Machine Words in Isabelle/HOL

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November 22, 2007

Abstract

A formalisation of generic, fixed size machine words in Isabelle/HOL. An earlier version of this formalisation is described in [1].

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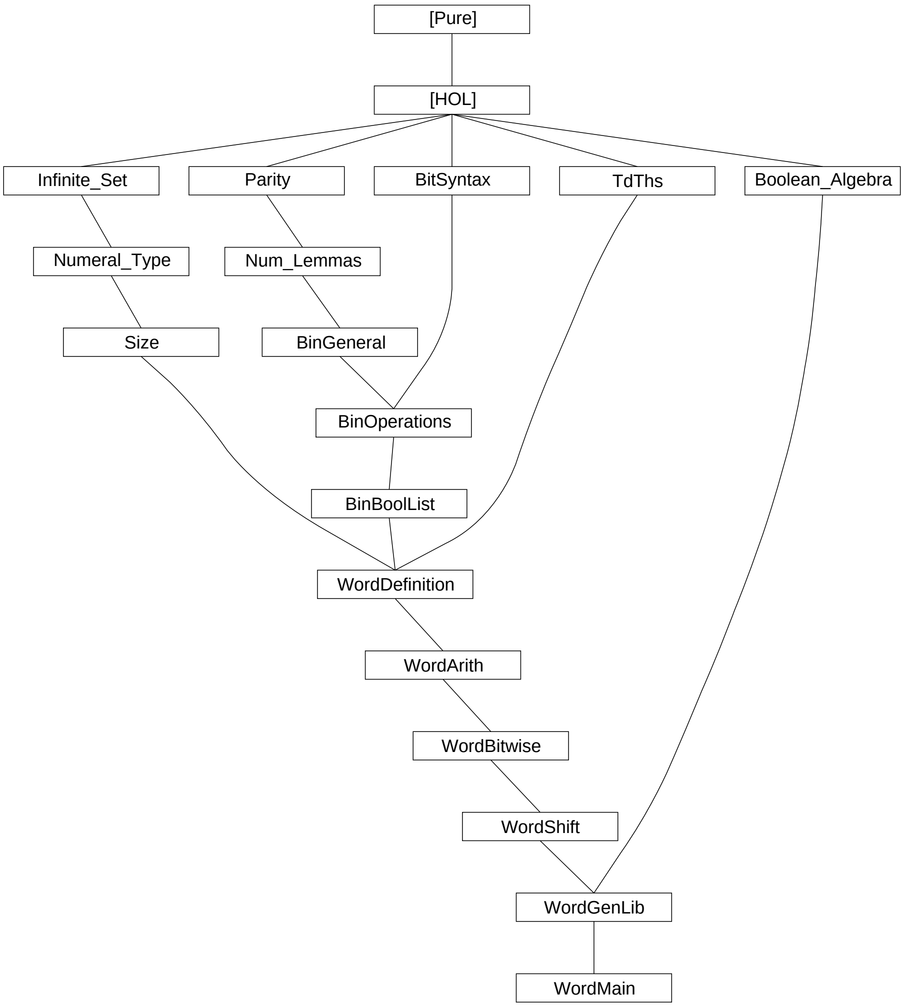
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1 Numeral-Type: Numeral Syntax for Types

```
theory Numeral-Type
  imports Infinite-Set
begin
```

1.1 Preliminary lemmas

```
lemma inj-Inl [simp]: inj-on Inl A
  by (rule inj-onI, simp)

lemma inj-Inr [simp]: inj-on Inr A
  by (rule inj-onI, simp)

lemma inj-Some [simp]: inj-on Some A
  by (rule inj-onI, simp)

lemma card-Plus:
  [| finite A; finite B |] ==> card (A <+> B) = card A + card B
  unfolding Plus-def
  apply (subgoal-tac Inl ‘ A ∩ Inr ‘ B = {})
  apply (simp add: card-Un-disjoint card-image)
  apply fast
  done

lemma (in type-definition) univ:
  UNIV = Abs ‘ A
proof
  show Abs ‘ A ⊆ UNIV by (rule subset-UNIV)
  show UNIV ⊆ Abs ‘ A
  proof
    fix x :: 'b
    have x = Abs (Rep x) by (rule Rep-inverse [symmetric])
    moreover have Rep x ∈ A by (rule Rep)
    ultimately show x ∈ Abs ‘ A by (rule image-eqI)
  qed
qed

lemma (in type-definition) card: card (UNIV :: 'b set) = card A
  by (simp add: univ card-image inj-on-def Abs-inject)
```

