ==> (a+b)\ div\ a = b\ div\ a + 1$
⟨proof⟩

lemma *zdiv-zadd-self2*[*simp*]: $a \neq (0::int) ==> (b+a)\ div\ a = b\ div\ a + 1$
⟨proof⟩

lemma *zmod-zadd-self1*[*simp*]: $(a+b)\ mod\ a = b\ mod\ (a::int)$
⟨proof⟩

lemma *zmod-zadd-self2*[*simp*]: $(b+a)\ mod\ a = b\ mod\ (a::int)$
⟨proof⟩

lemma *zmod-zdiff1-eq*: **fixes** $a::int$
shows $(a - b)\ mod\ c = (a\ mod\ c - b\ mod\ c)\ mod\ c$ (**is** $?l = ?r$)
⟨proof⟩

32.12 Proving $a\ div\ (b * c) = a\ div\ b\ div\ c$

first, four lemmas to bound the remainder for the cases b¡0 and b¿0

lemma *zmult2-lemma-aux1*: $[|\ (0::int) < c;\ \ b < r;\ \ r \leq 0\ |] ==> b*c < b*(q\ mod\ c) + r$
⟨proof⟩

lemma *zmult2-lemma-aux2*:
$[|\ (0::int) < c;\ \ b < r;\ \ r \leq 0\ |] ==> b * (q\ mod\ c) + r \leq 0$
⟨proof⟩

lemma *zmult2-lemma-aux3*: $[|\ (0::int) < c;\ \ 0 \leq r;\ \ r < b\ |] ==> 0 \leq b * (q\ mod\ c) + r$
⟨proof⟩

lemma *zmult2-lemma-aux4*: $[|\ (0::int) < c;\ 0 \leq r;\ r < b\ |] ==> b * (q\ mod\ c) + r < b * c$
⟨proof⟩

lemma *zmult2-lemma*: $[|\ quorem\ ((a,b),\ (q,r));\ \ b \neq 0;\ \ 0 < c\ |]$
$==> quorem\ ((a,\ b*c),\ (q\ div\ c,\ b*(q\ mod\ c) + r))$

⟨*proof*⟩

lemma *zdiv-zmult2-eq*: $(0::int) < c ==> a\ div\ (b*c) = (a\ div\ b)\ div\ c$
⟨*proof*⟩

lemma *zmod-zmult2-eq*:
$(0::int) < c ==> a\ mod\ (b*c) = b*(a\ div\ b\ mod\ c) + a\ mod\ b$
⟨*proof*⟩

32.13 Cancellation of Common Factors in div

lemma *zdiv-zmult-zmult1-aux1*:
$[|\ (0::int) < b;\ \ c \neq 0\ |] ==> (c*a)\ div\ (c*b) = a\ div\ b$
⟨*proof*⟩

lemma *zdiv-zmult-zmult1-aux2*:
$[|\ b < (0::int);\ \ c \neq 0\ |] ==> (c*a)\ div\ (c*b) = a\ div\ b$
⟨*proof*⟩

lemma *zdiv-zmult-zmult1*: $c \neq (0::int) ==> (c*a)\ div\ (c*b) = a\ div\ b$
⟨*proof*⟩

lemma *zdiv-zmult-zmult1-if* [*simp*]:
$(k*m)\ div\ (k*n) = (if\ k = (0::int)\ then\ 0\ else\ m\ div\ n)$
⟨*proof*⟩

32.14 Distribution of Factors over mod

lemma *zmod-zmult-zmult1-aux1*:
$[|\ (0::int) < b;\ \ c \neq 0\ |] ==> (c*a)\ mod\ (c*b) = c * (a\ mod\ b)$
⟨*proof*⟩

lemma *zmod-zmult-zmult1-aux2*:
$[|\ b < (0::int);\ \ c \neq 0\ |] ==> (c*a)\ mod\ (c*b) = c * (a\ mod\ b)$
⟨*proof*⟩

lemma *zmod-zmult-zmult1*: $(c*a)\ mod\ (c*b) = (c::int) * (a\ mod\ b)$
⟨*proof*⟩

lemma *zmod-zmult-zmult2*: $(a*c)\ mod\ (b*c) = (a\ mod\ b) * (c::int)$
⟨*proof*⟩

lemma *zmod-zmod-cancel*:
assumes *n dvd m* **shows** $(k::int)\ mod\ m\ mod\ n = k\ mod\ n$
⟨*proof*⟩

32.15 Splitting Rules for div and mod

The proofs of the two lemmas below are essentially identical

lemma *split-pos-lemma*:
$0<k ==>$
$P(n\ div\ k :: int)(n\ mod\ k) = (\forall i\ j.\ 0\leq j\ \&\ j<k\ \&\ n = k*i + j\ -->\ P\ i\ j)$
⟨*proof*⟩

lemma *split-neg-lemma*:
$k<0 ==>$
$P(n\ div\ k :: int)(n\ mod\ k) = (\forall i\ j.\ k<j\ \&\ j\leq 0\ \&\ n = k*i + j\ -->\ P\ i\ j)$
⟨*proof*⟩

lemma *split-zdiv*:
$P(n\ div\ k :: int) =$
$((k = 0\ -->\ P\ 0)\ \&$
$(0<k\ -->\ (\forall i\ j.\ 0\leq j\ \&\ j<k\ \&\ n = k*i + j\ -->\ P\ i))\ \&$
$(k<0\ -->\ (\forall i\ j.\ k<j\ \&\ j\leq 0\ \&\ n = k*i + j\ -->\ P\ i)))$
⟨*proof*⟩

lemma *split-zmod*:
$P(n\ mod\ k :: int) =$
$((k = 0\ -->\ P\ n)\ \&$
$(0<k\ -->\ (\forall i\ j.\ 0\leq j\ \&\ j<k\ \&\ n = k*i + j\ -->\ P\ j))\ \&$
$(k<0\ -->\ (\forall i\ j.\ k<j\ \&\ j\leq 0\ \&\ n = k*i + j\ -->\ P\ j)))$
⟨*proof*⟩

declare *split-zdiv* [*of - - number-of k*, *simplified*, *standard*, *arith-split*]
declare *split-zmod* [*of - - number-of k*, *simplified*, *standard*, *arith-split*]

32.16 Speeding up the Division Algorithm with Shifting

computing div by shifting

lemma *pos-zdiv-mult-2*: $(0::int) \leq a ==> (1 + 2*b)\ div\ (2*a) = b\ div\ a$
⟨*proof*⟩

lemma *neg-zdiv-mult-2*: $a \leq (0::int) ==> (1 + 2*b)\ div\ (2*a) = (b+1)\ div\ a$
⟨*proof*⟩

lemma *not-0-le-lemma*: $\sim\ 0 \leq x ==> x \leq (0::int)$
⟨*proof*⟩

lemma *zdiv-number-of-BIT*[*simp*]:
number-of (*v BIT b*) *div number-of* (*w BIT bit.B0*) =
(*if b=bit.B0* | (*0::int*) ≤ *number-of w*
then number-of v div (*number-of w*)
else (*number-of v* + (*1::int*)) *div* (*number-of w*))
⟨*proof*⟩

32.17 Computing mod by Shifting (proofs resemble those for div)

lemma *pos-zmod-mult-2*:
  $(0::int) \leq a ==> (1 + 2*b)\ mod\ (2*a) = 1 + 2 * (b\ mod\ a)$
⟨*proof*⟩

lemma *neg-zmod-mult-2*:
  $a \leq (0::int) ==> (1 + 2*b)\ mod\ (2*a) = 2 * ((b+1)\ mod\ a) - 1$
⟨*proof*⟩

lemma *zmod-number-of-BIT* [*simp*]:
  $number\text{-}of\ (v\ BIT\ b)\ mod\ number\text{-}of\ (w\ BIT\ bit.B0) =$
  $(case\ b\ of$
    $bit.B0 => 2 * (number\text{-}of\ v\ mod\ number\text{-}of\ w)$
   $|\ bit.B1 => if\ (0::int) \leq number\text{-}of\ w$
      $then\ 2 * (number\text{-}of\ v\ mod\ number\text{-}of\ w) + 1$
      $else\ 2 * ((number\text{-}of\ v + (1::int))\ mod\ number\text{-}of\ w) - 1)$
⟨*proof*⟩

32.18 Quotients of Signs

lemma *div-neg-pos-less0*: $[|\ a < (0::int);\ \ 0 < b\ |] ==> a\ div\ b < 0$
⟨*proof*⟩

lemma *div-nonneg-neg-le0*: $[|\ (0::int) \leq a;\ \ b < 0\ |] ==> a\ div\ b \leq 0$
⟨*proof*⟩

lemma *pos-imp-zdiv-nonneg-iff*: $(0::int) < b ==> (0 \leq a\ div\ b) = (0 \leq a)$
⟨*proof*⟩

lemma *neg-imp-zdiv-nonneg-iff*:
  $b < (0::int) ==> (0 \leq a\ div\ b) = (a \leq (0::int))$
⟨*proof*⟩

lemma *pos-imp-zdiv-neg-iff*: $(0::int) < b ==> (a\ div\ b < 0) = (a < 0)$
⟨*proof*⟩

lemma *neg-imp-zdiv-neg-iff*: $b < (0::int) ==> (a\ div\ b < 0) = (0 < a)$
⟨*proof*⟩

32.19 The Divides Relation

lemma *zdvd-iff-zmod-eq-0*: $(m\ dvd\ n) = (n\ mod\ m = (0::int))$
⟨*proof*⟩

instance *int* :: *dvd-mod*
⟨*proof*⟩

lemmas *zdvd-iff-zmod-eq-0-number-of* [*simp*] =
zdvd-iff-zmod-eq-0 [*of number-of x number-of y*, *standard*]

lemma *zdvd-0-right* [*iff*]: (*m*::*int*) *dvd 0*
⟨*proof*⟩

lemma *zdvd-0-left* [*iff*,*noatp*]: (*0 dvd* (*m*::*int*)) = (*m* = *0*)
⟨*proof*⟩

lemma *zdvd-1-left* [*iff*]: *1 dvd* (*m*::*int*)
⟨*proof*⟩

lemma *zdvd-refl* [*simp*]: *m dvd* (*m*::*int*)
⟨*proof*⟩

lemma *zdvd-trans*: *m dvd n* ==> *n dvd k* ==> *m dvd* (*k*::*int*)
⟨*proof*⟩

lemma *zdvd-zminus-iff*: (*m dvd* −*n*) = (*m dvd* (*n*::*int*))
⟨*proof*⟩

lemma *zdvd-zminus2-iff*: (−*m dvd n*) = (*m dvd* (*n*::*int*))
⟨*proof*⟩
lemma *zdvd-abs1*: (|*i*::*int*| *dvd j*) = (*i dvd j*)
⟨*proof*⟩
lemma *zdvd-abs2*: ((*i*::*int*) *dvd* |*j*|) = (*i dvd j*)
⟨*proof*⟩

lemma *zdvd-anti-sym*:
0 < *m* ==> *0* < *n* ==> *m dvd n* ==> *n dvd m* ==> *m* = (*n*::*int*)
⟨*proof*⟩

lemma *zdvd-zadd*: *k dvd m* ==> *k dvd n* ==> *k dvd* (*m* + *n* :: *int*)
⟨*proof*⟩

lemma *zdvd-dvd-eq*: **assumes** *anz*:*a* ≠ *0* **and** *ab*: (*a*::*int*) *dvd b* **and** *ba*:*b dvd a*
shows |*a*| = |*b*|
⟨*proof*⟩

lemma *zdvd-zdiff*: *k dvd m* ==> *k dvd n* ==> *k dvd* (*m* − *n* :: *int*)
⟨*proof*⟩

lemma *zdvd-zdiffD*: *k dvd m* − *n* ==> *k dvd n* ==> *k dvd* (*m*::*int*)
⟨*proof*⟩

lemma *zdvd-zmult*: *k dvd* (*n*::*int*) ==> *k dvd m* ∗ *n*
⟨*proof*⟩

lemma *zdvd-zmult2*: $k\ dvd\ (m::int) ==> k\ dvd\ m * n$
⟨*proof*⟩

lemma *zdvd-triv-right* [*iff*]: $(k::int)\ dvd\ m * k$
⟨*proof*⟩

lemma *zdvd-triv-left* [*iff*]: $(k::int)\ dvd\ k * m$
⟨*proof*⟩

lemma *zdvd-zmultD2*: $j * k\ dvd\ n ==> j\ dvd\ (n::int)$
⟨*proof*⟩

lemma *zdvd-zmultD*: $j * k\ dvd\ n ==> k\ dvd\ (n::int)$
⟨*proof*⟩

lemma *zdvd-zmult-mono*: $i\ dvd\ m ==> j\ dvd\ (n::int) ==> i * j\ dvd\ m * n$
⟨*proof*⟩

lemma *zdvd-reduce*: $(k\ dvd\ n + k * m) = (k\ dvd\ (n::int))$
⟨*proof*⟩

lemma *zdvd-zmod*: $f\ dvd\ m ==> f\ dvd\ (n::int) ==> f\ dvd\ m\ mod\ n$
⟨*proof*⟩

lemma *zdvd-zmod-imp-zdvd*: $k\ dvd\ m\ mod\ n ==> k\ dvd\ n ==> k\ dvd\ (m::int)$
⟨*proof*⟩

lemma *zdvd-not-zless*: $0 < m ==> m < n ==> \neg\ n\ dvd\ (m::int)$
⟨*proof*⟩
lemma *zmult-div-cancel*: $(n::int) * (m\ div\ n) = m - (m\ mod\ n)$
⟨*proof*⟩

lemma *zdvd-mult-div-cancel*:$(n::int)\ dvd\ m \Longrightarrow n * (m\ div\ n) = m$
⟨*proof*⟩

lemma *zdvd-mult-cancel*: **assumes** $d{:}k * m\ dvd\ k * n$ **and** $kz{:}k \neq (0::int)$
shows $m\ dvd\ n$
⟨*proof*⟩

lemma *zdvd-zmult-cancel-disj*[*simp*]:
$(k{*}m)\ dvd\ (k{*}n) = (k{=}0\ |\ m\ dvd\ (n::int))$
⟨*proof*⟩

theorem *ex-nat*: $(\exists x::nat.\ P\ x) = (\exists x::int.\ 0 <= x \wedge P\ (nat\ x))$
⟨*proof*⟩

theorem *zdvd-int*: $(x\ dvd\ y) = (int\ x\ dvd\ int\ y)$
⟨*proof*⟩

lemma *zdvd1-eq*[*simp*]: $(x::int)$ *dvd* $1 = (|x| = 1)$
⟨*proof*⟩
lemma *zdvd-mult-cancel1*:
assumes *mp*:$m \neq (0::int)$ **shows** $(m * n$ *dvd* $m) = (|n| = 1)$
⟨*proof*⟩

lemma *int-dvd-iff*: (*int m dvd z*) = (*m dvd nat* (*abs z*))
⟨*proof*⟩

lemma *dvd-int-iff*: (*z dvd int m*) = (*nat* (*abs z*) *dvd m*)
⟨*proof*⟩

lemma *nat-dvd-iff*: (*nat z dvd m*) = (*if* $0 \leq z$ *then* (*z dvd int m*) *else* $m = 0$)
⟨*proof*⟩

lemma *zminus-dvd-iff* [*iff*]: $(-z$ *dvd* $w) = (z$ *dvd* $(w::int))$
⟨*proof*⟩

lemma *dvd-zminus-iff* [*iff*]: $(z$ *dvd* $-w) = (z$ *dvd* $(w::int))$
⟨*proof*⟩

lemma *zdvd-imp-le*: [| *z dvd n*; $0 < n$ |] ==> $z \leq (n::int)$
⟨*proof*⟩

32.20 Integer Powers

instance *int* :: *power* ⟨*proof*⟩

primrec
p ^ $0 = 1$
p ^ (*Suc n*) = $(p::int) * (p$ ^ $n)$

instance *int* :: *recpower*
⟨*proof*⟩

lemma *of-int-power*:
of-int (z ^ n) = (*of-int* z ^ n :: $'a$::{*recpower*, *ring-1*})
⟨*proof*⟩

lemma *zpower-zmod*: $((x::int)$ *mod* $m)$^y *mod* $m = x$^y *mod* m
⟨*proof*⟩

lemma *zpower-zadd-distrib*: x^$(y+z) = ((x$^$y)*(x$^$z)::int)$
⟨*proof*⟩

lemma *zpower-zpower*: $(x$^$y)$^$z = (x$^$(y*z)::int)$
⟨*proof*⟩

lemma *zero-less-zpower-abs-iff* [*simp*]:
$(0 < (abs\ x)\hat{}n) = (x \neq (0::int) \mid n=0)$
⟨*proof*⟩

lemma *zero-le-zpower-abs* [*simp*]: $(0::int) <= (abs\ x)\hat{}n$
⟨*proof*⟩

lemma *int-power*: $int\ (m\hat{}n) = (int\ m)\ \hat{}\ n$
⟨*proof*⟩

Compatibility binding

lemmas *zpower-int* = *int-power* [*symmetric*]

lemma *zdiv-int*: $int\ (a\ div\ b) = (int\ a)\ div\ (int\ b)$
⟨*proof*⟩

lemma *zmod-int*: $int\ (a\ mod\ b) = (int\ a)\ mod\ (int\ b)$
⟨*proof*⟩

Suggested by Matthias Daum

lemma *int-power-div-base*:
$[\![0 < m;\ 0 < k]\!] \Longrightarrow k\ \hat{}\ m\ div\ k = (k::int)\ \hat{}\ (m - Suc\ 0)$
⟨*proof*⟩

by Brian Huffman

lemma *zminus-zmod*: $-\ ((x::int)\ mod\ m)\ mod\ m = -\ x\ mod\ m$
⟨*proof*⟩

lemma *zdiff-zmod-left*: $(x\ mod\ m - y)\ mod\ m = (x - y)\ mod\ (m::int)$
⟨*proof*⟩

lemma *zdiff-zmod-right*: $(x - y\ mod\ m)\ mod\ m = (x - y)\ mod\ (m::int)$
⟨*proof*⟩

lemmas *zmod-simps* =
IntDiv.zmod-zadd-left-eq [*symmetric*]
IntDiv.zmod-zadd-right-eq [*symmetric*]
IntDiv.zmod-zmult1-eq [*symmetric*]
IntDiv.zmod-zmult1-eq′ [*symmetric*]
IntDiv.zpower-zmod
zminus-zmod zdiff-zmod-left zdiff-zmod-right

code generator setup

code-modulename *SML*
IntDiv Integer

code-modulename *OCaml*

IntDiv Integer

code-modulename *Haskell*
IntDiv Integer

end

33 NatBin: Binary arithmetic for the natural numbers

theory *NatBin*
imports *IntDiv*
begin

Arithmetic for naturals is reduced to that for the non-negative integers.

instance *nat* :: *number*
nat-number-of-def [*code inline*]: *number-of v* == *nat* (*number-of* (*v*::*int*)) ⟨*proof*⟩

abbreviation (*xsymbols*)
square :: *'a*::*power* => *'a* ((-^{2}) [*1000*] *999*) **where**
x^2 == *x*ˆ*2*

notation (*latex* **output**)
square ((-^{2}) [*1000*] *999*)

notation (*HTML* **output**)
square ((-^{2}) [*1000*] *999*)

33.1 Function *nat*: Coercion from Type *int* to *nat*

declare *nat-0* [*simp*] *nat-1* [*simp*]

lemma *nat-number-of* [*simp*]: *nat* (*number-of w*) = *number-of w*
⟨*proof*⟩

lemma *nat-numeral-0-eq-0* [*simp*]: *Numeral0* = (*0*::*nat*)
⟨*proof*⟩

lemma *nat-numeral-1-eq-1* [*simp*]: *Numeral1* = (*1*::*nat*)
⟨*proof*⟩

lemma *numeral-1-eq-Suc-0*: *Numeral1* = *Suc 0*
⟨*proof*⟩

lemma *numeral-2-eq-2*: *2* = *Suc* (*Suc 0*)
⟨*proof*⟩

Distributive laws for type *nat*. The others are in theory *IntArith*, but these require div and mod to be defined for type "int". They also need some of the lemmas proved above.

lemma *nat-div-distrib*: $(0::int) <= z ==> nat\ (z\ div\ z') = nat\ z\ div\ nat\ z'$
⟨*proof*⟩

lemma *nat-mod-distrib*:
$[|\ (0::int) <= z;\ \ 0 <= z'\ |] ==> nat\ (z\ mod\ z') = nat\ z\ mod\ nat\ z'$
⟨*proof*⟩

Suggested by Matthias Daum

lemma *int-div-less-self*: $[\![0 < x;\ 1 < k]\!] \Longrightarrow x\ div\ k < (x::int)$
⟨*proof*⟩

33.2 Function *int*: Coercion from Type *nat* to *int*

lemma *int-nat-number-of* [*simp*]:
int (*number-of v*) =
(*if neg* (*number-of v* :: *int*) *then 0*
else (*number-of v* :: *int*))
⟨*proof*⟩

33.2.1 Successor

lemma *Suc-nat-eq-nat-zadd1*: $(0::int) <= z ==> Suc\ (nat\ z) = nat\ (1 + z)$
⟨*proof*⟩

lemma *Suc-nat-number-of-add*:
Suc (*number-of v* + *n*) =
(*if neg* (*number-of v* :: *int*) *then 1+n else number-of* (*Numeral.succ v*) +
n)
⟨*proof*⟩

lemma *Suc-nat-number-of* [*simp*]:
Suc (*number-of v*) =
(*if neg* (*number-of v* :: *int*) *then 1 else number-of* (*Numeral.succ v*))
⟨*proof*⟩

33.2.2 Addition

lemma *add-nat-number-of* [*simp*]:
(*number-of v* :: *nat*) + *number-of v′* =
(*if neg* (*number-of v* :: *int*) *then number-of v′*
else if neg (*number-of v′* :: *int*) *then number-of v*
else number-of (*v* + *v′*))
⟨*proof*⟩

33.2.3 Subtraction

lemma *diff-nat-eq-if*:

$nat\ z\ -\ nat\ z' =$
$(if\ neg\ z'\ then\ nat\ z$
$else\ let\ d = z-z'\ in$
$if\ neg\ d\ then\ 0\ else\ nat\ d)$
⟨*proof*⟩

lemma *diff-nat-number-of* [*simp*]:
$(number\text{-}of\ v :: nat) - number\text{-}of\ v' =$
$(if\ neg\ (number\text{-}of\ v' :: int)\ then\ number\text{-}of\ v$
$else\ let\ d = number\text{-}of\ (v + uminus\ v')\ in$
$if\ neg\ d\ then\ 0\ else\ nat\ d)$
⟨*proof*⟩

33.2.4 Multiplication

lemma *mult-nat-number-of* [*simp*]:
$(number\text{-}of\ v :: nat) * number\text{-}of\ v' =$
$(if\ neg\ (number\text{-}of\ v :: int)\ then\ 0\ else\ number\text{-}of\ (v * v'))$
⟨*proof*⟩

33.2.5 Quotient

lemma *div-nat-number-of* [*simp*]:
$(number\text{-}of\ v :: nat)\ \ div\ \ number\text{-}of\ v' =$
$(if\ neg\ (number\text{-}of\ v :: int)\ then\ 0$
$else\ nat\ (number\text{-}of\ v\ div\ number\text{-}of\ v'))$
⟨*proof*⟩

lemma *one-div-nat-number-of* [*simp*]:
$(Suc\ 0)\ \ div\ \ number\text{-}of\ v' = (nat\ (1\ div\ number\text{-}of\ v'))$
⟨*proof*⟩

33.2.6 Remainder

lemma *mod-nat-number-of* [*simp*]:
$(number\text{-}of\ v :: nat)\ \ mod\ \ number\text{-}of\ v' =$
$(if\ neg\ (number\text{-}of\ v :: int)\ then\ 0$
$else\ if\ neg\ (number\text{-}of\ v' :: int)\ then\ number\text{-}of\ v$
$else\ nat\ (number\text{-}of\ v\ mod\ number\text{-}of\ v'))$
⟨*proof*⟩

lemma *one-mod-nat-number-of* [*simp*]:
$(Suc\ 0)\ \ mod\ \ number\text{-}of\ v' =$
$(if\ neg\ (number\text{-}of\ v' :: int)\ then\ Suc\ 0$
$else\ nat\ (1\ mod\ number\text{-}of\ v'))$
⟨*proof*⟩

33.2.7 Divisibility

lemmas *dvd-eq-mod-eq-0-number-of* =

dvd-eq-mod-eq-0 [*of number-of x number-of y, standard*]

declare *dvd-eq-mod-eq-0-number-of* [*simp*]

⟨*ML*⟩

33.3 Comparisons

33.3.1 Equals (=)

lemma *eq-nat-nat-iff*:
[| (*0*::*int*) <= *z*; *0* <= *z'* |] ==> (*nat z* = *nat z'*) = (*z*=*z'*)
⟨*proof*⟩

lemma *eq-nat-number-of* [*simp*]:
((*number-of v* :: *nat*) = *number-of v'*) =
(*if neg* (*number-of v* :: *int*) *then* (*iszero* (*number-of v'* :: *int*) | *neg* (*number-of v'* :: *int*))
else if neg (*number-of v'* :: *int*) *then iszero* (*number-of v* :: *int*)
else iszero (*number-of* (*v* + *uminus v'*) :: *int*))
⟨*proof*⟩

33.3.2 Less-than (¡)

lemma *less-nat-number-of* [*simp*]:
((*number-of v* :: *nat*) < *number-of v'*) =
(*if neg* (*number-of v* :: *int*) *then neg* (*number-of* (*uminus v'*) :: *int*)
else neg (*number-of* (*v* + *uminus v'*) :: *int*))
⟨*proof*⟩

lemmas *numerals* = *nat-numeral-0-eq-0 nat-numeral-1-eq-1 numeral-2-eq-2*

33.4 Powers with Numeric Exponents

We cannot refer to the number *2*::'*a* in *Ring-and-Field.thy*. We cannot prove general results about the numeral −*1*::'*a*, so we have to use − (*1*::'*a*) instead.

lemma *power2-eq-square*: (a::'a::*recpower*)^2 = $a * a$
⟨*proof*⟩

lemma *zero-power2* [*simp*]: (*0*::'*a*::{*semiring-1*,*recpower*})^2 = *0*
⟨*proof*⟩

lemma *one-power2* [*simp*]: (*1*::'*a*::{*semiring-1*,*recpower*})^2 = *1*
⟨*proof*⟩

lemma *power3-eq-cube*: $(x::'a::recpower)$ ˆ $3 = x * x * x$
⟨*proof*⟩

Squares of literal numerals will be evaluated.

lemmas *power2-eq-square-number-of* =
power2-eq-square [*of number-of w*, *standard*]
declare *power2-eq-square-number-of* [*simp*]

lemma *zero-le-power2*[*simp*]: $0 \leq (a^2::'a::\{ordered\text{-}idom,recpower\})$
⟨*proof*⟩

lemma *zero-less-power2*[*simp*]:
$(0 < a^2) = (a \neq (0::'a::\{ordered\text{-}idom,recpower\}))$
⟨*proof*⟩

lemma *power2-less-0*[*simp*]:
fixes $a :: 'a::\{ordered\text{-}idom,recpower\}$
shows $\sim (a^2 < 0)$
⟨*proof*⟩

lemma *zero-eq-power2*[*simp*]:
$(a^2 = 0) = (a = (0::'a::\{ordered\text{-}idom,recpower\}))$
⟨*proof*⟩

lemma *abs-power2*[*simp*]:
$abs(a^2) = (a^2::'a::\{ordered\text{-}idom,recpower\})$
⟨*proof*⟩

lemma *power2-abs*[*simp*]:
$(abs\ a)^2 = (a^2::'a::\{ordered\text{-}idom,recpower\})$
⟨*proof*⟩

lemma *power2-minus*[*simp*]:
$(-\ a)^2 = (a^2::'a::\{comm\text{-}ring\text{-}1,recpower\})$
⟨*proof*⟩

lemma *power2-le-imp-le*:
fixes $x\ y :: 'a::\{ordered\text{-}semidom,recpower\}$
shows $[\![x^2 \leq y^2;\ 0 \leq y]\!] \Longrightarrow x \leq y$
⟨*proof*⟩

lemma *power2-less-imp-less*:
fixes $x\ y :: 'a::\{ordered\text{-}semidom,recpower\}$
shows $[\![x^2 < y^2;\ 0 \leq y]\!] \Longrightarrow x < y$
⟨*proof*⟩

lemma *power2-eq-imp-eq*:
fixes $x\ y :: 'a::\{ordered\text{-}semidom,recpower\}$

shows $[\![x^2 = y^2;\ 0 \le x;\ 0 \le y]\!] \Longrightarrow x = y$
⟨*proof*⟩

lemma *power-minus1-even*[*simp*]: $(-\ 1)\ \hat{}\ (2{*}n) = (1::'a::\{comm\text{-}ring\text{-}1,recpower\})$
⟨*proof*⟩

lemma *power-even-eq*: $(a::'a::recpower)\ \hat{}\ (2{*}n) = (a\hat{}n)\hat{}2$
⟨*proof*⟩

lemma *power-odd-eq*: $(a::int)\ \hat{}\ Suc(2{*}n) = a * (a\hat{}n)\hat{}2$
⟨*proof*⟩

lemma *power-minus-even* [*simp*]:
$(-a)\ \hat{}\ (2{*}n) = (a::'a::\{comm\text{-}ring\text{-}1,recpower\})\ \hat{}\ (2{*}n)$
⟨*proof*⟩

lemma *zero-le-even-power'*[*simp*]:
$0 \le (a::'a::\{ordered\text{-}idom,recpower\})\ \hat{}\ (2{*}n)$
⟨*proof*⟩

lemma *odd-power-less-zero*:
$(a::'a::\{ordered\text{-}idom,recpower\}) < 0 ==> a\ \hat{}\ Suc(2{*}n) < 0$
⟨*proof*⟩

lemma *odd-0-le-power-imp-0-le*:
$0 \le a\ \ \hat{}\ Suc(2{*}n) ==> 0 \le (a::'a::\{ordered\text{-}idom,recpower\})$
⟨*proof*⟩

Simprules for comparisons where common factors can be cancelled.

lemmas *zero-compare-simps* =
add-strict-increasing add-strict-increasing2 add-increasing
zero-le-mult-iff zero-le-divide-iff
zero-less-mult-iff zero-less-divide-iff
mult-le-0-iff divide-le-0-iff
mult-less-0-iff divide-less-0-iff
zero-le-power2 power2-less-0

33.4.1 Nat

lemma *Suc-pred'*: $0 < n ==> n = Suc(n - 1)$
⟨*proof*⟩

lemmas *expand-Suc* = *Suc-pred'* [*of number-of v*, *standard*]

33.4.2 Arith

lemma *Suc-eq-add-numeral-1*: $Suc\ n = n + 1$
⟨*proof*⟩

lemma *Suc-eq-add-numeral-1-left*: $Suc\ n = 1 + n$
⟨*proof*⟩

lemma *add-eq-if*: $(m::nat) + n = (if\ m{=}0\ then\ n\ else\ Suc\ ((m - 1) + n))$
⟨*proof*⟩

lemma *mult-eq-if*: $(m::nat) * n = (if\ m{=}0\ then\ 0\ else\ n + ((m - 1) * n))$
⟨*proof*⟩

lemma *power-eq-if*: $(p\ \hat{}\ m :: nat) = (if\ m{=}0\ then\ 1\ else\ p * (p\ \hat{}\ (m - 1)))$
⟨*proof*⟩

33.5 Comparisons involving (0::nat)

Simplification already does $n < (0::'a)$, $n \leq (0::'a)$ and $(0::'a) \leq n$.

lemma *eq-number-of-0* [*simp*]:
$(number\text{-}of\ v = (0::nat)) =$
$(if\ neg\ (number\text{-}of\ v :: int)\ then\ True\ else\ iszero\ (number\text{-}of\ v :: int))$
⟨*proof*⟩

lemma *eq-0-number-of* [*simp*]:
$((0::nat) = number\text{-}of\ v) =$
$(if\ neg\ (number\text{-}of\ v :: int)\ then\ True\ else\ iszero\ (number\text{-}of\ v :: int))$
⟨*proof*⟩

lemma *less-0-number-of* [*simp*]:
$((0::nat) < number\text{-}of\ v) = neg\ (number\text{-}of\ (uminus\ v) :: int)$
⟨*proof*⟩

lemma *neg-imp-number-of-eq-0*: $neg\ (number\text{-}of\ v :: int) ==> number\text{-}of\ v = (0::nat)$
⟨*proof*⟩

33.6 Comparisons involving *Suc*

lemma *eq-number-of-Suc* [*simp*]:
$(number\text{-}of\ v = Suc\ n) =$
$(let\ pv = number\text{-}of\ (Numeral.pred\ v)\ in$
$if\ neg\ pv\ then\ False\ else\ nat\ pv = n)$
⟨*proof*⟩

lemma *Suc-eq-number-of* [*simp*]:
$(Suc\ n = number\text{-}of\ v) =$
$(let\ pv = number\text{-}of\ (Numeral.pred\ v)\ in$
$if\ neg\ pv\ then\ False\ else\ nat\ pv = n)$
⟨*proof*⟩

lemma *less-number-of-Suc* [*simp*]:
(*number-of v* < *Suc n*) =
(*let pv* = *number-of* (*Numeral.pred v*) *in*
if neg pv then True else nat pv < *n*)
⟨*proof*⟩

lemma *less-Suc-number-of* [*simp*]:
(*Suc n* < *number-of v*) =
(*let pv* = *number-of* (*Numeral.pred v*) *in*
if neg pv then False else n < *nat pv*)
⟨*proof*⟩

lemma *le-number-of-Suc* [*simp*]:
(*number-of v* <= *Suc n*) =
(*let pv* = *number-of* (*Numeral.pred v*) *in*
if neg pv then True else nat pv <= *n*)
⟨*proof*⟩

lemma *le-Suc-number-of* [*simp*]:
(*Suc n* <= *number-of v*) =
(*let pv* = *number-of* (*Numeral.pred v*) *in*
if neg pv then False else n <= *nat pv*)
⟨*proof*⟩

lemma *lemma1*: (*m*+*m* = *n*+*n*) = (*m* = (*n*::*int*))
⟨*proof*⟩

lemma *lemma2*: *m*+*m* ~= (*1*::*int*) + (*n* + *n*)
⟨*proof*⟩

lemma *eq-number-of-BIT-BIT*:
((*number-of* (*v BIT x*) ::*int*) = *number-of* (*w BIT y*)) =
(*x*=*y* & (((*number-of v*) ::*int*) = *number-of w*))
⟨*proof*⟩

lemma *eq-number-of-BIT-Pls*:
((*number-of* (*v BIT x*) ::*int*) = *Numeral0*) =
(*x*=*bit.B0* & (((*number-of v*) ::*int*) = *Numeral0*))
⟨*proof*⟩

lemma *eq-number-of-BIT-Min*:
((*number-of* (*v BIT x*) ::*int*) = *number-of Numeral.Min*) =
(*x*=*bit.B1* & (((*number-of v*) ::*int*) = *number-of Numeral.Min*))
⟨*proof*⟩

lemma *eq-number-of-Pls-Min*: (*Numeral0* ::*int*) ~= *number-of Numeral.Min*
⟨*proof*⟩

33.7 Max and Min Combined with *Suc*

lemma *max-number-of-Suc* [*simp*]:
  max (*Suc n*) (*number-of v*) =
    (*let pv* = *number-of* (*Numeral.pred v*) *in*
     if neg pv then Suc n else Suc(*max n* (*nat pv*)))
⟨*proof*⟩

lemma *max-Suc-number-of* [*simp*]:
  max (*number-of v*) (*Suc n*) =
    (*let pv* = *number-of* (*Numeral.pred v*) *in*
     if neg pv then Suc n else Suc(*max* (*nat pv*) *n*))
⟨*proof*⟩

lemma *min-number-of-Suc* [*simp*]:
  min (*Suc n*) (*number-of v*) =
    (*let pv* = *number-of* (*Numeral.pred v*) *in*
     if neg pv then 0 else Suc(*min n* (*nat pv*)))
⟨*proof*⟩

lemma *min-Suc-number-of* [*simp*]:
  min (*number-of v*) (*Suc n*) =
    (*let pv* = *number-of* (*Numeral.pred v*) *in*
     if neg pv then 0 else Suc(*min* (*nat pv*) *n*))
⟨*proof*⟩

33.8 Literal arithmetic involving powers

lemma *nat-power-eq*: (*0::int*) <= *z* ==> *nat* (*z^n*) = *nat z* ^ *n*
⟨*proof*⟩

lemma *power-nat-number-of*:
  (*number-of v* :: *nat*) ^ *n* =
   (*if neg* (*number-of v* :: *int*) *then 0^n else nat* ((*number-of v* :: *int*) ^ *n*))
⟨*proof*⟩

lemmas *power-nat-number-of-number-of* = *power-nat-number-of* [*of - number-of w*, *standard*]
declare *power-nat-number-of-number-of* [*simp*]

For arbitrary rings

lemma *power-number-of-even*:
 fixes *z* :: *'a*::{*number-ring*,*recpower*}
 shows *z* ^ *number-of* (*w BIT bit.B0*) = (*let w* = *z* ^ (*number-of w*) *in w* * *w*)
⟨*proof*⟩

lemma *power-number-of-odd*:
 fixes *z* :: *'a*::{*number-ring*,*recpower*}
 shows *z* ^ *number-of* (*w BIT bit.B1*) = (*if* (*0::int*) <= *number-of w*

then (*let* $w = z$ ^ (*number-of* w) *in* $z * w * w$) *else* *1*)
⟨*proof*⟩

lemmas *zpower-number-of-even* = *power-number-of-even* [**where** $'a$=*int*]
lemmas *zpower-number-of-odd* = *power-number-of-odd* [**where** $'a$=*int*]

lemmas *power-number-of-even-number-of* [*simp*] =
power-number-of-even [*of number-of v*, *standard*]

lemmas *power-number-of-odd-number-of* [*simp*] =
power-number-of-odd [*of number-of v*, *standard*]

⟨*ML*⟩

declare *split-div*[*of - - number-of k*, *standard*, *arith-split*]
declare *split-mod*[*of - - number-of k*, *standard*, *arith-split*]

lemma *nat-number-of-Pls*: *Numeral0* = (*0*::*nat*)
⟨*proof*⟩

lemma *nat-number-of-Min*: *number-of Numeral.Min* = (*0*::*nat*)
⟨*proof*⟩

lemma *nat-number-of-BIT-1*:
number-of (*w BIT bit.B1*) =
(*if neg* (*number-of w* :: *int*) *then 0*
else let n = *number-of w in Suc* ($n + n$))
⟨*proof*⟩

lemma *nat-number-of-BIT-0*:
number-of (*w BIT bit.B0*) = (*let n*::*nat* = *number-of w in* $n + n$)
⟨*proof*⟩

lemmas *nat-number* =
nat-number-of-Pls nat-number-of-Min
nat-number-of-BIT-1 nat-number-of-BIT-0

lemma *Let-Suc* [*simp*]: *Let* (*Suc n*) *f* == *f* (*Suc n*)
⟨*proof*⟩

lemma *power-m1-even*: (-1) ^ ($2*n$) = (*1*::$'a$::{*number-ring*,*recpower*})
⟨*proof*⟩

lemma *power-m1-odd*: (-1) ^ *Suc*($2*n$) = (-1::$'a$::{*number-ring*,*recpower*})
⟨*proof*⟩

33.9 Literal arithmetic and *of-nat*

lemma *of-nat-double*:
$0 \leq x ==> \textit{of-nat}\ (\textit{nat}\ (2 * x)) = \textit{of-nat}\ (\textit{nat}\ x) + \textit{of-nat}\ (\textit{nat}\ x)$
⟨*proof*⟩

lemma *nat-numeral-m1-eq-0*: $-1 = (0::\textit{nat})$
⟨*proof*⟩

lemma *of-nat-number-of-lemma*:
of-nat (*number-of v* :: *nat*) =
(*if 0* ≤ (*number-of v* :: *int*)
then (*number-of v* :: ′*a* :: *number-ring*)
else 0)
⟨*proof*⟩

lemma *of-nat-number-of-eq* [*simp*]:
of-nat (*number-of v* :: *nat*) =
(*if neg* (*number-of v* :: *int*) *then 0*
else (*number-of v* :: ′*a* :: *number-ring*))
⟨*proof*⟩

33.10 Lemmas for the Combination and Cancellation Simprocs

lemma *nat-number-of-add-left*:
number-of v + (*number-of v*′ + (*k*::*nat*)) =
(*if neg* (*number-of v* :: *int*) *then number-of v*′ + *k*
else if neg (*number-of v*′ :: *int*) *then number-of v* + *k*
else number-of (*v* + *v*′) + *k*)
⟨*proof*⟩

lemma *nat-number-of-mult-left*:
number-of v ∗ (*number-of v*′ ∗ (*k*::*nat*)) =
(*if neg* (*number-of v* :: *int*) *then 0*
else number-of (*v* ∗ *v*′) ∗ *k*)
⟨*proof*⟩

33.10.1 For *combine-numerals*

lemma *left-add-mult-distrib*: $i*u + (j*u + k) = (i+j)*u + (k::\textit{nat})$
⟨*proof*⟩

33.10.2 For *cancel-numerals*

lemma *nat-diff-add-eq1*:
$j <= (i::\textit{nat}) ==> ((i*u + m) - (j*u + n)) = (((i-j)*u + m) - n)$
⟨*proof*⟩

lemma *nat-diff-add-eq2*:

$i <= (j::nat) ==> ((i*u + m) - (j*u + n)) = (m - ((j-i)*u + n))$
⟨proof⟩

lemma *nat-eq-add-iff1*:
$j <= (i::nat) ==> (i*u + m = j*u + n) = ((i-j)*u + m = n)$
⟨proof⟩

lemma *nat-eq-add-iff2*:
$i <= (j::nat) ==> (i*u + m = j*u + n) = (m = (j-i)*u + n)$
⟨proof⟩

lemma *nat-less-add-iff1*:
$j <= (i::nat) ==> (i*u + m < j*u + n) = ((i-j)*u + m < n)$
⟨proof⟩

lemma *nat-less-add-iff2*:
$i <= (j::nat) ==> (i*u + m < j*u + n) = (m < (j-i)*u + n)$
⟨proof⟩

lemma *nat-le-add-iff1*:
$j <= (i::nat) ==> (i*u + m <= j*u + n) = ((i-j)*u + m <= n)$
⟨proof⟩

lemma *nat-le-add-iff2*:
$i <= (j::nat) ==> (i*u + m <= j*u + n) = (m <= (j-i)*u + n)$
⟨proof⟩