1.2 Cardinalities of types

```
syntax -type-card :: type => nat ((1CARD/(1'(-'))))

translations CARD(t) => card (UNIV::t set)

typed-print-translation ⟪
let
  fun card-univ-tr' show-sorts - [Const (@{const-name UNIV}, Type(-,[T]))] =
```

```
  Syntax.const -type-card $ Syntax.term-of-typ show-sorts T;
in [(card, card-univ-tr')]
end
⟫
```

lemma *card-unit*: $CARD(unit) = 1$
 unfolding *univ-unit* **by** *simp*

lemma *card-bool*: $CARD(bool) = 2$
 unfolding *univ-bool* **by** *simp*

lemma *card-prod*: $CARD('a{::}finite \times 'b{::}finite) = CARD('a) * CARD('b)$
 unfolding *univ-prod* **by** (*simp only*: *card-cartesian-product*)

lemma *card-sum*: $CARD('a{::}finite + 'b{::}finite) = CARD('a) + CARD('b)$
 unfolding *univ-sum* **by** (*simp only*: *finite card-Plus*)

lemma *card-option*: $CARD('a{::}finite\ option) = Suc\ CARD('a)$
 unfolding *univ-option*
 apply (*subgoal-tac* $(None{::}'a\ option) \notin range\ Some$)
 apply (*simp add*: *finite card-image*)
 apply *fast*
 done

lemma *card-set*: $CARD('a{::}finite\ set) = 2\ \hat{}\ CARD('a)$
 unfolding *univ-set*
 by (*simp only*: *card-Pow finite numeral-2-eq-2*)

1.3 Numeral Types

typedef (**open**) *num0* = *UNIV* :: *nat set* **..**
typedef (**open**) *num1* = *UNIV* :: *unit set* **..**
typedef (**open**) *'a bit0* = *UNIV* :: (*bool* ∗ *'a*) *set* **..**
typedef (**open**) *'a bit1* = *UNIV* :: (*bool* ∗ *'a*) *option set* **..**

instance *num1* :: *finite*
proof
 show *finite* (*UNIV*::*num1 set*)
  unfolding *type-definition.univ* [*OF type-definition-num1*]
  using *finite* **by** (*rule finite-imageI*)
qed

instance *bit0* :: (*finite*) *finite*
proof
 show *finite* (*UNIV*::*'a bit0 set*)
  unfolding *type-definition.univ* [*OF type-definition-bit0*]
  using *finite* **by** (*rule finite-imageI*)
qed

instance *bit1* :: (*finite*) *finite*
proof
  show *finite* (*UNIV*::'*a bit1 set*)
    unfolding *type-definition.univ* [*OF type-definition-bit1*]
    using *finite* **by** (*rule finite-imageI*)
qed

lemma *card-num1*: *CARD*(*num1*) = *1*
  unfolding *type-definition.card* [*OF type-definition-num1*]
  by (*simp only*: *card-unit*)

lemma *card-bit0*: *CARD*('*a*::*finite bit0*) = *2* ∗ *CARD*('*a*)
  unfolding *type-definition.card* [*OF type-definition-bit0*]
  by (*simp only*: *card-prod card-bool*)

lemma *card-bit1*: *CARD*('*a*::*finite bit1*) = *Suc* (*2* ∗ *CARD*('*a*))
  unfolding *type-definition.card* [*OF type-definition-bit1*]
  by (*simp only*: *card-prod card-option card-bool*)

lemma *card-num0*: *CARD* (*num0*) = *0*
  by (*simp add*: *type-definition.card* [*OF type-definition-num0*])

lemmas *card-univ-simps* [*simp*] =
  card-unit
  card-bool
  card-prod
  card-sum
  card-option
  card-set
  card-num1
  card-bit0
  card-bit1
  card-num0

1.4 Syntax

syntax
  -NumeralType :: *num-const* => *type* (-)
  -NumeralType0 :: *type* (*0*)
  -NumeralType1 :: *type* (*1*)

translations
  -NumeralType1 == (*type*) *num1*
  -NumeralType0 == (*type*) *num0*

parse-translation ⟪
let

val num1-const = *Syntax.const Numeral-Type.num1*;

```
val num0-const = Syntax.const Numeral-Type.num0;
val B0-const = Syntax.const Numeral-Type.bit0;
val B1-const = Syntax.const Numeral-Type.bit1;

fun mk-bintype n =
  let
    fun mk-bit n = if n = 0 then B0-const else B1-const;
    fun bin-of n =
      if n = 1 then num1-const
      else if n = 0 then num0-const
      else if n = ~1 then raise TERM (negative type numeral, [])
      else
        let val (q, r) = Integer.div-mod n 2;
        in mk-bit r $ bin-of q end;
  in bin-of n end;

fun numeral-tr (*-NumeralType*) [Const (str, -)] =
      mk-bintype (valOf (Int.fromString str))
  | numeral-tr (*-NumeralType*) ts = raise TERM (numeral-tr, ts);

in [(-NumeralType, numeral-tr)] end;
⟫
```

print-translation ⟪

```
let
fun int-of [] = 0
  | int-of (b :: bs) = b + 2 * int-of bs;

fun bin-of (Const (num0, -)) = []
  | bin-of (Const (num1, -)) = [1]
  | bin-of (Const (bit0, -) $ bs) = 0 :: bin-of bs
  | bin-of (Const (bit1, -) $ bs) = 1 :: bin-of bs
  | bin-of t = raise TERM(bin-of, [t]);

fun bit-tr' b [t] =
  let
    val rev-digs = b :: bin-of t handle TERM - => raise Match
    val i = int-of rev-digs;
    val num = string-of-int (abs i);
  in
    Syntax.const -NumeralType $ Syntax.free num
  end
  | bit-tr' b - = raise Match;

in [(bit0, bit-tr' 0), (bit1, bit-tr' 1)] end;
⟫
```

1.5 Classes with at least 1 and 2

Class finite already captures "at least 1"

lemma *zero-less-card-finite* [*simp*]:
0 < *CARD*(*'a*::*finite*)
proof (*cases CARD*(*'a*::*finite*) = *0*)
case *False* **thus** *?thesis* **by** (*simp del*: *card-0-eq*)
next
case *True*
thus *?thesis* **by** (*simp add*: *finite*)
qed

lemma *one-le-card-finite* [*simp*]:
Suc 0 <= *CARD*(*'a*::*finite*)
by (*simp add*: *less-Suc-eq-le* [*symmetric*] *zero-less-card-finite*)

Class for cardinality "at least 2"

class *card2* = *finite* +
assumes *two-le-card*: *2* <= *CARD*(*'a*)

lemma *one-less-card*: *Suc 0* < *CARD*(*'a*::*card2*)
using *two-le-card* [**where** *'a*=*'a*] **by** *simp*

instance *bit0* :: (*finite*) *card2*
by *intro-classes* (*simp add*: *one-le-card-finite*)

instance *bit1* :: (*finite*) *card2*
by *intro-classes* (*simp add*: *one-le-card-finite*)

1.6 Examples

term *TYPE*(*10*)

lemma *CARD*(*0*) = *0* **by** *simp*
lemma *CARD*(*17*) = *17* **by** *simp*

end

2 Size: The size class

theory *Size*
imports *Numeral-Type*
begin

The aim of this is to allow any type as index type, but to provide a default instantiation for numeral types. This independence requires some duplication with the definitions in Numeral_Type.

axclass *len0* < *type*

consts
len-of :: (*'a* :: *len0 itself*) => *nat*

Some theorems are only true on words with length greater 0.