33.10.3 For *cancel-numeral-factors*

lemma *nat-mult-le-cancel1*: $(0::nat) < k ==> (k*m <= k*n) = (m<=n)$
⟨proof⟩

lemma *nat-mult-less-cancel1*: $(0::nat) < k ==> (k*m < k*n) = (m<n)$
⟨proof⟩

lemma *nat-mult-eq-cancel1*: $(0::nat) < k ==> (k*m = k*n) = (m=n)$
⟨proof⟩

lemma *nat-mult-div-cancel1*: $(0::nat) < k ==> (k*m)\ div\ (k*n) = (m\ div\ n)$
⟨proof⟩

lemma *nat-mult-dvd-cancel-disj*[*simp*]:
$(k*m)\ dvd\ (k*n) = (k=0 \mid m\ dvd\ (n::nat))$
⟨proof⟩

lemma *nat-mult-dvd-cancel1*: $0 < k \Longrightarrow (k*m)\ dvd\ (k*n::nat) = (m\ dvd\ n)$
⟨proof⟩

33.10.4 For *cancel-factor*

lemma *nat-mult-le-cancel-disj*: $(k*m <= k*n) = ((0::nat) < k --> m<=n)$
⟨*proof*⟩

lemma *nat-mult-less-cancel-disj*: $(k*m < k*n) = ((0::nat) < k \ \& \ m<n)$
⟨*proof*⟩

lemma *nat-mult-eq-cancel-disj*: $(k*m = k*n) = (k = (0::nat) \mid m=n)$
⟨*proof*⟩

lemma *nat-mult-div-cancel-disj*[*simp*]:
  $(k*m)$ *div* $(k*n) =$ (*if* $k = (0::nat)$ *then* 0 *else* m *div* n)
⟨*proof*⟩

33.11 legacy ML bindings

⟨*ML*⟩

end

34 Groebner-Basis: Semiring normalization and Groebner Bases

theory *Groebner-Basis*
imports *NatBin*
uses
  Tools/Groebner-Basis/misc.ML
  Tools/Groebner-Basis/normalizer-data.ML
  (*Tools/Groebner-Basis/normalizer.ML*)
  (*Tools/Groebner-Basis/groebner.ML*)
begin

34.1 Semiring normalization

⟨*ML*⟩

locale *gb-semiring* =
  fixes *add mul pwr r0 r1*
  assumes *add-a*:(*add x* (*add y z*) = *add* (*add x y*) *z*)
    and *add-c*: *add x y* = *add y x* **and** *add-0*:*add r0 x* = *x*
    and *mul-a*:*mul x* (*mul y z*) = *mul* (*mul x y*) *z* **and** *mul-c*:*mul x y* = *mul y x*
    and *mul-1*:*mul r1 x* = *x* **and** *mul-0*:*mul r0 x* = *r0*
    and *mul-d*:*mul x* (*add y z*) = *add* (*mul x y*) (*mul x z*)
    and *pwr-0*:*pwr x 0* = *r1* **and** *pwr-Suc*:*pwr x* (*Suc n*) = *mul x* (*pwr x n*)
begin

lemma *mul-pwr*:*mul* (*pwr x p*) (*pwr x q*) = *pwr x* (*p* + *q*)
⟨*proof*⟩

lemma *pwr-mul*: *pwr* (*mul x y*) *q* = *mul* (*pwr x q*) (*pwr y q*)
⟨*proof*⟩

lemma *pwr-pwr*: *pwr* (*pwr x p*) *q* = *pwr x* (*p* * *q*)
⟨*proof*⟩

34.1.1 Declaring the abstract theory

lemma *semiring-ops*:
  includes *meta-term-syntax*
  shows *TERM* (*add x y*) **and** *TERM* (*mul x y*) **and** *TERM* (*pwr x n*)
    and *TERM r0* **and** *TERM r1*
  ⟨*proof*⟩

lemma *semiring-rules*:
  add (*mul a m*) (*mul b m*) = *mul* (*add a b*) *m*
  add (*mul a m*) *m* = *mul* (*add a r1*) *m*
  add m (*mul a m*) = *mul* (*add a r1*) *m*
  add m m = *mul* (*add r1 r1*) *m*
  add r0 a = *a*
  add a r0 = *a*
  mul a b = *mul b a*
  mul (*add a b*) *c* = *add* (*mul a c*) (*mul b c*)
  mul r0 a = *r0*
  mul a r0 = *r0*
  mul r1 a = *a*
  mul a r1 = *a*
  mul (*mul lx ly*) (*mul rx ry*) = *mul* (*mul lx rx*) (*mul ly ry*)
  mul (*mul lx ly*) (*mul rx ry*) = *mul lx* (*mul ly* (*mul rx ry*))
  mul (*mul lx ly*) (*mul rx ry*) = *mul rx* (*mul* (*mul lx ly*) *ry*)
  mul (*mul lx ly*) *rx* = *mul* (*mul lx rx*) *ly*
  mul (*mul lx ly*) *rx* = *mul lx* (*mul ly rx*)
  mul lx (*mul rx ry*) = *mul* (*mul lx rx*) *ry*
  mul lx (*mul rx ry*) = *mul rx* (*mul lx ry*)
  add (*add a b*) (*add c d*) = *add* (*add a c*) (*add b d*)
  add (*add a b*) *c* = *add a* (*add b c*)
  add a (*add c d*) = *add c* (*add a d*)
  add (*add a b*) *c* = *add* (*add a c*) *b*
  add a c = *add c a*
  add a (*add c d*) = *add* (*add a c*) *d*
  mul (*pwr x p*) (*pwr x q*) = *pwr x* (*p* + *q*)
  mul x (*pwr x q*) = *pwr x* (*Suc q*)
  mul (*pwr x q*) *x* = *pwr x* (*Suc q*)
  mul x x = *pwr x 2*
  pwr (*mul x y*) *q* = *mul* (*pwr x q*) (*pwr y q*)
  pwr (*pwr x p*) *q* = *pwr x* (*p* * *q*)

pwr x 0 = r1
pwr x 1 = x
mul x (add y z) = add (mul x y) (mul x z)
pwr x (Suc q) = mul x (pwr x q)
pwr x (2∗n) = mul (pwr x n) (pwr x n)
pwr x (Suc (2∗n)) = mul x (mul (pwr x n) (pwr x n))
⟨*proof*⟩

lemma *axioms* [*normalizer*
semiring ops: *semiring-ops*
semiring rules: *semiring-rules*]:
gb-semiring add mul pwr r0 r1 ⟨*proof*⟩

end

interpretation *class-semiring*: *gb-semiring*
[*op + op ∗ op ˆ 0::'a::*{*comm-semiring-1*, *recpower*} *1*]
⟨*proof*⟩

lemmas *nat-arith* =
add-nat-number-of diff-nat-number-of mult-nat-number-of eq-nat-number-of less-nat-number-of

lemma *not-iszero-Numeral1*: ¬ *iszero* (*Numeral1::'a::number-ring*)
⟨*proof*⟩
lemmas *comp-arith* = *Let-def arith-simps nat-arith rel-simps if-False*
if-True add-0 add-Suc add-number-of-left mult-number-of-left
numeral-1-eq-1[*symmetric*] *Suc-eq-add-numeral-1*
numeral-0-eq-0[*symmetric*] *numerals*[*symmetric*] *not-iszero-1*
iszero-number-of-1 iszero-number-of-0 nonzero-number-of-Min
iszero-number-of-Pls iszero-0 not-iszero-Numeral1

lemmas *semiring-norm* = *comp-arith*

⟨*ML*⟩

locale *gb-ring* = *gb-semiring* +
fixes *sub* :: *'a ⇒ 'a ⇒ 'a*
and *neg* :: *'a ⇒ 'a*
assumes *neg-mul*: *neg x = mul (neg r1) x*
and *sub-add*: *sub x y = add x (neg y)*
begin

lemma *ring-ops*:
includes *meta-term-syntax*
shows *TERM* (*sub x y*) **and** *TERM* (*neg x*) ⟨*proof*⟩

lemmas *ring-rules* = *neg-mul sub-add*

lemma *axioms* [*normalizer*
semiring ops: *semiring-ops*
semiring rules: *semiring-rules*
ring ops: *ring-ops*
ring rules: *ring-rules*]:
gb-ring add mul pwr r0 r1 sub neg ⟨*proof*⟩

end

interpretation *class-ring*: *gb-ring* [*op* + *op* ∗ *op* ˆ
0::′*a*::{*comm-semiring-1*,*recpower*,*number-ring*} *1 op* − *uminus*]
⟨*proof*⟩

⟨*ML*⟩

locale *gb-field* = *gb-ring* +
fixes *divide* :: ′*a* ⇒ ′*a* ⇒ ′*a*
and *inverse*:: ′*a* ⇒ ′*a*
assumes *divide*: *divide x y* = *mul x* (*inverse y*)
and *inverse*: *inverse x* = *divide r1 x*
begin

lemma *axioms* [*normalizer*
semiring ops: *semiring-ops*
semiring rules: *semiring-rules*
ring ops: *ring-ops*
ring rules: *ring-rules*]:
gb-field add mul pwr r0 r1 sub neg divide inverse ⟨*proof*⟩

end

34.2 Groebner Bases

locale *semiringb* = *gb-semiring* +
assumes *add-cancel*: *add* (*x*::′*a*) *y* = *add x z* ⟷ *y* = *z*
and *add-mul-solve*: *add* (*mul w y*) (*mul x z*) =
add (*mul w z*) (*mul x y*) ⟷ *w* = *x* ∨ *y* = *z*
begin

lemma *noteq-reduce*: *a* ≠ *b* ∧ *c* ≠ *d* ⟷ *add* (*mul a c*) (*mul b d*) ≠ *add* (*mul a d*) (*mul b c*)
⟨*proof*⟩

lemma *add-scale-eq-noteq*: ⟦*r* ≠ *r0* ; (*a* = *b*) ∧ ~(*c* = *d*)⟧
⟹ *add a* (*mul r c*) ≠ *add b* (*mul r d*)

⟨*proof*⟩

lemma *add-r0-iff*: $x = add\ x\ a \longleftrightarrow a = r0$
⟨*proof*⟩

declare *axioms* [*normalizer del*]

lemma *axioms* [*normalizer*
semiring ops: *semiring-ops*
semiring rules: *semiring-rules*
idom rules: *noteq-reduce add-scale-eq-noteq*]:
semiringb add mul pwr r0 r1 ⟨*proof*⟩

end

locale *ringb* = *semiringb* + *gb-ring* +
assumes *subr0-iff*: $sub\ x\ y = r0 \longleftrightarrow x = y$
begin

declare *axioms* [*normalizer del*]

lemma *axioms* [*normalizer*
semiring ops: *semiring-ops*
semiring rules: *semiring-rules*
ring ops: *ring-ops*
ring rules: *ring-rules*
idom rules: *noteq-reduce add-scale-eq-noteq*
ideal rules: *subr0-iff add-r0-iff*]:
ringb add mul pwr r0 r1 sub neg ⟨*proof*⟩

end

lemma *no-zero-divirors-neq0*:
assumes *az*: $(a::'a::no\text{-}zero\text{-}divisors) \neq 0$
and *ab*: $a*b = 0$ **shows** $b = 0$
⟨*proof*⟩

interpretation *class-ringb*: *ringb*
[$op\ +\ op\ *\ op\ \hat{}\ 0::'a::\{idom,recpower,number\text{-}ring\}\ 1\ op\ -\ uminus$]
⟨*proof*⟩

⟨*ML*⟩

interpretation *natgb*: *semiringb*
[$op\ +\ op\ *\ op\ \hat{}\ 0::nat\ 1$]
⟨*proof*⟩

⟨*ML*⟩

locale *fieldgb* = *ringb* + *gb-field*
begin

declare *axioms* [*normalizer del*]

lemma *axioms* [*normalizer*
semiring ops: *semiring-ops*
semiring rules: *semiring-rules*
ring ops: *ring-ops*
ring rules: *ring-rules*
idom rules: *noteq-reduce add-scale-eq-noteq*
ideal rules: *subr0-iff add-r0-iff*]:
fieldgb add mul pwr r0 r1 sub neg divide inverse ⟨*proof*⟩
end

lemmas *bool-simps* = *simp-thms*(*1*−*34*)
lemma *dnf*:
$(P \ \& \ (Q \mid R)) = ((P\&Q) \mid (P\&R))$ $((Q \mid R) \ \& \ P) = ((Q\&P) \mid (R\&P))$
$(P \wedge Q) = (Q \wedge P)$ $(P \vee Q) = (Q \vee P)$
⟨*proof*⟩

lemmas *weak-dnf-simps* = *dnf bool-simps*

lemma *nnf-simps*:
$(\neg(P \wedge Q)) = (\neg P \vee \neg Q)$ $(\neg(P \vee Q)) = (\neg P \wedge \neg Q)$ $(P \longrightarrow Q) = (\neg P \vee Q)$
$(P = Q) = ((P \wedge Q) \vee (\neg P \wedge \neg Q))$ $(\neg \neg(P)) = P$
⟨*proof*⟩

lemma *PFalse*:
$P \equiv \mathit{False} \Longrightarrow \neg P$
$\neg P \Longrightarrow (P \equiv \mathit{False})$
⟨*proof*⟩

⟨*ML*⟩

end

35 Dense-Linear-Order: Dense linear order without endpoints and a quantifier elimination procedure in Ferrante and Rackoff style

theory *Dense-Linear-Order*
imports *Finite-Set*
uses
Tools/Qelim/qelim.ML

Tools/Qelim/langford-data.ML
Tools/Qelim/ferrante-rackoff-data.ML
(*Tools/Qelim/langford.ML*)
(*Tools/Qelim/ferrante-rackoff.ML*)
begin

⟨*ML*⟩

context *linorder*
begin

lemma *less-not-permute*: $\neg\ (x < y \wedge y < x)$ ⟨*proof*⟩

lemma *gather-simps*:
shows
$(\exists x.\ (\forall y \in L.\ y < x) \wedge (\forall y \in U.\ x < y) \wedge x < u \wedge P\ x) \longleftrightarrow (\exists x.\ (\forall y \in L.\ y < x) \wedge (\forall y \in (\mathit{insert}\ u\ U).\ x < y) \wedge P\ x)$
and $(\exists x.\ (\forall y \in L.\ y < x) \wedge (\forall y \in U.\ x < y) \wedge l < x \wedge P\ x) \longleftrightarrow (\exists x.\ (\forall y \in (\mathit{insert}\ l\ L).\ y < x) \wedge (\forall y \in U.\ x < y) \wedge P\ x)$
$(\exists x.\ (\forall y \in L.\ y < x) \wedge (\forall y \in U.\ x < y) \wedge x < u) \longleftrightarrow (\exists x.\ (\forall y \in L.\ y < x) \wedge (\forall y \in (\mathit{insert}\ u\ U).\ x < y))$
and $(\exists x.\ (\forall y \in L.\ y < x) \wedge (\forall y \in U.\ x < y) \wedge l < x) \longleftrightarrow (\exists x.\ (\forall y \in (\mathit{insert}\ l\ L).\ y < x) \wedge (\forall y \in U.\ x < y))$ ⟨*proof*⟩

lemma
gather-start: $(\exists x.\ P\ x) \equiv (\exists x.\ (\forall y \in \{\}.\ y < x) \wedge (\forall y \in \{\}.\ x < y) \wedge P\ x)$
⟨*proof*⟩

Theorems for $\exists z.\ \forall x.\ x < z \longrightarrow (P\ x \longleftrightarrow P_{-\infty})$

lemma *minf-lt*: $\exists z\ .\ \forall x.\ x < z \longrightarrow (x < t \longleftrightarrow \mathit{True})$ ⟨*proof*⟩
lemma *minf-gt*: $\exists z\ .\ \forall x.\ x < z \longrightarrow\ (t < x \longleftrightarrow\ \mathit{False})$
⟨*proof*⟩

lemma *minf-le*: $\exists z.\ \forall x.\ x < z \longrightarrow (x \le t \longleftrightarrow \mathit{True})$ ⟨*proof*⟩
lemma *minf-ge*: $\exists z.\ \forall x.\ x < z \longrightarrow (t \le x \longleftrightarrow \mathit{False})$
⟨*proof*⟩
lemma *minf-eq*: $\exists z.\ \forall x.\ x < z \longrightarrow (x = t \longleftrightarrow \mathit{False})$ ⟨*proof*⟩
lemma *minf-neq*: $\exists z.\ \forall x.\ x < z \longrightarrow (x \ne t \longleftrightarrow \mathit{True})$ ⟨*proof*⟩
lemma *minf-P*: $\exists z.\ \forall x.\ x < z \longrightarrow (P \longleftrightarrow P)$ ⟨*proof*⟩

Theorems for $\exists z.\ \forall x.\ x < z \longrightarrow (P\ x \longleftrightarrow P_{+\infty})$

lemma *pinf-gt*: $\exists z\ .\ \forall x.\ z < x \longrightarrow (t < x \longleftrightarrow \mathit{True})$ ⟨*proof*⟩
lemma *pinf-lt*: $\exists z\ .\ \forall x.\ z < x \longrightarrow\ (x < t \longleftrightarrow\ \mathit{False})$
⟨*proof*⟩

lemma *pinf-ge*: $\exists z.\ \forall x.\ z < x \longrightarrow (t \le x \longleftrightarrow \mathit{True})$ ⟨*proof*⟩
lemma *pinf-le*: $\exists z.\ \forall x.\ z < x \longrightarrow (x \le t \longleftrightarrow \mathit{False})$
⟨*proof*⟩
lemma *pinf-eq*: $\exists z.\ \forall x.\ z < x \longrightarrow (x = t \longleftrightarrow \mathit{False})$ ⟨*proof*⟩

lemma *pinf-neq*: $\exists z. \forall x. z < x \longrightarrow (x \neq t \longleftrightarrow True)$ ⟨*proof*⟩
lemma *pinf-P*: $\exists z. \forall x. z < x \longrightarrow (P \longleftrightarrow P)$ ⟨*proof*⟩

lemma *nmi-lt*: $t \in U \Longrightarrow \forall x. \neg True \wedge x < t \longrightarrow (\exists u \in U. u \leq x)$ ⟨*proof*⟩
lemma *nmi-gt*: $t \in U \Longrightarrow \forall x. \neg False \wedge t < x \longrightarrow (\exists u \in U. u \leq x)$
⟨*proof*⟩
lemma *nmi-le*: $t \in U \Longrightarrow \forall x. \neg True \wedge x \leq t \longrightarrow (\exists u \in U. u \leq x)$ ⟨*proof*⟩
lemma *nmi-ge*: $t \in U \Longrightarrow \forall x. \neg False \wedge t \leq x \longrightarrow (\exists u \in U. u \leq x)$ ⟨*proof*⟩
lemma *nmi-eq*: $t \in U \Longrightarrow \forall x. \neg False \wedge x = t \longrightarrow (\exists u \in U. u \leq x)$ ⟨*proof*⟩
lemma *nmi-neq*: $t \in U \Longrightarrow \forall x. \neg True \wedge x \neq t \longrightarrow (\exists u \in U. u \leq x)$ ⟨*proof*⟩
lemma *nmi-P*: $\forall x. \sim P \wedge P \longrightarrow (\exists u \in U. u \leq x)$ ⟨*proof*⟩
lemma *nmi-conj*: $[\![\forall x. \neg P1' \wedge P1\ x \longrightarrow (\exists u \in U. u \leq x)$;
$\forall x. \neg P2' \wedge P2\ x \longrightarrow (\exists u \in U. u \leq x)]\!] \Longrightarrow$
$\forall x. \neg(P1' \wedge P2') \wedge (P1\ x \wedge P2\ x) \longrightarrow (\exists u \in U. u \leq x)$ ⟨*proof*⟩
lemma *nmi-disj*: $[\![\forall x. \neg P1' \wedge P1\ x \longrightarrow (\exists u \in U. u \leq x)$;
$\forall x. \neg P2' \wedge P2\ x \longrightarrow (\exists u \in U. u \leq x)]\!] \Longrightarrow$
$\forall x. \neg(P1' \vee P2') \wedge (P1\ x \vee P2\ x) \longrightarrow (\exists u \in U. u \leq x)$ ⟨*proof*⟩

lemma *npi-lt*: $t \in U \Longrightarrow \forall x. \neg False \wedge x < t \longrightarrow (\exists u \in U. x \leq u)$ ⟨*proof*⟩
lemma *npi-gt*: $t \in U \Longrightarrow \forall x. \neg True \wedge t < x \longrightarrow (\exists u \in U. x \leq u)$ ⟨*proof*⟩
lemma *npi-le*: $t \in U \Longrightarrow \forall x. \neg False \wedge x \leq t \longrightarrow (\exists u \in U. x \leq u)$ ⟨*proof*⟩
lemma *npi-ge*: $t \in U \Longrightarrow \forall x. \neg True \wedge t \leq x \longrightarrow (\exists u \in U. x \leq u)$ ⟨*proof*⟩
lemma *npi-eq*: $t \in U \Longrightarrow \forall x. \neg False \wedge x = t \longrightarrow (\exists u \in U. x \leq u)$ ⟨*proof*⟩
lemma *npi-neq*: $t \in U \Longrightarrow \forall x. \neg True \wedge x \neq t \longrightarrow (\exists u \in U. x \leq u)$ ⟨*proof*⟩
lemma *npi-P*: $\forall x. \sim P \wedge P \longrightarrow (\exists u \in U. x \leq u)$ ⟨*proof*⟩
lemma *npi-conj*: $[\![\forall x. \neg P1' \wedge P1\ x \longrightarrow (\exists u \in U. x \leq u)$; $\forall x. \neg P2' \wedge P2\ x$
$\longrightarrow (\exists u \in U. x \leq u)]\!]$
$\Longrightarrow \forall x. \neg(P1' \wedge P2') \wedge (P1\ x \wedge P2\ x) \longrightarrow (\exists u \in U. x \leq u)$ ⟨*proof*⟩
lemma *npi-disj*: $[\![\forall x. \neg P1' \wedge P1\ x \longrightarrow (\exists u \in U. x \leq u)$; $\forall x. \neg P2' \wedge P2\ x$
$\longrightarrow (\exists u \in U. x \leq u)]\!]$
$\Longrightarrow \forall x. \neg(P1' \vee P2') \wedge (P1\ x \vee P2\ x) \longrightarrow (\exists u \in U. x \leq u)$ ⟨*proof*⟩

lemma *lin-dense-lt*: $t \in U \Longrightarrow \forall x\ l\ u. (\forall t. l < t \wedge t < u \longrightarrow t \notin U) \wedge l < x \wedge x < u \wedge x < t \longrightarrow (\forall y. l < y \wedge y < u \longrightarrow y < t)$
⟨*proof*⟩

lemma *lin-dense-gt*: $t \in U \Longrightarrow \forall x\ l\ u. (\forall t. l < t \wedge t < u \longrightarrow t \notin U) \wedge l < x \wedge x < u \wedge t < x \longrightarrow (\forall y. l < y \wedge y < u \longrightarrow t < y)$
⟨*proof*⟩

lemma *lin-dense-le*: $t \in U \Longrightarrow \forall x\ l\ u. (\forall t. l < t \wedge t < u \longrightarrow t \notin U) \wedge l < x \wedge x < u \wedge x \leq t \longrightarrow (\forall y. l < y \wedge y < u \longrightarrow y \leq t)$
⟨*proof*⟩

lemma *lin-dense-ge*: $t \in U \Longrightarrow \forall x\ l\ u. (\forall t. l < t \wedge t < u \longrightarrow t \notin U) \wedge l < x \wedge x < u \wedge t \leq x \longrightarrow (\forall y. l < y \wedge y < u \longrightarrow t \leq y)$
⟨*proof*⟩
lemma *lin-dense-eq*: $t \in U \Longrightarrow \forall x\ l\ u. (\forall t. l < t \wedge t < u \longrightarrow t \notin U) \wedge l < x \wedge x < u \wedge x = t \longrightarrow (\forall y. l < y \wedge y < u \longrightarrow y = t)$ ⟨*proof*⟩

lemma *lin-dense-neq*: $t \in U \Longrightarrow \forall x\ l\ u.\ (\forall\ t.\ l < t \wedge t< u \longrightarrow t \notin U) \wedge l< x \wedge x < u \wedge x \neq t \longrightarrow (\forall\ y.\ l < y \wedge y < u \longrightarrow y\neq t)$ ⟨*proof*⟩
lemma *lin-dense-P*: $\forall x\ l\ u.\ (\forall\ t.\ l < t \wedge t< u \longrightarrow t \notin U) \wedge l< x \wedge x < u \wedge P \longrightarrow (\forall\ y.\ l < y \wedge y < u \longrightarrow P)$ ⟨*proof*⟩

lemma *lin-dense-conj*:
$[\![\forall x\ l\ u.\ (\forall\ t.\ l < t \wedge t< u \longrightarrow t \notin U) \wedge l< x \wedge x < u \wedge P1\ x$
$\longrightarrow (\forall\ y.\ l < y \wedge y < u \longrightarrow P1\ y)$;
$\forall x\ l\ u.\ (\forall\ t.\ l < t \wedge t< u \longrightarrow t \notin U) \wedge l< x \wedge x < u \wedge P2\ x$
$\longrightarrow (\forall\ y.\ l < y \wedge y < u \longrightarrow P2\ y)]\!] \Longrightarrow$
$\forall x\ l\ u.\ (\forall\ t.\ l < t \wedge t< u \longrightarrow t \notin U) \wedge l< x \wedge x < u \wedge (P1\ x \wedge P2\ x)$
$\longrightarrow (\forall\ y.\ l < y \wedge y < u \longrightarrow (P1\ y \wedge P2\ y))$
⟨*proof*⟩
lemma *lin-dense-disj*:
$[\![\forall x\ l\ u.\ (\forall\ t.\ l < t \wedge t< u \longrightarrow t \notin U) \wedge l< x \wedge x < u \wedge P1\ x$
$\longrightarrow (\forall\ y.\ l < y \wedge y < u \longrightarrow P1\ y)$;
$\forall x\ l\ u.\ (\forall\ t.\ l < t \wedge t< u \longrightarrow t \notin U) \wedge l< x \wedge x < u \wedge P2\ x$
$\longrightarrow (\forall\ y.\ l < y \wedge y < u \longrightarrow P2\ y)]\!] \Longrightarrow$
$\forall x\ l\ u.\ (\forall\ t.\ l < t \wedge t< u \longrightarrow t \notin U) \wedge l< x \wedge x < u \wedge (P1\ x \vee P2\ x)$
$\longrightarrow (\forall\ y.\ l < y \wedge y < u \longrightarrow (P1\ y \vee P2\ y))$
⟨*proof*⟩

lemma *npmibnd*: $[\![\forall x.\ \neg\ MP \wedge P\ x \longrightarrow (\exists\ u\in\ U.\ u \leq x); \forall x.\ \neg PP \wedge P\ x \longrightarrow (\exists\ u\in\ U.\ x \leq u)]\!]$
$\Longrightarrow \forall x.\ \neg\ MP \wedge \neg PP \wedge P\ x \longrightarrow (\exists\ u\in\ U.\ \exists\ u' \in U.\ u \leq x \wedge x \leq u')$
⟨*proof*⟩

lemma *finite-set-intervals*:
assumes *px*: $P\ x$ **and** *lx*: $l \leq x$ **and** *xu*: $x \leq u$ **and** *linS*: $l\in S$
and *uinS*: $u \in S$ **and** *fS*:*finite* S **and** *lS*: $\forall\ x\in S.\ l \leq x$ **and** *Su*: $\forall\ x\in S.\ x \leq u$
shows $\exists\ a \in S.\ \exists\ b \in S.\ (\forall\ y.\ a < y \wedge y < b \longrightarrow y \notin S) \wedge a \leq x \wedge x \leq b \wedge P\ x$
⟨*proof*⟩

lemma *finite-set-intervals2*:
assumes *px*: $P\ x$ **and** *lx*: $l \leq x$ **and** *xu*: $x \leq u$ **and** *linS*: $l\in S$
and *uinS*: $u \in S$ **and** *fS*:*finite* S **and** *lS*: $\forall\ x\in S.\ l \leq x$ **and** *Su*: $\forall\ x\in S.\ x \leq u$
shows $(\exists\ s\in S.\ P\ s) \vee (\exists\ a \in S.\ \exists\ b \in S.\ (\forall\ y.\ a < y \wedge y < b \longrightarrow y \notin S) \wedge a < x \wedge x < b \wedge P\ x)$
⟨*proof*⟩

end

36 The classical QE after Langford for dense linear orders

context *dense-linear-order*
begin

lemma *dlo-qe-bnds*:
assumes *ne*: $L \neq \{\}$ **and** *neU*: $U \neq \{\}$ **and** *fL*: *finite* L **and** *fU*: *finite* U
shows $(\exists x. (\forall y \in L. y < x) \wedge (\forall y \in U. x < y)) \equiv (\forall\ l \in L. \forall u \in U. l < u)$
$\langle proof \rangle$

lemma *dlo-qe-noub*:
assumes *ne*: $L \neq \{\}$ **and** *fL*: *finite* L
shows $(\exists x. (\forall y \in L. y < x) \wedge (\forall y \in \{\}. x < y)) \equiv True$
$\langle proof \rangle$

lemma *dlo-qe-nolb*:
assumes *ne*: $U \neq \{\}$ **and** *fU*: *finite* U
shows $(\exists x. (\forall y \in \{\}. y < x) \wedge (\forall y \in U. x < y)) \equiv True$
$\langle proof \rangle$

lemma *exists-neq*: $\exists (x::'a).\ x \neq t$ $\exists (x::'a).\ t \neq x$
$\langle proof \rangle$

lemmas *dlo-simps* = *order-refl less-irrefl not-less not-le exists-neq le-less neq-iff linear less-not-permute*

lemma *axiom*: *dense-linear-order* $(op \leq)$ $(op <)$ $\langle proof \rangle$
lemma *atoms*:
includes *meta-term-syntax*
shows *TERM* $(less :: 'a \Rightarrow$ -)
and *TERM* $(less\text{-}eq :: 'a \Rightarrow$ -)
and *TERM* $(op = :: 'a \Rightarrow$ -) $\langle proof \rangle$

declare *axiom*[*langford qe*: *dlo-qe-bnds dlo-qe-nolb dlo-qe-noub gather*: *gather-start gather-simps atoms*: *atoms*]
declare *dlo-simps*[*langfordsimp*]

end

lemma *dnf*:
$(P \ \& \ (Q \mid R)) = ((P\&Q) \mid (P\&R))$
$((Q \mid R) \ \& \ P) = ((Q\&P) \mid (R\&P))$
$\langle proof \rangle$

lemmas *weak-dnf-simps* = *simp-thms dnf*

lemma *nnf-simps*:

$(\neg(P \wedge Q)) = (\neg P \vee \neg Q)$ $(\neg(P \vee Q)) = (\neg P \wedge \neg Q)$ $(P \longrightarrow Q) = (\neg P \vee Q)$
$(P = Q) = ((P \wedge Q) \vee (\neg P \wedge \neg\, Q))$ $(\neg\, \neg(P)) = P$
⟨*proof*⟩

lemma *ex-distrib*: $(\exists\, x.\ P\ x \vee Q\ x) \longleftrightarrow ((\exists\, x.\ P\ x) \vee (\exists\, x.\ Q\ x))$ ⟨*proof*⟩

lemmas *dnf-simps* = *weak-dnf-simps nnf-simps ex-distrib*

⟨*ML*⟩

37 Contructive dense linear orders yield QE for linear arithmetic over ordered Fields – see *Arith-Tools.thy*

Linear order without upper bounds

class *linorder-no-ub* = *linorder* +
assumes *gt-ex*: $\exists\, y.\ x < y$
begin

lemma *ge-ex*: $\exists\, y.\ x \leq y$ ⟨*proof*⟩

Theorems for $\exists\, z.\ \forall\, x.\ z < x \longrightarrow (P\ x \longleftrightarrow P_{+\infty})$

lemma *pinf-conj*:
assumes *ex1*: $\exists\, z1.\ \forall\, x.\ z1 < x \longrightarrow (P1\ x \longleftrightarrow P1')$
and *ex2*: $\exists\, z2.\ \forall\, x.\ z2 < x \longrightarrow (P2\ x \longleftrightarrow P2')$
shows $\exists\, z.\ \forall\, x.\ z < x \longrightarrow ((P1\ x \wedge P2\ x) \longleftrightarrow (P1' \wedge P2'))$
⟨*proof*⟩

lemma *pinf-disj*:
assumes *ex1*: $\exists\, z1.\ \forall\, x.\ z1 < x \longrightarrow (P1\ x \longleftrightarrow P1')$
and *ex2*: $\exists\, z2.\ \forall\, x.\ z2 < x \longrightarrow (P2\ x \longleftrightarrow P2')$
shows $\exists\, z.\ \forall\, x.\ z < x \longrightarrow ((P1\ x \vee P2\ x) \longleftrightarrow (P1' \vee P2'))$
⟨*proof*⟩

lemma *pinf-ex*: **assumes** *ex*:$\exists\, z.\ \forall\, x.\ z < x \longrightarrow (P\ x \longleftrightarrow P1)$ **and** *p1*: *P1* **shows** $\exists\ x.\ P\ x$
⟨*proof*⟩

end

Linear order without upper bounds

class *linorder-no-lb* = *linorder* +
assumes *lt-ex*: $\exists\, y.\ y < x$
begin

lemma *le-ex*: $\exists\, y.\ y \leq x$ ⟨*proof*⟩

Theorems for $\exists\, z.\ \forall\, x.\ x < z \longrightarrow (P\ x \longleftrightarrow P_{-\infty})$

lemma *minf-conj*:
  assumes *ex1*: $\exists z1. \forall x. x < z1 \longrightarrow (P1\ x \longleftrightarrow P1')$
  and *ex2*: $\exists z2. \forall x. x < z2 \longrightarrow (P2\ x \longleftrightarrow P2')$
  shows $\exists z. \forall x. x < z \longrightarrow ((P1\ x \wedge P2\ x) \longleftrightarrow (P1' \wedge P2'))$
$\langle proof \rangle$

lemma *minf-disj*:
  assumes *ex1*: $\exists z1. \forall x. x < z1 \longrightarrow (P1\ x \longleftrightarrow P1')$
  and *ex2*: $\exists z2. \forall x. x < z2 \longrightarrow (P2\ x \longleftrightarrow P2')$
  shows $\exists z. \forall x. x < z \longrightarrow ((P1\ x \vee P2\ x) \longleftrightarrow (P1' \vee P2'))$
$\langle proof \rangle$

lemma *minf-ex*: **assumes** *ex*:$\exists z. \forall x. x < z \longrightarrow (P\ x \longleftrightarrow P1)$ **and** *p1*: $P1$ **shows** $\exists\ x. P\ x$
$\langle proof \rangle$

end

class *constr-dense-linear-order* = *linorder-no-lb* + *linorder-no-ub* +
  fixes *between*
  assumes *between-less*: $x < y \Longrightarrow x < between\ x\ y \wedge between\ x\ y < y$
    and *between-same*: $between\ x\ x = x$
begin

subclass *dense-linear-order*
  $\langle proof \rangle$

lemma *rinf-U*:
  assumes *fU*: *finite U*
  and *lin-dense*: $\forall x\ l\ u. (\forall\ t. l < t \wedge t< u \longrightarrow t \notin U) \wedge l< x \wedge x < u \wedge P\ x \longrightarrow (\forall\ y. l < y \wedge y < u \longrightarrow P\ y\ )$
  and *nmpiU*: $\forall x. \neg\ MP \wedge \neg PP \wedge P\ x \longrightarrow (\exists\ u \in U. \exists\ u' \in U. u \leq x \wedge x \leq u')$
  and *nmi*: $\neg\ MP$ **and** *npi*: $\neg\ PP$ **and** *ex*: $\exists\ x.\ P\ x$
  shows $\exists\ u \in U. \exists\ u' \in U. P\ (between\ u\ u')$
$\langle proof \rangle$

theorem *fr-eq*:
  assumes *fU*: *finite U*
  and *lin-dense*: $\forall x\ l\ u. (\forall\ t. l < t \wedge t< u \longrightarrow t \notin U) \wedge l< x \wedge x < u \wedge P\ x \longrightarrow (\forall\ y. l < y \wedge y < u \longrightarrow P\ y\ )$
  and *nmibnd*: $\forall x. \neg\ MP \wedge P\ x \longrightarrow (\exists\ u \in U. u \leq x)$
  and *npibnd*: $\forall x. \neg PP \wedge P\ x \longrightarrow (\exists\ u \in U. x \leq u)$
  and *mi*: $\exists z. \forall x. x < z \longrightarrow (P\ x = MP)$ **and** *pi*: $\exists z. \forall x. z < x \longrightarrow (P\ x = PP)$
  shows $(\exists\ x. P\ x) \equiv (MP \vee PP \vee (\exists\ u \in U. \exists\ u' \in U. P\ (between\ u\ u')))$
  (**is** - $\equiv$ (- $\vee$ - $\vee$ *?F*) **is** *?E* $\equiv$ *?D*)
$\langle proof \rangle$

lemmas *minf-thms* = *minf-conj minf-disj minf-eq minf-neq minf-lt minf-le minf-gt minf-ge minf-P*
lemmas *pinf-thms* = *pinf-conj pinf-disj pinf-eq pinf-neq pinf-lt pinf-le pinf-gt pinf-ge pinf-P*

lemmas *nmi-thms* = *nmi-conj nmi-disj nmi-eq nmi-neq nmi-lt nmi-le nmi-gt nmi-ge nmi-P*
lemmas *npi-thms* = *npi-conj npi-disj npi-eq npi-neq npi-lt npi-le npi-gt npi-ge npi-P*
lemmas *lin-dense-thms* = *lin-dense-conj lin-dense-disj lin-dense-eq lin-dense-neq lin-dense-lt lin-dense-le lin-dense-gt lin-dense-ge lin-dense-P*

lemma *ferrack-axiom*: *constr-dense-linear-order less-eq less between* ⟨*proof*⟩
lemma *atoms*:
  includes *meta-term-syntax*
  shows *TERM* (*less* :: ′*a* ⇒ -)
    and *TERM* (*less-eq* :: ′*a* ⇒ -)
    and *TERM* (*op* = :: ′*a* ⇒ -) ⟨*proof*⟩

declare *ferrack-axiom* [*ferrack minf*: *minf-thms pinf*: *pinf-thms*
    nmi: *nmi-thms npi*: *npi-thms lindense*:
    lin-dense-thms qe: *fr-eq atoms*: *atoms*]

⟨*ML*⟩

end

⟨*ML*⟩

end

38 Arith-Tools: Setup of arithmetic tools

theory *Arith-Tools*
imports *Groebner-Basis Dense-Linear-Order*
uses
  ~~/src/Provers/Arith/cancel-numeral-factor.ML
  ~~/src/Provers/Arith/extract-common-term.ML
  int-factor-simprocs.ML
  nat-simprocs.ML
begin

38.1 Simprocs for the Naturals

⟨*ML*⟩

38.1.1 For simplifying *Suc m − K* and *K − Suc m*

Where K above is a literal

lemma *Suc-diff-eq-diff-pred*: *Numeral0 < n ==> Suc m − n = m − (n − Numeral1)*
⟨*proof*⟩

Now just instantiating *n* to *number-of v* does the right simplification, but with some redundant inequality tests.

lemma *neg-number-of-pred-iff-0*:
neg (number-of (Numeral.pred v)::int) = (number-of v = (0::nat))
⟨*proof*⟩

No longer required as a simprule because of the *inverse-fold* simproc

lemma *Suc-diff-number-of*:
neg (number-of (uminus v)::int) ==>
Suc m − (number-of v) = m − (number-of (Numeral.pred v))
⟨*proof*⟩

lemma *diff-Suc-eq-diff-pred*: *m − Suc n = (m − 1) − n*
⟨*proof*⟩

38.1.2 For *nat-case* and *nat-rec*

lemma *nat-case-number-of* [*simp*]:
nat-case a f (number-of v) =
(let pv = number-of (Numeral.pred v) in
if neg pv then a else f (nat pv))
⟨*proof*⟩

lemma *nat-case-add-eq-if* [*simp*]:
nat-case a f ((number-of v) + n) =
(let pv = number-of (Numeral.pred v) in
if neg pv then nat-case a f n else f (nat pv + n))
⟨*proof*⟩

lemma *nat-rec-number-of* [*simp*]:
nat-rec a f (number-of v) =
(let pv = number-of (Numeral.pred v) in
if neg pv then a else f (nat pv) (nat-rec a f (nat pv)))
⟨*proof*⟩

lemma *nat-rec-add-eq-if* [*simp*]:
nat-rec a f (number-of v + n) =
(let pv = number-of (Numeral.pred v) in
if neg pv then nat-rec a f n
else f (nat pv + n) (nat-rec a f (nat pv + n)))
⟨*proof*⟩

38.1.3 Various Other Lemmas

Evens and Odds, for Mutilated Chess Board

Lemmas for specialist use, NOT as default simprules

lemma *nat-mult-2*: $2 * z = (z+z::nat)$
⟨*proof*⟩

lemma *nat-mult-2-right*: $z * 2 = (z+z::nat)$
⟨*proof*⟩

Case analysis on $n < (2::'a)$

lemma *less-2-cases*: $(n::nat) < 2 ==> n = 0 \mid n = Suc\ 0$
⟨*proof*⟩

lemma *div2-Suc-Suc* [*simp*]: $Suc(Suc\ m)\ div\ 2 = Suc\ (m\ div\ 2)$
⟨*proof*⟩

lemma *add-self-div-2* [*simp*]: $(m + m)\ div\ 2 = (m::nat)$
⟨*proof*⟩

lemma *mod2-Suc-Suc* [*simp*]: $Suc(Suc(m))\ mod\ 2 = m\ mod\ 2$
⟨*proof*⟩

lemma *mod2-gr-0* [*simp*]: $!!m::nat.\ (0 < m\ mod\ 2) = (m\ mod\ 2 = 1)$
⟨*proof*⟩

Removal of Small Numerals: 0, 1 and (in additive positions) 2

lemma *add-2-eq-Suc* [*simp*]: $2 + n = Suc\ (Suc\ n)$
⟨*proof*⟩

lemma *add-2-eq-Suc'* [*simp*]: $n + 2 = Suc\ (Suc\ n)$
⟨*proof*⟩

Can be used to eliminate long strings of Sucs, but not by default

lemma *Suc3-eq-add-3*: $Suc\ (Suc\ (Suc\ n)) = 3 + n$
⟨*proof*⟩

These lemmas collapse some needless occurrences of Suc: at least three Sucs, since two and fewer are rewritten back to Suc again! We already have some rules to simplify operands smaller than 3.

lemma *div-Suc-eq-div-add3* [*simp*]: $m\ div\ (Suc\ (Suc\ (Suc\ n))) = m\ div\ (3+n)$
⟨*proof*⟩

lemma *mod-Suc-eq-mod-add3* [*simp*]: $m\ mod\ (Suc\ (Suc\ (Suc\ n))) = m\ mod\ (3+n)$
⟨*proof*⟩

lemma *Suc-div-eq-add3-div*: $(Suc\ (Suc\ (Suc\ m)))\ div\ n = (3+m)\ div\ n$

⟨*proof*⟩

lemma *Suc-mod-eq-add3-mod*: (*Suc* (*Suc* (*Suc m*))) *mod n* = (*3*+*m*) *mod n*
⟨*proof*⟩

lemmas *Suc-div-eq-add3-div-number-of* =
Suc-div-eq-add3-div [*of - number-of v*, *standard*]
declare *Suc-div-eq-add3-div-number-of* [*simp*]

lemmas *Suc-mod-eq-add3-mod-number-of* =
Suc-mod-eq-add3-mod [*of - number-of v*, *standard*]
declare *Suc-mod-eq-add3-mod-number-of* [*simp*]

38.1.4 Special Simplification for Constants

These belong here, late in the development of HOL, to prevent their interfering with proofs of abstract properties of instances of the function *number-of*

These distributive laws move literals inside sums and differences.

lemmas *left-distrib-number-of* = *left-distrib* [*of - - number-of v*, *standard*]
declare *left-distrib-number-of* [*simp*]

lemmas *right-distrib-number-of* = *right-distrib* [*of number-of v*, *standard*]
declare *right-distrib-number-of* [*simp*]

lemmas *left-diff-distrib-number-of* =
left-diff-distrib [*of - - number-of v*, *standard*]
declare *left-diff-distrib-number-of* [*simp*]

lemmas *right-diff-distrib-number-of* =
right-diff-distrib [*of number-of v*, *standard*]
declare *right-diff-distrib-number-of* [*simp*]

These are actually for fields, like real: but where else to put them?

lemmas *zero-less-divide-iff-number-of* =
zero-less-divide-iff [*of number-of w*, *standard*]
declare *zero-less-divide-iff-number-of* [*simp*,*noatp*]

lemmas *divide-less-0-iff-number-of* =
divide-less-0-iff [*of number-of w*, *standard*]
declare *divide-less-0-iff-number-of* [*simp*,*noatp*]

lemmas *zero-le-divide-iff-number-of* =
zero-le-divide-iff [*of number-of w*, *standard*]
declare *zero-le-divide-iff-number-of* [*simp*,*noatp*]

lemmas *divide-le-0-iff-number-of* =
divide-le-0-iff [*of number-of w*, *standard*]

declare *divide-le-0-iff-number-of* [*simp*,*noatp*]

Replaces *inverse #nn* by *1/#nn*. It looks strange, but then other simprocs simplify the quotient.

lemmas *inverse-eq-divide-number-of* =
  inverse-eq-divide [*of number-of w*, *standard*]
declare *inverse-eq-divide-number-of* [*simp*]

These laws simplify inequalities, moving unary minus from a term into the literal.

lemmas *less-minus-iff-number-of* =
  less-minus-iff [*of number-of v*, *standard*]
declare *less-minus-iff-number-of* [*simp*,*noatp*]

lemmas *le-minus-iff-number-of* =
  le-minus-iff [*of number-of v*, *standard*]
declare *le-minus-iff-number-of* [*simp*,*noatp*]

lemmas *equation-minus-iff-number-of* =
  equation-minus-iff [*of number-of v*, *standard*]
declare *equation-minus-iff-number-of* [*simp*,*noatp*]

lemmas *minus-less-iff-number-of* =
  minus-less-iff [*of - number-of v*, *standard*]
declare *minus-less-iff-number-of* [*simp*,*noatp*]

lemmas *minus-le-iff-number-of* =
  minus-le-iff [*of - number-of v*, *standard*]
declare *minus-le-iff-number-of* [*simp*,*noatp*]

lemmas *minus-equation-iff-number-of* =
  minus-equation-iff [*of - number-of v*, *standard*]
declare *minus-equation-iff-number-of* [*simp*,*noatp*]

To Simplify Inequalities Where One Side is the Constant 1

lemma *less-minus-iff-1* [*simp*,*noatp*]:
  fixes b::$'b$::{*ordered-idom*,*number-ring*}
  shows $(1 < - b) = (b < -1)$
⟨*proof*⟩

lemma *le-minus-iff-1* [*simp*,*noatp*]:
  fixes b::$'b$::{*ordered-idom*,*number-ring*}
  shows $(1 \le - b) = (b \le -1)$
⟨*proof*⟩

lemma *equation-minus-iff-1* [*simp*,*noatp*]:
  fixes b::$'b$::*number-ring*

shows $(1 = - b) = (b = -1)$
⟨*proof*⟩

lemma *minus-less-iff-1* [*simp*,*noatp*]:
fixes $a::'b::\{ordered\text{-}idom,number\text{-}ring\}$
shows $(- a < 1) = (-1 < a)$
⟨*proof*⟩

lemma *minus-le-iff-1* [*simp*,*noatp*]:
fixes $a::'b::\{ordered\text{-}idom,number\text{-}ring\}$
shows $(- a \leq 1) = (-1 \leq a)$
⟨*proof*⟩

lemma *minus-equation-iff-1* [*simp*,*noatp*]:
fixes $a::'b::number\text{-}ring$
shows $(- a = 1) = (a = -1)$
⟨*proof*⟩

Cancellation of constant factors in comparisons ($<$ and $\leq$)

lemmas *mult-less-cancel-left-number-of* =
mult-less-cancel-left [*of number-of v*, *standard*]
declare *mult-less-cancel-left-number-of* [*simp*,*noatp*]

lemmas *mult-less-cancel-right-number-of* =
mult-less-cancel-right [*of - number-of v*, *standard*]
declare *mult-less-cancel-right-number-of* [*simp*,*noatp*]

lemmas *mult-le-cancel-left-number-of* =
mult-le-cancel-left [*of number-of v*, *standard*]
declare *mult-le-cancel-left-number-of* [*simp*,*noatp*]

lemmas *mult-le-cancel-right-number-of* =
mult-le-cancel-right [*of - number-of v*, *standard*]
declare *mult-le-cancel-right-number-of* [*simp*,*noatp*]

Multiplying out constant divisors in comparisons ($<$, $\leq$ and $=$)

lemmas *le-divide-eq-number-of* = *le-divide-eq* [*of - - number-of w*, *standard*]
declare *le-divide-eq-number-of* [*simp*]

lemmas *divide-le-eq-number-of* = *divide-le-eq* [*of - number-of w*, *standard*]
declare *divide-le-eq-number-of* [*simp*]

lemmas *less-divide-eq-number-of* = *less-divide-eq* [*of - - number-of w*, *standard*]
declare *less-divide-eq-number-of* [*simp*]

lemmas *divide-less-eq-number-of* = *divide-less-eq* [*of - number-of w*, *standard*]
declare *divide-less-eq-number-of* [*simp*]

lemmas *eq-divide-eq-number-of* = *eq-divide-eq* [*of - - number-of w*, *standard*]

declare *eq-divide-eq-number-of* [*simp*]

lemmas *divide-eq-eq-number-of* = *divide-eq-eq* [*of - number-of w*, *standard*]
declare *divide-eq-eq-number-of* [*simp*]

38.1.5 Optional Simplification Rules Involving Constants

Simplify quotients that are compared with a literal constant.

lemmas *le-divide-eq-number-of* = *le-divide-eq* [*of number-of w*, *standard*]
lemmas *divide-le-eq-number-of* = *divide-le-eq* [*of - - number-of w*, *standard*]
lemmas *less-divide-eq-number-of* = *less-divide-eq* [*of number-of w*, *standard*]
lemmas *divide-less-eq-number-of* = *divide-less-eq* [*of - - number-of w*, *standard*]
lemmas *eq-divide-eq-number-of* = *eq-divide-eq* [*of number-of w*, *standard*]
lemmas *divide-eq-eq-number-of* = *divide-eq-eq* [*of - - number-of w*, *standard*]

Not good as automatic simprules because they cause case splits.

lemmas *divide-const-simps* =
le-divide-eq-number-of divide-le-eq-number-of less-divide-eq-number-of
divide-less-eq-number-of eq-divide-eq-number-of divide-eq-eq-number-of
le-divide-eq-1 divide-le-eq-1 less-divide-eq-1 divide-less-eq-1

Division By -1

lemma *divide-minus1* [*simp*]:
$x/-1 = -(x::'a::\{field,division\text{-}by\text{-}zero,number\text{-}ring\})$
⟨*proof*⟩

lemma *minus1-divide* [*simp*]:
$-1 \;/\; (x::'a::\{field,division\text{-}by\text{-}zero,number\text{-}ring\}) = -\ (1/x)$
⟨*proof*⟩

lemma *half-gt-zero-iff*:
$(0 < r/2) = (0 < (r::'a::\{ordered\text{-}field,division\text{-}by\text{-}zero,number\text{-}ring\}))$
⟨*proof*⟩

lemmas *half-gt-zero* = *half-gt-zero-iff* [*THEN iffD2*, *standard*]
declare *half-gt-zero* [*simp*]

lemma *nat-dvd-not-less*:
$[|\ 0 < m;\ m < n\ |] ==> \neg\ n\ dvd\ (m::nat)$
⟨*proof*⟩

⟨*ML*⟩

38.2 Groebner Bases for fields

interpretation *class-fieldgb*:

fieldgb[*op* + *op* $*$ *op* ^ *0*::′*a*::{*field*,*recpower*,*number-ring*} *1* *op* − *uminus* *op* / *inverse*] ⟨*proof*⟩

lemma *divide-Numeral1*: (*x*::′*a*::{*field*,*number-ring*}) / *Numeral1* = *x* ⟨*proof*⟩
lemma *divide-Numeral0*: (*x*::′*a*::{*field*,*number-ring*, *division-by-zero*}) / *Numeral0* = *0*
⟨*proof*⟩
lemma *mult-frac-frac*: ((*x*::′*a*::{*field*,*division-by-zero*}) / *y*) $*$ (*z* / *w*) = (*x*$*$*z*) / (*y*$*$*w*)
⟨*proof*⟩
lemma *mult-frac-num*: ((*x*::′*a*::{*field*, *division-by-zero*}) / *y*) $*$ *z* = (*x*$*$*z*) / *y*
⟨*proof*⟩
lemma *mult-num-frac*: ((*x*::′*a*::{*field*, *division-by-zero*}) / *y*) $*$ *z* = (*x*$*$*z*) / *y*
⟨*proof*⟩

lemma *Numeral1-eq1-nat*: (*1*::*nat*) = *Numeral1* ⟨*proof*⟩

lemma *add-frac-num*: *y*≠ *0* ⟹ (*x*::′*a*::{*field*, *division-by-zero*}) / *y* + *z* = (*x* + *z*$*$*y*) / *y*
⟨*proof*⟩
lemma *add-num-frac*: *y*≠ *0* ⟹ *z* + (*x*::′*a*::{*field*, *division-by-zero*}) / *y* = (*x* + *z*$*$*y*) / *y*
⟨*proof*⟩

⟨*ML*⟩

38.3 Ferrante and Rackoff algorithm over ordered fields

lemma *neg-prod-lt*:(*c*::′*a*::*ordered-field*) < *0* ⟹ ((*c*$*$*x* < *0*) == (*x* > *0*))
⟨*proof*⟩

lemma *pos-prod-lt*:(*c*::′*a*::*ordered-field*) > *0* ⟹ ((*c*$*$*x* < *0*) == (*x* < *0*))
⟨*proof*⟩

lemma *neg-prod-sum-lt*: (*c*::′*a*::*ordered-field*) < *0* ⟹ ((*c*$*$*x* + *t*< *0*) == (*x* > (− *1*/*c*)$*$*t*))
⟨*proof*⟩

lemma *pos-prod-sum-lt*:(*c*::′*a*::*ordered-field*) > *0* ⟹ ((*c*$*$*x* + *t* < *0*) == (*x* < (− *1*/*c*)$*$*t*))
⟨*proof*⟩

lemma *sum-lt*:((*x*::′*a*::*pordered-ab-group-add*) + *t* < *0*) == (*x* < − *t*)
⟨*proof*⟩

lemma *neg-prod-le*:(*c*::′*a*::*ordered-field*) < *0* ⟹ ((*c*$*$*x* <= *0*) == (*x* >= *0*))
⟨*proof*⟩

lemma *pos-prod-le*:$(c::'a::ordered\text{-}field) > 0 \Longrightarrow ((c*x <= 0) == (x <= 0))$
⟨*proof*⟩

lemma *neg-prod-sum-le*: $(c::'a::ordered\text{-}field) < 0 \Longrightarrow ((c*x + t <= 0) == (x >= (- 1/c)*t))$
⟨*proof*⟩

lemma *pos-prod-sum-le*:$(c::'a::ordered\text{-}field) > 0 \Longrightarrow ((c*x + t <= 0) == (x <= (- 1/c)*t))$
⟨*proof*⟩

lemma *sum-le*:$((x::'a::pordered\text{-}ab\text{-}group\text{-}add) + t <= 0) == (x <= - t)$
⟨*proof*⟩

lemma *nz-prod-eq*:$(c::'a::ordered\text{-}field) \neq 0 \Longrightarrow ((c*x = 0) == (x = 0))$ ⟨*proof*⟩
lemma *nz-prod-sum-eq*: $(c::'a::ordered\text{-}field) \neq 0 \Longrightarrow ((c*x + t = 0) == (x = (- 1/c)*t))$
⟨*proof*⟩
lemma *sum-eq*:$((x::'a::pordered\text{-}ab\text{-}group\text{-}add) + t = 0) == (x = - t)$
⟨*proof*⟩

interpretation *class-ordered-field-dense-linear-order*: *constr-dense-linear-order*
[*op* <= *op* <
λ *x y*. $1/2 * ((x::'a::\{ordered\text{-}field,recpower,number\text{-}ring\}) + y)$]
⟨*proof*⟩

⟨*ML*⟩

end

39 SetInterval: Set intervals

theory *SetInterval*
imports *IntArith*
begin

context *ord*
begin
definition
lessThan :: *'a => 'a set ((1{..<-}))* **where**
$\{..<u\} == \{x.\ x < u\}$

definition
atMost :: *'a => 'a set ((1{..-}))* **where**
$\{..u\} == \{x.\ x \leq u\}$

definition

greaterThan :: *'a* => *'a set* ((*1*{-<..})) **where**
{*l*<..} == {*x*. *l*<*x*}

definition
atLeast :: *'a* => *'a set* ((*1*{-..})) **where**
{*l*..} == {*x*. *l*≤*x*}

definition
greaterThanLessThan :: *'a* => *'a* => *'a set* ((*1*{-<..<-})) **where**
{*l*<..<*u*} == {*l*<..} *Int* {..<*u*}

definition
atLeastLessThan :: *'a* => *'a* => *'a set* ((*1*{-..<-})) **where**
{*l*..<*u*} == {*l*..} *Int* {..<*u*}

definition
greaterThanAtMost :: *'a* => *'a* => *'a set* ((*1*{-<..-})) **where**
{*l*<..*u*} == {*l*<..} *Int* {..*u*}

definition
atLeastAtMost :: *'a* => *'a* => *'a set* ((*1*{-..-})) **where**
{*l*..*u*} == {*l*..} *Int* {..*u*}

end

A note of warning when using {..<n} on type *nat*: it is equivalent to {*0*..<n} but some lemmas involving {m..<n} may not exist in {..<n}-form as well.

syntax
@*UNION-le* :: *nat* => *nat* => *'b set* => *'b set* ((*3UN* -<=-./ -) *10*)
@*UNION-less* :: *nat* => *nat* => *'b set* => *'b set* ((*3UN* -<-./ -) *10*)
@*INTER-le* :: *nat* => *nat* => *'b set* => *'b set* ((*3INT* -<=-./ -) *10*)
@*INTER-less* :: *nat* => *nat* => *'b set* => *'b set* ((*3INT* -<-./ -) *10*)

syntax (*input*)
@*UNION-le* :: *nat* => *nat* => *'b set* => *'b set* ((*3*⋃ -≤-./ -) *10*)
@*UNION-less* :: *nat* => *nat* => *'b set* => *'b set* ((*3*⋃ -<-./ -) *10*)
@*INTER-le* :: *nat* => *nat* => *'b set* => *'b set* ((*3*⋂ -≤-./ -) *10*)
@*INTER-less* :: *nat* => *nat* => *'b set* => *'b set* ((*3*⋂ -<-./ -) *10*)

syntax (*xsymbols*)
@*UNION-le* :: *nat* ⇒ *nat* => *'b set* => *'b set* ((*3*⋃(*00*$_{- \le -}$)/ -) *10*)
@*UNION-less* :: *nat* ⇒ *nat* => *'b set* => *'b set* ((*3*⋃(*00*$_{- < -}$)/ -) *10*)
@*INTER-le* :: *nat* ⇒ *nat* => *'b set* => *'b set* ((*3*⋂(*00*$_{- \le -}$)/ -) *10*)
@*INTER-less* :: *nat* ⇒ *nat* => *'b set* => *'b set* ((*3*⋂(*00*$_{- < -}$)/ -) *10*)

translations
UN i<=*n*. *A* == *UN i*:{..*n*}. *A*
UN i<*n*. *A* == *UN i*:{..<*n*}. *A*
INT i<=*n*. *A* == *INT i*:{..*n*}. *A*

$INT\ i{<}n.\ A\ \ == INT\ i{:}\{..{<}n\}.\ A$

39.1 Various equivalences

lemma (**in** *ord*) *lessThan-iff* [*iff*]: $(i{:}\ lessThan\ k) = (i{<}k)$
⟨*proof*⟩

lemma *Compl-lessThan* [*simp*]:
$!!k{::}\ 'a{::}linorder.\ -lessThan\ k = atLeast\ k$
⟨*proof*⟩

lemma *single-Diff-lessThan* [*simp*]: $!!k{::}\ 'a{::}order.\ \{k\} - lessThan\ k = \{k\}$
⟨*proof*⟩

lemma (**in** *ord*) *greaterThan-iff* [*iff*]: $(i{:}\ greaterThan\ k) = (k{<}i)$
⟨*proof*⟩

lemma *Compl-greaterThan* [*simp*]:
$!!k{::}\ 'a{::}linorder.\ -greaterThan\ k = atMost\ k$
⟨*proof*⟩

lemma *Compl-atMost* [*simp*]: $!!k{::}\ 'a{::}linorder.\ -atMost\ k = greaterThan\ k$
⟨*proof*⟩

lemma (**in** *ord*) *atLeast-iff* [*iff*]: $(i{:}\ atLeast\ k) = (k{<=}i)$
⟨*proof*⟩

lemma *Compl-atLeast* [*simp*]:
$!!k{::}\ 'a{::}linorder.\ -atLeast\ k = lessThan\ k$
⟨*proof*⟩

lemma (**in** *ord*) *atMost-iff* [*iff*]: $(i{:}\ atMost\ k) = (i{<=}k)$
⟨*proof*⟩

lemma *atMost-Int-atLeast*: $!!n{::}\ 'a{::}order.\ atMost\ n\ Int\ atLeast\ n = \{n\}$
⟨*proof*⟩

39.2 Logical Equivalences for Set Inclusion and Equality

lemma *atLeast-subset-iff* [*iff*]:
$(atLeast\ x \subseteq atLeast\ y) = (y \le (x{::}'a{::}order))$
⟨*proof*⟩

lemma *atLeast-eq-iff* [*iff*]:
$(atLeast\ x = atLeast\ y) = (x = (y{::}'a{::}linorder))$
⟨*proof*⟩

lemma *greaterThan-subset-iff* [*iff*]:
$(greaterThan\ x \subseteq greaterThan\ y) = (y \le (x{::}'a{::}linorder))$
⟨*proof*⟩

lemma *greaterThan-eq-iff* [*iff*]:
$(greaterThan\ x = greaterThan\ y) = (x = (y::'a::linorder))$
⟨*proof*⟩

lemma *atMost-subset-iff* [*iff*]: $(atMost\ x \subseteq atMost\ y) = (x \leq (y::'a::order))$
⟨*proof*⟩

lemma *atMost-eq-iff* [*iff*]: $(atMost\ x = atMost\ y) = (x = (y::'a::linorder))$
⟨*proof*⟩

lemma *lessThan-subset-iff* [*iff*]:
$(lessThan\ x \subseteq lessThan\ y) = (x \leq (y::'a::linorder))$
⟨*proof*⟩

lemma *lessThan-eq-iff* [*iff*]:
$(lessThan\ x = lessThan\ y) = (x = (y::'a::linorder))$
⟨*proof*⟩

39.3 Two-sided intervals

context *ord*
begin

lemma *greaterThanLessThan-iff* [*simp*,*noatp*]:
$(i : \{l<..<u\}) = (l < i \ \&\ i < u)$
⟨*proof*⟩

lemma *atLeastLessThan-iff* [*simp*,*noatp*]:
$(i : \{l..<u\}) = (l <= i \ \&\ i < u)$
⟨*proof*⟩

lemma *greaterThanAtMost-iff* [*simp*,*noatp*]:
$(i : \{l<..u\}) = (l < i \ \&\ i <= u)$
⟨*proof*⟩

lemma *atLeastAtMost-iff* [*simp*,*noatp*]:
$(i : \{l..u\}) = (l <= i \ \&\ i <= u)$
⟨*proof*⟩

The above four lemmas could be declared as iffs. If we do so, a call to blast in Hyperreal/Star.ML, lemma *STAR-Int* seems to take forever (more than one hour).

end

39.3.1 Emptyness and singletons

context *order*
begin

lemma *atLeastAtMost-empty* [*simp*]: $n < m$ ==> $\{m..n\} = \{\}$
⟨*proof*⟩

lemma *atLeastLessThan-empty*[*simp*]: $n \le m$ ==> $\{m..<n\} = \{\}$
⟨*proof*⟩

lemma *greaterThanAtMost-empty*[*simp*]:$l \le k$ ==> $\{k<..l\} = \{\}$
⟨*proof*⟩

lemma *greaterThanLessThan-empty*[*simp*]:$l \le k$ ==> $\{k<..l\} = \{\}$
⟨*proof*⟩

lemma *atLeastAtMost-singleton* [*simp*]: $\{a..a\} = \{a\}$
⟨*proof*⟩

end

39.4 Intervals of natural numbers

39.4.1 The Constant *lessThan*

lemma *lessThan-0* [*simp*]: *lessThan* (*0*::*nat*) = {}
⟨*proof*⟩

lemma *lessThan-Suc*: *lessThan* (*Suc k*) = *insert k* (*lessThan k*)
⟨*proof*⟩

lemma *lessThan-Suc-atMost*: *lessThan* (*Suc k*) = *atMost k*
⟨*proof*⟩

lemma *UN-lessThan-UNIV*: (*UN m*::*nat*. *lessThan m*) = *UNIV*
⟨*proof*⟩

39.4.2 The Constant *greaterThan*

lemma *greaterThan-0* [*simp*]: *greaterThan 0* = *range Suc*
⟨*proof*⟩

lemma *greaterThan-Suc*: *greaterThan* (*Suc k*) = *greaterThan k* − {*Suc k*}
⟨*proof*⟩

lemma *INT-greaterThan-UNIV*: (*INT m*::*nat*. *greaterThan m*) = {}
⟨*proof*⟩

39.4.3 The Constant *atLeast*

lemma *atLeast-0* [*simp*]: *atLeast* (*0*::*nat*) = *UNIV*
⟨*proof*⟩

lemma *atLeast-Suc*: $atLeast\ (Suc\ k) = atLeast\ k - \{k\}$
⟨*proof*⟩

lemma *atLeast-Suc-greaterThan*: $atLeast\ (Suc\ k) = greaterThan\ k$
⟨*proof*⟩

lemma *UN-atLeast-UNIV*: $(UN\ m::nat.\ atLeast\ m) = UNIV$
⟨*proof*⟩

39.4.4 The Constant *atMost*

lemma *atMost-0* [*simp*]: $atMost\ (0::nat) = \{0\}$
⟨*proof*⟩

lemma *atMost-Suc*: $atMost\ (Suc\ k) = insert\ (Suc\ k)\ (atMost\ k)$
⟨*proof*⟩

lemma *UN-atMost-UNIV*: $(UN\ m::nat.\ atMost\ m) = UNIV$
⟨*proof*⟩

39.4.5 The Constant *atLeastLessThan*

The orientation of the following rule is tricky. The lhs is defined in terms of the rhs. Hence the chosen orientation makes sense in this theory — the reverse orientation complicates proofs (eg nontermination). But outside, when the definition of the lhs is rarely used, the opposite orientation seems preferable because it reduces a specific concept to a more general one.

lemma *atLeast0LessThan*: $\{0::nat..{<}n\} = \{..{<}n\}$
⟨*proof*⟩

declare *atLeast0LessThan*[*symmetric*, *code unfold*]

lemma *atLeastLessThan0*: $\{m..{<}0::nat\} = \{\}$
⟨*proof*⟩

39.4.6 Intervals of nats with *Suc*

Not a simprule because the RHS is too messy.

lemma *atLeastLessThanSuc*:
$\{m..{<}Suc\ n\} = (if\ m \le n\ then\ insert\ n\ \{m..{<}n\}\ else\ \{\})$
⟨*proof*⟩

lemma *atLeastLessThan-singleton* [*simp*]: $\{m..{<}Suc\ m\} = \{m\}$
⟨*proof*⟩

lemma *atLeastLessThanSuc-atLeastAtMost*: $\{l..{<}Suc\ u\} = \{l..u\}$
⟨*proof*⟩

lemma *atLeastSucAtMost-greaterThanAtMost*: $\{Suc\ l..u\} = \{l{<}..u\}$
⟨*proof*⟩

lemma *atLeastSucLessThan-greaterThanLessThan*: $\{Suc\ l..{<}u\} = \{l{<}..{<}u\}$
⟨*proof*⟩

lemma *atLeastAtMostSuc-conv*: $m \leq Suc\ n \Longrightarrow \{m..Suc\ n\} = insert\ (Suc\ n)\ \{m..n\}$
⟨*proof*⟩

39.4.7 Image

lemma *image-add-atLeastAtMost*:
$(\%n{::}nat.\ n{+}k)$ ` $\{i..j\} = \{i{+}k..j{+}k\}$ (**is** $?A = ?B$)
⟨*proof*⟩

lemma *image-add-atLeastLessThan*:
$(\%n{::}nat.\ n{+}k)$ ` $\{i..{<}j\} = \{i{+}k..{<}j{+}k\}$ (**is** $?A = ?B$)
⟨*proof*⟩

corollary *image-Suc-atLeastAtMost*[*simp*]:
Suc ` $\{i..j\} = \{Suc\ i..Suc\ j\}$
⟨*proof*⟩

corollary *image-Suc-atLeastLessThan*[*simp*]:
Suc ` $\{i..{<}j\} = \{Suc\ i..{<}Suc\ j\}$
⟨*proof*⟩

lemma *image-add-int-atLeastLessThan*:
$(\%x.\ x + (l{::}int))$ ` $\{0..{<}u{-}l\} = \{l..{<}u\}$
⟨*proof*⟩

39.4.8 Finiteness

lemma *finite-lessThan* [*iff*]: **fixes** $k :: nat$ **shows** *finite* $\{..{<}k\}$
⟨*proof*⟩

lemma *finite-atMost* [*iff*]: **fixes** $k :: nat$ **shows** *finite* $\{..k\}$
⟨*proof*⟩

lemma *finite-greaterThanLessThan* [*iff*]:
fixes $l :: nat$ **shows** *finite* $\{l{<}..{<}u\}$
⟨*proof*⟩

lemma *finite-atLeastLessThan* [*iff*]:
fixes $l :: nat$ **shows** *finite* $\{l..{<}u\}$
⟨*proof*⟩

lemma *finite-greaterThanAtMost* [*iff*]:
fixes $l :: nat$ **shows** *finite* $\{l{<}..u\}$

⟨*proof*⟩

lemma *finite-atLeastAtMost* [*iff*]:
fixes $l :: nat$ **shows** *finite* $\{l..u\}$
⟨*proof*⟩

lemma *bounded-nat-set-is-finite*:
$(ALL\ i{:}N.\ i < (n{::}nat))\ ==>\ finite\ N$
— A bounded set of natural numbers is finite.
⟨*proof*⟩

Any subset of an interval of natural numbers the size of the subset is exactly that interval.

lemma *subset-card-intvl-is-intvl*:
$A <= \{k..<k+card\ A\} \Longrightarrow A = \{k..<k+card\ A\}$ (**is** *PROP ?P*)
⟨*proof*⟩

39.4.9 Cardinality

lemma *card-lessThan* [*simp*]: $card\ \{..<u\} = u$
⟨*proof*⟩

lemma *card-atMost* [*simp*]: $card\ \{..u\} = Suc\ u$
⟨*proof*⟩

lemma *card-atLeastLessThan* [*simp*]: $card\ \{l..<u\} = u - l$
⟨*proof*⟩

lemma *card-atLeastAtMost* [*simp*]: $card\ \{l..u\} = Suc\ u - l$
⟨*proof*⟩

lemma *card-greaterThanAtMost* [*simp*]: $card\ \{l<..u\} = u - l$
⟨*proof*⟩

lemma *card-greaterThanLessThan* [*simp*]: $card\ \{l<..<u\} = u - Suc\ l$
⟨*proof*⟩

39.5 Intervals of integers

lemma *atLeastLessThanPlusOne-atLeastAtMost-int*: $\{l..<u+1\} = \{l..(u{::}int)\}$
⟨*proof*⟩

lemma *atLeastPlusOneAtMost-greaterThanAtMost-int*: $\{l+1..u\} = \{l<..(u{::}int)\}$
⟨*proof*⟩

lemma *atLeastPlusOneLessThan-greaterThanLessThan-int*:
$\{l+1..<u\} = \{l<..<u{::}int\}$
⟨*proof*⟩

39.5.1 Finiteness

lemma *image-atLeastZeroLessThan-int*: $0 \le u ==>$
$\{(0::int)..<u\} = int\ `\ \{..<nat\ u\}$
⟨*proof*⟩

lemma *finite-atLeastZeroLessThan-int*: $finite\ \{(0::int)..<u\}$
⟨*proof*⟩

lemma *finite-atLeastLessThan-int* [*iff*]: $finite\ \{l..<u::int\}$
⟨*proof*⟩

lemma *finite-atLeastAtMost-int* [*iff*]: $finite\ \{l..(u::int)\}$
⟨*proof*⟩

lemma *finite-greaterThanAtMost-int* [*iff*]: $finite\ \{l<..(u::int)\}$
⟨*proof*⟩

lemma *finite-greaterThanLessThan-int* [*iff*]: $finite\ \{l<..<u::int\}$
⟨*proof*⟩

39.5.2 Cardinality

lemma *card-atLeastZeroLessThan-int*: $card\ \{(0::int)..<u\} = nat\ u$
⟨*proof*⟩

lemma *card-atLeastLessThan-int* [*simp*]: $card\ \{l..<u\} = nat\ (u - l)$
⟨*proof*⟩

lemma *card-atLeastAtMost-int* [*simp*]: $card\ \{l..u\} = nat\ (u - l + 1)$
⟨*proof*⟩

lemma *card-greaterThanAtMost-int* [*simp*]: $card\ \{l<..u\} = nat\ (u - l)$
⟨*proof*⟩

lemma *card-greaterThanLessThan-int* [*simp*]: $card\ \{l<..<u\} = nat\ (u - (l + 1))$
⟨*proof*⟩

39.6 Lemmas useful with the summation operator setsum

For examples, see Algebra/poly/UnivPoly2.thy

39.6.1 Disjoint Unions

Singletons and open intervals

lemma *ivl-disj-un-singleton*:
$\{l::'a::linorder\}\ Un\ \{l<..\} = \{l..\}$
$\{..<u\}\ Un\ \{u::'a::linorder\} = \{..u\}$
$(l::'a::linorder) < u ==> \{l\}\ Un\ \{l<..<u\} = \{l..<u\}$

$(l::'a::linorder) < u ==> \{l<..<u\}\ \mathit{Un}\ \{u\} = \{l<..u\}$
$(l::'a::linorder) <= u ==> \{l\}\ \mathit{Un}\ \{l<..u\} = \{l..u\}$
$(l::'a::linorder) <= u ==> \{l..<u\}\ \mathit{Un}\ \{u\} = \{l..u\}$
⟨*proof*⟩

One- and two-sided intervals

lemma *ivl-disj-un-one*:
$(l::'a::linorder) < u ==> \{..l\}\ \mathit{Un}\ \{l<..<u\} = \{..<u\}$
$(l::'a::linorder) <= u ==> \{..<l\}\ \mathit{Un}\ \{l..<u\} = \{..<u\}$
$(l::'a::linorder) <= u ==> \{..l\}\ \mathit{Un}\ \{l<..u\} = \{..u\}$
$(l::'a::linorder) <= u ==> \{..<l\}\ \mathit{Un}\ \{l..u\} = \{..u\}$
$(l::'a::linorder) <= u ==> \{l<..u\}\ \mathit{Un}\ \{u<..\} = \{l<..\}$
$(l::'a::linorder) < u ==> \{l<..<u\}\ \mathit{Un}\ \{u..\} = \{l<..\}$
$(l::'a::linorder) <= u ==> \{l..u\}\ \mathit{Un}\ \{u<..\} = \{l..\}$
$(l::'a::linorder) <= u ==> \{l..<u\}\ \mathit{Un}\ \{u..\} = \{l..\}$
⟨*proof*⟩

Two- and two-sided intervals

lemma *ivl-disj-un-two*:
$[|\ (l::'a::linorder) < m;\ m <= u\ |] ==> \{l<..<m\}\ \mathit{Un}\ \{m..<u\} = \{l<..<u\}$
$[|\ (l::'a::linorder) <= m;\ m < u\ |] ==> \{l<..m\}\ \mathit{Un}\ \{m<..<u\} = \{l<..<u\}$
$[|\ (l::'a::linorder) <= m;\ m <= u\ |] ==> \{l..<m\}\ \mathit{Un}\ \{m..<u\} = \{l..<u\}$
$[|\ (l::'a::linorder) <= m;\ m < u\ |] ==> \{l..m\}\ \mathit{Un}\ \{m<..<u\} = \{l..<u\}$
$[|\ (l::'a::linorder) < m;\ m <= u\ |] ==> \{l<..<m\}\ \mathit{Un}\ \{m..u\} = \{l<..u\}$
$[|\ (l::'a::linorder) <= m;\ m <= u\ |] ==> \{l<..m\}\ \mathit{Un}\ \{m<..u\} = \{l<..u\}$
$[|\ (l::'a::linorder) <= m;\ m <= u\ |] ==> \{l..<m\}\ \mathit{Un}\ \{m..u\} = \{l..u\}$
$[|\ (l::'a::linorder) <= m;\ m <= u\ |] ==> \{l..m\}\ \mathit{Un}\ \{m<..u\} = \{l..u\}$
⟨*proof*⟩

lemmas *ivl-disj-un* = *ivl-disj-un-singleton ivl-disj-un-one ivl-disj-un-two*

39.6.2 Disjoint Intersections

Singletons and open intervals

lemma *ivl-disj-int-singleton*:
$\{l::'a::order\}\ \mathit{Int}\ \{l<..\} = \{\}$
$\{..<u\}\ \mathit{Int}\ \{u\} = \{\}$
$\{l\}\ \mathit{Int}\ \{l<..<u\} = \{\}$
$\{l<..<u\}\ \mathit{Int}\ \{u\} = \{\}$
$\{l\}\ \mathit{Int}\ \{l<..u\} = \{\}$
$\{l..<u\}\ \mathit{Int}\ \{u\} = \{\}$
⟨*proof*⟩

One- and two-sided intervals

lemma *ivl-disj-int-one*:
$\{..l::'a::order\}\ \mathit{Int}\ \{l<..<u\} = \{\}$
$\{..<l\}\ \mathit{Int}\ \{l..<u\} = \{\}$
$\{..l\}\ \mathit{Int}\ \{l<..u\} = \{\}$

$\{..<l\}$ *Int* $\{l..u\} = \{\}$
$\{l<..u\}$ *Int* $\{u<..\} = \{\}$
$\{l<..<u\}$ *Int* $\{u..\} = \{\}$
$\{l..u\}$ *Int* $\{u<..\} = \{\}$
$\{l..<u\}$ *Int* $\{u..\} = \{\}$
⟨*proof*⟩

Two- and two-sided intervals

lemma *ivl-disj-int-two*:
$\{l::'a::order<..<m\}$ *Int* $\{m..<u\} = \{\}$
$\{l<..m\}$ *Int* $\{m<..<u\} = \{\}$
$\{l..<m\}$ *Int* $\{m..<u\} = \{\}$
$\{l..m\}$ *Int* $\{m<..<u\} = \{\}$
$\{l<..<m\}$ *Int* $\{m..u\} = \{\}$
$\{l<..m\}$ *Int* $\{m<..u\} = \{\}$
$\{l..<m\}$ *Int* $\{m..u\} = \{\}$
$\{l..m\}$ *Int* $\{m<..u\} = \{\}$
⟨*proof*⟩

lemmas *ivl-disj-int* = *ivl-disj-int-singleton ivl-disj-int-one ivl-disj-int-two*

39.6.3 Some Differences

lemma *ivl-diff* [*simp*]:
$i \le n \Longrightarrow \{i..<m\} - \{i..<n\} = \{n..<(m::'a::linorder)\}$
⟨*proof*⟩

39.6.4 Some Subset Conditions

lemma *ivl-subset* [*simp*,*noatp*]:
$(\{i..<j\} \subseteq \{m..<n\}) = (j \le i \mid m \le i \;\&\; j \le (n::'a::linorder))$
⟨*proof*⟩

39.7 Summation indexed over intervals

syntax
-from-to-setsum :: *idt* ⇒ *'a* ⇒ *'a* ⇒ *'b* ⇒ *'b* ((*SUM* - = -..-./ -) [*0*,*0*,*0*,*10*] *10*)
-from-upto-setsum :: *idt* ⇒ *'a* ⇒ *'a* ⇒ *'b* ⇒ *'b* ((*SUM* - = -..<-./ -) [*0*,*0*,*0*,*10*] *10*)
-upt-setsum :: *idt* ⇒ *'a* ⇒ *'b* ⇒ *'b* ((*SUM* -<-./ -) [*0*,*0*,*10*] *10*)
-upto-setsum :: *idt* ⇒ *'a* ⇒ *'b* ⇒ *'b* ((*SUM* -<=-./ -) [*0*,*0*,*10*] *10*)
syntax (*xsymbols*)
-from-to-setsum :: *idt* ⇒ *'a* ⇒ *'a* ⇒ *'b* ⇒ *'b* ((*3*∑- = -..-./ -) [*0*,*0*,*0*,*10*] *10*)
-from-upto-setsum :: *idt* ⇒ *'a* ⇒ *'a* ⇒ *'b* ⇒ *'b* ((*3*∑- = -..<-./ -) [*0*,*0*,*0*,*10*] *10*)
-upt-setsum :: *idt* ⇒ *'a* ⇒ *'b* ⇒ *'b* ((*3*∑-<-./ -) [*0*,*0*,*10*] *10*)
-upto-setsum :: *idt* ⇒ *'a* ⇒ *'b* ⇒ *'b* ((*3*∑-≤-./ -) [*0*,*0*,*10*] *10*)
syntax (*HTML* **output**)
-from-to-setsum :: *idt* ⇒ *'a* ⇒ *'a* ⇒ *'b* ⇒ *'b* ((*3*∑- = -..-./ -) [*0*,*0*,*0*,*10*] *10*)

-from-upto-setsum :: *idt* ⇒ *'a* ⇒ *'a* ⇒ *'b* ⇒ *'b* ((*3*∑ - = -..<-./ -) [*0*,*0*,*0*,*10*] *10*)
-upt-setsum :: *idt* ⇒ *'a* ⇒ *'b* ⇒ *'b* ((*3*∑ -<-./ -) [*0*,*0*,*10*] *10*)
-upto-setsum :: *idt* ⇒ *'a* ⇒ *'b* ⇒ *'b* ((*3*∑ -≤-./ -) [*0*,*0*,*10*] *10*)
syntax (*latex-sum* **output**)
-from-to-setsum :: *idt* ⇒ *'a* ⇒ *'a* ⇒ *'b* ⇒ *'b*
(($3\sum_{-\ =\ -}^{-}$ -) [*0*,*0*,*0*,*10*] *10*)
-from-upto-setsum :: *idt* ⇒ *'a* ⇒ *'a* ⇒ *'b* ⇒ *'b*
(($3\sum_{-\ =\ -}^{<-}$ -) [*0*,*0*,*0*,*10*] *10*)
-upt-setsum :: *idt* ⇒ *'a* ⇒ *'b* ⇒ *'b*
(($3\sum_{-\ <\ -}$ -) [*0*,*0*,*10*] *10*)
-upto-setsum :: *idt* ⇒ *'a* ⇒ *'b* ⇒ *'b*
(($3\sum_{-\ \leq\ -}$ -) [*0*,*0*,*10*] *10*)

translations
$\sum x{=}a..b.\ t$ == *setsum* (%*x*. *t*) $\{a..b\}$
$\sum x{=}a..{<}b.\ t$ == *setsum* (%*x*. *t*) $\{a..{<}b\}$
$\sum i{\leq}n.\ t$ == *setsum* ($\lambda i.\ t$) $\{..n\}$
$\sum i{<}n.\ t$ == *setsum* ($\lambda i.\ t$) $\{..{<}n\}$

The above introduces some pretty alternative syntaxes for summation over intervals:

Old	New	LaTeX
$\sum x{\in}\{a..b\}.\ e$	$\sum x = a..b.\ e$	$\sum_{x\ =\ a}^{b} e$
$\sum x{\in}\{a..{<}b\}.\ e$	$\sum x = a..{<}b.\ e$	$\sum_{x\ =\ a}^{<b} e$
$\sum x{\in}\{..b\}.\ e$	$\sum x{\leq}b.\ e$	$\sum_{x\ \leq\ b} e$
$\sum x{\in}\{..{<}b\}.\ e$	$\sum x{<}b.\ e$	$\sum_{x\ <\ b} e$

The left column shows the term before introduction of the new syntax, the middle column shows the new (default) syntax, and the right column shows a special syntax. The latter is only meaningful for latex output and has to be activated explicitly by setting the print mode to *latex-sum* (e.g. via *mode* = *latex-sum* in antiquotations). It is not the default LaTeX output because it only works well with italic-style formulae, not tt-style.
Note that for uniformity on *nat* it is better to use $\sum x = 0..{<}n.\ e$ rather than $\sum x{<}n.\ e$: *setsum* may not provide all lemmas available for $\{m..{<}n\}$ also in the special form for $\{..{<}n\}$.