axclass *len* < *len0*
len-gt-0 [*iff*]: *0* < *len-of TYPE* (*'a* :: *len0*)

instance *num0* :: *len0* **..**
instance *num1* :: *len0* **..**
instance *bit0* :: (*len0*) *len0* **..**
instance *bit1* :: (*len0*) *len0* **..**

defs (**overloaded**)
len-num0: *len-of* (*x*::*num0 itself*) == *0*
len-num1: *len-of* (*x*::*num1 itself*) == *1*
len-bit0: *len-of* (*x*::*'a*::*len0 bit0 itself*) == *2* ∗ *len-of TYPE* (*'a*)
len-bit1: *len-of* (*x*::*'a*::*len0 bit1 itself*) == *2* ∗ *len-of TYPE* (*'a*) + *1*

lemmas *len-of-numeral-defs* [*simp*] = *len-num0 len-num1 len-bit0 len-bit1*

instance *num1* :: *len* **by** (*intro-classes*) *simp*
instance *bit0* :: (*len*) *len* **by** (*intro-classes*) *simp*
instance *bit1* :: (*len0*) *len* **by** (*intro-classes*) *simp*

— Examples:
lemma *len-of TYPE*(*17*) = *17* **by** *simp*
lemma *len-of TYPE*(*0*) = *0* **by** *simp*

— not simplified:
lemma *len-of TYPE*(*'a*::*len0*) = *x*
oops

end

3 Num-Lemmas: Useful Numerical Lemmas

theory *Num-Lemmas* **imports** *Parity* **begin**

lemma *contentsI*: *y* = {*x*} ==> *contents y* = *x*
unfolding *contents-def* **by** *auto*

lemma *prod-case-split*: *prod-case* = *split*
by (*rule ext*)+ *auto*

lemmas *split-split* = *prod.split* [*unfolded prod-case-split*]

lemmas *split-split-asm* = *prod.split-asm* [*unfolded prod-case-split*]
lemmas *split.splits* = *split-split split-split-asm*

lemmas *funpow-0* = *funpow.simps*(*1*)
lemmas *funpow-Suc* = *funpow.simps*(*2*)

lemma *nonemptyE*: S ~= {} ==> (!!x. x : S ==> R) ==> R
  apply (*erule contrapos-np*)
  apply (*rule equals0I*)
  apply *auto*
  done

lemma *gt-or-eq-0*: $0 < y \vee 0 = (y::nat)$ **by** *auto*

constdefs
  mod-alt :: 'a => 'a => 'a :: *Divides.div*
  mod-alt n m == *n mod m*

  — alternative way of defining *bin-last*, *bin-rest*
  bin-rl :: *int* => *int* ∗ *bit*
  bin-rl w == *SOME* (*r*, *l*). *w* = *r BIT l*

declare *iszero-0* [*iff*]

lemmas *xtr1* = *xtrans*(*1*)
lemmas *xtr2* = *xtrans*(*2*)
lemmas *xtr3* = *xtrans*(*3*)
lemmas *xtr4* = *xtrans*(*4*)
lemmas *xtr5* = *xtrans*(*5*)
lemmas *xtr6* = *xtrans*(*6*)
lemmas *xtr7* = *xtrans*(*7*)
lemmas *xtr8* = *xtrans*(*8*)

lemma *Min-ne-Pls* [*iff*]:
  Numeral.Min ~= *Numeral.Pls*
  unfolding *Min-def Pls-def* **by** *auto*

lemmas *Pls-ne-Min* [*iff*] = *Min-ne-Pls* [*symmetric*]

lemmas *PlsMin-defs* [*intro*!] =
  Pls-def Min-def Pls-def [*symmetric*] *Min-def* [*symmetric*]

lemmas *PlsMin-simps* [*simp*] = *PlsMin-defs* [*THEN Eq-TrueI*]

lemma *number-of-False-cong*:
  False ==> *number-of x* = *number-of y*
  by *auto*

lemmas *nat-simps* = *diff-add-inverse2 diff-add-inverse*

lemmas *nat-iffs* = *le-add1 le-add2*

lemma *sum-imp-diff*: *j* = *k* + *i* ==> *j* − *i* = (*k* :: *nat*)
  by (*clarsimp simp add*: *nat-simps*)

lemma *nobm1*:
  0 < (*number-of w* :: *nat*) ==>
   number-of w − (*1* :: *nat*) = *number-of* (*Numeral.pred w*)
  apply (*unfold nat-number-of-def One-nat-def nat-1* [*symmetric*] *pred-def*)
  apply (*simp add*: *number-of-eq nat-diff-distrib* [*symmetric*])
  done