This congruence rule should be used for sums over intervals as the standard theorem *setsum-cong* does not work well with the simplifier who adds the unsimplified premise $x \in B$ to the context.

lemma *setsum-ivl-cong*:
$[\![a = c;\ b = d;\ !!x.\ [\![\ c \leq x;\ x < d\ ]\!] \Longrightarrow f\,x = g\ x\ ]\!] \Longrightarrow$
$setsum\ f\ \{a..{<}b\} = setsum\ g\ \{c..{<}d\}$
⟨*proof*⟩

lemma *setsum-atMost-Suc*[*simp*]: $(\sum i \leq Suc\ n.\ f\ i) = (\sum i \leq n.\ f\ i) + f(Suc\ n)$
⟨*proof*⟩

lemma *setsum-lessThan-Suc*[*simp*]: $(\sum i < Suc\ n.\ f\ i) = (\sum i < n.\ f\ i) + f\ n$
⟨*proof*⟩

lemma *setsum-cl-ivl-Suc*[*simp*]:
$setsum\ f\ \{m..Suc\ n\} = (if\ Suc\ n < m\ then\ 0\ else\ setsum\ f\ \{m..n\} + f(Suc\ n))$
⟨*proof*⟩

lemma *setsum-op-ivl-Suc*[*simp*]:
$setsum\ f\ \{m..<Suc\ n\} = (if\ n < m\ then\ 0\ else\ setsum\ f\ \{m..<n\} + f(n))$
⟨*proof*⟩

lemma *setsum-add-nat-ivl*: $[\![\ m \leq n;\ n \leq p\ ]\!] \Longrightarrow$
$setsum\ f\ \{m..<n\} + setsum\ f\ \{n..<p\} = setsum\ f\ \{m..<p::nat\}$
⟨*proof*⟩

lemma *setsum-diff-nat-ivl*:
fixes f :: $nat \Rightarrow {}'a::ab\text{-}group\text{-}add$
shows $[\![\ m \leq n;\ n \leq p\ ]\!] \Longrightarrow$
$setsum\ f\ \{m..<p\} - setsum\ f\ \{m..<n\} = setsum\ f\ \{n..<p\}$
⟨*proof*⟩

39.8 Shifting bounds

lemma *setsum-shift-bounds-nat-ivl*:
$setsum\ f\ \{m+k..<n+k\} = setsum\ (\%i.\ f(i + k))\{m..<n::nat\}$
⟨*proof*⟩

lemma *setsum-shift-bounds-cl-nat-ivl*:
$setsum\ f\ \{m+k..n+k\} = setsum\ (\%i.\ f(i + k))\{m..n::nat\}$
⟨*proof*⟩

corollary *setsum-shift-bounds-cl-Suc-ivl*:
$setsum\ f\ \{Suc\ m..Suc\ n\} = setsum\ (\%i.\ f(Suc\ i))\{m..n\}$
⟨*proof*⟩

corollary *setsum-shift-bounds-Suc-ivl*:
$setsum\ f\ \{Suc\ m..<Suc\ n\} = setsum\ (\%i.\ f(Suc\ i))\{m..<n\}$
⟨*proof*⟩

lemma *setsum-head*:
fixes n :: nat
assumes mn: $m <= n$
shows $(\sum x \in \{m..n\}.\ P\ x) = P\ m + (\sum x \in \{m<..n\}.\ P\ x)$ (**is** *?lhs = ?rhs*)
⟨*proof*⟩

lemma *setsum-head-upt*:
  fixes m::*nat*
  assumes m: $0 < m$
  shows $(\sum x{<}m.\ P\ x) = P\ 0 + (\sum x{\in}\{1..{<}m\}.\ P\ x)$
⟨*proof*⟩

39.9 The formula for geometric sums

lemma *geometric-sum*:
  x ~= 1 ==> $(\sum i{=}0..{<}n.\ x\ \hat{}\ i) =$
  $(x\ \hat{}\ n - 1)\ /\ (x - 1$::'a::{*field*, *recpower*})
⟨*proof*⟩

39.10 The formula for arithmetic sums

lemma *gauss-sum*:
  $((1$::'a::*comm-semiring-1*$) + 1)*(\sum i{\in}\{1..n\}.$ *of-nat* $i) =$
   of-nat $n*(($*of-nat* $n)+1)$
⟨*proof*⟩

theorem *arith-series-general*:
  $((1$::'a::*comm-semiring-1*$) + 1) * (\sum i{\in}\{..{<}n\}.\ a +$ *of-nat* $i * d) =$
  of-nat $n * (a + (a +$ *of-nat*$(n - 1)*d))$
⟨*proof*⟩

lemma *arith-series-nat*:
  Suc (*Suc* 0) $* (\sum i{\in}\{..{<}n\}.\ a{+}i{*}d) = n * (a + (a{+}(n - 1){*}d))$
⟨*proof*⟩

lemma *arith-series-int*:
  $(2$::*int*$) * (\sum i{\in}\{..{<}n\}.\ a +$ *of-nat* $i * d) =$
  of-nat $n * (a + (a +$ *of-nat*$(n - 1)*d))$
⟨*proof*⟩

lemma *sum-diff-distrib*:
  fixes P::*nat*⇒*nat*
  shows
  $\forall x.\ Q\ x \leq P\ x \Longrightarrow$
  $(\sum x{<}n.\ P\ x) - (\sum x{<}n.\ Q\ x) = (\sum x{<}n.\ P\ x - Q\ x)$
⟨*proof*⟩

⟨*ML*⟩

end

40 Presburger: Decision Procedure for Presburger Arithmetic

theory *Presburger*
imports *Arith-Tools SetInterval*
uses
Tools/Qelim/cooper-data.ML
Tools/Qelim/generated-cooper.ML
(*Tools/Qelim/cooper.ML*)
(*Tools/Qelim/presburger.ML*)
begin

⟨*ML*⟩

40.1 The $-\infty$ and $+\infty$ Properties

lemma *minf*:
$[\![\exists (z ::'a::linorder).\forall x<z.\ P\ x = P'\ x;\ \exists z.\forall x<z.\ Q\ x = Q'\ x]\!]$
$\Longrightarrow \exists z.\forall x<z.\ (P\ x \wedge Q\ x) = (P'\ x \wedge Q'\ x)$
$[\![\exists (z ::'a::linorder).\forall x<z.\ P\ x = P'\ x;\ \exists z.\forall x<z.\ Q\ x = Q'\ x]\!]$
$\Longrightarrow \exists z.\forall x<z.\ (P\ x \vee Q\ x) = (P'\ x \vee Q'\ x)$
$\exists (z ::'a::\{linorder\}).\forall x<z.(x = t) = False$
$\exists (z ::'a::\{linorder\}).\forall x<z.(x \neq t) = True$
$\exists (z ::'a::\{linorder\}).\forall x<z.(x < t) = True$
$\exists (z ::'a::\{linorder\}).\forall x<z.(x \leq t) = True$
$\exists (z ::'a::\{linorder\}).\forall x<z.(x > t) = False$
$\exists (z ::'a::\{linorder\}).\forall x<z.(x \geq t) = False$
$\exists z.\forall (x::'a::\{linorder,plus,Divides.div\})<z.\ (d\ dvd\ x + s) = (d\ dvd\ x + s)$
$\exists z.\forall (x::'a::\{linorder,plus,Divides.div\})<z.\ (\neg\ d\ dvd\ x + s) = (\neg\ d\ dvd\ x + s)$
$\exists z.\forall x<z.\ F = F$
⟨*proof*⟩

lemma *pinf*:
$[\![\exists (z ::'a::linorder).\forall x>z.\ P\ x = P'\ x;\ \exists z.\forall x>z.\ Q\ x = Q'\ x]\!]$
$\Longrightarrow \exists z.\forall x>z.\ (P\ x \wedge Q\ x) = (P'\ x \wedge Q'\ x)$
$[\![\exists (z ::'a::linorder).\forall x>z.\ P\ x = P'\ x;\ \exists z.\forall x>z.\ Q\ x = Q'\ x]\!]$
$\Longrightarrow \exists z.\forall x>z.\ (P\ x \vee Q\ x) = (P'\ x \vee Q'\ x)$
$\exists (z ::'a::\{linorder\}).\forall x>z.(x = t) = False$
$\exists (z ::'a::\{linorder\}).\forall x>z.(x \neq t) = True$
$\exists (z ::'a::\{linorder\}).\forall x>z.(x < t) = False$
$\exists (z ::'a::\{linorder\}).\forall x>z.(x \leq t) = False$
$\exists (z ::'a::\{linorder\}).\forall x>z.(x > t) = True$
$\exists (z ::'a::\{linorder\}).\forall x>z.(x \geq t) = True$
$\exists z.\forall (x::'a::\{linorder,plus,Divides.div\})>z.\ (d\ dvd\ x + s) = (d\ dvd\ x + s)$
$\exists z.\forall (x::'a::\{linorder,plus,Divides.div\})>z.\ (\neg\ d\ dvd\ x + s) = (\neg\ d\ dvd\ x + s)$
$\exists z.\forall x>z.\ F = F$
⟨*proof*⟩

lemma *inf-period*:

$[\![\forall x\, k.\ P\ x = P\ (x - k{*}D);\ \forall x\, k.\ Q\ x = Q\ (x - k{*}D)]\!]$
$\Longrightarrow \forall x\, k.\ (P\ x \wedge Q\ x) = (P\ (x - k{*}D) \wedge Q\ (x - k{*}D))$
$[\![\forall x\, k.\ P\ x = P\ (x - k{*}D);\ \forall x\, k.\ Q\ x = Q\ (x - k{*}D)]\!]$
$\Longrightarrow \forall x\, k.\ (P\ x \vee Q\ x) = (P\ (x - k{*}D) \vee Q\ (x - k{*}D))$
$(d{::}'a{::}\{comm\text{-}ring,Divides.div\})\ dvd\ D \Longrightarrow \forall x\, k.\ (d\ dvd\ x + t) = (d\ dvd\ (x - k{*}D) + t)$
$(d{::}'a{::}\{comm\text{-}ring,Divides.div\})\ dvd\ D \Longrightarrow \forall x\, k.\ (\neg d\ dvd\ x + t) = (\neg d\ dvd\ (x - k{*}D) + t)$
$\forall x\, k.\ F = F$
$\langle proof\rangle$

40.2 The A and B sets

lemma *bset*:
$[\![\forall x.(\forall j \in \{1\ ..\ D\}.\ \forall b{\in}B.\ x \neq b + j) \longrightarrow P\ x \longrightarrow P(x - D)\ ;$
$\forall x.(\forall j{\in}\{1\ ..\ D\}.\ \forall b{\in}B.\ x \neq b + j) \longrightarrow Q\ x \longrightarrow Q(x - D)]\!] \Longrightarrow$
$\forall x.(\forall j{\in}\{1\ ..\ D\}.\ \forall b{\in}B.\ x \neq b + j) \longrightarrow (P\ x \wedge Q\ x) \longrightarrow (P(x - D) \wedge Q\ (x - D))$
$[\![\forall x.(\forall j{\in}\{1\ ..\ D\}.\ \forall b{\in}B.\ x \neq b + j) \longrightarrow P\ x \longrightarrow P(x - D)\ ;$
$\forall x.(\forall j{\in}\{1\ ..\ D\}.\ \forall b{\in}B.\ x \neq b + j) \longrightarrow Q\ x \longrightarrow Q(x - D)]\!] \Longrightarrow$
$\forall x.(\forall j{\in}\{1\ ..\ D\}.\ \forall b{\in}B.\ x \neq b + j) \longrightarrow (P\ x \vee Q\ x) \longrightarrow (P(x - D) \vee Q\ (x - D))$
$[\![D{>}0;\ t - 1{\in}\ B]\!] \Longrightarrow (\forall x.(\forall j{\in}\{1\ ..\ D\}.\ \forall b{\in}B.\ x \neq b + j) \longrightarrow (x = t) \longrightarrow (x - D = t))$
$[\![D{>}0\ ;\ t \in B]\!] \Longrightarrow (\forall (x{::}int).(\forall j{\in}\{1\ ..\ D\}.\ \forall b{\in}B.\ x \neq b + j) \longrightarrow (x \neq t) \longrightarrow (x - D \neq t))$
$D{>}0 \Longrightarrow (\forall (x{::}int).(\forall j{\in}\{1\ ..\ D\}.\ \forall b{\in}B.\ x \neq b + j) \longrightarrow (x < t) \longrightarrow (x - D < t))$
$D{>}0 \Longrightarrow (\forall (x{::}int).(\forall j{\in}\{1\ ..\ D\}.\ \forall b{\in}B.\ x \neq b + j) \longrightarrow (x \leq t) \longrightarrow (x - D \leq t))$
$[\![D{>}0\ ;\ t \in B]\!] \Longrightarrow (\forall (x{::}int).(\forall j{\in}\{1\ ..\ D\}.\ \forall b{\in}B.\ x \neq b + j) \longrightarrow (x > t) \longrightarrow (x - D > t))$
$[\![D{>}0\ ;\ t - 1 \in B]\!] \Longrightarrow (\forall (x{::}int).(\forall j{\in}\{1\ ..\ D\}.\ \forall b{\in}B.\ x \neq b + j) \longrightarrow (x \geq t) \longrightarrow (x - D \geq t))$
$d\ dvd\ D \Longrightarrow (\forall (x{::}int).(\forall j{\in}\{1\ ..\ D\}.\ \forall b{\in}B.\ x \neq b + j) \longrightarrow (d\ dvd\ x{+}t) \longrightarrow (d\ dvd\ (x - D) + t))$
$d\ dvd\ D \Longrightarrow (\forall (x{::}int).(\forall j{\in}\{1\ ..\ D\}.\ \forall b{\in}B.\ x \neq b + j) \longrightarrow (\neg d\ dvd\ x{+}t) \longrightarrow (\neg\ d\ dvd\ (x - D) + t))$
$\forall x.(\forall j{\in}\{1\ ..\ D\}.\ \forall b{\in}B.\ x \neq b + j) \longrightarrow F \longrightarrow F$
$\langle proof\rangle$

lemma *aset*:
$[\![\forall x.(\forall j{\in}\{1\ ..\ D\}.\ \forall b{\in}A.\ x \neq b - j) \longrightarrow P\ x \longrightarrow P(x + D)\ ;$
$\forall x.(\forall j{\in}\{1\ ..\ D\}.\ \forall b{\in}A.\ x \neq b - j) \longrightarrow Q\ x \longrightarrow Q(x + D)]\!] \Longrightarrow$
$\forall x.(\forall j{\in}\{1\ ..\ D\}.\ \forall b{\in}A.\ x \neq b - j) \longrightarrow (P\ x \wedge Q\ x) \longrightarrow (P(x + D) \wedge Q\ (x + D))$
$[\![\forall x.(\forall j{\in}\{1\ ..\ D\}.\ \forall b{\in}A.\ x \neq b - j) \longrightarrow P\ x \longrightarrow P(x + D)\ ;$
$\forall x.(\forall j{\in}\{1\ ..\ D\}.\ \forall b{\in}A.\ x \neq b - j) \longrightarrow Q\ x \longrightarrow Q(x + D)]\!] \Longrightarrow$
$\forall x.(\forall j{\in}\{1\ ..\ D\}.\ \forall b{\in}A.\ x \neq b - j) \longrightarrow (P\ x \vee Q\ x) \longrightarrow (P(x + D) \vee Q\ (x +$

$D))$
$[\![D{>}0;\ t + 1{\in}\ A]\!] \Longrightarrow (\forall x.(\forall j{\in}\{1\ ..\ D\}.\ \forall b{\in}A.\ x \neq b - j) \longrightarrow (x = t) \longrightarrow (x + D = t))$
$[\![D{>}0\ ;\ t \in A]\!] \Longrightarrow (\forall (x{::}int).(\forall j{\in}\{1\ ..\ D\}.\ \forall b{\in}A.\ x \neq b - j) \longrightarrow (x \neq t) \longrightarrow (x + D \neq t))$
$[\![D{>}0;\ t{\in}\ A]\!] \Longrightarrow (\forall (x{::}int).\ (\forall j{\in}\{1\ ..\ D\}.\ \forall b{\in}A.\ x \neq b - j) \longrightarrow (x < t) \longrightarrow (x + D < t))$
$[\![D{>}0;\ t + 1 \in A]\!] \Longrightarrow (\forall (x{::}int).(\forall j{\in}\{1\ ..\ D\}.\ \forall b{\in}A.\ x \neq b - j) \longrightarrow (x \leq t) \longrightarrow (x + D \leq t))$
$D{>}0 \Longrightarrow (\forall (x{::}int).(\forall j{\in}\{1\ ..\ D\}.\ \forall b{\in}A.\ x \neq b - j) \longrightarrow (x > t) \longrightarrow (x + D > t))$
$D{>}0 \Longrightarrow (\forall (x{::}int).(\forall j{\in}\{1\ ..\ D\}.\ \forall b{\in}A.\ x \neq b - j) \longrightarrow (x \geq t) \longrightarrow (x + D \geq t))$
$d\ dvd\ D \Longrightarrow (\forall (x{::}int).(\forall j{\in}\{1\ ..\ D\}.\ \forall b{\in}A.\ x \neq b - j) \longrightarrow (d\ dvd\ x{+}t) \longrightarrow (d\ dvd\ (x + D) + t))$
$d\ dvd\ D \Longrightarrow (\forall (x{::}int).(\forall j{\in}\{1\ ..\ D\}.\ \forall b{\in}A.\ x \neq b - j) \longrightarrow (\neg d\ dvd\ x{+}t) \longrightarrow (\neg\ d\ dvd\ (x + D) + t))$
$\forall x.(\forall j{\in}\{1\ ..\ D\}.\ \forall b{\in}A.\ x \neq b - j) \longrightarrow F \longrightarrow F$
⟨*proof*⟩

40.3 Cooper's Theorem $-\infty$ and $+\infty$ Version

40.3.1 First some trivial facts about periodic sets or predicates

lemma *periodic-finite-ex*:
assumes *dpos*: $(0{::}int) < d$ **and** *modd*: $ALL\ x\ k.\ P\ x = P(x - k{*}d)$
shows $(EX\ x.\ P\ x) = (EX\ j : \{1..d\}.\ P\ j)$
(**is** $?LHS = ?RHS$)
⟨*proof*⟩

40.3.2 The $-\infty$ Version

lemma *decr-lemma*: $0 < (d{::}int) \Longrightarrow x - (abs(x{-}z){+}1) * d < z$
⟨*proof*⟩

lemma *incr-lemma*: $0 < (d{::}int) \Longrightarrow z < x + (abs(x{-}z){+}1) * d$
⟨*proof*⟩

theorem *int-induct*[*case-names base step1 step2*]:
assumes
base: $P(k{::}int)$ **and** *step1*: $\bigwedge i.\ [\![k \leq i;\ P\ i]\!] \Longrightarrow P(i{+}1)$ **and**
step2: $\bigwedge i.\ [\![k \geq i;\ P\ i]\!] \Longrightarrow P(i - 1)$
shows $P\ i$
⟨*proof*⟩

lemma *decr-mult-lemma*:
assumes *dpos*: $(0{::}int) < d$ **and** *minus*: $\forall x.\ P\ x \longrightarrow P(x - d)$ **and** *knneg*: $0 <= k$
shows $ALL\ x.\ P\ x \longrightarrow P(x - k{*}d)$
⟨*proof*⟩

lemma *minusinfinity*:
assumes *dpos*: $0 < d$ **and**
$P1eqP1$: $ALL\ x\ k.\ P1\ x = P1(x - k*d)$ **and** *ePeqP1*: $EX\ z::int.\ ALL\ x.\ x < z \longrightarrow (P\ x = P1\ x)$
shows $(EX\ x.\ P1\ x) \longrightarrow (EX\ x.\ P\ x)$
⟨*proof*⟩

lemma *cpmi*:
assumes *dp*: $0 < D$ **and** $p1$:$\exists z.\ \forall\ x< z.\ P\ x = P'\ x$
and nb:$\forall x.(\forall\ j\in \{1..D\}.\ \forall (b::int) \in B.\ x \neq b+j) --> P\ (x) --> P\ (x - D)$
and *pd*: $\forall\ x\ k.\ P'\ x = P'\ (x-k*D)$
shows $(\exists x.\ P\ x) = ((\exists\ j\in \{1..D\}\ .\ P'\ j) \mid (\exists\ j \in \{1..D\}.\exists\ b\in B.\ P\ (b+j)))$
(**is** $?L = (?R1 \vee ?R2)$)
⟨*proof*⟩

40.3.3 The $+\infty$ Version

lemma *plusinfinity*:
assumes *dpos*: $(0::int) < d$ **and**
$P1eqP1$: $\forall x\ k.\ P'\ x = P'(x - k*d)$ **and** *ePeqP1*: $\exists\ z.\ \forall\ x>z.\ P\ x = P'\ x$
shows $(\exists\ x.\ P'\ x) \longrightarrow (\exists\ x.\ P\ x)$
⟨*proof*⟩

lemma *incr-mult-lemma*:
assumes *dpos*: $(0::int) < d$ **and** *plus*: $ALL\ x::int.\ P\ x \longrightarrow P(x + d)$ **and** *knneg*: $0 <= k$
shows $ALL\ x.\ P\ x \longrightarrow P(x + k*d)$
⟨*proof*⟩

lemma *cppi*:
assumes *dp*: $0 < D$ **and** $p1$:$\exists z.\ \forall\ x> z.\ P\ x = P'\ x$
and nb:$\forall x.(\forall\ j\in \{1..D\}.\ \forall (b::int) \in A.\ x \neq b - j) --> P\ (x) --> P\ (x + D)$
and *pd*: $\forall\ x\ k.\ P'\ x= P'\ (x-k*D)$
shows $(\exists x.\ P\ x) = ((\exists\ j\in \{1..D\}\ .\ P'\ j) \mid (\exists\ j \in \{1..D\}.\exists\ b\in A.\ P\ (b - j)))$
(**is** $?L = (?R1 \vee ?R2)$)
⟨*proof*⟩

lemma *simp-from-to*: $\{i..j::int\} = (if\ j < i\ then\ \{\}\ else\ insert\ i\ \{i+1..j\})$
⟨*proof*⟩

theorem *unity-coeff-ex*: $(\exists (x::'a::\{semiring\text{-}0,Divides.div\}).\ P\ (l * x)) \equiv (\exists x.\ l\ dvd\ (x + 0) \wedge P\ x)$
⟨*proof*⟩

lemma *zdvd-mono*: **assumes** *not0*: $(k::int) \neq 0$
shows $((m::int)\ dvd\ t) \equiv (k*m\ dvd\ k*t)$

⟨*proof*⟩

lemma *uminus-dvd-conv*: $(d\ dvd\ (t::int)) \equiv (-d\ dvd\ t)$ $(d\ dvd\ (t::int)) \equiv (d\ dvd\ -t)$
⟨*proof*⟩

Theorems for transforming predicates on nat to predicates on *int*

lemma *all-nat*: $(\forall x::nat.\ P\ x) = (\forall x::int.\ 0 <= x \longrightarrow P\ (nat\ x))$
⟨*proof*⟩

lemma *ex-nat*: $(\exists x::nat.\ P\ x) = (\exists x::int.\ 0 <= x \wedge P\ (nat\ x))$
⟨*proof*⟩

lemma *zdiff-int-split*: $P\ (int\ (x - y)) =$
$((y \leq x \longrightarrow P\ (int\ x - int\ y)) \wedge (x < y \longrightarrow P\ 0))$
⟨*proof*⟩

lemma *number-of1*: $(0::int) <= number\text{-}of\ n \Longrightarrow (0::int) <= number\text{-}of\ (n\ BIT\ b)$ ⟨*proof*⟩
lemma *number-of2*: $(0::int) <= Numeral0$ ⟨*proof*⟩
lemma *Suc-plus1*: $Suc\ n = n + 1$ ⟨*proof*⟩

Specific instances of congruence rules, to prevent simplifier from looping.

theorem *imp-le-cong*: $(0 <= x \Longrightarrow P = P') \Longrightarrow (0 <= (x::int) \longrightarrow P) = (0 <= x \longrightarrow P')$ ⟨*proof*⟩

theorem *conj-le-cong*: $(0 <= x \Longrightarrow P = P') \Longrightarrow (0 <= (x::int) \wedge P) = (0 <= x \wedge P')$
⟨*proof*⟩
lemma *int-eq-number-of-eq*:
$(((number\text{-}of\ v)::int) = (number\text{-}of\ w)) = iszero\ ((number\text{-}of\ (v + (uminus\ w)))::int)$
⟨*proof*⟩

lemma *mod-eq0-dvd-iff* [*presburger*]: $(m::nat)\ mod\ n = 0 \longleftrightarrow n\ dvd\ m$
⟨*proof*⟩

lemma *zmod-eq0-zdvd-iff* [*presburger*]: $(m::int)\ mod\ n = 0 \longleftrightarrow n\ dvd\ m$
⟨*proof*⟩
declare *mod-1* [*presburger*]
declare *mod-0* [*presburger*]
declare *zmod-1* [*presburger*]
declare *zmod-zero* [*presburger*]
declare *zmod-self* [*presburger*]
declare *mod-self* [*presburger*]
declare *DIVISION-BY-ZERO-MOD* [*presburger*]
declare *nat-mod-div-trivial* [*presburger*]
declare *div-mod-equality2* [*presburger*]

declare *div-mod-equality*[*presburger*]
declare *mod-div-equality2*[*presburger*]
declare *mod-div-equality*[*presburger*]
declare *mod-mult-self1*[*presburger*]
declare *mod-mult-self2*[*presburger*]
declare *zdiv-zmod-equality2*[*presburger*]
declare *zdiv-zmod-equality*[*presburger*]
declare *mod2-Suc-Suc*[*presburger*]
lemma [*presburger*]: $(a::int)\ div\ 0 = 0$ **and** [*presburger*]: $a\ mod\ 0 = a$
⟨*proof*⟩

⟨*ML*⟩

lemma [*presburger*]: $m\ mod\ 2 = (1::nat) \longleftrightarrow \neg\ 2\ dvd\ m$ ⟨*proof*⟩
lemma [*presburger*]: $m\ mod\ 2 = Suc\ 0 \longleftrightarrow \neg\ 2\ dvd\ m$ ⟨*proof*⟩
lemma [*presburger*]: $m\ mod\ (Suc\ (Suc\ 0)) = (1::nat) \longleftrightarrow \neg\ 2\ dvd\ m$ ⟨*proof*⟩
lemma [*presburger*]: $m\ mod\ (Suc\ (Suc\ 0)) = Suc\ 0 \longleftrightarrow \neg\ 2\ dvd\ m$ ⟨*proof*⟩
lemma [*presburger*]: $m\ mod\ 2 = (1::int) \longleftrightarrow \neg\ 2\ dvd\ m$ ⟨*proof*⟩

lemma *zdvd-period*:
fixes $a\ d :: int$
assumes *advdd*: $a\ dvd\ d$
shows $a\ dvd\ (x + t) \longleftrightarrow a\ dvd\ ((x + c * d) + t)$
⟨*proof*⟩

40.4 Code generator setup

Presburger arithmetic is convenient to prove some of the following code lemmas on integer numerals:

lemma *eq-Pls-Pls*:
Numeral.Pls = *Numeral.Pls* $\longleftrightarrow$ *True* ⟨*proof*⟩

lemma *eq-Pls-Min*:
Numeral.Pls = *Numeral.Min* $\longleftrightarrow$ *False*
⟨*proof*⟩

lemma *eq-Pls-Bit0*:
Numeral.Pls = *Numeral.Bit k bit.B0* $\longleftrightarrow$ *Numeral.Pls* = *k*
⟨*proof*⟩

lemma *eq-Pls-Bit1*:
Numeral.Pls = *Numeral.Bit k bit.B1* $\longleftrightarrow$ *False*
⟨*proof*⟩

lemma *eq-Min-Pls*:
Numeral.Min = *Numeral.Pls* $\longleftrightarrow$ *False*
⟨*proof*⟩

lemma *eq-Min-Min*:
$Numeral.Min = Numeral.Min \longleftrightarrow True$ ⟨*proof*⟩

lemma *eq-Min-Bit0*:
$Numeral.Min = Numeral.Bit\ k\ bit.B0 \longleftrightarrow False$
⟨*proof*⟩

lemma *eq-Min-Bit1*:
$Numeral.Min = Numeral.Bit\ k\ bit.B1 \longleftrightarrow Numeral.Min = k$
⟨*proof*⟩

lemma *eq-Bit0-Pls*:
$Numeral.Bit\ k\ bit.B0 = Numeral.Pls \longleftrightarrow Numeral.Pls = k$
⟨*proof*⟩

lemma *eq-Bit1-Pls*:
$Numeral.Bit\ k\ bit.B1 = Numeral.Pls \longleftrightarrow False$
⟨*proof*⟩

lemma *eq-Bit0-Min*:
$Numeral.Bit\ k\ bit.B0 = Numeral.Min \longleftrightarrow False$
⟨*proof*⟩

lemma *eq-Bit1-Min*:
$(Numeral.Bit\ k\ bit.B1) = Numeral.Min \longleftrightarrow Numeral.Min = k$
⟨*proof*⟩

lemma *eq-Bit-Bit*:
$Numeral.Bit\ k1\ v1 = Numeral.Bit\ k2\ v2 \longleftrightarrow$
$v1 = v2 \wedge k1 = k2$
⟨*proof*⟩

lemma *eq-number-of*:
$(number\text{-}of\ k :: int) = number\text{-}of\ l \longleftrightarrow k = l$
⟨*proof*⟩

lemma *less-eq-Pls-Pls*:
$Numeral.Pls \leq Numeral.Pls \longleftrightarrow True$ ⟨*proof*⟩

lemma *less-eq-Pls-Min*:
$Numeral.Pls \leq Numeral.Min \longleftrightarrow False$
⟨*proof*⟩

lemma *less-eq-Pls-Bit*:
$Numeral.Pls \leq Numeral.Bit\ k\ v \longleftrightarrow Numeral.Pls \leq k$
⟨*proof*⟩

lemma *less-eq-Min-Pls*:

$Numeral.Min \le Numeral.Pls \longleftrightarrow True$
$\langle proof \rangle$

lemma *less-eq-Min-Min*:
$Numeral.Min \le Numeral.Min \longleftrightarrow True$ $\langle proof \rangle$

lemma *less-eq-Min-Bit0*:
$Numeral.Min \le Numeral.Bit\ k\ bit.B0 \longleftrightarrow Numeral.Min < k$
$\langle proof \rangle$

lemma *less-eq-Min-Bit1*:
$Numeral.Min \le Numeral.Bit\ k\ bit.B1 \longleftrightarrow Numeral.Min \le k$
$\langle proof \rangle$

lemma *less-eq-Bit0-Pls*:
$Numeral.Bit\ k\ bit.B0 \le Numeral.Pls \longleftrightarrow k \le Numeral.Pls$
$\langle proof \rangle$

lemma *less-eq-Bit1-Pls*:
$Numeral.Bit\ k\ bit.B1 \le Numeral.Pls \longleftrightarrow k < Numeral.Pls$
$\langle proof \rangle$

lemma *less-eq-Bit-Min*:
$Numeral.Bit\ k\ v \le Numeral.Min \longleftrightarrow k \le Numeral.Min$
$\langle proof \rangle$

lemma *less-eq-Bit0-Bit*:
$Numeral.Bit\ k1\ bit.B0 \le Numeral.Bit\ k2\ v \longleftrightarrow k1 \le k2$
$\langle proof \rangle$

lemma *less-eq-Bit-Bit1*:
$Numeral.Bit\ k1\ v \le Numeral.Bit\ k2\ bit.B1 \longleftrightarrow k1 \le k2$
$\langle proof \rangle$

lemma *less-eq-Bit1-Bit0*:
$Numeral.Bit\ k1\ bit.B1 \le Numeral.Bit\ k2\ bit.B0 \longleftrightarrow k1 < k2$
$\langle proof \rangle$

lemma *less-eq-number-of*:
$(number\text{-}of\ k :: int) \le number\text{-}of\ l \longleftrightarrow k \le l$
$\langle proof \rangle$

lemma *less-Pls-Pls*:
$Numeral.Pls < Numeral.Pls \longleftrightarrow False$ $\langle proof \rangle$

lemma *less-Pls-Min*:
$Numeral.Pls < Numeral.Min \longleftrightarrow False$
$\langle proof \rangle$

lemma *less-Pls-Bit0*:
$Numeral.Pls < Numeral.Bit\ k\ bit.B0 \longleftrightarrow Numeral.Pls < k$
⟨*proof*⟩

lemma *less-Pls-Bit1*:
$Numeral.Pls < Numeral.Bit\ k\ bit.B1 \longleftrightarrow Numeral.Pls \leq k$
⟨*proof*⟩

lemma *less-Min-Pls*:
$Numeral.Min < Numeral.Pls \longleftrightarrow True$
⟨*proof*⟩

lemma *less-Min-Min*:
$Numeral.Min < Numeral.Min \longleftrightarrow False$ ⟨*proof*⟩

lemma *less-Min-Bit*:
$Numeral.Min < Numeral.Bit\ k\ v \longleftrightarrow Numeral.Min < k$
⟨*proof*⟩

lemma *less-Bit-Pls*:
$Numeral.Bit\ k\ v < Numeral.Pls \longleftrightarrow k < Numeral.Pls$
⟨*proof*⟩

lemma *less-Bit0-Min*:
$Numeral.Bit\ k\ bit.B0 < Numeral.Min \longleftrightarrow k \leq Numeral.Min$
⟨*proof*⟩

lemma *less-Bit1-Min*:
$Numeral.Bit\ k\ bit.B1 < Numeral.Min \longleftrightarrow k < Numeral.Min$
⟨*proof*⟩

lemma *less-Bit-Bit0*:
$Numeral.Bit\ k1\ v < Numeral.Bit\ k2\ bit.B0 \longleftrightarrow k1 < k2$
⟨*proof*⟩

lemma *less-Bit1-Bit*:
$Numeral.Bit\ k1\ bit.B1 < Numeral.Bit\ k2\ v \longleftrightarrow k1 < k2$
⟨*proof*⟩

lemma *less-Bit0-Bit1*:
$Numeral.Bit\ k1\ bit.B0 < Numeral.Bit\ k2\ bit.B1 \longleftrightarrow k1 \leq k2$
⟨*proof*⟩

lemma *less-number-of*:
$(number\text{-}of\ k :: int) < number\text{-}of\ l \longleftrightarrow k < l$
⟨*proof*⟩

lemmas *pred-succ-numeral-code* [*code func*] =

arith-simps(*5*−*12*)

lemmas *plus-numeral-code* [*code func*] =
arith-simps(*13*−*17*)
arith-simps(*26*−*27*)
arith-extra-simps(*1*) [**where** ′*a* = *int*]

lemmas *minus-numeral-code* [*code func*] =
arith-simps(*18*−*21*)
arith-extra-simps(*2*) [**where** ′*a* = *int*]
arith-extra-simps(*5*) [**where** ′*a* = *int*]

lemmas *times-numeral-code* [*code func*] =
arith-simps(*22*−*25*)
arith-extra-simps(*4*) [**where** ′*a* = *int*]

lemmas *eq-numeral-code* [*code func*] =
eq-Pls-Pls eq-Pls-Min eq-Pls-Bit0 eq-Pls-Bit1
eq-Min-Pls eq-Min-Min eq-Min-Bit0 eq-Min-Bit1
eq-Bit0-Pls eq-Bit1-Pls eq-Bit0-Min eq-Bit1-Min eq-Bit-Bit
eq-number-of

lemmas *less-eq-numeral-code* [*code func*] = *less-eq-Pls-Pls less-eq-Pls-Min less-eq-Pls-Bit*
less-eq-Min-Pls less-eq-Min-Min less-eq-Min-Bit0 less-eq-Min-Bit1
less-eq-Bit0-Pls less-eq-Bit1-Pls less-eq-Bit-Min less-eq-Bit0-Bit less-eq-Bit-Bit1
less-eq-Bit1-Bit0
less-eq-number-of

lemmas *less-numeral-code* [*code func*] = *less-Pls-Pls less-Pls-Min less-Pls-Bit0*
less-Pls-Bit1 less-Min-Pls less-Min-Min less-Min-Bit less-Bit-Pls
less-Bit0-Min less-Bit1-Min less-Bit-Bit0 less-Bit1-Bit less-Bit0-Bit1
less-number-of

context *ring-1*
begin

lemma *of-int-num* [*code func*]:
of-int k = (*if k* = *0 then 0 else if k* < *0 then*
− *of-int* (− *k*) *else let*
(*l*, *m*) = *divAlg* (*k*, *2*);
l′ = *of-int l*
in if m = *0 then l*′ + *l*′ *else l*′ + *l*′ + *1*)
⟨*proof*⟩

end

end

41 Relation-Power: Powers of Relations and Functions

theory *Relation-Power*
imports *Power*
begin

instance
  set :: (*type*) *power* ⟨*proof*⟩

primrec (**unchecked** *relpow*)
  $R\hat{}0 = Id$
  $R\hat{}(Suc\ n) = R\ O\ (R\hat{}n)$

instance
  fun :: (*type*, *type*) *power* ⟨*proof*⟩

primrec (**unchecked** *funpow*)
  $f\hat{}0 = id$
  $f\hat{}(Suc\ n) = f\ o\ (f\hat{}n)$

WARNING: due to the limits of Isabelle's type classes, exponentiation on functions and relations has too general a domain, namely $('a \times 'b)\ set$ and $'a \Rightarrow 'b$. Explicit type constraints may therefore be necessary. For example, $range\ (f\ \hat{}\ n) = A$ and $Range\ (R\ \hat{}\ n) = B$ need constraints.

Circumvent this problem for code generation:

definition
  $funpow :: nat \Rightarrow ('a \Rightarrow 'a) \Rightarrow 'a \Rightarrow 'a$
where
  $funpow\text{-}def$: $funpow\ n\ f = f\ \hat{}\ n$

lemmas [*code inline*] = *funpow-def* [*symmetric*]

lemma [*code func*]:
  $funpow\ 0\ f = id$
  $funpow\ (Suc\ n)\ f = f\ o\ funpow\ n\ f$
  ⟨*proof*⟩

lemma *funpow-add*: $f\ \hat{}\ (m+n) = f\hat{}m\ o\ f\hat{}n$
  ⟨*proof*⟩

lemma *funpow-swap1*: $f((f\hat{}n)\ x) = (f\hat{}n)(f\ x)$
⟨*proof*⟩

lemma *rel-pow-1* [*simp*]:

fixes $R :: ('a*'a)set$
shows $R\hat{\ }1 = R$
$\langle proof \rangle$

lemma *rel-pow-0-I*: $(x,x) : R\hat{\ }0$
$\langle proof \rangle$

lemma *rel-pow-Suc-I*: $[| (x,y) : R\hat{\ }n; (y,z):R |] ==> (x,z):R\hat{\ }(Suc\ n)$
$\langle proof \rangle$

lemma *rel-pow-Suc-I2*:
$(x,\ y) : R \Longrightarrow (y,\ z) : R\hat{\ }n \Longrightarrow (x,z) : R\hat{\ }(Suc\ n)$
$\langle proof \rangle$

lemma *rel-pow-0-E*: $[| (x,y) : R\hat{\ }0; x{=}y ==> P |] ==> P$
$\langle proof \rangle$

lemma *rel-pow-Suc-E*:
$[| (x,z) : R\hat{\ }(Suc\ n);\ \ !!y.\ [| (x,y) : R\hat{\ }n; (y,z) : R |] ==> P |] ==> P$
$\langle proof \rangle$

lemma *rel-pow-E*:
$[| (x,z) : R\hat{\ }n;\ \ [| n{=}0; x = z |] ==> P;$
$!!y\ m.\ [| n = Suc\ m; (x,y) : R\hat{\ }m; (y,z) : R |] ==> P$
$|] ==> P$
$\langle proof \rangle$

lemma *rel-pow-Suc-D2*:
$(x,\ z) : R\hat{\ }(Suc\ n) \Longrightarrow (\exists\, y.\ (x,y) : R\ \&\ (y,z) : R\hat{\ }n)$
$\langle proof \rangle$

lemma *rel-pow-Suc-D2'*:
$\forall\, x\ y\ z.\ (x,y) : R\hat{\ }n\ \&\ (y,z) : R\ {-}{-}{>}\ (\exists\, w.\ (x,w) : R\ \&\ (w,z) : R\hat{\ }n)$
$\langle proof \rangle$

lemma *rel-pow-E2*:
$[| (x,z) : R\hat{\ }n;\ \ [| n{=}0; x = z |] ==> P;$
$!!y\ m.\ [| n = Suc\ m; (x,y) : R; (y,z) : R\hat{\ }m |] ==> P$
$|] ==> P$
$\langle proof \rangle$

lemma *rtrancl-imp-UN-rel-pow*: $!!p.\ p{:}R\hat{\ }{*} ==> p : (UN\ n.\ R\hat{\ }n)$
$\langle proof \rangle$

lemma *rel-pow-imp-rtrancl*: $!!p.\ p{:}R\hat{\ }n ==> p{:}R\hat{\ }{*}$
$\langle proof \rangle$

lemma *rtrancl-is-UN-rel-pow*: $R\hat{\ }{*} = (UN\ n.\ R\hat{\ }n)$
$\langle proof \rangle$

lemma *trancl-power*:
$x \in r\hat{}+ = (\exists\, n > 0.\ x \in r\hat{}n)$
⟨*proof*⟩

lemma *single-valued-rel-pow*:
$!!r::('a * 'a)set.\ single\text{-}valued\ r ==> single\text{-}valued\ (r\hat{}n)$
⟨*proof*⟩

⟨*ML*⟩

end

42 Refute: Refute

theory *Refute*
imports *Datatype*
uses *Tools/prop-logic.ML*
Tools/sat-solver.ML
Tools/refute.ML
Tools/refute-isar.ML
begin

⟨*ML*⟩

```
(* ------------------------------------------------------------------------- *)
(* REFUTE                                                                    *)
(*                                                                           *)
(* We use a SAT solver to search for a (finite) model that refutes a given   *)
(* HOL formula.                                                              *)
(* ------------------------------------------------------------------------- *)

(* ------------------------------------------------------------------------- *)
(* NOTE                                                                      *)
(*                                                                           *)
(* I strongly recommend that you install a stand-alone SAT solver if you     *)
(* want to use 'refute'.  For details see 'HOL/Tools/sat_solver.ML'.  If you *)
(* have installed (a supported version of) zChaff, simply set 'ZCHAFF_HOME'  *)
(* in 'etc/settings'.                                                        *)
(* ------------------------------------------------------------------------- *)

(* ------------------------------------------------------------------------- *)
(* USAGE                                                                     *)
(*                                                                           *)
(* See the file 'HOL/ex/Refute_Examples.thy' for examples.  The supported    *)
(* parameters are explained below.                                           *)
(* ------------------------------------------------------------------------- *)
```

```
(* ------------------------------------------------------------------------- *)
(* CURRENT LIMITATIONS                                                       *)
(*                                                                           *)
(* 'refute' currently accepts formulas of higher-order predicate logic (with *)
(* equality), including free/bound/schematic variables, lambda abstractions, *)
(* sets and set membership, "arbitrary", "The", "Eps", records and          *)
(* inductively defined sets.  Constants are unfolded automatically, and sort *)
(* axioms are added as well.  Other, user-asserted axioms however are       *)
(* ignored.  Inductive datatypes and recursive functions are supported, but  *)
(* may lead to spurious countermodels.                                       *)
(*                                                                           *)
(* The (space) complexity of the algorithm is non-elementary.               *)
(*                                                                           *)
(* Schematic type variables are not supported.                              *)
(* ------------------------------------------------------------------------- *)

(* ------------------------------------------------------------------------- *)
(* PARAMETERS                                                                *)
(*                                                                           *)
(* The following global parameters are currently supported (and required):  *)
(*                                                                           *)
(* Name          Type    Description                                        *)
(*                                                                           *)
(* "minsize"     int     Only search for models with size at least          *)
(*                       'minsize'.                                          *)
(* "maxsize"     int     If >0, only search for models with size at most    *)
(*                       'maxsize'.                                          *)
(* "maxvars"     int     If >0, use at most 'maxvars' boolean variables     *)
(*                       when transforming the term into a propositional    *)
(*                       formula.                                            *)
(* "maxtime"     int     If >0, terminate after at most 'maxtime' seconds.  *)
(*                       This value is ignored under some ML compilers.     *)
(* "satsolver"   string  Name of the SAT solver to be used.                 *)
(*                                                                           *)
(* See 'HOL/SAT.thy' for default values.                                    *)
(*                                                                           *)
(* The size of particular types can be specified in the form type=size      *)
(* (where 'type' is a string, and 'size' is an int).  Examples:             *)
(* "'a"=1                                                                    *)
(* "List.list"=2                                                             *)
(* ------------------------------------------------------------------------- *)

(* ------------------------------------------------------------------------- *)
(* FILES                                                                     *)
(*                                                                           *)
(* HOL/Tools/prop_logic.ML    Propositional logic                           *)
(* HOL/Tools/sat_solver.ML    SAT solvers                                   *)
(* HOL/Tools/refute.ML        Translation HOL -> propositional logic and    *)
```

```
(*                               Boolean assignment -> HOL model                *)
(* HOL/Tools/refute_isar.ML      Adds 'refute'/'refute_params' to Isabelle's    *)
(*                               syntax                                         *)
(* HOL/Refute.thy                This file: loads the ML files, basic setup,    *)
(*                               documentation                                  *)
(* HOL/SAT.thy                   Sets default parameters                        *)
(* HOL/ex/RefuteExamples.thy     Examples                                       *)
(* ------------------------------------------------------------------------ *)
```

end

43 SAT: Reconstructing external resolution proofs for propositional logic

theory *SAT* **imports** *Refute*

uses
Tools/cnf-funcs.ML
Tools/sat-funcs.ML

begin

Late package setup: default values for refute, see also theory *Refute*.

refute-params
[*itself=1*,
minsize=1,
maxsize=8,
maxvars=10000,
maxtime=60,
satsolver=auto]

⟨*ML*⟩

end

44 Recdef: TFL: recursive function definitions

theory *Recdef*
imports *Wellfounded-Relations FunDef*
uses
(*Tools/TFL/casesplit.ML*)
(*Tools/TFL/utils.ML*)
(*Tools/TFL/usyntax.ML*)
(*Tools/TFL/dcterm.ML*)

(*Tools/TFL/thms.ML*)
(*Tools/TFL/rules.ML*)
(*Tools/TFL/thry.ML*)
(*Tools/TFL/tfl.ML*)
(*Tools/TFL/post.ML*)
(*Tools/recdef-package.ML*)
begin

lemma *tfl-eq-True*: $(x = \mathit{True}) \dashrightarrow x$
⟨*proof*⟩

lemma *tfl-rev-eq-mp*: $(x = y) \dashrightarrow y \dashrightarrow x$
⟨*proof*⟩

lemma *tfl-simp-thm*: $(x \dashrightarrow y) \dashrightarrow (x = x') \dashrightarrow (x' \dashrightarrow y)$
⟨*proof*⟩

lemma *tfl-P-imp-P-iff-True*: $P \Longrightarrow P = \mathit{True}$
⟨*proof*⟩

lemma *tfl-imp-trans*: $(A \dashrightarrow B) \Longrightarrow (B \dashrightarrow C) \Longrightarrow (A \dashrightarrow C)$
⟨*proof*⟩

lemma *tfl-disj-assoc*: $(a \lor b) \lor c == a \lor (b \lor c)$
⟨*proof*⟩

lemma *tfl-disjE*: $P \lor Q \Longrightarrow P \dashrightarrow R \Longrightarrow Q \dashrightarrow R \Longrightarrow R$
⟨*proof*⟩

lemma *tfl-exE*: $\exists\, x.\ P\ x \Longrightarrow \forall\, x.\ P\ x \dashrightarrow Q \Longrightarrow Q$
⟨*proof*⟩

⟨*ML*⟩

lemmas [*recdef-simp*] =
inv-image-def
measure-def
lex-prod-def
same-fst-def
less-Suc-eq [*THEN iffD2*]

lemmas [*recdef-cong*] =
if-cong let-cong image-cong INT-cong UN-cong bex-cong ball-cong imp-cong

lemmas [*recdef-wf*] =
wf-trancl
wf-less-than
wf-lex-prod
wf-inv-image

wf-measure
wf-pred-nat
wf-same-fst
wf-empty

end

45 Extraction: Program extraction for HOL

theory *Extraction*
imports *Datatype*
uses *Tools/rewrite-hol-proof.ML*
begin

45.1 Setup

⟨*ML*⟩

lemmas [*extraction-expand*] =
meta-spec atomize-eq atomize-all atomize-imp atomize-conj
allE rev-mp conjE Eq-TrueI Eq-FalseI eqTrueI eqTrueE eq-cong2
notE′ impE′ impE iffE imp-cong simp-thms eq-True eq-False
induct-forall-eq induct-implies-eq induct-equal-eq induct-conj-eq
induct-forall-def induct-implies-def induct-equal-def induct-conj-def
induct-atomize induct-rulify induct-rulify-fallback
True-implies-equals TrueE

datatype *sumbool* = *Left* | *Right*

45.2 Type of extracted program

extract-type

$typeof\ (Trueprop\ P) \equiv typeof\ P$

$typeof\ P \equiv Type\ (TYPE(Null)) \Longrightarrow typeof\ Q \equiv Type\ (TYPE('Q)) \Longrightarrow$
$\quad typeof\ (P \longrightarrow Q) \equiv Type\ (TYPE('Q))$

$typeof\ Q \equiv Type\ (TYPE(Null)) \Longrightarrow typeof\ (P \longrightarrow Q) \equiv Type\ (TYPE(Null))$

$typeof\ P \equiv Type\ (TYPE('P)) \Longrightarrow typeof\ Q \equiv Type\ (TYPE('Q)) \Longrightarrow$
$\quad typeof\ (P \longrightarrow Q) \equiv Type\ (TYPE('P \Rightarrow 'Q))$

$(\lambda x.\ typeof\ (P\ x)) \equiv (\lambda x.\ Type\ (TYPE(Null))) \Longrightarrow$
$\quad typeof\ (\forall x.\ P\ x) \equiv Type\ (TYPE(Null))$

$(\lambda x.\ typeof\ (P\ x)) \equiv (\lambda x.\ Type\ (TYPE('P))) \Longrightarrow$
$\quad typeof\ (\forall x{::}'a.\ P\ x) \equiv Type\ (TYPE('a \Rightarrow 'P))$

$(\lambda x.\ typeof\ (P\ x)) \equiv (\lambda x.\ Type\ (TYPE(Null))) \Longrightarrow$
$\quad typeof\ (\exists x::'a.\ P\ x) \equiv Type\ (TYPE('a))$

$(\lambda x.\ typeof\ (P\ x)) \equiv (\lambda x.\ Type\ (TYPE('P))) \Longrightarrow$
$\quad typeof\ (\exists x::'a.\ P\ x) \equiv Type\ (TYPE('a \times 'P))$

$typeof\ P \equiv Type\ (TYPE(Null)) \Longrightarrow typeof\ Q \equiv Type\ (TYPE(Null)) \Longrightarrow$
$\quad typeof\ (P \vee Q) \equiv Type\ (TYPE(sumbool))$

$typeof\ P \equiv Type\ (TYPE(Null)) \Longrightarrow typeof\ Q \equiv Type\ (TYPE('Q)) \Longrightarrow$
$\quad typeof\ (P \vee Q) \equiv Type\ (TYPE('Q\ option))$

$typeof\ P \equiv Type\ (TYPE('P)) \Longrightarrow typeof\ Q \equiv Type\ (TYPE(Null)) \Longrightarrow$
$\quad typeof\ (P \vee Q) \equiv Type\ (TYPE('P\ option))$

$typeof\ P \equiv Type\ (TYPE('P)) \Longrightarrow typeof\ Q \equiv Type\ (TYPE('Q)) \Longrightarrow$
$\quad typeof\ (P \vee Q) \equiv Type\ (TYPE('P + 'Q))$

$typeof\ P \equiv Type\ (TYPE(Null)) \Longrightarrow typeof\ Q \equiv Type\ (TYPE('Q)) \Longrightarrow$
$\quad typeof\ (P \wedge Q) \equiv Type\ (TYPE('Q))$

$typeof\ P \equiv Type\ (TYPE('P)) \Longrightarrow typeof\ Q \equiv Type\ (TYPE(Null)) \Longrightarrow$
$\quad typeof\ (P \wedge Q) \equiv Type\ (TYPE('P))$

$typeof\ P \equiv Type\ (TYPE('P)) \Longrightarrow typeof\ Q \equiv Type\ (TYPE('Q)) \Longrightarrow$
$\quad typeof\ (P \wedge Q) \equiv Type\ (TYPE('P \times 'Q))$

$typeof\ (P = Q) \equiv typeof\ ((P \longrightarrow Q) \wedge (Q \longrightarrow P))$

$typeof\ (x \in P) \equiv typeof\ P$

45.3 Realizability

realizability

$(realizes\ t\ (Trueprop\ P)) \equiv (Trueprop\ (realizes\ t\ P))$

$(typeof\ P) \equiv (Type\ (TYPE(Null))) \Longrightarrow$
$\quad (realizes\ t\ (P \longrightarrow Q)) \equiv (realizes\ Null\ P \longrightarrow realizes\ t\ Q)$

$(typeof\ P) \equiv (Type\ (TYPE('P))) \Longrightarrow$
$(typeof\ Q) \equiv (Type\ (TYPE(Null))) \Longrightarrow$
$\quad (realizes\ t\ (P \longrightarrow Q)) \equiv (\forall x::'P.\ realizes\ x\ P \longrightarrow realizes\ Null\ Q)$

$(realizes\ t\ (P \longrightarrow Q)) \equiv (\forall x.\ realizes\ x\ P \longrightarrow realizes\ (t\ x)\ Q)$

$(\lambda x.\ typeof\ (P\ x)) \equiv (\lambda x.\ Type\ (TYPE(Null))) \Longrightarrow$
$\quad (realizes\ t\ (\forall x.\ P\ x)) \equiv (\forall x.\ realizes\ Null\ (P\ x))$

$(realizes\ t\ (\forall x.\ P\ x)) \equiv (\forall x.\ realizes\ (t\ x)\ (P\ x))$

$(\lambda x.\ typeof\ (P\ x)) \equiv (\lambda x.\ Type\ (TYPE(Null))) \Longrightarrow$
$\quad (realizes\ t\ (\exists x.\ P\ x)) \equiv (realizes\ Null\ (P\ t))$

$(realizes\ t\ (\exists x.\ P\ x)) \equiv (realizes\ (snd\ t)\ (P\ (fst\ t)))$

$(typeof\ P) \equiv (Type\ (TYPE(Null))) \Longrightarrow$
$(typeof\ Q) \equiv (Type\ (TYPE(Null))) \Longrightarrow$
$\quad (realizes\ t\ (P \vee Q)) \equiv$
$\quad (case\ t\ of\ Left \Rightarrow realizes\ Null\ P \mid Right \Rightarrow realizes\ Null\ Q)$

$(typeof\ P) \equiv (Type\ (TYPE(Null))) \Longrightarrow$
$\quad (realizes\ t\ (P \vee Q)) \equiv$
$\quad (case\ t\ of\ None \Rightarrow realizes\ Null\ P \mid Some\ q \Rightarrow realizes\ q\ Q)$

$(typeof\ Q) \equiv (Type\ (TYPE(Null))) \Longrightarrow$
$\quad (realizes\ t\ (P \vee Q)) \equiv$
$\quad (case\ t\ of\ None \Rightarrow realizes\ Null\ Q \mid Some\ p \Rightarrow realizes\ p\ P)$

$(realizes\ t\ (P \vee Q)) \equiv$
$(case\ t\ of\ Inl\ p \Rightarrow realizes\ p\ P \mid Inr\ q \Rightarrow realizes\ q\ Q)$

$(typeof\ P) \equiv (Type\ (TYPE(Null))) \Longrightarrow$
$\quad (realizes\ t\ (P \wedge Q)) \equiv (realizes\ Null\ P \wedge realizes\ t\ Q)$

$(typeof\ Q) \equiv (Type\ (TYPE(Null))) \Longrightarrow$
$\quad (realizes\ t\ (P \wedge Q)) \equiv (realizes\ t\ P \wedge realizes\ Null\ Q)$

$(realizes\ t\ (P \wedge Q)) \equiv (realizes\ (fst\ t)\ P \wedge realizes\ (snd\ t)\ Q)$

$typeof\ P \equiv Type\ (TYPE(Null)) \Longrightarrow$
$\quad realizes\ t\ (\neg\ P) \equiv \neg\ realizes\ Null\ P$

$typeof\ P \equiv Type\ (TYPE('P)) \Longrightarrow$
$\quad realizes\ t\ (\neg\ P) \equiv (\forall x::'P.\ \neg\ realizes\ x\ P)$

$typeof\ (P::bool) \equiv Type\ (TYPE(Null)) \Longrightarrow$
$typeof\ Q \equiv Type\ (TYPE(Null)) \Longrightarrow$
$\quad realizes\ t\ (P = Q) \equiv realizes\ Null\ P = realizes\ Null\ Q$

$(realizes\ t\ (P = Q)) \equiv (realizes\ t\ ((P \longrightarrow Q) \wedge (Q \longrightarrow P)))$

45.4 Computational content of basic inference rules

theorem *disjE-realizer*:
assumes r: $case\ x\ of\ Inl\ p \Rightarrow P\ p \mid Inr\ q \Rightarrow Q\ q$
and $r1$: $\bigwedge p.\ P\ p \Longrightarrow R\ (f\ p)$ **and** $r2$: $\bigwedge q.\ Q\ q \Longrightarrow R\ (g\ q)$
shows $R\ (case\ x\ of\ Inl\ p \Rightarrow f\ p \mid Inr\ q \Rightarrow g\ q)$
$\langle proof \rangle$

theorem *disjE-realizer2*:
  assumes *r*: $case\ x\ of\ None \Rightarrow P \mid Some\ q \Rightarrow Q\ q$
  and *r1*: $P \Longrightarrow R\ f$ **and** *r2*: $\bigwedge q.\ Q\ q \Longrightarrow R\ (g\ q)$
  shows $R\ (case\ x\ of\ None \Rightarrow f \mid Some\ q \Rightarrow g\ q)$
⟨*proof*⟩

theorem *disjE-realizer3*:
  assumes *r*: $case\ x\ of\ Left \Rightarrow P \mid Right \Rightarrow Q$
  and *r1*: $P \Longrightarrow R\ f$ **and** *r2*: $Q \Longrightarrow R\ g$
  shows $R\ (case\ x\ of\ Left \Rightarrow f \mid Right \Rightarrow g)$
⟨*proof*⟩

theorem *conjI-realizer*:
  $P\ p \Longrightarrow Q\ q \Longrightarrow P\ (fst\ (p,\ q)) \wedge Q\ (snd\ (p,\ q))$
  ⟨*proof*⟩

theorem *exI-realizer*:
  $P\ y\ x \Longrightarrow P\ (snd\ (x,\ y))\ (fst\ (x,\ y))$ ⟨*proof*⟩

theorem *exE-realizer*: $P\ (snd\ p)\ (fst\ p) \Longrightarrow$
  $(\bigwedge x\ y.\ P\ y\ x \Longrightarrow Q\ (f\ x\ y)) \Longrightarrow Q\ (let\ (x,\ y) = p\ in\ f\ x\ y)$
  ⟨*proof*⟩

theorem *exE-realizer'*: $P\ (snd\ p)\ (fst\ p) \Longrightarrow$
  $(\bigwedge x\ y.\ P\ y\ x \Longrightarrow Q) \Longrightarrow Q$ ⟨*proof*⟩

realizers
  $impI\ (P,\ Q)$: $\lambda pq.\ pq$
    $\Lambda\ P\ Q\ pq\ (h\text{: -}).\ allI \cdot - \bullet (\Lambda\ x.\ impI \cdot - \cdot - \bullet (h \cdot x))$

  $impI\ (P)$: *Null*
    $\Lambda\ P\ Q\ (h\text{: -}).\ allI \cdot - \bullet (\Lambda\ x.\ impI \cdot - \cdot - \bullet (h \cdot x))$

  $impI\ (Q)$: $\lambda q.\ q\ \Lambda\ P\ Q\ q.\ impI \cdot - \cdot -$

  impI: *Null impI*

  $mp\ (P,\ Q)$: $\lambda pq.\ pq$
    $\Lambda\ P\ Q\ pq\ (h\text{: -})\ p.\ mp \cdot - \cdot - \bullet (spec \cdot - \cdot p \bullet h)$

  $mp\ (P)$: *Null*
    $\Lambda\ P\ Q\ (h\text{: -})\ p.\ mp \cdot - \cdot - \bullet (spec \cdot - \cdot p \bullet h)$

  $mp\ (Q)$: $\lambda q.\ q\ \Lambda\ P\ Q\ q.\ mp \cdot - \cdot -$

  mp: *Null mp*

  $allI\ (P)$: $\lambda p.\ p\ \Lambda\ P\ p.\ allI \cdot -$

allI: *Null allI*

spec (P): $\lambda x\ p.\ p\ x\ \Lambda\ P\ x\ p.\ spec \cdot \cdot \cdot x$

spec: *Null spec*

exI (P): $\lambda x\ p.\ (x,\ p)\ \Lambda\ P\ x\ p.\ exI\text{-}realizer \cdot P \cdot p \cdot x$

exI: $\lambda x.\ x\ \Lambda\ P\ x\ (h\text{: -}).\ h$

exE (P, Q): $\lambda p\ pq.\ let\ (x,\ y) = p\ in\ pq\ x\ y$
$\Lambda\ P\ Q\ p\ (h\text{: -})\ pq.\ exE\text{-}realizer \cdot P \cdot p \cdot Q \cdot pq \bullet h$

exE (P): *Null*
$\Lambda\ P\ Q\ p.\ exE\text{-}realizer' \cdot \cdot \cdot \cdot \cdot \cdot$

exE (Q): $\lambda x\ pq.\ pq\ x$
$\Lambda\ P\ Q\ x\ (h1\text{: -})\ pq\ (h2\text{: -}).\ h2 \cdot x \bullet h1$

exE: *Null*
$\Lambda\ P\ Q\ x\ (h1\text{: -})\ (h2\text{: -}).\ h2 \cdot x \bullet h1$

conjI (P, Q): *Pair*
$\Lambda\ P\ Q\ p\ (h\text{: -})\ q.\ conjI\text{-}realizer \cdot P \cdot p \cdot Q \cdot q \bullet h$

conjI (P): $\lambda p.\ p$
$\Lambda\ P\ Q\ p.\ conjI \cdot \cdot \cdot \cdot$

conjI (Q): $\lambda q.\ q$
$\Lambda\ P\ Q\ (h\text{: -})\ q.\ conjI \cdot \cdot \cdot \cdot \bullet h$

conjI: *Null conjI*

conjunct1 (P, Q): *fst*
$\Lambda\ P\ Q\ pq.\ conjunct1 \cdot \cdot \cdot \cdot$

conjunct1 (P): $\lambda p.\ p$
$\Lambda\ P\ Q\ p.\ conjunct1 \cdot \cdot \cdot \cdot$

conjunct1 (Q): *Null*
$\Lambda\ P\ Q\ q.\ conjunct1 \cdot \cdot \cdot \cdot$

conjunct1: *Null conjunct1*

conjunct2 (P, Q): *snd*
$\Lambda\ P\ Q\ pq.\ conjunct2 \cdot \cdot \cdot \cdot$

conjunct2 (P): *Null*

Λ *P Q p. conjunct2* $\cdot$ - $\cdot$ -

conjunct2 (Q): $\lambda p.\ p$
Λ *P Q p. conjunct2* $\cdot$ - $\cdot$ -

conjunct2: *Null conjunct2*

disjI1 $(P,\ Q)$: *Inl*
Λ *P Q p. iffD2* $\cdot$ - $\cdot$ - $\bullet$ (*sum.cases-1* $\cdot$ *P* $\cdot$ - $\cdot$ *p*)

disjI1 (P): *Some*
Λ *P Q p. iffD2* $\cdot$ - $\cdot$ - $\bullet$ (*option.cases-2* $\cdot$ - $\cdot$ *P* $\cdot$ *p*)

disjI1 (Q): *None*
Λ *P Q. iffD2* $\cdot$ - $\cdot$ - $\bullet$ (*option.cases-1* $\cdot$ - $\cdot$ -)

disjI1: *Left*
Λ *P Q. iffD2* $\cdot$ - $\cdot$ - $\bullet$ (*sumbool.cases-1* $\cdot$ - $\cdot$ -)

disjI2 $(P,\ Q)$: *Inr*
Λ *Q P q. iffD2* $\cdot$ - $\cdot$ - $\bullet$ (*sum.cases-2* $\cdot$ - $\cdot$ *Q* $\cdot$ *q*)

disjI2 (P): *None*
Λ *Q P. iffD2* $\cdot$ - $\cdot$ - $\bullet$ (*option.cases-1* $\cdot$ - $\cdot$ -)

disjI2 (Q): *Some*
Λ *Q P q. iffD2* $\cdot$ - $\cdot$ - $\bullet$ (*option.cases-2* $\cdot$ - $\cdot$ *Q* $\cdot$ *q*)

disjI2: *Right*
Λ *Q P. iffD2* $\cdot$ - $\cdot$ - $\bullet$ (*sumbool.cases-2* $\cdot$ - $\cdot$ -)

disjE $(P,\ Q,\ R)$: $\lambda pq\ pr\ qr.$
(*case pq of Inl p* $\Rightarrow$ *pr p* | *Inr q* $\Rightarrow$ *qr q*)
Λ *P Q R pq* (*h1*: -) *pr* (*h2*: -) *qr*.
disjE-realizer $\cdot$ - $\cdot$ - $\cdot$ *pq* $\cdot$ *R* $\cdot$ *pr* $\cdot$ *qr* $\bullet$ *h1* $\bullet$ *h2*

disjE $(Q,\ R)$: $\lambda pq\ pr\ qr.$
(*case pq of None* $\Rightarrow$ *pr* | *Some q* $\Rightarrow$ *qr q*)
Λ *P Q R pq* (*h1*: -) *pr* (*h2*: -) *qr*.
disjE-realizer2 $\cdot$ - $\cdot$ - $\cdot$ *pq* $\cdot$ *R* $\cdot$ *pr* $\cdot$ *qr* $\bullet$ *h1* $\bullet$ *h2*

disjE $(P,\ R)$: $\lambda pq\ pr\ qr.$
(*case pq of None* $\Rightarrow$ *qr* | *Some p* $\Rightarrow$ *pr p*)
Λ *P Q R pq* (*h1*: -) *pr* (*h2*: -) *qr* (*h3*: -).
disjE-realizer2 $\cdot$ - $\cdot$ - $\cdot$ *pq* $\cdot$ *R* $\cdot$ *qr* $\cdot$ *pr* $\bullet$ *h1* $\bullet$ *h3* $\bullet$ *h2*

disjE (R): $\lambda pq\ pr\ qr.$
(*case pq of Left* $\Rightarrow$ *pr* | *Right* $\Rightarrow$ *qr*)
Λ *P Q R pq* (*h1*: -) *pr* (*h2*: -) *qr*.

disjE-realizer3 · - · - · - · - · *pq* · *R* · *pr* · *qr* • *h1* • *h2*

disjE (*P*, *Q*): *Null*
Λ *P* *Q* *R* *pq*. *disjE-realizer* · - · - · - · - · *pq* · (λ*x*. *R*) · - · - · -

disjE (*Q*): *Null*
Λ *P* *Q* *R* *pq*. *disjE-realizer2* · - · - · - · - · *pq* · (λ*x*. *R*) · - · - · -

disjE (*P*): *Null*
Λ *P* *Q* *R* *pq* (*h1*: -) (*h2*: -) (*h3*: -).
disjE-realizer2 · - · - · - · - · *pq* · (λ*x*. *R*) · - · - · - · - • *h1* • *h3* • *h2*

disjE: *Null*
Λ *P* *Q* *R* *pq*. *disjE-realizer3* · - · - · - · - · *pq* · (λ*x*. *R*) · - · - · -

FalseE (*P*): *arbitrary*
Λ *P*. *FalseE* · -

FalseE: *Null* *FalseE*

notI (*P*): *Null*
Λ *P* (*h*: -). *allI* · - • (Λ *x*. *notI* · - • (*h* · *x*))

notI: *Null* *notI*

notE (*P*, *R*): λ*p*. *arbitrary*
Λ *P* *R* (*h*: -) *p*. *notE* · - · - · - • (*spec* · - · *p* • *h*)

notE (*P*): *Null*
Λ *P* *R* (*h*: -) *p*. *notE* · - · - · - • (*spec* · - · *p* • *h*)

notE (*R*): *arbitrary*
Λ *P* *R*. *notE* · - · - · -

notE: *Null* *notE*

subst (*P*): λ*s* *t* *ps*. *ps*
Λ *s* *t* *P* (*h*: -) *ps*. *subst* · *s* · *t* · *P* *ps* • *h*

subst: *Null* *subst*

iffD1 (*P*, *Q*): *fst*
Λ *Q* *P* *pq* (*h*: -) *p*.
mp · - · - · - • (*spec* · - · *p* • (*conjunct1* · - · - · - • *h*))

iffD1 (*P*): λ*p*. *p*
Λ *Q* *P* *p* (*h*: -). *mp* · - · - · - • (*conjunct1* · - · - · - • *h*)

iffD1 (*Q*): *Null*

Λ *Q P q1* (*h*: -) *q2*.
mp · - - - • (*spec* · - · *q2* • (*conjunct1* · - - - • *h*))

iffD1: *Null iffD1*

iffD2 (*P*, *Q*): *snd*
Λ *P Q pq* (*h*: -) *q*.
mp · - - - • (*spec* · - · *q* • (*conjunct2* · - - - • *h*))

iffD2 (*P*): $\lambda p.\ p$
Λ *P Q p* (*h*: -). *mp* · - - - • (*conjunct2* · - - - • *h*)

iffD2 (*Q*): *Null*
Λ *P Q q1* (*h*: -) *q2*.
mp · - - - • (*spec* · - · *q2* • (*conjunct2* · - - - • *h*))

iffD2: *Null iffD2*

iffI (*P*, *Q*): *Pair*
Λ *P Q pq* (*h1* : -) *qp* (*h2* : -). *conjI-realizer* ·
($\lambda pq.\ \forall x.\ P\ x \longrightarrow Q\ (pq\ x)$) · *pq* ·
($\lambda qp.\ \forall x.\ Q\ x \longrightarrow P\ (qp\ x)$) · *qp* •
(*allI* · - • (Λ *x*. *impI* · - - - • (*h1* · *x*))) •
(*allI* · - • (Λ *x*. *impI* · - - - • (*h2* · *x*)))

iffI (*P*): $\lambda p.\ p$
Λ *P Q* (*h1* : -) *p* (*h2* : -). *conjI* · - - - •
(*allI* · - • (Λ *x*. *impI* · - - - • (*h1* · *x*))) •
(*impI* · - - - • *h2*)

iffI (*Q*): $\lambda q.\ q$
Λ *P Q q* (*h1* : -) (*h2* : -). *conjI* · - - - •
(*impI* · - - - • *h1*) •
(*allI* · - • (Λ *x*. *impI* · - - - • (*h2* · *x*)))

iffI: *Null iffI*

end

46 ATP-Linkup: The Isabelle-ATP Linkup

theory *ATP-Linkup*
imports *Divides Record Hilbert-Choice Presburger Relation-Power SAT Recdef Extraction*

uses

Tools/polyhash.ML
Tools/res-clause.ML
(*Tools/res-hol-clause.ML*)
(*Tools/res-axioms.ML*)
(*Tools/res-reconstruct.ML*)
(*Tools/watcher.ML*)
(*Tools/res-atp.ML*)
(*Tools/res-atp-provers.ML*)
(*Tools/res-atp-methods.ML*)
~~/src/Tools/Metis/metis.ML
(*Tools/metis-tools.ML*)
begin

definition *COMBI* :: *'a => 'a*
where *COMBI P == P*

definition *COMBK* :: *'a => 'b => 'a*
where *COMBK P Q == P*

definition *COMBB* :: *('b => 'c) => ('a => 'b) => 'a => 'c*
where *COMBB P Q R == P (Q R)*

definition *COMBC* :: *('a => 'b => 'c) => 'b => 'a => 'c*
where *COMBC P Q R == P R Q*

definition *COMBS* :: *('a => 'b => 'c) => ('a => 'b) => 'a => 'c*
where *COMBS P Q R == P R (Q R)*

definition *fequal* :: *'a => 'a => bool*
where *fequal X Y == (X=Y)*

lemma *fequal-imp-equal*: *fequal X Y ==> X=Y*
⟨*proof*⟩

lemma *equal-imp-fequal*: *X=Y ==> fequal X Y*
⟨*proof*⟩

These two represent the equivalence between Boolean equality and iff. They can't be converted to clauses automatically, as the iff would be expanded...

lemma *iff-positive*: *P | Q | P=Q*
⟨*proof*⟩

lemma *iff-negative*: *~P | ~Q | P=Q*
⟨*proof*⟩

Theorems for translation to combinators

lemma *abs-S*: *(%x. (f x) (g x)) == COMBS f g*
⟨*proof*⟩

lemma *abs-I*: $(\%x.\ x) == COMBI$
⟨*proof*⟩

lemma *abs-K*: $(\%x.\ y) == COMBK\ y$
⟨*proof*⟩

lemma *abs-B*: $(\%x.\ a\ (g\ x)) == COMBB\ a\ g$
⟨*proof*⟩

lemma *abs-C*: $(\%x.\ (f\ x)\ b) == COMBC\ f\ b$
⟨*proof*⟩

⟨*ML*⟩

46.1 Setup for Vampire, E prover and SPASS

⟨*ML*⟩

46.2 The Metis prover

⟨*ML*⟩

end

47 PreList: A Basis for Building the Theory of Lists

theory *PreList*
imports *ATP-Linkup*
uses *Tools/function-package/lexicographic-order.ML*
Tools/function-package/fundef-datatype.ML
begin

This is defined separately to serve as a basis for theory ToyList in the documentation.

⟨*ML*⟩

end

48 List: The datatype of finite lists

theory *List*
imports *PreList*
uses *Tools/string-syntax.ML*
begin

datatype 'a list =
Nil ([])
| Cons 'a 'a list (**infixr** # 65)

48.1 Basic list processing functions

consts
filter:: ('a => bool) => 'a list => 'a list
concat:: 'a list list => 'a list
foldl :: ('b => 'a => 'b) => 'b => 'a list => 'b
foldr :: ('a => 'b => 'b) => 'a list => 'b => 'b
hd:: 'a list => 'a
tl:: 'a list => 'a list
last:: 'a list => 'a
butlast :: 'a list => 'a list
set :: 'a list => 'a set
map :: ('a=>'b) => ('a list => 'b list)
listsum :: 'a list => 'a::monoid-add
nth :: 'a list => nat => 'a (**infixl** ! 100)
list-update :: 'a list => nat => 'a => 'a list
take:: nat => 'a list => 'a list
drop:: nat => 'a list => 'a list
takeWhile :: ('a => bool) => 'a list => 'a list
dropWhile :: ('a => bool) => 'a list => 'a list
rev :: 'a list => 'a list
zip :: 'a list => 'b list => ('a * 'b) list
upt :: nat => nat => nat list ((1[-..</-']))
remdups :: 'a list => 'a list
remove1 :: 'a => 'a list => 'a list
distinct:: 'a list => bool
replicate :: nat => 'a => 'a list
splice :: 'a list ⇒ 'a list ⇒ 'a list

nonterminals lupdbinds lupdbind

syntax
— list Enumeration
@list :: args => 'a list ([(-)])

— Special syntax for filter
@filter :: [pttrn, 'a list, bool] => 'a list ((1[-<−-./ -]))

— list update
-lupdbind:: ['a, 'a] => lupdbind ((2- :=/ -))
:: lupdbind => lupdbinds (-)
-lupdbinds :: [lupdbind, lupdbinds] => lupdbinds (-,/ -)
-LUpdate :: ['a, lupdbinds] => 'a (-/[(-)] [900,0] 900)

translations
  $[x, xs]$ == $x\#[xs]$
  $[x]$ == $x\#[]$
  $[x<-xs\ .\ P]$== *filter* $(\%x.\ P)$ *xs*

  -LUpdate xs (*-lupdbinds b bs*)== *-LUpdate* (*-LUpdate xs b*) *bs*
  $xs[i{:=}x]$ == *list-update xs i x*

syntax (*xsymbols*)
  @filter :: [*pttrn*, *'a list*, *bool*] => *'a list*((*1*[-←- ./ -]))
syntax (*HTML* **output**)
  @filter :: [*pttrn*, *'a list*, *bool*] => *'a list*((*1*[-←- ./ -]))

Function *size* is overloaded for all datatypes. Users may refer to the list version as *length*.

abbreviation
  length :: *'a list* => *nat* **where**
  length == *size*

primrec
  hd($x\#xs$) = x

primrec
  tl([]) = []
  tl($x\#xs$) = *xs*

primrec
  last($x\#xs$) = (*if xs*=[] *then x else last xs*)

primrec
  butlast []= []
  butlast($x\#xs$) = (*if xs*=[] *then* [] *else x#butlast xs*)

primrec
  set [] = {}
  set ($x\#xs$) = *insert x* (*set xs*)

primrec
  map f [] = []
  map f ($x\#xs$) = $f(x)\#$*map f xs*

⟨*ML*⟩
primrec
  append-Nil:[]@*ys* = *ys*
  append-Cons: ($x\#xs$)@*ys* = $x\#$(*xs*@*ys*)

primrec

$rev([]) = []$
$rev(x\#xs) = rev(xs)\ @\ [x]$

primrec
$filter\ P\ []\ =\ []$
$filter\ P\ (x\#xs) = (if\ P\ x\ then\ x\#filter\ P\ xs\ else\ filter\ P\ xs)$

primrec
$foldl\text{-}Nil{:}foldl\ f\ a\ []\ =\ a$
$foldl\text{-}Cons{:}\ foldl\ f\ a\ (x\#xs) = foldl\ f\ (f\ a\ x)\ xs$

primrec
$foldr\ f\ []\ a = a$
$foldr\ f\ (x\#xs)\ a = f\ x\ (foldr\ f\ xs\ a)$

primrec
$concat([]) = []$
$concat(x\#xs) = x\ @\ concat(xs)$

primrec
$listsum\ []\ =\ 0$
$listsum\ (x\ \#\ xs) = x + listsum\ xs$

primrec
$drop\text{-}Nil{:}drop\ n\ []\ =\ []$
$drop\text{-}Cons{:}\ drop\ n\ (x\#xs) = (case\ n\ of\ 0\ =>\ x\#xs\ |\ Suc(m)\ =>\ drop\ m\ xs)$
— Warning: simpset does not contain this definition, but separate theorems for $n = 0$ and $n = Suc\ k$

primrec
$take\text{-}Nil{:}take\ n\ []\ =\ []$
$take\text{-}Cons{:}\ take\ n\ (x\#xs) = (case\ n\ of\ 0\ =>\ []\ |\ Suc(m)\ =>\ x\ \#\ take\ m\ xs)$
— Warning: simpset does not contain this definition, but separate theorems for $n = 0$ and $n = Suc\ k$

primrec
$nth\text{-}Cons{:}(x\#xs)!n = (case\ n\ of\ 0\ =>\ x\ |\ (Suc\ k)\ =>\ xs!k)$
— Warning: simpset does not contain this definition, but separate theorems for $n = 0$ and $n = Suc\ k$

primrec
$[][i{:=}v] = []$
$(x\#xs)[i{:=}v] = (case\ i\ of\ 0\ =>\ v\ \#\ xs\ |\ Suc\ j\ =>\ x\ \#\ xs[j{:=}v])$

primrec
$takeWhile\ P\ []\ =\ []$
$takeWhile\ P\ (x\#xs) = (if\ P\ x\ then\ x\#takeWhile\ P\ xs\ else\ [])$

primrec

dropWhile P [] = []
dropWhile P (*x#xs*) = (*if P x then dropWhile P xs else x#xs*)

primrec
zip xs [] = []
zip-Cons: *zip xs* (*y#ys*) = (*case xs of* [] => [] | *z#zs* => (*z*,*y*)*#zip zs ys*)
— Warning: simpset does not contain this definition, but separate theorems for *xs* = [] and *xs* = *z* # *zs*

primrec
upt-0: [*i*..<*0*] = []
upt-Suc: [*i*..<(*Suc j*)] = (*if i* <= *j then* [*i*..<*j*] @ [*j*] *else* [])

primrec
distinct [] = *True*
distinct (*x#xs*) = (*x* ~: *set xs* ∧ *distinct xs*)

primrec
remdups [] = []
remdups (*x#xs*) = (*if x* : *set xs then remdups xs else x* # *remdups xs*)

primrec
remove1 x [] = []
remove1 x (*y#xs*) = (*if x*=*y then xs else y* # *remove1 x xs*)

primrec
replicate-0: *replicate 0 x* = []
replicate-Suc: *replicate* (*Suc n*) *x* = *x* # *replicate n x*

definition
rotate1 :: *'a list* ⇒ *'a list* **where**
rotate1 xs = (*case xs of* [] ⇒ [] | *x#xs* ⇒ *xs* @ [*x*])

definition
rotate :: *nat* ⇒ *'a list* ⇒ *'a list* **where**
rotate n = *rotate1* ^ *n*

definition
list-all2 :: (*'a* => *'b* => *bool*) => *'a list* => *'b list* => *bool* **where**
list-all2 P xs ys =
(*length xs* = *length ys* ∧ (∀ (*x*, *y*) ∈ *set* (*zip xs ys*). *P x y*))

definition
sublist :: *'a list* => *nat set* => *'a list* **where**
sublist xs A = *map fst* (*filter* (λ*p*. *snd p* ∈ *A*) (*zip xs* [*0*..<*size xs*]))

primrec
splice [] *ys* = *ys*
splice (*x#xs*) *ys* = (*if ys*=[] *then x#xs else x* # *hd ys* # *splice xs* (*tl ys*))

— Warning: simpset does not contain the second eqn but a derived one.

The following simple sort functions are intended for proofs, not for efficient implementations.

context *linorder*
begin

fun *sorted* :: $'a\ list \Rightarrow bool$ **where**
$sorted\ [] \longleftrightarrow True$ |
$sorted\ [x] \longleftrightarrow True$ |
$sorted\ (x\#y\#zs) \longleftrightarrow x <= y \wedge sorted\ (y\#zs)$

fun *insort* :: $'a \Rightarrow 'a\ list \Rightarrow 'a\ list$ **where**
$insort\ x\ [] = [x]$ |
$insort\ x\ (y\#ys) = (if\ x <= y\ then\ (x\#y\#ys)\ else\ y\#(insort\ x\ ys))$

fun *sort* :: $'a\ list \Rightarrow 'a\ list$ **where**
$sort\ [] = []$ |
$sort\ (x\#xs) = insort\ x\ (sort\ xs)$

end

48.1.1 List comprehension

Input syntax for Haskell-like list comprehension notation. Typical example: $[(x,y).\ x \leftarrow xs,\ y \leftarrow ys,\ x \neq y]$, the list of all pairs of distinct elements from *xs* and *ys*. The syntax is as in Haskell, except that | becomes a dot (like in Isabelle's set comprehension): $[e.\ x \leftarrow xs,\ \ldots]$ rather than `[e| x <- xs, ...]`.

The qualifiers after the dot are

generators $p \leftarrow xs$, where p is a pattern and xs an expression of list type, or

guards b, where b is a boolean expression.

Just like in Haskell, list comprehension is just a shorthand. To avoid misunderstandings, the translation into desugared form is not reversed upon output. Note that the translation of $[e.\ x \leftarrow xs]$ is optmized to $map\ (\lambda x.\ e)\ xs$.

It is easy to write short list comprehensions which stand for complex expressions. During proofs, they may become unreadable (and mangled). In such cases it can be advisable to introduce separate definitions for the list comprehensions in question.

nonterminals *lc-qual lc-quals*

syntax
-listcompr :: $'a \Rightarrow$ *lc-qual* $\Rightarrow$ *lc-quals* $\Rightarrow$ $'a$ *list* ([- . --)
-lc-gen :: $'a \Rightarrow 'a$ *list* $\Rightarrow$ *lc-qual* (- <− -)
-lc-test :: *bool* $\Rightarrow$ *lc-qual* (-)

-lc-end :: *lc-quals* (])
-lc-quals :: *lc-qual* $\Rightarrow$ *lc-quals* $\Rightarrow$ *lc-quals* (, --)
-lc-abs :: $'a$ => $'b$ *list* => $'b$ *list*

syntax (*xsymbols*)
-lc-gen :: $'a \Rightarrow 'a$ *list* $\Rightarrow$ *lc-qual* (- $\leftarrow$ -)
syntax (*HTML* **output**)
-lc-gen :: $'a \Rightarrow 'a$ *list* $\Rightarrow$ *lc-qual* (- $\leftarrow$ -)

$\langle ML \rangle$

48.1.2 [] and *op #*

lemma *not-Cons-self* [*simp*]:
  $xs \neq x \# xs$
$\langle proof \rangle$

lemmas *not-Cons-self2* [*simp*] = *not-Cons-self* [*symmetric*]

lemma *neq-Nil-conv*: $(xs \neq []) = (\exists\, y\ ys.\ xs = y \# ys)$
$\langle proof \rangle$

lemma *length-induct*:
  $(\bigwedge xs.\ \forall\, ys.\ length\ ys < length\ xs \longrightarrow P\ ys \Longrightarrow P\ xs) \Longrightarrow P\ xs$
$\langle proof \rangle$

48.1.3 *length*

Needs to come before @ because of theorem *append-eq-append-conv*.

lemma *length-append* [*simp*]: $length\ (xs\ @\ ys) = length\ xs + length\ ys$
$\langle proof \rangle$

lemma *length-map* [*simp*]: $length\ (map\ f\ xs) = length\ xs$
$\langle proof \rangle$

lemma *length-rev* [*simp*]: $length\ (rev\ xs) = length\ xs$
$\langle proof \rangle$

lemma *length-tl* [*simp*]: $length\ (tl\ xs) = length\ xs - 1$
$\langle proof \rangle$

lemma *length-0-conv* [*iff*]: $(length\ xs = 0) = (xs = [])$

⟨proof⟩

lemma *length-greater-0-conv* [*iff*]: $(0 < length\ xs) = (xs \neq [])$
⟨proof⟩

lemma *length-pos-if-in-set*: $x : set\ xs \Longrightarrow length\ xs > 0$
⟨proof⟩

lemma *length-Suc-conv*:
$(length\ xs = Suc\ n) = (\exists\, y\ ys.\ xs = y\ \#\ ys \wedge length\ ys = n)$
⟨proof⟩

lemma *Suc-length-conv*:
$(Suc\ n = length\ xs) = (\exists\, y\ ys.\ xs = y\ \#\ ys \wedge length\ ys = n)$
⟨proof⟩

lemma *impossible-Cons*: $length\ xs <= length\ ys ==> xs = x\ \#\ ys = False$
⟨proof⟩

lemma *list-induct2* [*consumes 1*]:
$\llbracket$ $length\ xs = length\ ys$;
$P\ []\ []$;
$\bigwedge x\ xs\ y\ ys.\ \llbracket\ length\ xs = length\ ys;\ P\ xs\ ys\ \rrbracket \Longrightarrow P\ (x\#xs)\ (y\#ys)\ \rrbracket$
$\Longrightarrow P\ xs\ ys$
⟨proof⟩

lemma *list-induct2'*:
$\llbracket\ P\ []\ []$;
$\bigwedge x\ xs.\ P\ (x\#xs)\ []$;
$\bigwedge y\ ys.\ P\ []\ (y\#ys)$;
$\bigwedge x\ xs\ y\ ys.\ P\ xs\ ys \Longrightarrow P\ (x\#xs)\ (y\#ys)\ \rrbracket$
$\Longrightarrow P\ xs\ ys$
⟨proof⟩

lemma *neq-if-length-neq*: $length\ xs \neq length\ ys \Longrightarrow (xs = ys) == False$
⟨proof⟩

⟨ML⟩

48.1.4 @ – append

lemma *append-assoc* [*simp*]: $(xs\ @\ ys)\ @\ zs = xs\ @\ (ys\ @\ zs)$
⟨proof⟩

lemma *append-Nil2* [*simp*]: $xs\ @\ [] = xs$
⟨proof⟩

interpretation *semigroup-append*: *semigroup-add* [*op* @]
⟨proof⟩

interpretation *monoid-append*: *monoid-add* [[] *op* @]
⟨*proof*⟩

lemma *append-is-Nil-conv* [*iff*]: $(xs\ @\ ys = []) = (xs = [] \wedge ys = [])$
⟨*proof*⟩

lemma *Nil-is-append-conv* [*iff*]: $([] = xs\ @\ ys) = (xs = [] \wedge ys = [])$
⟨*proof*⟩

lemma *append-self-conv* [*iff*]: $(xs\ @\ ys = xs) = (ys = [])$
⟨*proof*⟩

lemma *self-append-conv* [*iff*]: $(xs = xs\ @\ ys) = (ys = [])$
⟨*proof*⟩

lemma *append-eq-append-conv* [*simp*, *noatp*]:
$length\ xs = length\ ys \vee length\ us = length\ vs$
$==> (xs@us = ys@vs) = (xs{=}ys \wedge us{=}vs)$
⟨*proof*⟩

lemma *append-eq-append-conv2*: $(xs\ @\ ys = zs\ @\ ts) =$
$(EX\ us.\ xs = zs\ @\ us\ \&\ us\ @\ ys = ts \mid xs\ @\ us = zs\ \&\ ys = us@\ ts)$
⟨*proof*⟩

lemma *same-append-eq* [*iff*]: $(xs\ @\ ys = xs\ @\ zs) = (ys = zs)$
⟨*proof*⟩

lemma *append1-eq-conv* [*iff*]: $(xs\ @\ [x] = ys\ @\ [y]) = (xs = ys \wedge x = y)$
⟨*proof*⟩

lemma *append-same-eq* [*iff*]: $(ys\ @\ xs = zs\ @\ xs) = (ys = zs)$
⟨*proof*⟩

lemma *append-self-conv2* [*iff*]: $(xs\ @\ ys = ys) = (xs = [])$
⟨*proof*⟩

lemma *self-append-conv2* [*iff*]: $(ys = xs\ @\ ys) = (xs = [])$
⟨*proof*⟩

lemma *hd-Cons-tl* [*simp*,*noatp*]: $xs \neq [] ==> hd\ xs\ \#\ tl\ xs = xs$
⟨*proof*⟩

lemma *hd-append*: $hd\ (xs\ @\ ys) = (if\ xs = []\ then\ hd\ ys\ else\ hd\ xs)$
⟨*proof*⟩

lemma *hd-append2* [*simp*]: $xs \neq [] ==> hd\ (xs\ @\ ys) = hd\ xs$
⟨*proof*⟩

lemma *tl-append*: $tl\ (xs\ @\ ys) = (case\ xs\ of\ [] => tl\ ys \mid z\#zs => zs\ @\ ys)$

⟨*proof*⟩

lemma *tl-append2* [*simp*]: $xs \neq [] ==> tl\ (xs\ @\ ys) = tl\ xs\ @\ ys$
⟨*proof*⟩

lemma *Cons-eq-append-conv*: $x\#xs = ys@zs =$
$(ys = []\ \&\ x\#xs = zs\ |\ (EX\ ys'.\ x\#ys' = ys\ \&\ xs = ys'@zs))$
⟨*proof*⟩

lemma *append-eq-Cons-conv*: $(ys@zs = x\#xs) =$
$(ys = []\ \&\ zs = x\#xs\ |\ (EX\ ys'.\ ys = x\#ys'\ \&\ ys'@zs = xs))$
⟨*proof*⟩

Trivial rules for solving @-equations automatically.

lemma *eq-Nil-appendI*: $xs = ys ==> xs = []\ @\ ys$
⟨*proof*⟩

lemma *Cons-eq-appendI*:
$[|\ x\ \#\ xs1 = ys;\ xs = xs1\ @\ zs\ |] ==> x\ \#\ xs = ys\ @\ zs$
⟨*proof*⟩

lemma *append-eq-appendI*:
$[|\ xs\ @\ xs1 = zs;\ ys = xs1\ @\ us\ |] ==> xs\ @\ ys = zs\ @\ us$
⟨*proof*⟩

Simplification procedure for all list equalities. Currently only tries to rearrange @ to see if - both lists end in a singleton list, - or both lists end in the same list.

⟨*ML*⟩

48.1.5 *map*

lemma *map-ext*: $(!!x.\ x : set\ xs --> f\ x = g\ x) ==> map\ f\ xs = map\ g\ xs$
⟨*proof*⟩

lemma *map-ident* [*simp*]: $map\ (\lambda x.\ x) = (\lambda xs.\ xs)$
⟨*proof*⟩

lemma *map-append* [*simp*]: $map\ f\ (xs\ @\ ys) = map\ f\ xs\ @\ map\ f\ ys$
⟨*proof*⟩

lemma *map-compose*: $map\ (f\ o\ g)\ xs = map\ f\ (map\ g\ xs)$
⟨*proof*⟩

lemma *rev-map*: $rev\ (map\ f\ xs) = map\ f\ (rev\ xs)$
⟨*proof*⟩

lemma *map-eq-conv*[*simp*]: $(map\ f\ xs = map\ g\ xs) = (!x : set\ xs.\ f\ x = g\ x)$

⟨*proof*⟩

lemma *map-cong* [*fundef-cong*, *recdef-cong*]:
$xs = ys ==> (!!x.\ x : set\ ys ==> f\ x = g\ x) ==> map\ f\ xs = map\ g\ ys$
— a congruence rule for *map*
⟨*proof*⟩

lemma *map-is-Nil-conv* [*iff*]: $(map\ f\ xs = []) = (xs = [])$
⟨*proof*⟩

lemma *Nil-is-map-conv* [*iff*]: $([] = map\ f\ xs) = (xs = [])$
⟨*proof*⟩

lemma *map-eq-Cons-conv*:
$(map\ f\ xs = y\#ys) = (\exists\ z\ zs.\ xs = z\#zs \wedge f\ z = y \wedge map\ f\ zs = ys)$
⟨*proof*⟩

lemma *Cons-eq-map-conv*:
$(x\#xs = map\ f\ ys) = (\exists\ z\ zs.\ ys = z\#zs \wedge x = f\ z \wedge xs = map\ f\ zs)$
⟨*proof*⟩

lemmas *map-eq-Cons-D* = *map-eq-Cons-conv* [*THEN iffD1*]
lemmas *Cons-eq-map-D* = *Cons-eq-map-conv* [*THEN iffD1*]
declare *map-eq-Cons-D* [*dest*!] *Cons-eq-map-D* [*dest*!]

lemma *ex-map-conv*:
$(EX\ xs.\ ys = map\ f\ xs) = (ALL\ y : set\ ys.\ EX\ x.\ y = f\ x)$
⟨*proof*⟩

lemma *map-eq-imp-length-eq*:
$map\ f\ xs = map\ f\ ys ==> length\ xs = length\ ys$
⟨*proof*⟩

lemma *map-inj-on*:
$[|\ map\ f\ xs = map\ f\ ys;\ inj\text{-}on\ f\ (set\ xs\ Un\ set\ ys)\ |]$
$==> xs = ys$
⟨*proof*⟩

lemma *inj-on-map-eq-map*:
$inj\text{-}on\ f\ (set\ xs\ Un\ set\ ys) \Longrightarrow (map\ f\ xs = map\ f\ ys) = (xs = ys)$
⟨*proof*⟩

lemma *map-injective*:
$map\ f\ xs = map\ f\ ys ==> inj\ f ==> xs = ys$
⟨*proof*⟩

lemma *inj-map-eq-map*[*simp*]: $inj\ f \Longrightarrow (map\ f\ xs = map\ f\ ys) = (xs = ys)$
⟨*proof*⟩

lemma *inj-mapI*: $inj\ f\ {=}{=}{>}\ inj\ (map\ f)$
⟨*proof*⟩

lemma *inj-mapD*: $inj\ (map\ f)\ {=}{=}{>}\ inj\ f$
⟨*proof*⟩

lemma *inj-map*[*iff*]: $inj\ (map\ f) = inj\ f$
⟨*proof*⟩

lemma *inj-on-mapI*: $inj\text{-}on\ f\ (\bigcup(set\ `\ A)) \Longrightarrow inj\text{-}on\ (map\ f)\ A$
⟨*proof*⟩

lemma *map-idI*: $(\bigwedge x.\ x \in set\ xs \Longrightarrow f\ x = x) \Longrightarrow map\ f\ xs = xs$
⟨*proof*⟩

lemma *map-fun-upd* [*simp*]: $y \notin set\ xs \Longrightarrow map\ (f(y{:=}v))\ xs = map\ f\ xs$
⟨*proof*⟩

lemma *map-fst-zip*[*simp*]:
$length\ xs = length\ ys \Longrightarrow map\ fst\ (zip\ xs\ ys) = xs$
⟨*proof*⟩

lemma *map-snd-zip*[*simp*]:
$length\ xs = length\ ys \Longrightarrow map\ snd\ (zip\ xs\ ys) = ys$
⟨*proof*⟩

48.1.6 *rev*

lemma *rev-append* [*simp*]: $rev\ (xs\ @\ ys) = rev\ ys\ @\ rev\ xs$
⟨*proof*⟩

lemma *rev-rev-ident* [*simp*]: $rev\ (rev\ xs) = xs$
⟨*proof*⟩

lemma *rev-swap*: $(rev\ xs = ys) = (xs = rev\ ys)$
⟨*proof*⟩

lemma *rev-is-Nil-conv* [*iff*]: $(rev\ xs = []) = (xs = [])$
⟨*proof*⟩

lemma *Nil-is-rev-conv* [*iff*]: $([] = rev\ xs) = (xs = [])$
⟨*proof*⟩

lemma *rev-singleton-conv* [*simp*]: $(rev\ xs = [x]) = (xs = [x])$
⟨*proof*⟩

lemma *singleton-rev-conv* [*simp*]: $([x] = rev\ xs) = (xs = [x])$
⟨*proof*⟩

lemma *rev-is-rev-conv* [*iff*]: (*rev xs* = *rev ys*) = (*xs* = *ys*)
⟨*proof*⟩

lemma *inj-on-rev*[*iff*]: *inj-on rev A*
⟨*proof*⟩

lemma *rev-induct* [*case-names Nil snoc*]:
[| *P* []; !!*x xs*. *P xs* ==> *P* (*xs* @ [*x*]) |] ==> *P xs*
⟨*proof*⟩

lemma *rev-exhaust* [*case-names Nil snoc*]:
(*xs* = [] ==> *P*) ==>(!!*ys y*. *xs* = *ys* @ [*y*] ==> *P*) ==> *P*
⟨*proof*⟩

lemmas *rev-cases* = *rev-exhaust*

lemma *rev-eq-Cons-iff*[*iff*]: (*rev xs* = *y*#*ys*) = (*xs* = *rev ys* @ [*y*])
⟨*proof*⟩

48.1.7 *set*

lemma *finite-set* [*iff*]: *finite* (*set xs*)
⟨*proof*⟩

lemma *set-append* [*simp*]: *set* (*xs* @ *ys*) = (*set xs* ∪ *set ys*)
⟨*proof*⟩

lemma *hd-in-set*[*simp*]: *xs* ≠ [] ⟹ *hd xs* : *set xs*
⟨*proof*⟩

lemma *set-subset-Cons*: *set xs* ⊆ *set* (*x* # *xs*)
⟨*proof*⟩

lemma *set-ConsD*: *y* ∈ *set* (*x* # *xs*) ⟹ *y*=*x* ∨ *y* ∈ *set xs*
⟨*proof*⟩

lemma *set-empty* [*iff*]: (*set xs* = {}) = (*xs* = [])
⟨*proof*⟩

lemma *set-empty2*[*iff*]: ({} = *set xs*) = (*xs* = [])
⟨*proof*⟩

lemma *set-rev* [*simp*]: *set* (*rev xs*) = *set xs*
⟨*proof*⟩

lemma *set-map* [*simp*]: *set* (*map f xs*) = *f*'(*set xs*)
⟨*proof*⟩

lemma *set-filter* [*simp*]: *set* (*filter P xs*) = {*x*. *x* : *set xs* ∧ *P x*}

⟨*proof*⟩

lemma *set-upt* [*simp*]: $set[i..<j] = \{k.\ i \leq k \wedge k < j\}$
⟨*proof*⟩

lemma *in-set-conv-decomp*: $(x : set\ xs) = (\exists\, ys\ zs.\ xs = ys\ @\ x\ \#\ zs)$
⟨*proof*⟩

lemma *split-list*: $x : set\ xs \Longrightarrow \exists\, ys\ zs.\ xs = ys\ @\ x\ \#\ zs$
⟨*proof*⟩

lemma *in-set-conv-decomp-first*:
$(x : set\ xs) = (\exists\, ys\ zs.\ xs = ys\ @\ x\ \#\ zs \wedge x \notin set\ ys)$
⟨*proof*⟩

lemma *split-list-first*: $x : set\ xs \Longrightarrow \exists\, ys\ zs.\ xs = ys\ @\ x\ \#\ zs \wedge x \notin set\ ys$
⟨*proof*⟩

lemma *finite-list*: $finite\ A ==> EX\ l.\ set\ l = A$
⟨*proof*⟩

lemma *card-length*: $card\ (set\ xs) \leq length\ xs$
⟨*proof*⟩

48.1.8 *filter*

lemma *filter-append* [*simp*]: $filter\ P\ (xs\ @\ ys) = filter\ P\ xs\ @\ filter\ P\ ys$
⟨*proof*⟩

lemma *rev-filter*: $rev\ (filter\ P\ xs) = filter\ P\ (rev\ xs)$
⟨*proof*⟩

lemma *filter-filter* [*simp*]: $filter\ P\ (filter\ Q\ xs) = filter\ (\lambda x.\ Q\ x \wedge P\ x)\ xs$
⟨*proof*⟩

lemma *length-filter-le* [*simp*]: $length\ (filter\ P\ xs) \leq length\ xs$
⟨*proof*⟩

lemma *sum-length-filter-compl*:
$length(filter\ P\ xs) + length(filter\ (\%x.\ {}^\sim P\ x)\ xs) = length\ xs$
⟨*proof*⟩

lemma *filter-True* [*simp*]: $\forall\, x \in set\ xs.\ P\ x ==> filter\ P\ xs = xs$
⟨*proof*⟩

lemma *filter-False* [*simp*]: $\forall\, x \in set\ xs.\ \neg\ P\ x ==> filter\ P\ xs = []$
⟨*proof*⟩

lemma *filter-empty-conv*: $(filter\ P\ xs = []) = (\forall x \in set\ xs.\ \neg\ P\ x)$
⟨*proof*⟩

lemma *filter-id-conv*: $(filter\ P\ xs = xs) = (\forall x \in set\ xs.\ P\ x)$
⟨*proof*⟩

lemma *filter-map*:
$filter\ P\ (map\ f\ xs) = map\ f\ (filter\ (P\ o\ f)\ xs)$
⟨*proof*⟩

lemma *length-filter-map*[*simp*]:
$length\ (filter\ P\ (map\ f\ xs)) = length(filter\ (P\ o\ f)\ xs)$
⟨*proof*⟩

lemma *filter-is-subset* [*simp*]: $set\ (filter\ P\ xs) \leq set\ xs$
⟨*proof*⟩

lemma *length-filter-less*:
$[\![\ x : set\ xs;\ {}^{\sim}\ P\ x\ ]\!] \Longrightarrow length(filter\ P\ xs) < length\ xs$
⟨*proof*⟩

lemma *length-filter-conv-card*:
$length(filter\ p\ xs) = card\{i.\ i < length\ xs\ \&\ p(xs!i)\}$
⟨*proof*⟩

lemma *Cons-eq-filterD*:
$x\#xs = filter\ P\ ys \Longrightarrow$
$\exists\, us\ vs.\ ys = us\ @\ x\ \#\ vs \wedge (\forall u \in set\ us.\ \neg\ P\ u) \wedge P\ x \wedge xs = filter\ P\ vs$
(**is** - $\Longrightarrow \exists\, us\ vs.\ ?P\ ys\ us\ vs$)
⟨*proof*⟩

lemma *filter-eq-ConsD*:
$filter\ P\ ys = x\#xs \Longrightarrow$
$\exists\, us\ vs.\ ys = us\ @\ x\ \#\ vs \wedge (\forall u \in set\ us.\ \neg\ P\ u) \wedge P\ x \wedge xs = filter\ P\ vs$
⟨*proof*⟩

lemma *filter-eq-Cons-iff*:
$(filter\ P\ ys = x\#xs) =$
$(\exists\, us\ vs.\ ys = us\ @\ x\ \#\ vs \wedge (\forall u \in set\ us.\ \neg\ P\ u) \wedge P\ x \wedge xs = filter\ P\ vs)$
⟨*proof*⟩

lemma *Cons-eq-filter-iff*:
$(x\#xs = filter\ P\ ys) =$
$(\exists\, us\ vs.\ ys = us\ @\ x\ \#\ vs \wedge (\forall u \in set\ us.\ \neg\ P\ u) \wedge P\ x \wedge xs = filter\ P\ vs)$
⟨*proof*⟩

lemma *filter-cong*[*fundef-cong*, *recdef-cong*]:
$xs = ys \Longrightarrow (\bigwedge x.\ x \in set\ ys \Longrightarrow P\ x = Q\ x) \Longrightarrow filter\ P\ xs = filter\ Q\ ys$
⟨*proof*⟩

48.1.9 *concat*

lemma *concat-append* [*simp*]: *concat* (*xs* @ *ys*) = *concat xs* @ *concat ys*
⟨*proof*⟩

lemma *concat-eq-Nil-conv* [*simp*]: (*concat xss* = []) = (∀ *xs* ∈ *set xss*. *xs* = [])
⟨*proof*⟩

lemma *Nil-eq-concat-conv* [*simp*]: ([] = *concat xss*) = (∀ *xs* ∈ *set xss*. *xs* = [])
⟨*proof*⟩

lemma *set-concat* [*simp*]: *set* (*concat xs*) = (*UN x*:*set xs*. *set x*)
⟨*proof*⟩

lemma *concat-map-singleton*[*simp*]: *concat*(*map* (%*x*. [*f x*]) *xs*) = *map f xs*
⟨*proof*⟩

lemma *map-concat*: *map f* (*concat xs*) = *concat* (*map* (*map f*) *xs*)
⟨*proof*⟩

lemma *filter-concat*: *filter p* (*concat xs*) = *concat* (*map* (*filter p*) *xs*)
⟨*proof*⟩

lemma *rev-concat*: *rev* (*concat xs*) = *concat* (*map rev* (*rev xs*))
⟨*proof*⟩

48.1.10 *nth*

lemma *nth-Cons-0* [*simp*]: (*x* # *xs*)!*0* = *x*
⟨*proof*⟩

lemma *nth-Cons-Suc* [*simp*]: (*x* # *xs*)!(*Suc n*) = *xs*!*n*
⟨*proof*⟩

declare *nth.simps* [*simp del*]

lemma *nth-append*:
(*xs* @ *ys*)!*n* = (*if n* < *length xs then xs*!*n else ys*!(*n* − *length xs*))
⟨*proof*⟩

lemma *nth-append-length* [*simp*]: (*xs* @ *x* # *ys*) ! *length xs* = *x*
⟨*proof*⟩

lemma *nth-append-length-plus*[*simp*]: (*xs* @ *ys*) ! (*length xs* + *n*) = *ys* ! *n*
⟨*proof*⟩

lemma *nth-map* [*simp*]: *n* < *length xs* ==> (*map f xs*)!*n* = *f*(*xs*!*n*)
⟨*proof*⟩

lemma *hd-conv-nth*: *xs* ≠ [] ⟹ *hd xs* = *xs*!*0*

⟨*proof*⟩

lemma *list-eq-iff-nth-eq*:
$(xs = ys) = (length\ xs = length\ ys \wedge (ALL\ i<length\ xs.\ xs!i = ys!i))$
⟨*proof*⟩

lemma *set-conv-nth*: $set\ xs = \{xs!i \mid i.\ i < length\ xs\}$
⟨*proof*⟩

lemma *in-set-conv-nth*: $(x \in set\ xs) = (\exists\, i < length\ xs.\ xs!i = x)$
⟨*proof*⟩

lemma *list-ball-nth*: $[|\ n < length\ xs;\ !x : set\ xs.\ P\ x|] ==> P(xs!n)$
⟨*proof*⟩

lemma *nth-mem* [*simp*]: $n < length\ xs ==> xs!n : set\ xs$
⟨*proof*⟩

lemma *all-nth-imp-all-set*:
$[|\ !i < length\ xs.\ P(xs!i);\ x : set\ xs|] ==> P\ x$
⟨*proof*⟩

lemma *all-set-conv-all-nth*:
$(\forall\, x \in set\ xs.\ P\ x) = (\forall\, i.\ i < length\ xs \dashrightarrow P\ (xs\ !\ i))$
⟨*proof*⟩

lemma *rev-nth*:
$n < size\ xs \Longrightarrow rev\ xs\ !\ n = xs\ !\ (length\ xs - Suc\ n)$
⟨*proof*⟩

48.1.11 *list-update*

lemma *length-list-update* [*simp*]: $length(xs[i:=x]) = length\ xs$
⟨*proof*⟩

lemma *nth-list-update*:
$i < length\ xs==> (xs[i:=x])!j = (if\ i = j\ then\ x\ else\ xs!j)$
⟨*proof*⟩

lemma *nth-list-update-eq* [*simp*]: $i < length\ xs ==> (xs[i:=x])!i = x$
⟨*proof*⟩

lemma *nth-list-update-neq* [*simp*]: $i \neq j ==> xs[i:=x]!j = xs!j$
⟨*proof*⟩

lemma *list-update-overwrite* [*simp*]:
$i < size\ xs ==> xs[i:=x,\ i:=y] = xs[i:=y]$
⟨*proof*⟩

lemma *list-update-id*[*simp*]: $xs[i := xs!i] = xs$
⟨*proof*⟩

lemma *list-update-beyond*[*simp*]: $length\ xs \leq i \Longrightarrow xs[i{:=}x] = xs$
⟨*proof*⟩

lemma *list-update-same-conv*:
$i < length\ xs ==> (xs[i := x] = xs) = (xs!i = x)$
⟨*proof*⟩

lemma *list-update-append1*:
$i < size\ xs \Longrightarrow (xs\ @\ ys)[i{:=}x] = xs[i{:=}x]\ @\ ys$
⟨*proof*⟩

lemma *list-update-append*:
$(xs\ @\ ys)\ [n{:=}\ x] =$
$(if\ n < length\ xs\ then\ xs[n{:=}\ x]\ @\ ys\ else\ xs\ @\ (ys\ [n-length\ xs{:=}\ x]))$
⟨*proof*⟩

lemma *list-update-length* [*simp*]:
$(xs\ @\ x\ \#\ ys)[length\ xs := y] = (xs\ @\ y\ \#\ ys)$
⟨*proof*⟩

lemma *update-zip*:
$length\ xs = length\ ys ==>$
$(zip\ xs\ ys)[i{:=}xy] = zip\ (xs[i{:=}fst\ xy])\ (ys[i{:=}snd\ xy])$
⟨*proof*⟩

lemma *set-update-subset-insert*: $set(xs[i{:=}x]) <= insert\ x\ (set\ xs)$
⟨*proof*⟩

lemma *set-update-subsetI*: $[|\ set\ xs <= A;\ x{:}A\ |] ==> set(xs[i := x]) <= A$
⟨*proof*⟩

lemma *set-update-memI*: $n < length\ xs \Longrightarrow x \in set\ (xs[n := x])$
⟨*proof*⟩

lemma *list-update-overwrite*:
$xs\ [i := x,\ i := y] = xs\ [i := y]$
⟨*proof*⟩

lemma *list-update-swap*:
$i \neq i' \Longrightarrow xs\ [i := x,\ i' := x'] = xs\ [i' := x',\ i := x]$
⟨*proof*⟩

48.1.12 *last* and *butlast*

lemma *last-snoc* [*simp*]: $last\ (xs\ @\ [x]) = x$

⟨*proof*⟩

lemma *butlast-snoc* [*simp*]: *butlast* (*xs* @ [*x*]) = *xs*
⟨*proof*⟩

lemma *last-ConsL*: $xs = [] \Longrightarrow last(x\#xs) = x$
⟨*proof*⟩

lemma *last-ConsR*: $xs \neq [] \Longrightarrow last(x\#xs) = last\ xs$
⟨*proof*⟩

lemma *last-append*: *last*(*xs* @ *ys*) = (*if ys* = [] *then last xs else last ys*)
⟨*proof*⟩

lemma *last-appendL*[*simp*]: $ys = [] \Longrightarrow$ *last*(*xs* @ *ys*) = *last xs*
⟨*proof*⟩

lemma *last-appendR*[*simp*]: $ys \neq [] \Longrightarrow$ *last*(*xs* @ *ys*) = *last ys*
⟨*proof*⟩

lemma *hd-rev*: $xs \neq [] \Longrightarrow hd(rev\ xs) = last\ xs$
⟨*proof*⟩

lemma *last-rev*: $xs \neq [] \Longrightarrow last(rev\ xs) = hd\ xs$
⟨*proof*⟩

lemma *last-in-set*[*simp*]: $as \neq [] \Longrightarrow last\ as \in set\ as$
⟨*proof*⟩

lemma *length-butlast* [*simp*]: *length* (*butlast xs*) = *length xs* − *1*
⟨*proof*⟩

lemma *butlast-append*:
butlast (*xs* @ *ys*) = (*if ys* = [] *then butlast xs else xs* @ *butlast ys*)
⟨*proof*⟩

lemma *append-butlast-last-id* [*simp*]:
xs ≠ [] ==> *butlast xs* @ [*last xs*] = *xs*
⟨*proof*⟩

lemma *in-set-butlastD*: *x* : *set* (*butlast xs*) ==> *x* : *set xs*
⟨*proof*⟩

lemma *in-set-butlast-appendI*:
x : *set* (*butlast xs*) | *x* : *set* (*butlast ys*) ==> *x* : *set* (*butlast* (*xs* @ *ys*))
⟨*proof*⟩

lemma *last-drop*[*simp*]: $n < length\ xs \Longrightarrow last\ (drop\ n\ xs) = last\ xs$
⟨*proof*⟩

lemma *last-conv-nth*: $xs \neq [] \Longrightarrow last\ xs = xs!(length\ xs - 1)$
⟨*proof*⟩

48.1.13 *take* and *drop*

lemma *take-0* [*simp*]: $take\ 0\ xs = []$
⟨*proof*⟩

lemma *drop-0* [*simp*]: $drop\ 0\ xs = xs$
⟨*proof*⟩

lemma *take-Suc-Cons* [*simp*]: $take\ (Suc\ n)\ (x\ \#\ xs) = x\ \#\ take\ n\ xs$
⟨*proof*⟩

lemma *drop-Suc-Cons* [*simp*]: $drop\ (Suc\ n)\ (x\ \#\ xs) = drop\ n\ xs$
⟨*proof*⟩

declare *take-Cons* [*simp del*] **and** *drop-Cons* [*simp del*]

lemma *take-Suc*: $xs\ {}^{\sim}{=}\ []\ {=}{=}{>}\ take\ (Suc\ n)\ xs = hd\ xs\ \#\ take\ n\ (tl\ xs)$
⟨*proof*⟩

lemma *drop-Suc*: $drop\ (Suc\ n)\ xs = drop\ n\ (tl\ xs)$
⟨*proof*⟩

lemma *drop-tl*: $drop\ n\ (tl\ xs) = tl(drop\ n\ xs)$
⟨*proof*⟩

lemma *nth-via-drop*: $drop\ n\ xs = y\#ys \Longrightarrow xs!n = y$
⟨*proof*⟩

lemma *take-Suc-conv-app-nth*:
$i < length\ xs \Longrightarrow take\ (Suc\ i)\ xs = take\ i\ xs\ @\ [xs!i]$
⟨*proof*⟩

lemma *drop-Suc-conv-tl*:
$i < length\ xs \Longrightarrow (xs!i)\ \#\ (drop\ (Suc\ i)\ xs) = drop\ i\ xs$
⟨*proof*⟩

lemma *length-take* [*simp*]: $length\ (take\ n\ xs) = min\ (length\ xs)\ n$
⟨*proof*⟩

lemma *length-drop* [*simp*]: $length\ (drop\ n\ xs) = (length\ xs - n)$
⟨*proof*⟩

lemma *take-all* [*simp*]: $length\ xs <= n\ {=}{=}{>}\ take\ n\ xs = xs$
⟨*proof*⟩

lemma *drop-all* [*simp*]: *length xs <= n ==> drop n xs = []*
⟨*proof*⟩

lemma *take-append* [*simp*]:
take n (xs @ ys) = (take n xs @ take (n − length xs) ys)
⟨*proof*⟩

lemma *drop-append* [*simp*]:
drop n (xs @ ys) = drop n xs @ drop (n − length xs) ys
⟨*proof*⟩

lemma *take-take* [*simp*]: *take n (take m xs) = take (min n m) xs*
⟨*proof*⟩

lemma *drop-drop* [*simp*]: *drop n (drop m xs) = drop (n + m) xs*
⟨*proof*⟩

lemma *take-drop*: *take n (drop m xs) = drop m (take (n + m) xs)*
⟨*proof*⟩

lemma *drop-take*: *drop n (take m xs) = take (m−n) (drop n xs)*
⟨*proof*⟩

lemma *append-take-drop-id* [*simp*]: *take n xs @ drop n xs = xs*
⟨*proof*⟩

lemma *take-eq-Nil*[*simp*]: *(take n xs = []) = (n = 0 ∨ xs = [])*
⟨*proof*⟩

lemma *drop-eq-Nil*[*simp*]: *(drop n xs = []) = (length xs <= n)*
⟨*proof*⟩

lemma *take-map*: *take n (map f xs) = map f (take n xs)*
⟨*proof*⟩

lemma *drop-map*: *drop n (map f xs) = map f (drop n xs)*
⟨*proof*⟩

lemma *rev-take*: *rev (take i xs) = drop (length xs − i) (rev xs)*
⟨*proof*⟩

lemma *rev-drop*: *rev (drop i xs) = take (length xs − i) (rev xs)*
⟨*proof*⟩

lemma *nth-take* [*simp*]: *i < n ==> (take n xs)!i = xs!i*
⟨*proof*⟩

lemma *nth-drop* [*simp*]:
n + i <= length xs ==> (drop n xs)!i = xs!(n + i)

$\langle proof \rangle$

lemma *hd-drop-conv-nth*: $[\![\ xs \neq []; n < length\ xs\ ]\!] \Longrightarrow hd(drop\ n\ xs) = xs!n$
$\langle proof \rangle$

lemma *set-take-subset*: $set(take\ n\ xs) \subseteq set\ xs$
$\langle proof \rangle$

lemma *set-drop-subset*: $set(drop\ n\ xs) \subseteq set\ xs$
$\langle proof \rangle$

lemma *in-set-takeD*: $x : set(take\ n\ xs) \Longrightarrow x : set\ xs$
$\langle proof \rangle$

lemma *in-set-dropD*: $x : set(drop\ n\ xs) \Longrightarrow x : set\ xs$
$\langle proof \rangle$

lemma *append-eq-conv-conj*:
$(xs\ @\ ys = zs) = (xs = take\ (length\ xs)\ zs \wedge ys = drop\ (length\ xs)\ zs)$
$\langle proof \rangle$

lemma *take-add*:
$i{+}j \leq length(xs) \Longrightarrow take\ (i{+}j)\ xs = take\ i\ xs\ @\ take\ j\ (drop\ i\ xs)$
$\langle proof \rangle$

lemma *append-eq-append-conv-if*:
$(xs_1\ @\ xs_2 = ys_1\ @\ ys_2) =$
$(if\ size\ xs_1 \leq size\ ys_1$
$then\ xs_1 = take\ (size\ xs_1)\ ys_1 \wedge xs_2 = drop\ (size\ xs_1)\ ys_1\ @\ ys_2$
$else\ take\ (size\ ys_1)\ xs_1 = ys_1 \wedge drop\ (size\ ys_1)\ xs_1\ @\ xs_2 = ys_2)$
$\langle proof \rangle$

lemma *take-hd-drop*:
$n < length\ xs \Longrightarrow take\ n\ xs\ @\ [hd\ (drop\ n\ xs)] = take\ (n{+}1)\ xs$
$\langle proof \rangle$

lemma *id-take-nth-drop*:
$i < length\ xs \Longrightarrow xs = take\ i\ xs\ @\ xs!i\ \#\ drop\ (Suc\ i)\ xs$
$\langle proof \rangle$

lemma *upd-conv-take-nth-drop*:
$i < length\ xs \Longrightarrow xs[i{:=}a] = take\ i\ xs\ @\ a\ \#\ drop\ (Suc\ i)\ xs$
$\langle proof \rangle$

lemma *nth-drop'*:
$i < length\ xs \Longrightarrow xs\ !\ i\ \#\ drop\ (Suc\ i)\ xs = drop\ i\ xs$
$\langle proof \rangle$

48.1.14 *takeWhile* and *dropWhile*

lemma *takeWhile-dropWhile-id* [*simp*]: *takeWhile P xs @ dropWhile P xs = xs*
⟨*proof*⟩

lemma *takeWhile-append1* [*simp*]:
[| *x*:*set xs*; ~*P*(*x*)|] ==> *takeWhile P* (*xs @ ys*) = *takeWhile P xs*
⟨*proof*⟩

lemma *takeWhile-append2* [*simp*]:
(!!*x*. *x* : *set xs* ==> *P x*) ==> *takeWhile P* (*xs @ ys*) = *xs @ takeWhile P ys*
⟨*proof*⟩

lemma *takeWhile-tail*: $\neg$ *P x* ==> *takeWhile P* (*xs @* (*x#l*)) = *takeWhile P xs*
⟨*proof*⟩

lemma *dropWhile-append1* [*simp*]:
[| *x* : *set xs*; ~*P*(*x*)|] ==> *dropWhile P* (*xs @ ys*) = (*dropWhile P xs*)@*ys*
⟨*proof*⟩

lemma *dropWhile-append2* [*simp*]:
(!!*x*. *x*:*set xs* ==> *P*(*x*)) ==> *dropWhile P* (*xs @ ys*) = *dropWhile P ys*
⟨*proof*⟩

lemma *set-takeWhileD*: *x* : *set* (*takeWhile P xs*) ==> *x* : *set xs* $\wedge$ *P x*
⟨*proof*⟩

lemma *takeWhile-eq-all-conv*[*simp*]:
(*takeWhile P xs = xs*) = ($\forall$ *x* $\in$ *set xs*. *P x*)
⟨*proof*⟩

lemma *dropWhile-eq-Nil-conv*[*simp*]:
(*dropWhile P xs* = []) = ($\forall$ *x* $\in$ *set xs*. *P x*)
⟨*proof*⟩

lemma *dropWhile-eq-Cons-conv*:
(*dropWhile P xs = y#ys*) = (*xs = takeWhile P xs @ y # ys* & $\neg$ *P y*)
⟨*proof*⟩

The following two lemmmas could be generalized to an arbitrary property.

lemma *takeWhile-neq-rev*: ⟦*distinct xs*; *x* $\in$ *set xs*⟧ $\Longrightarrow$
takeWhile (λ*y*. *y* $\neq$ *x*) (*rev xs*) = *rev* (*tl* (*dropWhile* (λ*y*. *y* $\neq$ *x*) *xs*))
⟨*proof*⟩

lemma *dropWhile-neq-rev*: ⟦*distinct xs*; *x* $\in$ *set xs*⟧ $\Longrightarrow$
dropWhile (λ*y*. *y* $\neq$ *x*) (*rev xs*) = *x # rev* (*takeWhile* (λ*y*. *y* $\neq$ *x*) *xs*)
⟨*proof*⟩

lemma *takeWhile-not-last*:
⟦ *xs* $\neq$ []; *distinct xs*⟧ $\Longrightarrow$ *takeWhile* (λ*y*. *y* $\neq$ *last xs*) *xs* = *butlast xs*

⟨*proof*⟩

lemma *takeWhile-cong* [*fundef-cong*, *recdef-cong*]:
[| $l = k$; !!x. x : *set* l ==> $P\ x = Q\ x$ |]
==> *takeWhile* $P\ l$ = *takeWhile* $Q\ k$
⟨*proof*⟩

lemma *dropWhile-cong* [*fundef-cong*, *recdef-cong*]:
[| $l = k$; !!x. x : *set* l ==> $P\ x = Q\ x$ |]
==> *dropWhile* $P\ l$ = *dropWhile* $Q\ k$
⟨*proof*⟩

48.1.15 *zip*

lemma *zip-Nil* [*simp*]: *zip* [] *ys* = []
⟨*proof*⟩

lemma *zip-Cons-Cons* [*simp*]: *zip* (x # *xs*) (y # *ys*) = (x, y) # *zip xs ys*
⟨*proof*⟩

declare *zip-Cons* [*simp del*]

lemma *zip-Cons1*:
zip (x#*xs*) *ys* = (*case ys of* [] ⇒ [] | y#*ys* ⇒ (x,y)#*zip xs ys*)
⟨*proof*⟩

lemma *length-zip* [*simp*]:
length (*zip xs ys*) = *min* (*length xs*) (*length ys*)
⟨*proof*⟩

lemma *zip-append1*:
zip (*xs* @ *ys*) *zs* =
zip xs (*take* (*length xs*) *zs*) @ *zip ys* (*drop* (*length xs*) *zs*)
⟨*proof*⟩

lemma *zip-append2*:
zip xs (*ys* @ *zs*) =
zip (*take* (*length ys*) *xs*) *ys* @ *zip* (*drop* (*length ys*) *xs*) *zs*
⟨*proof*⟩

lemma *zip-append* [*simp*]:
[| *length xs* = *length us*; *length ys* = *length vs* |] ==>
zip (*xs*@*ys*) (*us*@*vs*) = *zip xs us* @ *zip ys vs*
⟨*proof*⟩

lemma *zip-rev*:
length xs = *length ys* ==> *zip* (*rev xs*) (*rev ys*) = *rev* (*zip xs ys*)
⟨*proof*⟩

lemma *map-zip-map*:
$map\ f\ (zip\ (map\ g\ xs)\ ys) = map\ (\%(x,y).\ f(g\ x,\ y))\ (zip\ xs\ ys)$
$\langle proof \rangle$

lemma *map-zip-map2*:
$map\ f\ (zip\ xs\ (map\ g\ ys)) = map\ (\%(x,y).\ f(x,\ g\ y))\ (zip\ xs\ ys)$
$\langle proof \rangle$

lemma *nth-zip* [*simp*]:
$[|\ i < length\ xs;\ i < length\ ys|] ==> (zip\ xs\ ys)!i = (xs!i,\ ys!i)$
$\langle proof \rangle$

lemma *set-zip*:
$set\ (zip\ xs\ ys) = \{(xs!i,\ ys!i)\ |\ i.\ i < min\ (length\ xs)\ (length\ ys)\}$
$\langle proof \rangle$

lemma *zip-update*:
$length\ xs = length\ ys ==> zip\ (xs[i:=x])\ (ys[i:=y]) = (zip\ xs\ ys)[i:=(x,y)]$
$\langle proof \rangle$

lemma *zip-replicate* [*simp*]:
$zip\ (replicate\ i\ x)\ (replicate\ j\ y) = replicate\ (min\ i\ j)\ (x,y)$
$\langle proof \rangle$

lemma *take-zip*:
$take\ n\ (zip\ xs\ ys) = zip\ (take\ n\ xs)\ (take\ n\ ys)$
$\langle proof \rangle$

lemma *drop-zip*:
$drop\ n\ (zip\ xs\ ys) = zip\ (drop\ n\ xs)\ (drop\ n\ ys)$
$\langle proof \rangle$

lemma *set-zip-leftD*:
$(x,y) \in set\ (zip\ xs\ ys) \Longrightarrow x \in set\ xs$
$\langle proof \rangle$

lemma *set-zip-rightD*:
$(x,y) \in set\ (zip\ xs\ ys) \Longrightarrow y \in set\ ys$
$\langle proof \rangle$

lemma *in-set-zipE*:
$(x,y) : set(zip\ xs\ ys) \Longrightarrow (\llbracket\ x : set\ xs;\ y : set\ ys\ \rrbracket \Longrightarrow R) \Longrightarrow R$
$\langle proof \rangle$

48.1.16 *list-all2*

lemma *list-all2-lengthD* [*intro?*]:
$list\text{-}all2\ P\ xs\ ys ==> length\ xs = length\ ys$
$\langle proof \rangle$

lemma *list-all2-Nil* [*iff*, *code*]: $list\text{-}all2\ P\ []\ ys = (ys = [])$
⟨*proof*⟩

lemma *list-all2-Nil2* [*iff*, *code*]: $list\text{-}all2\ P\ xs\ []\ = (xs = [])$
⟨*proof*⟩

lemma *list-all2-Cons* [*iff*, *code*]:
$list\text{-}all2\ P\ (x\ \#\ xs)\ (y\ \#\ ys) = (P\ x\ y \wedge list\text{-}all2\ P\ xs\ ys)$
⟨*proof*⟩

lemma *list-all2-Cons1*:
$list\text{-}all2\ P\ (x\ \#\ xs)\ ys = (\exists\, z\ zs.\ ys = z\ \#\ zs \wedge P\ x\ z \wedge list\text{-}all2\ P\ xs\ zs)$
⟨*proof*⟩

lemma *list-all2-Cons2*:
$list\text{-}all2\ P\ xs\ (y\ \#\ ys) = (\exists\, z\ zs.\ xs = z\ \#\ zs \wedge P\ z\ y \wedge list\text{-}all2\ P\ zs\ ys)$
⟨*proof*⟩

lemma *list-all2-rev* [*iff*]:
$list\text{-}all2\ P\ (rev\ xs)\ (rev\ ys) = list\text{-}all2\ P\ xs\ ys$
⟨*proof*⟩

lemma *list-all2-rev1*:
$list\text{-}all2\ P\ (rev\ xs)\ ys = list\text{-}all2\ P\ xs\ (rev\ ys)$
⟨*proof*⟩

lemma *list-all2-append1*:
$list\text{-}all2\ P\ (xs\ @\ ys)\ zs =$
$(EX\ us\ vs.\ zs = us\ @\ vs \wedge length\ us = length\ xs \wedge length\ vs = length\ ys \wedge$
$list\text{-}all2\ P\ xs\ us \wedge list\text{-}all2\ P\ ys\ vs)$
⟨*proof*⟩

lemma *list-all2-append2*:
$list\text{-}all2\ P\ xs\ (ys\ @\ zs) =$
$(EX\ us\ vs.\ xs = us\ @\ vs \wedge length\ us = length\ ys \wedge length\ vs = length\ zs \wedge$
$list\text{-}all2\ P\ us\ ys \wedge list\text{-}all2\ P\ vs\ zs)$
⟨*proof*⟩

lemma *list-all2-append*:
$length\ xs = length\ ys \Longrightarrow$
$list\text{-}all2\ P\ (xs@us)\ (ys@vs) = (list\text{-}all2\ P\ xs\ ys \wedge list\text{-}all2\ P\ us\ vs)$
⟨*proof*⟩

lemma *list-all2-appendI* [*intro?*, *trans*]:
$[\![\ list\text{-}all2\ P\ a\ b;\ list\text{-}all2\ P\ c\ d\ ]\!] \Longrightarrow list\text{-}all2\ P\ (a@c)\ (b@d)$
⟨*proof*⟩

lemma *list-all2-conv-all-nth*:

$list\text{-}all2\ P\ xs\ ys =$
$(length\ xs = length\ ys \wedge (\forall\, i < length\ xs.\ P\ (xs!i)\ (ys!i)))$
$\langle proof \rangle$

lemma *list-all2-trans*:
  assumes *tr*: $!!a\ b\ c.\ P1\ a\ b ==> P2\ b\ c ==> P3\ a\ c$
  shows $!!bs\ cs.\ list\text{-}all2\ P1\ as\ bs ==> list\text{-}all2\ P2\ bs\ cs ==> list\text{-}all2\ P3\ as\ cs$
    (**is** $!!bs\ cs.\ PROP\ ?Q\ as\ bs\ cs$)
$\langle proof \rangle$

lemma *list-all2-all-nthI* [*intro?*]:
  $length\ a = length\ b \Longrightarrow (\bigwedge n.\ n < length\ a \Longrightarrow P\ (a!n)\ (b!n)) \Longrightarrow list\text{-}all2\ P\ a\ b$
$\langle proof \rangle$

lemma *list-all2I*:
  $\forall\, x \in set\ (zip\ a\ b).\ split\ P\ x \Longrightarrow length\ a = length\ b \Longrightarrow list\text{-}all2\ P\ a\ b$
$\langle proof \rangle$

lemma *list-all2-nthD*:
  $[\![\ list\text{-}all2\ P\ xs\ ys;\ p < size\ xs\ ]\!] \Longrightarrow P\ (xs!p)\ (ys!p)$
$\langle proof \rangle$

lemma *list-all2-nthD2*:
  $[\![list\text{-}all2\ P\ xs\ ys;\ p < size\ ys]\!] \Longrightarrow P\ (xs!p)\ (ys!p)$
$\langle proof \rangle$

lemma *list-all2-map1*:
  $list\text{-}all2\ P\ (map\ f\ as)\ bs = list\text{-}all2\ (\lambda x\ y.\ P\ (f\ x)\ y)\ as\ bs$
$\langle proof \rangle$

lemma *list-all2-map2*:
  $list\text{-}all2\ P\ as\ (map\ f\ bs) = list\text{-}all2\ (\lambda x\ y.\ P\ x\ (f\ y))\ as\ bs$
$\langle proof \rangle$

lemma *list-all2-refl* [*intro?*]:
  $(\bigwedge x.\ P\ x\ x) \Longrightarrow list\text{-}all2\ P\ xs\ xs$
$\langle proof \rangle$

lemma *list-all2-update-cong*:
  $[\![\ i{<}size\ xs;\ list\text{-}all2\ P\ xs\ ys;\ P\ x\ y\ ]\!] \Longrightarrow list\text{-}all2\ P\ (xs[i{:=}x])\ (ys[i{:=}y])$
$\langle proof \rangle$

lemma *list-all2-update-cong2*:
  $[\![list\text{-}all2\ P\ xs\ ys;\ P\ x\ y;\ i < length\ ys]\!] \Longrightarrow list\text{-}all2\ P\ (xs[i{:=}x])\ (ys[i{:=}y])$
$\langle proof \rangle$

lemma *list-all2-takeI* [*simp,intro?*]:
  $list\text{-}all2\ P\ xs\ ys \Longrightarrow list\text{-}all2\ P\ (take\ n\ xs)\ (take\ n\ ys)$
$\langle proof \rangle$

lemma *list-all2-dropI* [*simp*,*intro?*]:
list-all2 P as bs $\Longrightarrow$ *list-all2 P* (*drop n as*) (*drop n bs*)
⟨*proof*⟩

lemma *list-all2-mono* [*intro?*]:
list-all2 P xs ys $\Longrightarrow$ ($\bigwedge$*xs ys*. *P xs ys* $\Longrightarrow$ *Q xs ys*) $\Longrightarrow$ *list-all2 Q xs ys*
⟨*proof*⟩

lemma *list-all2-eq*:
xs = *ys* $\longleftrightarrow$ *list-all2* (*op* =) *xs ys*
⟨*proof*⟩

48.1.17 *foldl* and *foldr*

lemma *foldl-append* [*simp*]:
foldl f a (*xs* @ *ys*) = *foldl f* (*foldl f a xs*) *ys*
⟨*proof*⟩

lemma *foldr-append*[*simp*]: *foldr f* (*xs* @ *ys*) *a* = *foldr f xs* (*foldr f ys a*)
⟨*proof*⟩

lemma *foldr-map*: *foldr g* (*map f xs*) *a* = *foldr* (*g o f*) *xs a*
⟨*proof*⟩

For efficient code generation: avoid intermediate list.

lemma *foldl-map*[*code unfold*]:
foldl g a (*map f xs*) = *foldl* (%*a x*. *g a* (*f x*)) *a xs*
⟨*proof*⟩

lemma *foldl-cong* [*fundef-cong*, *recdef-cong*]:
[| *a* = *b*; *l* = *k*; !!*a x*. *x* : *set l* ==> *f a x* = *g a x* |]
==> *foldl f a l* = *foldl g b k*
⟨*proof*⟩

lemma *foldr-cong* [*fundef-cong*, *recdef-cong*]:
[| *a* = *b*; *l* = *k*; !!*a x*. *x* : *set l* ==> *f x a* = *g x a* |]
==> *foldr f l a* = *foldr g k b*
⟨*proof*⟩

lemma (**in** *semigroup-add*) *foldl-assoc*:
shows *foldl op+* (*x*+*y*) *zs* = *x* + (*foldl op+ y zs*)
⟨*proof*⟩

lemma (**in** *monoid-add*) *foldl-absorb0*:
shows *x* + (*foldl op+ 0 zs*) = *foldl op+ x zs*
⟨*proof*⟩

The "First Duality Theorem" in Bird & Wadler:

lemma *foldl-foldr1-lemma*:
$foldl\ op\ +\ a\ xs = a + foldr\ op\ +\ xs\ (0::'a::monoid\text{-}add)$
$\langle proof \rangle$

corollary *foldl-foldr1*:
$foldl\ op\ +\ 0\ xs = foldr\ op\ +\ xs\ (0::'a::monoid\text{-}add)$
$\langle proof \rangle$

The "Third Duality Theorem" in Bird & Wadler:

lemma *foldr-foldl*: $foldr\ f\ xs\ a = foldl\ (\%x\ y.\ f\ y\ x)\ a\ (rev\ xs)$
$\langle proof \rangle$

lemma *foldl-foldr*: $foldl\ f\ a\ xs = foldr\ (\%x\ y.\ f\ y\ x)\ (rev\ xs)\ a$
$\langle proof \rangle$

lemma (**in** *ab-semigroup-add*) *foldr-conv-foldl*: $foldr\ op\ +\ xs\ a = foldl\ op\ +\ a\ xs$
$\langle proof \rangle$

Note: $n \leq foldl\ (op\ +)\ n\ ns$ looks simpler, but is more difficult to use because it requires an additional transitivity step.

lemma *start-le-sum*: $(m::nat) <= n ==> m <= foldl\ (op\ +)\ n\ ns$
$\langle proof \rangle$

lemma *elem-le-sum*: $(n::nat) : set\ ns ==> n <= foldl\ (op\ +)\ 0\ ns$
$\langle proof \rangle$

lemma *sum-eq-0-conv* [*iff*]:
$(foldl\ (op\ +)\ (m::nat)\ ns = 0) = (m = 0 \wedge (\forall n \in set\ ns.\ n = 0))$
$\langle proof \rangle$

lemma *foldr-invariant*:
$[\![Q\ x\ ;\ \forall\ x \in set\ xs.\ P\ x;\ \forall\ x\ y.\ P\ x \wedge Q\ y \longrightarrow Q\ (f\ x\ y)\ ]\!] \Longrightarrow Q\ (foldr\ f\ xs\ x)$
$\langle proof \rangle$

lemma *foldl-invariant*:
$[\![Q\ x\ ;\ \forall\ x \in set\ xs.\ P\ x;\ \forall\ x\ y.\ P\ x \wedge Q\ y \longrightarrow Q\ (f\ y\ x)\ ]\!] \Longrightarrow Q\ (foldl\ f\ x\ xs)$
$\langle proof \rangle$

foldl and *concat*

lemma *concat-conv-foldl*: $concat\ xss = foldl\ op@\ []\ xss$
$\langle proof \rangle$

lemma *foldl-conv-concat*:
$foldl\ (op\ @)\ xs\ xxs = xs\ @\ (concat\ xxs)$
$\langle proof \rangle$

48.1.18 List summation: *listsum* and $\sum$

lemma *listsum-append*[*simp*]: $listsum\ (xs\ @\ ys) = listsum\ xs + listsum\ ys$

⟨*proof*⟩

lemma *listsum-rev*[*simp*]:
fixes *xs* :: *'a*::*comm-monoid-add list*
shows *listsum* (*rev xs*) = *listsum xs*
⟨*proof*⟩

lemma *listsum-foldr*:
listsum xs = *foldr* (*op* +) *xs 0*
⟨*proof*⟩

For efficient code generation — *listsum* is not tail recursive but *foldl* is.

lemma *listsum*[*code unfold*]: *listsum xs* = *foldl* (*op* +) *0 xs*
⟨*proof*⟩

Some syntactic sugar for summing a function over a list:

syntax
  -listsum :: *pttrn* => *'a list* => *'b* => *'b*   ((*3SUM* -<--. -) [*0*, *51*, *10*] *10*)
syntax (*xsymbols*)
  -listsum :: *pttrn* => *'a list* => *'b* => *'b*   ((*3*∑-←-. -) [*0*, *51*, *10*] *10*)
syntax (*HTML* **output**)
  -listsum :: *pttrn* => *'a list* => *'b* => *'b*   ((*3*∑-←-. -) [*0*, *51*, *10*] *10*)

translations — Beware of argument permutation!
  SUM x<-*xs*. *b* == *CONST listsum* (*map* (%*x*. *b*) *xs*)
  ∑*x*←*xs*. *b* == *CONST listsum* (*map* (%*x*. *b*) *xs*)

lemma *listsum-0* [*simp*]: (∑*x*←*xs*. *0*) = *0*
⟨*proof*⟩

For non-Abelian groups *xs* needs to be reversed on one side:

lemma *uminus-listsum-map*:
− *listsum* (*map f xs*) = (*listsum* (*map* (*uminus o f*) *xs*) :: *'a*::*ab-group-add*)
⟨*proof*⟩

48.1.19 *upt*

lemma *upt-rec*[*code*]: [*i*..<*j*] = (*if i*<*j then i*#[*Suc i*..<*j*] *else* [])
— simp does not terminate!
⟨*proof*⟩

lemma *upt-conv-Nil* [*simp*]: *j* <= *i* ==> [*i*..<*j*] = []
⟨*proof*⟩

lemma *upt-eq-Nil-conv*[*simp*]: ([*i*..<*j*] = []) = (*j* = *0* ∨ *j* <= *i*)
⟨*proof*⟩

lemma *upt-eq-Cons-conv*:
([*i*..<*j*] = *x*#*xs*) = (*i* < *j* & *i* = *x* & [*i*+*1*..<*j*] = *xs*)

⟨*proof*⟩

lemma *upt-Suc-append*: *i* <= *j* ==> [*i*..<(*Suc j*)] = [*i*..<*j*]@[*j*]
— Only needed if *upt-Suc* is deleted from the simpset.
⟨*proof*⟩

lemma *upt-conv-Cons*: *i* < *j* ==> [*i*..<*j*] = *i* # [*Suc i*..<*j*]
⟨*proof*⟩

lemma *upt-add-eq-append*: *i*<=*j* ==> [*i*..<*j*+*k*] = [*i*..<*j*]@[*j*..<*j*+*k*]
— LOOPS as a simprule, since *j* <= *j*.
⟨*proof*⟩

lemma *length-upt* [*simp*]: *length* [*i*..<*j*] = *j* − *i*
⟨*proof*⟩

lemma *nth-upt* [*simp*]: *i* + *k* < *j* ==> [*i*..<*j*] ! *k* = *i* + *k*
⟨*proof*⟩

lemma *hd-upt*[*simp*]: *i* < *j* ⟹ *hd*[*i*..<*j*] = *i*
⟨*proof*⟩

lemma *last-upt*[*simp*]: *i* < *j* ⟹ *last*[*i*..<*j*] = *j* − *1*
⟨*proof*⟩

lemma *take-upt* [*simp*]: *i*+*m* <= *n* ==> *take m* [*i*..<*n*] = [*i*..<*i*+*m*]
⟨*proof*⟩

lemma *drop-upt*[*simp*]: *drop m* [*i*..<*j*] = [*i*+*m*..<*j*]
⟨*proof*⟩

lemma *map-Suc-upt*: *map Suc* [*m*..<*n*] = [*Suc m*..<*Suc n*]
⟨*proof*⟩

lemma *nth-map-upt*: *i* < *n*−*m* ==> (*map f* [*m*..<*n*]) ! *i* = *f*(*m*+*i*)
⟨*proof*⟩

lemma *nth-take-lemma*:
k <= *length xs* ==> *k* <= *length ys* ==>
(!!*i*. *i* < *k* −−> *xs*!*i* = *ys*!*i*) ==> *take k xs* = *take k ys*
⟨*proof*⟩

lemma *nth-equalityI*:
[| *length xs* = *length ys*; *ALL i* < *length xs*. *xs*!*i* = *ys*!*i* |] ==> *xs* = *ys*
⟨*proof*⟩

lemma *map-nth*:
map (λ*i*. *xs* ! *i*) [*0*..<*length xs*] = *xs*

⟨*proof*⟩

lemma *list-all2-antisym*:
$[\![\ (\bigwedge x\ y.\ [\![P\ x\ y;\ Q\ y\ x]\!] \Longrightarrow x = y);\ list\text{-}all2\ P\ xs\ ys;\ list\text{-}all2\ Q\ ys\ xs\ ]\!]$
$\Longrightarrow xs = ys$
⟨*proof*⟩

lemma *take-equalityI*: $(\forall\, i.\ take\ i\ xs = take\ i\ ys) ==> xs = ys$
— The famous take-lemma.
⟨*proof*⟩

lemma *take-Cons'*:
$take\ n\ (x\ \#\ xs) = (if\ n = 0\ then\ []\ else\ x\ \#\ take\ (n - 1)\ xs)$
⟨*proof*⟩

lemma *drop-Cons'*:
$drop\ n\ (x\ \#\ xs) = (if\ n = 0\ then\ x\ \#\ xs\ else\ drop\ (n - 1)\ xs)$
⟨*proof*⟩

lemma *nth-Cons'*: $(x\ \#\ xs)!n = (if\ n = 0\ then\ x\ else\ xs!(n - 1))$
⟨*proof*⟩

lemmas *take-Cons-number-of* = *take-Cons'*[*of number-of v*,*standard*]
lemmas *drop-Cons-number-of* = *drop-Cons'*[*of number-of v*,*standard*]
lemmas *nth-Cons-number-of* = *nth-Cons'*[*of - - number-of v*,*standard*]

declare *take-Cons-number-of* [*simp*]
drop-Cons-number-of [*simp*]
nth-Cons-number-of [*simp*]

48.1.20 *distinct* and *remdups*

lemma *distinct-append* [*simp*]:
$distinct\ (xs\ @\ ys) = (distinct\ xs \wedge distinct\ ys \wedge set\ xs \cap set\ ys = \{\})$
⟨*proof*⟩

lemma *distinct-rev*[*simp*]: $distinct(rev\ xs) = distinct\ xs$
⟨*proof*⟩

lemma *set-remdups* [*simp*]: $set\ (remdups\ xs) = set\ xs$
⟨*proof*⟩

lemma *distinct-remdups* [*iff*]: $distinct\ (remdups\ xs)$
⟨*proof*⟩

lemma *distinct-remdups-id*: $distinct\ xs ==> remdups\ xs = xs$
⟨*proof*⟩

lemma *remdups-id-iff-distinct*[*simp*]: (*remdups xs* = *xs*) = *distinct xs*
⟨*proof*⟩

lemma *finite-distinct-list*: *finite A* ⟹ *EX xs. set xs* = *A* & *distinct xs*
⟨*proof*⟩

lemma *remdups-eq-nil-iff* [*simp*]: (*remdups x* = []) = (*x* = [])
⟨*proof*⟩

lemma *remdups-eq-nil-right-iff* [*simp*]: ([] = *remdups x*) = (*x* = [])
⟨*proof*⟩

lemma *length-remdups-leq*[*iff*]: *length*(*remdups xs*) <= *length xs*
⟨*proof*⟩

lemma *length-remdups-eq*[*iff*]:
(*length* (*remdups xs*) = *length xs*) = (*remdups xs* = *xs*)
⟨*proof*⟩

lemma *distinct-map*:
distinct(*map f xs*) = (*distinct xs* & *inj-on f* (*set xs*))
⟨*proof*⟩

lemma *distinct-filter* [*simp*]: *distinct xs* ==> *distinct* (*filter P xs*)
⟨*proof*⟩

lemma *distinct-upt*[*simp*]: *distinct*[*i*..<*j*]
⟨*proof*⟩

lemma *distinct-take*[*simp*]: *distinct xs* ⟹ *distinct* (*take i xs*)
⟨*proof*⟩

lemma *distinct-drop*[*simp*]: *distinct xs* ⟹ *distinct* (*drop i xs*)
⟨*proof*⟩

lemma *distinct-list-update*:
assumes *d*: *distinct xs* **and** *a*: $a \notin$ *set xs* − {*xs*!*i*}
shows *distinct* (*xs*[*i*:=*a*])
⟨*proof*⟩

It is best to avoid this indexed version of distinct, but sometimes it is useful.

lemma *distinct-conv-nth*:
distinct xs = (∀ *i* < *size xs*. ∀ *j* < *size xs*. *i* ≠ *j* −−> *xs*!*i* ≠ *xs*!*j*)
⟨*proof*⟩

lemma *nth-eq-iff-index-eq*:

$[\![$ *distinct xs*; $i <$ *length xs*; $j <$ *length xs* $]\!] \Longrightarrow (xs!i = xs!j) = (i = j)$
⟨*proof*⟩

lemma *distinct-card*: *distinct xs ==> card (set xs) = size xs*
⟨*proof*⟩

lemma *card-distinct*: *card (set xs) = size xs ==> distinct xs*
⟨*proof*⟩

lemma *not-distinct-decomp*: ~ *distinct ws ==> EX xs ys zs y. ws = xs@[y]@ys@[y]@zs*
⟨*proof*⟩

lemma *length-remdups-concat*:
length(remdups(concat xss)) = *card*($\bigcup$*xs* ∈ *set xss. set xs*)
⟨*proof*⟩

48.1.21 *remove1*

lemma *remove1-append*:
remove1 x (xs @ ys) =
(*if x* ∈ *set xs then remove1 x xs @ ys else xs @ remove1 x ys*)
⟨*proof*⟩

lemma *in-set-remove1*[*simp*]:
$a \neq b \Longrightarrow$ *a : set(remove1 b xs) = (a : set xs)*
⟨*proof*⟩

lemma *set-remove1-subset*: *set(remove1 x xs) <= set xs*
⟨*proof*⟩

lemma *set-remove1-eq* [*simp*]: *distinct xs ==> set(remove1 x xs) = set xs* − {*x*}
⟨*proof*⟩

lemma *length-remove1*:
length(remove1 x xs) = (if x : set xs then length xs − 1 else length xs)
⟨*proof*⟩

lemma *remove1-filter-not*[*simp*]:
¬ *P x* $\Longrightarrow$ *remove1 x (filter P xs) = filter P xs*
⟨*proof*⟩

lemma *notin-set-remove1*[*simp*]: *x ~: set xs ==> x ~: set(remove1 y xs)*
⟨*proof*⟩

lemma *distinct-remove1*[*simp*]: *distinct xs ==> distinct(remove1 x xs)*
⟨*proof*⟩

48.1.22 *replicate*

lemma *length-replicate* [*simp*]: *length (replicate n x) = n*

⟨*proof*⟩

lemma *map-replicate* [*simp*]: *map f* (*replicate n x*) = *replicate n* (*f x*)
⟨*proof*⟩

lemma *replicate-app-Cons-same*:
(*replicate n x*) @ (*x* # *xs*) = *x* # *replicate n x* @ *xs*
⟨*proof*⟩

lemma *rev-replicate* [*simp*]: *rev* (*replicate n x*) = *replicate n x*
⟨*proof*⟩

lemma *replicate-add*: *replicate* (*n* + *m*) *x* = *replicate n x* @ *replicate m x*
⟨*proof*⟩

Courtesy of Matthias Daum:

lemma *append-replicate-commute*:
replicate n x @ *replicate k x* = *replicate k x* @ *replicate n x*
⟨*proof*⟩

lemma *hd-replicate* [*simp*]: *n* ≠ *0* ==> *hd* (*replicate n x*) = *x*
⟨*proof*⟩

lemma *tl-replicate* [*simp*]: *n* ≠ *0* ==> *tl* (*replicate n x*) = *replicate* (*n* − *1*) *x*
⟨*proof*⟩

lemma *last-replicate* [*simp*]: *n* ≠ *0* ==> *last* (*replicate n x*) = *x*
⟨*proof*⟩

lemma *nth-replicate*[*simp*]: *i* < *n* ==> (*replicate n x*)!*i* = *x*
⟨*proof*⟩

Courtesy of Matthias Daum (2 lemmas):

lemma *take-replicate*[*simp*]: *take i* (*replicate k x*) = *replicate* (*min i k*) *x*
⟨*proof*⟩

lemma *drop-replicate*[*simp*]: *drop i* (*replicate k x*) = *replicate* (*k*−*i*) *x*
⟨*proof*⟩

lemma *set-replicate-Suc*: *set* (*replicate* (*Suc n*) *x*) = {*x*}
⟨*proof*⟩

lemma *set-replicate* [*simp*]: *n* ≠ *0* ==> *set* (*replicate n x*) = {*x*}
⟨*proof*⟩

lemma *set-replicate-conv-if*: *set* (*replicate n x*) = (*if n* = *0 then* {} *else* {*x*})
⟨*proof*⟩

lemma *in-set-replicateD*: $x : set\ (replicate\ n\ y) ==> x = y$
⟨*proof*⟩

lemma *replicate-append-same*:
$replicate\ i\ x\ @\ [x] = x\ \#\ replicate\ i\ x$
⟨*proof*⟩

lemma *map-replicate-trivial*:
$map\ (\lambda i.\ x)\ [0..<i] = replicate\ i\ x$
⟨*proof*⟩

48.1.23 *rotate1* and *rotate*

lemma *rotate-simps*[*simp*]: $rotate1\ [] = [] \wedge rotate1\ (x\#xs) = xs\ @\ [x]$
⟨*proof*⟩

lemma *rotate0*[*simp*]: $rotate\ 0 = id$
⟨*proof*⟩

lemma *rotate-Suc*[*simp*]: $rotate\ (Suc\ n)\ xs = rotate1(rotate\ n\ xs)$
⟨*proof*⟩

lemma *rotate-add*:
$rotate\ (m+n) = rotate\ m\ o\ rotate\ n$
⟨*proof*⟩

lemma *rotate-rotate*: $rotate\ m\ (rotate\ n\ xs) = rotate\ (m+n)\ xs$
⟨*proof*⟩

lemma *rotate1-rotate-swap*: $rotate1\ (rotate\ n\ xs) = rotate\ n\ (rotate1\ xs)$
⟨*proof*⟩

lemma *rotate1-length01*[*simp*]: $length\ xs <= 1 \Longrightarrow rotate1\ xs = xs$
⟨*proof*⟩

lemma *rotate-length01*[*simp*]: $length\ xs <= 1 \Longrightarrow rotate\ n\ xs = xs$
⟨*proof*⟩

lemma *rotate1-hd-tl*: $xs \neq [] \Longrightarrow rotate1\ xs = tl\ xs\ @\ [hd\ xs]$
⟨*proof*⟩

lemma *rotate-drop-take*:
$rotate\ n\ xs = drop\ (n\ mod\ length\ xs)\ xs\ @\ take\ (n\ mod\ length\ xs)\ xs$
⟨*proof*⟩

lemma *rotate-conv-mod*: $rotate\ n\ xs = rotate\ (n\ mod\ length\ xs)\ xs$
⟨*proof*⟩

lemma *rotate-id*[*simp*]: $n\ mod\ length\ xs = 0 \Longrightarrow rotate\ n\ xs = xs$

⟨*proof*⟩

lemma *length-rotate1*[*simp*]: $length(rotate1\ xs) = length\ xs$
⟨*proof*⟩

lemma *length-rotate*[*simp*]: $length(rotate\ n\ xs) = length\ xs$
⟨*proof*⟩

lemma *distinct1-rotate*[*simp*]: $distinct(rotate1\ xs) = distinct\ xs$
⟨*proof*⟩

lemma *distinct-rotate*[*simp*]: $distinct(rotate\ n\ xs) = distinct\ xs$
⟨*proof*⟩

lemma *rotate-map*: $rotate\ n\ (map\ f\ xs) = map\ f\ (rotate\ n\ xs)$
⟨*proof*⟩

lemma *set-rotate1*[*simp*]: $set(rotate1\ xs) = set\ xs$
⟨*proof*⟩

lemma *set-rotate*[*simp*]: $set(rotate\ n\ xs) = set\ xs$
⟨*proof*⟩

lemma *rotate1-is-Nil-conv*[*simp*]: $(rotate1\ xs = []) = (xs = [])$
⟨*proof*⟩

lemma *rotate-is-Nil-conv*[*simp*]: $(rotate\ n\ xs = []) = (xs = [])$
⟨*proof*⟩

lemma *rotate-rev*:
$rotate\ n\ (rev\ xs) = rev(rotate\ (length\ xs - (n\ mod\ length\ xs))\ xs)$
⟨*proof*⟩

lemma *hd-rotate-conv-nth*: $xs \neq [] \Longrightarrow hd(rotate\ n\ xs) = xs!(n\ mod\ length\ xs)$
⟨*proof*⟩

48.1.24 *sublist* — a generalization of *nth* to sets

lemma *sublist-empty* [*simp*]: $sublist\ xs\ \{\} = []$
⟨*proof*⟩

lemma *sublist-nil* [*simp*]: $sublist\ []\ A = []$
⟨*proof*⟩

lemma *length-sublist*:
$length(sublist\ xs\ I) = card\{i.\ i < length\ xs \wedge i : I\}$
⟨*proof*⟩

lemma *sublist-shift-lemma-Suc*:

$map\ fst\ (filter\ (\%p.\ P(Suc(snd\ p)))\ (zip\ xs\ is)) =$
$map\ fst\ (filter\ (\%p.\ P(snd\ p))\ (zip\ xs\ (map\ Suc\ is)))$
⟨proof⟩

lemma *sublist-shift-lemma*:
$map\ fst\ [p{<}{-}zip\ xs\ [i..{<}i + length\ xs]\ .\ snd\ p : A] =$
$map\ fst\ [p{<}{-}zip\ xs\ [0..{<}length\ xs]\ .\ snd\ p + i : A]$
⟨proof⟩

lemma *sublist-append*:
$sublist\ (l\ @\ l')\ A = sublist\ l\ A\ @\ sublist\ l'\ \{j.\ j + length\ l : A\}$
⟨proof⟩

lemma *sublist-Cons*:
$sublist\ (x\ \#\ l)\ A = (if\ 0{:}A\ then\ [x]\ else\ [])\ @\ sublist\ l\ \{j.\ Suc\ j : A\}$
⟨proof⟩

lemma *set-sublist*: $set(sublist\ xs\ I) = \{xs!i|i.\ i{<}size\ xs \wedge i \in I\}$
⟨proof⟩

lemma *set-sublist-subset*: $set(sublist\ xs\ I) \subseteq set\ xs$
⟨proof⟩

lemma *notin-set-sublistI*[*simp*]: $x \notin set\ xs \Longrightarrow x \notin set(sublist\ xs\ I)$
⟨proof⟩

lemma *in-set-sublistD*: $x \in set(sublist\ xs\ I) \Longrightarrow x \in set\ xs$
⟨proof⟩

lemma *sublist-singleton* [*simp*]: $sublist\ [x]\ A = (if\ 0 : A\ then\ [x]\ else\ [])$
⟨proof⟩

lemma *distinct-sublistI*[*simp*]: $distinct\ xs \Longrightarrow distinct(sublist\ xs\ I)$
⟨proof⟩

lemma *sublist-upt-eq-take* [*simp*]: $sublist\ l\ \{..{<}n\} = take\ n\ l$
⟨proof⟩

lemma *filter-in-sublist*:
$distinct\ xs \Longrightarrow filter\ (\%x.\ x \in set(sublist\ xs\ s))\ xs = sublist\ xs\ s$
⟨proof⟩

48.1.25 *splice*

lemma *splice-Nil2* [*simp*, *code*]:
$splice\ xs\ [] = xs$
⟨proof⟩

lemma *splice-Cons-Cons* [*simp*, *code*]:
splice $(x\#xs)$ $(y\#ys) = x \# y \#$ *splice xs ys*
⟨*proof*⟩

declare *splice.simps*(*2*) [*simp del*, *code del*]

lemma *length-splice*[*simp*]: *length*(*splice xs ys*) = *length xs* + *length ys*
⟨*proof*⟩

48.2 Sorting

Currently it is not shown that *sort* returns a permutation of its input because the nicest proof is via multisets, which are not yet available. Alternatively one could define a function that counts the number of occurrences of an element in a list and use that instead of multisets to state the correctness property.

context *linorder*
begin

lemma *sorted-Cons*: *sorted* $(x\#xs) = ($*sorted xs* & $(ALL\ y{:}set\ xs.\ x <= y))$
⟨*proof*⟩

lemma *sorted-append*:
sorted $(xs@ys) = ($*sorted xs* & *sorted ys* & $(\forall x \in set\ xs.\ \forall y \in set\ ys.\ x \le y))$
⟨*proof*⟩

lemma *set-insort*: *set*(*insort x xs*) = *insert x* (*set xs*)
⟨*proof*⟩

lemma *set-sort*[*simp*]: *set*(*sort xs*) = *set xs*
⟨*proof*⟩

lemma *distinct-insort*: *distinct* (*insort x xs*) = $(x \notin set\ xs \wedge distinct\ xs)$
⟨*proof*⟩

lemma *distinct-sort*[*simp*]: *distinct* (*sort xs*) = *distinct xs*
⟨*proof*⟩

lemma *sorted-insort*: *sorted* (*insort x xs*) = *sorted xs*
⟨*proof*⟩

theorem *sorted-sort*[*simp*]: *sorted* (*sort xs*)
⟨*proof*⟩

lemma *sorted-distinct-set-unique*:
assumes *sorted xs distinct xs sorted ys distinct ys set xs* = *set ys*
shows *xs* = *ys*

⟨*proof*⟩

lemma *finite-sorted-distinct-unique*:
shows *finite A* $\Longrightarrow$ *EX*! *xs*. *set xs* = *A* & *sorted xs* & *distinct xs*
⟨*proof*⟩

end

lemma *sorted-upt*[*simp*]: *sorted*[*i*..<*j*]
⟨*proof*⟩

48.2.1 *sorted-list-of-set*

This function maps (finite) linearly ordered sets to sorted lists. Warning: in most cases it is not a good idea to convert from sets to lists but one should convert in the other direction (via *set*).

context *linorder*
begin

definition
sorted-list-of-set :: *'a set* $\Rightarrow$ *'a list* **where**
sorted-list-of-set A == *THE xs*. *set xs* = *A* & *sorted xs* & *distinct xs*

lemma *sorted-list-of-set*[*simp*]: *finite A* $\Longrightarrow$
set(*sorted-list-of-set A*) = *A* &
sorted(*sorted-list-of-set A*) & *distinct*(*sorted-list-of-set A*)
⟨*proof*⟩

lemma *sorted-list-of-empty*[*simp*]: *sorted-list-of-set* {} = []
⟨*proof*⟩

end

48.2.2 *upto*: the generic interval-list

class *finite-intvl-succ* = *linorder* +
fixes *successor* :: *'a* $\Rightarrow$ *'a*
assumes *finite-intvl*: *finite*{*a*..*b*}
and *successor-incr*: *a* < *successor a*
and *ord-discrete*: $\neg$($\exists$ *x*. *a* < *x* & *x* < *successor a*)

context *finite-intvl-succ*
begin

definition
upto :: *'a* $\Rightarrow$ *'a* $\Rightarrow$ *'a list* ((*1*[-../-])) **where**
upto i j == *sorted-list-of-set* {*i*..*j*}

lemma *upto*[*simp*]: *set*[*a*..*b*] = {*a*..*b*} & *sorted*[*a*..*b*] & *distinct*[*a*..*b*]

⟨*proof*⟩

lemma *insert-intvl*: $i \leq j \Longrightarrow insert\ i\ \{successor\ i..j\} = \{i..j\}$
⟨*proof*⟩

lemma *sorted-list-of-set-rec*: $i \leq j \Longrightarrow$
$sorted\text{-}list\text{-}of\text{-}set\ \{i..j\} = i\ \#\ sorted\text{-}list\text{-}of\text{-}set\ \{successor\ i..j\}$
⟨*proof*⟩

lemma *upto-rec*[*code*]: $[i..j] = (if\ i \leq j\ then\ i\ \#\ [successor\ i..j]\ else\ [])$
⟨*proof*⟩

end

The integers are an instance of the above class:

instance *int*:: *finite-intvl-succ*
successor-int-def: *successor* == (%*i*. *i*+*1*)
⟨*proof*⟩

Now $[i..j]$ is defined for integers.

hide (**open**) *const successor*

48.2.3 *lists*: the list-forming operator over sets

inductive-set
lists :: *'a set* => *'a list set*
for *A* :: *'a set*
where
Nil [*intro*!]: []: *lists A*
| *Cons* [*intro*!,*noatp*]: [| *a*: *A*;*l*: *lists A*|] ==> *a*#*l* : *lists A*

inductive-cases *listsE* [*elim*!,*noatp*]: *x*#*l* : *lists A*
inductive-cases *listspE* [*elim*!,*noatp*]: *listsp A* (*x* # *l*)

lemma *listsp-mono* [*mono*]: $A \leq B$ ==> *listsp A* $\leq$ *listsp B*
⟨*proof*⟩

lemmas *lists-mono* = *listsp-mono* [*to-set*]

lemma *listsp-infI*:
assumes *l*: *listsp A l* **shows** *listsp B l* ==> *listsp* (*inf A B*) *l* ⟨*proof*⟩

lemmas *lists-IntI* = *listsp-infI* [*to-set*]

lemma *listsp-inf-eq* [*simp*]: *listsp* (*inf A B*) = *inf* (*listsp A*) (*listsp B*)
⟨*proof*⟩

lemmas *listsp-conj-eq* [*simp*] = *listsp-inf-eq* [*simplified inf-fun-eq inf-bool-eq*]

lemmas *lists-Int-eq* [*simp*] = *listsp-inf-eq* [*to-set*]

lemma *append-in-listsp-conv* [*iff*]:
  (*listsp A* (*xs* @ *ys*)) = (*listsp A xs* $\wedge$ *listsp A ys*)
⟨*proof*⟩

lemmas *append-in-lists-conv* [*iff*] = *append-in-listsp-conv* [*to-set*]

lemma *in-listsp-conv-set*: (*listsp A xs*) = ($\forall x \in$ *set xs*. *A x*)
— eliminate *listsp* in favour of *set*
⟨*proof*⟩

lemmas *in-lists-conv-set* = *in-listsp-conv-set* [*to-set*]

lemma *in-listspD* [*dest*!,*noatp*]: *listsp A xs* ==> $\forall x \in$*set xs*. *A x*
⟨*proof*⟩

lemmas *in-listsD* [*dest*!,*noatp*] = *in-listspD* [*to-set*]

lemma *in-listspI* [*intro*!,*noatp*]: $\forall x \in$*set xs*. *A x* ==> *listsp A xs*
⟨*proof*⟩

lemmas *in-listsI* [*intro*!,*noatp*] = *in-listspI* [*to-set*]

lemma *lists-UNIV* [*simp*]: *lists UNIV* = *UNIV*
⟨*proof*⟩

48.2.4 Inductive definition for membership

inductive *ListMem* :: *'a* $\Rightarrow$ *'a list* $\Rightarrow$ *bool*
where
  elem: *ListMem x* (*x* # *xs*)
| *insert*: *ListMem x xs* $\Longrightarrow$ *ListMem x* (*y* # *xs*)

lemma *ListMem-iff*: (*ListMem x xs*) = (*x* $\in$ *set xs*)
⟨*proof*⟩

48.2.5 Lists as Cartesian products

set-Cons A Xs: the set of lists with head drawn from *A* and tail drawn from *Xs*.

constdefs
  set-Cons :: *'a set* $\Rightarrow$ *'a list set* $\Rightarrow$ *'a list set*
  set-Cons A XS == {*z*. $\exists$ *x xs*. *z* = *x*#*xs* & *x* $\in$ *A* & *xs* $\in$ *XS*}

lemma *set-Cons-sing-Nil* [*simp*]: *set-Cons A* {[]} = (%*x*. [*x*])'*A*
⟨*proof*⟩

Yields the set of lists, all of the same length as the argument and with elements drawn from the corresponding element of the argument.

consts *listset* :: *'a set list* ⇒ *'a list set*
primrec
listset [] = {[]}
listset(*A*#*As*) = *set-Cons A* (*listset As*)

48.3 Relations on Lists

48.3.1 Length Lexicographic Ordering

These orderings preserve well-foundedness: shorter lists precede longer lists. These ordering are not used in dictionaries.

consts *lexn* :: (*'a* ∗ *'a*)*set* => *nat* => (*'a list* ∗ *'a list*)*set*
— The lexicographic ordering for lists of the specified length
primrec
lexn r 0 = {}
lexn r (*Suc n*) =
(*prod-fun* (%(*x*,*xs*). *x*#*xs*) (%(*x*,*xs*). *x*#*xs*) ' (*r* <∗*lex*∗> *lexn r n*)) *Int*
{(*xs*,*ys*). *length xs* = *Suc n* ∧ *length ys* = *Suc n*}

constdefs
lex :: (*'a* × *'a*) *set* => (*'a list* × *'a list*) *set*
lex r == ⋃*n*. *lexn r n*
— Holds only between lists of the same length

lenlex :: (*'a* × *'a*) *set* => (*'a list* × *'a list*) *set*
lenlex r == *inv-image* (*less-than* <∗*lex*∗> *lex r*) (%*xs*. (*length xs*, *xs*))
— Compares lists by their length and then lexicographically

lemma *wf-lexn*: *wf r* ==> *wf* (*lexn r n*)
⟨*proof*⟩

lemma *lexn-length*:
(*xs*, *ys*) : *lexn r n* ==> *length xs* = *n* ∧ *length ys* = *n*
⟨*proof*⟩

lemma *wf-lex* [*intro*!]: *wf r* ==> *wf* (*lex r*)
⟨*proof*⟩

lemma *lexn-conv*:
lexn r n =
{(*xs*,*ys*). *length xs* = *n* ∧ *length ys* = *n* ∧
(∃ *xys x y xs' ys'*. *xs*= *xys* @ *x*#*xs'* ∧ *ys*= *xys* @ *y* # *ys'* ∧ (*x*, *y*):*r*)}
⟨*proof*⟩

lemma *lex-conv*:
lex r =
{(*xs*,*ys*). *length xs* = *length ys* ∧

$(\exists xys\ x\ y\ xs'\ ys'.\ xs = xys\ @\ x\ \#\ xs' \wedge ys = xys\ @\ y\ \#\ ys' \wedge (x,\ y){:}r)\}$
⟨proof⟩

lemma *wf-lenlex* [*intro*!]: *wf r ==> wf (lenlex r)*
⟨proof⟩

lemma *lenlex-conv*:
$lenlex\ r = \{(xs,ys).\ length\ xs < length\ ys\ |$
$length\ xs = length\ ys \wedge (xs,\ ys) : lex\ r\}$
⟨proof⟩

lemma *Nil-notin-lex* [*iff*]: $([],\ ys) \notin lex\ r$
⟨proof⟩

lemma *Nil2-notin-lex* [*iff*]: $(xs,\ []) \notin lex\ r$
⟨proof⟩

lemma *Cons-in-lex* [*simp*]:
$((x\ \#\ xs,\ y\ \#\ ys) : lex\ r) =$
$((x,\ y) : r \wedge length\ xs = length\ ys\ |\ x = y \wedge (xs,\ ys) : lex\ r)$
⟨proof⟩

48.3.2 Lexicographic Ordering

Classical lexicographic ordering on lists, ie. "a" ¡ "ab" ¡ "b". This ordering does *not* preserve well-foundedness. Author: N. Voelker, March 2005.

constdefs
$lexord :: ('a * 'a)set \Rightarrow ('a\ list * 'a\ list)\ set$
$lexord\ \ r == \{(x,y).\ \exists\ a\ v.\ y = x\ @\ a\ \#\ v \vee$
$(\exists\ u\ a\ b\ v\ w.\ (a,b) \in r \wedge x = u\ @\ (a\ \#\ v) \wedge y = u\ @\ (b\ \#\ w))\}$

lemma *lexord-Nil-left*[*simp*]: $([],y) \in lexord\ r = (\exists\ a\ x.\ y = a\ \#\ x)$
⟨proof⟩

lemma *lexord-Nil-right*[*simp*]: $(x,[]) \notin lexord\ r$
⟨proof⟩

lemma *lexord-cons-cons*[*simp*]:
$((a\ \#\ x,\ b\ \#\ y) \in lexord\ r) = ((a,b){\in}\ r\ |\ (a = b\ \&\ (x,y){\in}\ lexord\ r))$
⟨proof⟩

lemmas *lexord-simps* = *lexord-Nil-left lexord-Nil-right lexord-cons-cons*

lemma *lexord-append-rightI*: $\exists\ b\ z.\ y = b\ \#\ z \Longrightarrow (x,\ x\ @\ y) \in lexord\ r$
⟨proof⟩

lemma *lexord-append-left-rightI*:
$(a,b) \in r \Longrightarrow (u\ @\ a\ \#\ x,\ u\ @\ b\ \#\ y) \in lexord\ r$
⟨proof⟩

lemma *lexord-append-leftI*: $(u,v) \in lexord\ r \Longrightarrow (x\ @\ u,\ x\ @\ v) \in lexord\ r$
⟨*proof*⟩

lemma *lexord-append-leftD*:
$[\![(x\ @\ u,\ x\ @\ v) \in lexord\ r;\ (!\ a.\ (a,a) \notin r)]\!] \Longrightarrow (u,v) \in lexord\ r$
⟨*proof*⟩

lemma *lexord-take-index-conv*:
$((x,y) : lexord\ r) =$
$((length\ x < length\ y \wedge take\ (length\ x)\ y = x) \vee$
$(\exists i.\ i < min(length\ x)(length\ y)\ \&\ take\ i\ x = take\ i\ y\ \&\ (x!i,y!i) \in r))$
⟨*proof*⟩
lemma *lexord-lex*: $(x,y) \in lex\ r = ((x,y) \in lexord\ r \wedge length\ x = length\ y)$
⟨*proof*⟩

lemma *lexord-irreflexive*: $(!\ x.\ (x,x) \notin r) \Longrightarrow (y,y) \notin lexord\ r$
⟨*proof*⟩

lemma *lexord-trans*:
$[\![(x,\ y) \in lexord\ r;\ (y,\ z) \in lexord\ r;\ trans\ r]\!] \Longrightarrow (x,\ z) \in lexord\ r$
⟨*proof*⟩

lemma *lexord-transI*: $trans\ r \Longrightarrow trans\ (lexord\ r)$
⟨*proof*⟩

lemma *lexord-linear*: $(!\ a\ b.\ (a,b)\in r \mid a = b \mid (b,a) \in r) \Longrightarrow (x,y) : lexord\ r \mid x = y \mid (y,x) : lexord\ r$
⟨*proof*⟩

48.4 Lexicographic combination of measure functions

These are useful for termination proofs

definition
$measures\ fs = inv\text{-}image\ (lex\ less\text{-}than)\ (\%a.\ map\ (\%f.\ f\ a)\ fs)$

lemma *wf-measures*[*recdef-wf*, *simp*]: $wf\ (measures\ fs)$
⟨*proof*⟩

lemma *in-measures*[*simp*]:
$(x,\ y) \in measures\ [] = False$
$(x,\ y) \in measures\ (f\ \#\ fs)$
$= (f\ x < f\ y \vee (f\ x = f\ y \wedge (x,\ y) \in measures\ fs))$
⟨*proof*⟩

lemma *measures-less*: $f\ x < f\ y ==> (x,\ y) \in measures\ (f\#fs)$
⟨*proof*⟩

lemma *measures-lesseq*: $f\ x <= f\ y ==> (x,\ y) \in measures\ fs ==> (x,\ y) \in$

measures $(f\#fs)$
⟨*proof*⟩

48.4.1 Lifting a Relation on List Elements to the Lists

inductive-set
listrel :: $('a * 'a)set => ('a\ list * 'a\ list)set$
for r :: $('a * 'a)set$
where
Nil: $([],[]) \in listrel\ r$
| *Cons*: $[|\ (x,y) \in r;\ (xs,ys) \in listrel\ r\ |] ==> (x\#xs,\ y\#ys) \in listrel\ r$

inductive-cases *listrel-Nil1* [*elim*!]: $([],xs) \in listrel\ r$
inductive-cases *listrel-Nil2* [*elim*!]: $(xs,[]) \in listrel\ r$
inductive-cases *listrel-Cons1* [*elim*!]: $(y\#ys,xs) \in listrel\ r$
inductive-cases *listrel-Cons2* [*elim*!]: $(xs,y\#ys) \in listrel\ r$

lemma *listrel-mono*: $r \subseteq s \Longrightarrow listrel\ r \subseteq listrel\ s$
⟨*proof*⟩

lemma *listrel-subset*: $r \subseteq A \times A \Longrightarrow listrel\ r \subseteq lists\ A \times lists\ A$
⟨*proof*⟩

lemma *listrel-refl*: $refl\ A\ r \Longrightarrow refl\ (lists\ A)\ (listrel\ r)$
⟨*proof*⟩

lemma *listrel-sym*: $sym\ r \Longrightarrow sym\ (listrel\ r)$
⟨*proof*⟩

lemma *listrel-trans*: $trans\ r \Longrightarrow trans\ (listrel\ r)$
⟨*proof*⟩

theorem *equiv-listrel*: $equiv\ A\ r \Longrightarrow equiv\ (lists\ A)\ (listrel\ r)$
⟨*proof*⟩

lemma *listrel-Nil* [*simp*]: $listrel\ r\ ``\ \{[]\} = \{[]\}$
⟨*proof*⟩

lemma *listrel-Cons*:
$listrel\ r\ ``\ \{x\#xs\} = set\text{-}Cons\ (r``\{x\})\ (listrel\ r\ ``\ \{xs\})$
⟨*proof*⟩

48.5 Miscellany

48.5.1 Characters and strings

datatype *nibble* =
Nibble0 | *Nibble1* | *Nibble2* | *Nibble3* | *Nibble4* | *Nibble5* | *Nibble6* | *Nibble7*
| *Nibble8* | *Nibble9* | *NibbleA* | *NibbleB* | *NibbleC* | *NibbleD* | *NibbleE* | *NibbleF*

datatype *char* = *Char nibble nibble*
— Note: canonical order of character encoding coincides with standard term ordering

types *string* = *char list*

syntax
-*Char* :: *xstr* => *char* (*CHR* -)
-*String* :: *xstr* => *string* (-)

⟨*ML*⟩

48.6 Code generator

48.6.1 Setup

types-code
list (- *list*)
attach (*term-of*) ⟪
fun term-of-list f T = *HOLogic.mk-list T o map f*;
⟫
attach (*test*) ⟪
fun gen-list′ aG i j = *frequency*
[(*i*, *fn* () => *aG j* :: *gen-list′ aG* (*i*−*1*) *j*), (*1*, *fn* () => [])] ()
and gen-list aG i = *gen-list′ aG i i*;
⟫
char (*string*)
attach (*term-of*) ⟪
val term-of-char = *HOLogic.mk-char o ord*;
⟫
attach (*test*) ⟪
fun gen-char i = *chr* (*random-range* (*ord a*) (*Int.min* (*ord a* + *i*, *ord z*)));
⟫

consts-code *Cons* ((- ::/ -))

code-type *list*
(*SML* - *list*)
(*OCaml* - *list*)
(*Haskell* ![-])

code-reserved *SML*
list

code-reserved *OCaml*
list

code-const *Nil*
(*SML* [])

$(OCaml\ [])$
$(Haskell\ [])$

$\langle ML \rangle$

code-instance *list* :: *eq*
$(Haskell\ -)$

code-const $op = :: {}'a::eq\ list \Rightarrow {}'a\ list \Rightarrow bool$
$(Haskell$ **infixl** $4\ ==)$

$\langle ML \rangle$

48.6.2 Generation of efficient code

consts
$null :: {}'a\ list \Rightarrow bool$
$list\text{-}inter :: {}'a\ list \Rightarrow {}'a\ list \Rightarrow {}'a\ list$
$list\text{-}ex :: ({}'a \Rightarrow bool) \Rightarrow {}'a\ list \Rightarrow bool$
$list\text{-}all :: ({}'a \Rightarrow bool) \Rightarrow ({}'a\ list \Rightarrow bool)$
$filtermap :: ({}'a \Rightarrow {}'b\ option) \Rightarrow {}'a\ list \Rightarrow {}'b\ list$
$map\text{-}filter :: ({}'a \Rightarrow {}'b) \Rightarrow ({}'a \Rightarrow bool) \Rightarrow {}'a\ list \Rightarrow {}'b\ list$

$\langle ML \rangle$
primrec
$x\ mem\ [] = False$
$x\ mem\ (y\#ys) = (if\ y=x\ then\ True\ else\ x\ mem\ ys)$

primrec
$null\ [] = True$
$null\ (x\#xs) = False$

primrec
$list\text{-}inter\ []\ bs = []$
$list\text{-}inter\ (a\#as)\ bs =$
$(if\ a \in set\ bs\ then\ a\ \#\ list\text{-}inter\ as\ bs\ else\ list\text{-}inter\ as\ bs)$

primrec
$list\text{-}all\ P\ [] = True$
$list\text{-}all\ P\ (x\#xs) = (P\ x \wedge list\text{-}all\ P\ xs)$

primrec
$list\text{-}ex\ P\ [] = False$
$list\text{-}ex\ P\ (x\#xs) = (P\ x \vee list\text{-}ex\ P\ xs)$

primrec
$filtermap\ f\ [] = []$
$filtermap\ f\ (x\#xs) =$
$(case\ f\ x\ of\ None \Rightarrow filtermap\ f\ xs$

  $|$ *Some y* $\Rightarrow$ *y* # *filtermap f xs*)

primrec
  map-filter f P [] = []
  map-filter f P (*x*#*xs*) =
    (*if P x then f x* # *map-filter f P xs else map-filter f P xs*)

Only use *mem* for generating executable code. Otherwise use $x \in$ *set xs* instead — it is much easier to reason about. The same is true for *list-all* and *list-ex*: write $\forall$ *x*$\in$*set xs* and $\exists$ *x*$\in$*set xs* instead because the HOL quantifiers are aleady known to the automatic provers. In fact, the declarations in the code subsection make sure that $\in$, $\forall$ *x*$\in$*set xs* and $\exists$ *x*$\in$*set xs* are implemented efficiently.

Efficient emptyness check is implemented by *null*.

The functions *filtermap* and *map-filter* are just there to generate efficient code. Do not use them for modelling and proving.

lemma *rev-foldl-cons* [*code*]:
  rev xs = *foldl* (λ*xs x*. *x* # *xs*) [] *xs*
⟨*proof*⟩

lemma *mem-iff* [*code post*]:
  x mem xs $\longleftrightarrow$ *x* $\in$ *set xs*
⟨*proof*⟩

lemmas *in-set-code* [*code unfold*] = *mem-iff* [*symmetric*]

lemma *empty-null* [*code inline*]:
  xs = [] $\longleftrightarrow$ *null xs*
⟨*proof*⟩

lemmas *null-empty* [*code post*] =
  empty-null [*symmetric*]

lemma *list-inter-conv*:
  set (*list-inter xs ys*) = *set xs* $\cap$ *set ys*
⟨*proof*⟩

lemma *list-all-iff* [*code post*]:
  list-all P xs $\longleftrightarrow$ ($\forall$ *x* $\in$ *set xs*. *P x*)
⟨*proof*⟩

lemmas *list-ball-code* [*code unfold*] = *list-all-iff* [*symmetric*]

lemma *list-all-append* [*simp*]:
  list-all P (*xs* @ *ys*) $\longleftrightarrow$ (*list-all P xs* $\wedge$ *list-all P ys*)
⟨*proof*⟩

lemma *list-all-rev* [*simp*]:

list-all P (*rev xs*) $\longleftrightarrow$ *list-all P xs*
⟨*proof*⟩

lemma *list-all-length*:
list-all P xs $\longleftrightarrow$ ($\forall$ *n* < *length xs*. *P* (*xs* ! *n*))
⟨*proof*⟩

lemma *list-ex-iff* [*code post*]:
list-ex P xs $\longleftrightarrow$ ($\exists$ *x* $\in$ *set xs*. *P x*)
⟨*proof*⟩

lemmas *list-bex-code* [*code unfold*] =
list-ex-iff [*symmetric*]

lemma *list-ex-length*:
list-ex P xs $\longleftrightarrow$ ($\exists$ *n* < *length xs*. *P* (*xs* ! *n*))
⟨*proof*⟩

lemma *filtermap-conv*:
filtermap f xs = *map* (λx. *the* (*f x*)) (*filter* (λx. *f x* $\neq$ *None*) *xs*)
⟨*proof*⟩

lemma *map-filter-conv* [*simp*]:
map-filter f P xs = *map f* (*filter P xs*)
⟨*proof*⟩

Code for bounded quantification and summation over nats.

lemma *atMost-upto* [*code unfold*]:
{..*n*} = *set* [*0*..<*Suc n*]
⟨*proof*⟩

lemma *atLeast-upt* [*code unfold*]:
{..<*n*} = *set* [*0*..<*n*]
⟨*proof*⟩

lemma *greaterThanLessThan-upt* [*code unfold*]:
{*n*<..<*m*} = *set* [*Suc n*..<*m*]
⟨*proof*⟩

lemma *atLeastLessThan-upt* [*code unfold*]:
{*n*..<*m*} = *set* [*n*..<*m*]
⟨*proof*⟩

lemma *greaterThanAtMost-upto* [*code unfold*]:
{*n*<..*m*} = *set* [*Suc n*..<*Suc m*]
⟨*proof*⟩

lemma *atLeastAtMost-upto* [*code unfold*]:
{*n*..*m*} = *set* [*n*..<*Suc m*]

⟨*proof*⟩

lemma *all-nat-less-eq* [*code unfold*]:
$(\forall m<n::nat.\ P\ m) \longleftrightarrow (\forall m \in \{0..<n\}.\ P\ m)$
⟨*proof*⟩

lemma *ex-nat-less-eq* [*code unfold*]:
$(\exists m<n::nat.\ P\ m) \longleftrightarrow (\exists m \in \{0..<n\}.\ P\ m)$
⟨*proof*⟩

lemma *all-nat-less* [*code unfold*]:
$(\forall m\leq n::nat.\ P\ m) \longleftrightarrow (\forall m \in \{0..n\}.\ P\ m)$
⟨*proof*⟩

lemma *ex-nat-less* [*code unfold*]:
$(\exists m\leq n::nat.\ P\ m) \longleftrightarrow (\exists m \in \{0..n\}.\ P\ m)$
⟨*proof*⟩

lemma *setsum-set-upt-conv-listsum*[*code unfold*]:
$setsum\ f\ (set[k..<n]) = listsum\ (map\ f\ [k..<n])$
⟨*proof*⟩

48.6.3 List partitioning

consts
$partition :: ('a \Rightarrow bool) \Rightarrow 'a\ list \Rightarrow 'a\ list \times 'a\ list$
primrec
$partition\ P\ [] = ([], [])$
$partition\ P\ (x\ \#\ xs) =$
$(let\ (yes, no) = partition\ P\ xs$
$in\ if\ P\ x\ then\ (x\ \#\ yes, no)\ else\ (yes, x\ \#\ no))$

lemma *partition-P*:
$partition\ P\ xs = (yes, no) \Longrightarrow (\forall p\in set\ yes.\ P\ p) \wedge (\forall p\in set\ no.\ \neg P\ p)$
⟨*proof*⟩

lemma *partition-filter1*:
$fst\ (partition\ P\ xs) = filter\ P\ xs$
⟨*proof*⟩

lemma *partition-filter2*:
$snd\ (partition\ P\ xs) = filter\ (Not\ o\ P)\ xs$
⟨*proof*⟩

lemma *partition-set*:
assumes $partition\ P\ xs = (yes, no)$
shows $set\ yes \cup set\ no = set\ xs$
⟨*proof*⟩

end

49 Map: Maps

theory *Map*
imports *List*
begin

types $('a,'b)$ ~=> = 'a => 'b *option* (**infixr** *0*)
translations (*type*) *a* ~=> *b* <= (*type*) *a* => *b option*

syntax (*xsymbols*)
~=> :: [*type*, *type*] => *type* (**infixr** $\rightharpoonup$ *0*)

abbreviation
empty :: 'a ~=> 'b **where**
empty == %x. *None*

definition
map-comp :: ('b ~=> 'c) => ('a ~=> 'b) => ('a ~=> 'c) (**infixl** *o'-m 55*)
where
f o-m g = (λk. *case g k of None* $\Rightarrow$ *None* | *Some v* $\Rightarrow$ *f v*)

notation (*xsymbols*)
map-comp (**infixl** $\circ_m$ *55*)

definition
map-add :: ('a ~=> 'b) => ('a ~=> 'b) => ('a ~=> 'b) (**infixl** ++ *100*)
where
m1 ++ *m2* = (λx. *case m2 x of None* => *m1 x* | *Some y* => *Some y*)

definition
restrict-map :: ('a ~=> 'b) => 'a *set* => ('a ~=> 'b) (**infixl** |' *110*) **where**
m|'*A* = (λx. *if x* : *A then m x else None*)

notation (*latex* **output**)
restrict-map (-$\restriction$- [*111*,*110*] *110*)

definition
dom :: ('a ~=> 'b) => 'a *set* **where**
dom m = {*a*. *m a* ~= *None*}

definition
ran :: ('a ~=> 'b) => 'b *set* **where**
ran m = {*b*. *EX a*. *m a* = *Some b*}

definition
map-le :: ('a ~=> 'b) => ('a ~=> 'b) => *bool* (**infix** $\subseteq_m$ *50*) **where**

$(m_1 \subseteq_m m_2) = (\forall a \in dom\ m_1.\ m_1\ a = m_2\ a)$

consts
map-of :: ('a * 'b) *list* => 'a ~=> 'b
map-upds :: ('a ~=> 'b) => 'a *list* => 'b *list* => ('a ~=> 'b)

nonterminals
maplets maplet

syntax
-maplet :: ['a, 'a] => *maplet* (- /|->/ -)
-maplets :: ['a, 'a] => *maplet* (- /[|->]/ -)
:: *maplet* => *maplets* (-)
-Maplets :: [*maplet*, *maplets*] => *maplets* (-,/ -)
-MapUpd :: ['a ~=> 'b, *maplets*] => 'a ~=> 'b (-/'(-') [*900*,*0*]*900*)
-Map :: *maplets* => 'a ~=> 'b ((*1*[-]))

syntax (*xsymbols*)
-maplet :: ['a, 'a] => *maplet* (- /↦/ -)
-maplets :: ['a, 'a] => *maplet* (- /[↦]/ -)

translations
-MapUpd m (*-Maplets xy ms*) == *-MapUpd* (*-MapUpd m xy*) *ms*
-MapUpd m (*-maplet x y*) == *m*(*x*:=*Some y*)
-MapUpd m (*-maplets x y*) == *map-upds m x y*
-Map ms == *-MapUpd* (*CONST empty*) *ms*
-Map (*-Maplets ms1 ms2*) <= *-MapUpd* (*-Map ms1*) *ms2*
-Maplets ms1 (*-Maplets ms2 ms3*) <= *-Maplets* (*-Maplets ms1 ms2*) *ms3*

primrec
map-of [] = *empty*
map-of (*p*#*ps*) = (*map-of ps*)(*fst p* |-> *snd p*)

defs
map-upds-def [*code func*]: *m*(*xs* [|->] *ys*) == *m* ++ *map-of* (*rev*(*zip xs ys*))

49.1 *empty*

lemma *empty-upd-none* [*simp*]: *empty*(*x* := *None*) = *empty*
⟨*proof*⟩

49.2 *map-upd*

lemma *map-upd-triv*: *t k* = *Some x* ==> *t*(*k*|->*x*) = *t*
⟨*proof*⟩

lemma *map-upd-nonempty* [*simp*]: *t*(*k*|->*x*) ~= *empty*
⟨*proof*⟩

lemma *map-upd-eqD1*:

assumes $m(a{\mapsto}x) = n(a{\mapsto}y)$
shows $x = y$
⟨*proof*⟩

lemma *map-upd-Some-unfold*:
$((m(a|{-}{>}b))\ x = Some\ y) = (x = a \wedge b = y \vee x \neq a \wedge m\ x = Some\ y)$
⟨*proof*⟩

lemma *image-map-upd* [*simp*]: $x \notin A \Longrightarrow m(x \mapsto y)$ ‘ $A = m$ ‘ A
⟨*proof*⟩

lemma *finite-range-updI*: $finite\ (range\ f) ==> finite\ (range\ (f(a|{-}{>}b)))$
⟨*proof*⟩

49.3 *map-of*

lemma *map-of-eq-None-iff*:
$(map\text{-}of\ xys\ x = None) = (x \notin fst$ ‘ $(set\ xys))$
⟨*proof*⟩

lemma *map-of-is-SomeD*: $map\text{-}of\ xys\ x = Some\ y \Longrightarrow (x,y) \in set\ xys$
⟨*proof*⟩

lemma *map-of-eq-Some-iff* [*simp*]:
$distinct(map\ fst\ xys) \Longrightarrow (map\text{-}of\ xys\ x = Some\ y) = ((x,y) \in set\ xys)$
⟨*proof*⟩

lemma *Some-eq-map-of-iff* [*simp*]:
$distinct(map\ fst\ xys) \Longrightarrow (Some\ y = map\text{-}of\ xys\ x) = ((x,y) \in set\ xys)$
⟨*proof*⟩

lemma *map-of-is-SomeI* [*simp*]: $[\![\ distinct(map\ fst\ xys);\ (x,y) \in set\ xys\ ]\!]$
$\Longrightarrow map\text{-}of\ xys\ x = Some\ y$
⟨*proof*⟩

lemma *map-of-zip-is-None* [*simp*]:
$length\ xs = length\ ys \Longrightarrow (map\text{-}of\ (zip\ xs\ ys)\ x = None) = (x \notin set\ xs)$
⟨*proof*⟩

lemma *finite-range-map-of*: $finite\ (range\ (map\text{-}of\ xys))$
⟨*proof*⟩

lemma *map-of-SomeD*: $map\text{-}of\ xs\ k = Some\ y \Longrightarrow (k,\ y) \in set\ xs$
⟨*proof*⟩

lemma *map-of-mapk-SomeI*:
$inj\ f ==> map\text{-}of\ t\ k = Some\ x ==>$
$map\text{-}of\ (map\ (split\ (\%k.\ Pair\ (f\ k)))\ t)\ (f\ k) = Some\ x$
⟨*proof*⟩

lemma *weak-map-of-SomeI*: $(k,\ x)$: *set l* ==> $\exists\, x$. *map-of l k* = *Some x*
⟨*proof*⟩

lemma *map-of-filter-in*:
map-of xs k = *Some z* $\Longrightarrow$ *P k z* $\Longrightarrow$ *map-of* (*filter* (*split P*) *xs*) *k* = *Some z*
⟨*proof*⟩

lemma *map-of-map*: *map-of* (*map* (%(*a*,*b*). (*a*,*f b*)) *xs*) *x* = *option-map f* (*map-of xs x*)
⟨*proof*⟩

49.4 *option-map* related

lemma *option-map-o-empty* [*simp*]: *option-map f o empty* = *empty*
⟨*proof*⟩

lemma *option-map-o-map-upd* [*simp*]:
option-map f o m(*a*|−>*b*) = (*option-map f o m*)(*a*|−>*f b*)
⟨*proof*⟩

49.5 *map-comp* related

lemma *map-comp-empty* [*simp*]:
$m \circ_m$ *empty* = *empty*
empty $\circ_m$ m = *empty*
⟨*proof*⟩

lemma *map-comp-simps* [*simp*]:
m2 k = *None* $\Longrightarrow$ (*m1* $\circ_m$ *m2*) *k* = *None*
m2 k = *Some k'* $\Longrightarrow$ (*m1* $\circ_m$ *m2*) *k* = *m1 k'*
⟨*proof*⟩

lemma *map-comp-Some-iff*:
((*m1* $\circ_m$ *m2*) *k* = *Some v*) = ($\exists\, k'$. *m2 k* = *Some k'* $\wedge$ *m1 k'* = *Some v*)
⟨*proof*⟩

lemma *map-comp-None-iff*:
((*m1* $\circ_m$ *m2*) *k* = *None*) = (*m2 k* = *None* $\vee$ ($\exists\, k'$. *m2 k* = *Some k'* $\wedge$ *m1 k'* = *None*))
⟨*proof*⟩

49.6 ++

lemma *map-add-empty*[*simp*]: *m* ++ *empty* = *m*
⟨*proof*⟩

lemma *empty-map-add*[*simp*]: *empty* ++ *m* = *m*
⟨*proof*⟩

lemma *map-add-assoc*[*simp*]: $m1 ++ (m2 ++ m3) = (m1 ++ m2) ++ m3$
⟨*proof*⟩

lemma *map-add-Some-iff*:
$((m ++ n)\ k = Some\ x) = (n\ k = Some\ x \mid n\ k = None\ \&\ m\ k = Some\ x)$
⟨*proof*⟩

lemma *map-add-SomeD* [*dest*!]:
$(m ++ n)\ k = Some\ x \Longrightarrow n\ k = Some\ x \lor n\ k = None \land m\ k = Some\ x$
⟨*proof*⟩

lemma *map-add-find-right* [*simp*]: $!!xx.\ n\ k = Some\ xx ==> (m ++ n)\ k = Some\ xx$
⟨*proof*⟩

lemma *map-add-None* [*iff*]: $((m ++ n)\ k = None) = (n\ k = None\ \&\ m\ k = None)$
⟨*proof*⟩

lemma *map-add-upd*[*simp*]: $f ++ g(x|->y) = (f ++ g)(x|->y)$
⟨*proof*⟩

lemma *map-add-upds*[*simp*]: $m1 ++ (m2(xs[\mapsto]ys)) = (m1{+}{+}m2)(xs[\mapsto]ys)$
⟨*proof*⟩

lemma *map-of-append*[*simp*]: $map\text{-}of\ (xs\ @\ ys) = map\text{-}of\ ys ++ map\text{-}of\ xs$
⟨*proof*⟩

lemma *finite-range-map-of-map-add*:
$finite\ (range\ f) ==> finite\ (range\ (f ++ map\text{-}of\ l))$
⟨*proof*⟩

lemma *inj-on-map-add-dom* [*iff*]:
$inj\text{-}on\ (m ++ m')\ (dom\ m') = inj\text{-}on\ m'\ (dom\ m')$
⟨*proof*⟩

49.7 *restrict-map*

lemma *restrict-map-to-empty* [*simp*]: $m|`\{\} = empty$
⟨*proof*⟩

lemma *restrict-map-empty* [*simp*]: $empty|`D = empty$
⟨*proof*⟩

lemma *restrict-in* [*simp*]: $x \in A \Longrightarrow (m|`A)\ x = m\ x$
⟨*proof*⟩

lemma *restrict-out* [*simp*]: $x \notin A \Longrightarrow (m|`A)\ x = None$
⟨*proof*⟩

lemma *ran-restrictD*: $y \in ran\ (m|`A) \Longrightarrow \exists x\in A.\ m\ x = Some\ y$
⟨*proof*⟩

lemma *dom-restrict* [*simp*]: $dom\ (m|`A) = dom\ m \cap A$
⟨*proof*⟩

lemma *restrict-upd-same* [*simp*]: $m(x\mapsto y)|`(-\{x\}) = m|`(-\{x\})$
⟨*proof*⟩

lemma *restrict-restrict* [*simp*]: $m|`A|`B = m|`(A\cap B)$
⟨*proof*⟩

lemma *restrict-fun-upd* [*simp*]:
$m(x := y)|`D = (if\ x \in D\ then\ (m|`(D-\{x\}))(x := y)\ else\ m|`D)$
⟨*proof*⟩

lemma *fun-upd-None-restrict* [*simp*]:
$(m|`D)(x := None) = (if\ x{:}D\ then\ m|`(D - \{x\})\ else\ m|`D)$
⟨*proof*⟩

lemma *fun-upd-restrict*: $(m|`D)(x := y) = (m|`(D-\{x\}))(x := y)$
⟨*proof*⟩

lemma *fun-upd-restrict-conv* [*simp*]:
$x \in D \Longrightarrow (m|`D)(x := y) = (m|`(D-\{x\}))(x := y)$
⟨*proof*⟩

49.8 *map-upds*

lemma *map-upds-Nil1* [*simp*]: $m([]\ [|->]\ bs) = m$
⟨*proof*⟩

lemma *map-upds-Nil2* [*simp*]: $m(as\ [|->]\ []) = m$
⟨*proof*⟩

lemma *map-upds-Cons* [*simp*]: $m(a\#as\ [|->]\ b\#bs) = (m(a|->b))(as[|->]bs)$
⟨*proof*⟩

lemma *map-upds-append1* [*simp*]: $\bigwedge ys\ m.\ size\ xs < size\ ys \Longrightarrow$
$m(xs@[x]\ [\mapsto]\ ys) = m(xs\ [\mapsto]\ ys)(x \mapsto ys!size\ xs)$
⟨*proof*⟩

lemma *map-upds-list-update2-drop* [*simp*]:
$[\![size\ xs \le i;\ i < size\ ys]\!]$
$\Longrightarrow m(xs[\mapsto]ys[i{:=}y]) = m(xs[\mapsto]ys)$
⟨*proof*⟩

lemma *map-upd-upds-conv-if*:

$(f(x|{-}{>}y))(xs\ [|{-}{>}]\ ys) =$
$(if\ x : set(take\ (length\ ys)\ xs)\ then\ f(xs\ [|{-}{>}]\ ys)$
$else\ (f(xs\ [|{-}{>}]\ ys))(x|{-}{>}y))$
⟨*proof*⟩

lemma *map-upds-twist* [*simp*]:
$a\ {\sim}{:}\ set\ as\ {=}{=}{>}\ m(a|{-}{>}b)(as[|{-}{>}]bs) = m(as[|{-}{>}]bs)(a|{-}{>}b)$
⟨*proof*⟩

lemma *map-upds-apply-nontin* [*simp*]:
$x\ {\sim}{:}\ set\ xs\ {=}{=}{>}\ (f(xs[|{-}{>}]ys))\ x = f\ x$
⟨*proof*⟩

lemma *fun-upds-append-drop* [*simp*]:
$size\ xs = size\ ys \Longrightarrow m(xs@zs[\mapsto]ys) = m(xs[\mapsto]ys)$
⟨*proof*⟩

lemma *fun-upds-append2-drop* [*simp*]:
$size\ xs = size\ ys \Longrightarrow m(xs[\mapsto]ys@zs) = m(xs[\mapsto]ys)$
⟨*proof*⟩

lemma *restrict-map-upds*[*simp*]:
$[\![\ length\ xs = length\ ys;\ set\ xs \subseteq D\ ]\!]$
$\Longrightarrow m(xs\ [\mapsto]\ ys)|`D = (m|`(D - set\ xs))(xs\ [\mapsto]\ ys)$
⟨*proof*⟩

49.9 *dom*

lemma *domI*: $m\ a = Some\ b\ {=}{=}{>}\ a : dom\ m$
⟨*proof*⟩

lemma *domD*: $a : dom\ m\ {=}{=}{>}\ \exists\, b.\ m\ a = Some\ b$
⟨*proof*⟩

lemma *domIff* [*iff*, *simp del*]: $(a : dom\ m) = (m\ a\ {\sim}{=}\ None)$
⟨*proof*⟩

lemma *dom-empty* [*simp*]: $dom\ empty = \{\}$
⟨*proof*⟩

lemma *dom-fun-upd* [*simp*]:
$dom(f(x := y)) = (if\ y{=}None\ then\ dom\ f - \{x\}\ else\ insert\ x\ (dom\ f))$
⟨*proof*⟩

lemma *dom-map-of*: $dom(map\text{-}of\ xys) = \{x.\ \exists\, y.\ (x,y) : set\ xys\}$
⟨*proof*⟩

lemma *dom-map-of-conv-image-fst*:
dom(*map-of xys*) = *fst* ' (*set xys*)
⟨*proof*⟩

lemma *dom-map-of-zip* [*simp*]: [| *length xs* = *length ys*; *distinct xs* |] ==>
dom(*map-of*(*zip xs ys*)) = *set xs*
⟨*proof*⟩

lemma *finite-dom-map-of*: *finite* (*dom* (*map-of l*))
⟨*proof*⟩

lemma *dom-map-upds* [*simp*]:
dom(*m*(*xs*[|->]*ys*)) = *set*(*take* (*length ys*) *xs*) *Un dom m*
⟨*proof*⟩

lemma *dom-map-add* [*simp*]: *dom*(*m*++*n*) = *dom n Un dom m*
⟨*proof*⟩

lemma *dom-override-on* [*simp*]:
dom(*override-on f g A*) =
(*dom f* − {*a*. *a* : *A* − *dom g*}) *Un* {*a*. *a* : *A Int dom g*}
⟨*proof*⟩

lemma *map-add-comm*: $dom\ m1 \cap dom\ m2 = \{\} \Longrightarrow m1{+}{+}m2 = m2{+}{+}m1$
⟨*proof*⟩

lemma *finite-map-freshness*:
$finite\ (dom\ (f :: 'a \rightharpoonup 'b)) \Longrightarrow \neg\ finite\ (UNIV :: 'a\ set) \Longrightarrow$
$\exists\, x.\ f\ x = None$
⟨*proof*⟩

49.10 *ran*

lemma *ranI*: *m a* = *Some b* ==> *b* : *ran m*
⟨*proof*⟩

lemma *ran-empty* [*simp*]: *ran empty* = {}
⟨*proof*⟩

lemma *ran-map-upd* [*simp*]: *m a* = *None* ==> *ran*(*m*(*a*|->*b*)) = *insert b* (*ran m*)
⟨*proof*⟩

49.11 *map-le*

lemma *map-le-empty* [*simp*]: $empty \subseteq_m g$
⟨*proof*⟩

lemma *upd-None-map-le* [*simp*]: $f(x := None) \subseteq_m f$
⟨*proof*⟩

lemma *map-le-upd*[*simp*]: $f \subseteq_m g$ ==> $f(a := b) \subseteq_m g(a := b)$
⟨*proof*⟩

lemma *map-le-imp-upd-le* [*simp*]: $m1 \subseteq_m m2 \Longrightarrow m1(x := None) \subseteq_m m2(x \mapsto y)$
⟨*proof*⟩

lemma *map-le-upds* [*simp*]:
$f \subseteq_m g$ ==> $f(as\ [|->]\ bs) \subseteq_m g(as\ [|->]\ bs)$
⟨*proof*⟩

lemma *map-le-implies-dom-le*: $(f \subseteq_m g) \Longrightarrow (dom\ f \subseteq dom\ g)$
⟨*proof*⟩

lemma *map-le-refl* [*simp*]: $f \subseteq_m f$
⟨*proof*⟩

lemma *map-le-trans*[*trans*]: $[\![m1 \subseteq_m m2;\ m2 \subseteq_m m3]\!] \Longrightarrow m1 \subseteq_m m3$
⟨*proof*⟩

lemma *map-le-antisym*: $[\![f \subseteq_m g;\ g \subseteq_m f]\!] \Longrightarrow f = g$
⟨*proof*⟩

lemma *map-le-map-add* [*simp*]: $f \subseteq_m (g\ ++\ f)$
⟨*proof*⟩

lemma *map-le-iff-map-add-commute*: $(f \subseteq_m f\ ++\ g) = (f{+}{+}g = g{+}{+}f)$
⟨*proof*⟩

lemma *map-add-le-mapE*: $f{+}{+}g \subseteq_m h \Longrightarrow g \subseteq_m h$
⟨*proof*⟩

lemma *map-add-le-mapI*: $[\![f \subseteq_m h;\ g \subseteq_m h;\ f \subseteq_m f{+}{+}g]\!] \Longrightarrow f{+}{+}g \subseteq_m h$
⟨*proof*⟩

end

50 Main: Main HOL

theory *Main*
imports *Map*
begin

Theory *Main* includes everything. Note that theory *PreList* already includes most HOL theories.

$\langle ML \rangle$

end

References