lemma *of-int-power*:
  of-int (*a* ^ *n*) = (*of-int a* ^ *n* :: *'a* :: {*recpower*, *comm-ring-1*})
  by (*induct n*) (*auto simp add*: *power-Suc*)

lemma *zless2*: *0* < (*2* :: *int*)
  by *auto*

lemmas *zless2p* [*simp*] = *zless2* [*THEN zero-less-power*]
lemmas *zle2p* [*simp*] = *zless2p* [*THEN order-less-imp-le*]

lemmas *pos-mod-sign2* = *zless2* [*THEN pos-mod-sign* [**where** *b* = *2*::*int*]]
lemmas *pos-mod-bound2* = *zless2* [*THEN pos-mod-bound* [**where** *b* = *2*::*int*]]

— the inverse(s) of *number-of*
lemma *nmod2*: *n mod* (*2*::*int*) = *0* | *n mod 2* = *1*
  using *pos-mod-sign2* [*of n*] *pos-mod-bound2* [*of n*]
  unfolding *mod-alt-def* [*symmetric*] **by** *auto*

lemma *emep1*:
  even n ==> *even d* ==> *0* <= *d* ==> (*n* + *1*) *mod* (*d* :: *int*) = (*n mod d*) +
1
  apply (*simp add*: *add-commute*)
  apply (*safe dest*!: *even-equiv-def* [*THEN iffD1*])
  apply (*subst pos-zmod-mult-2*)
   apply *arith*
  apply (*simp add*: *zmod-zmult-zmult1*)
 done

lemmas *eme1p* = *emep1* [*simplified add-commute*]

lemma *le-diff-eq'*: ($a \leq c - b$) = ($b + a \leq$ (*c*::*int*))
  by (*simp add*: *le-diff-eq add-commute*)

lemma *less-diff-eq'*: ($a < c - b$) = ($b + a <$ (*c*::*int*))
  by (*simp add*: *less-diff-eq add-commute*)

lemma *diff-le-eq'*: ($a - b \leq c$) = ($a \leq b +$ (*c*::*int*))

by (*simp add*: *diff-le-eq add-commute*)

lemma *diff-less-eq'*: $(a - b < c) = (a < b + (c::int))$
by (*simp add*: *diff-less-eq add-commute*)

lemmas *m1mod2k* = *zless2p* [*THEN zmod-minus1*]
lemmas *m1mod22k* = *mult-pos-pos* [*OF zless2 zless2p*, *THEN zmod-minus1*]
lemmas *p1mod22k'* = *zless2p* [*THEN order-less-imp-le*, *THEN pos-zmod-mult-2*]
lemmas *z1pmod2'* = *zero-le-one* [*THEN pos-zmod-mult-2*, *simplified*]
lemmas *z1pdiv2'* = *zero-le-one* [*THEN pos-zdiv-mult-2*, *simplified*]

lemma *p1mod22k*:
$(2 * b + 1)\ mod\ (2 * 2\ \hat{}\ n) = 2 * (b\ mod\ 2\ \hat{}\ n) + (1::int)$
by (*simp add*: *p1mod22k' add-commute*)

lemma *z1pmod2*:
$(2 * b + 1)\ mod\ 2 = (1::int)$
by (*simp add*: *z1pmod2' add-commute*)

lemma *z1pdiv2*:
$(2 * b + 1)\ div\ 2 = (b::int)$
by (*simp add*: *z1pdiv2' add-commute*)

lemmas *zdiv-le-dividend* = *xtr3* [*OF zdiv-1* [*symmetric*] *zdiv-mono2*,
simplified int-one-le-iff-zero-less, *simplified*, *standard*]

lemma *BIT-eq*: *u BIT b* = *v BIT c* ==> *u* = *v* & *b* = *c*
apply (*unfold Bit-def*)
apply (*simp* (*no-asm-use*) *split*: *bit.split-asm*)
apply *simp-all*
apply (*drule-tac f*=*even* **in** *arg-cong*, *clarsimp*)+
done

lemmas *BIT-eqE* [*elim*!] = *BIT-eq* [*THEN conjE*, *standard*]

lemma *BIT-eq-iff* [*simp*]:
(*u BIT b* = *v BIT c*) = ($u = v \wedge b = c$)
by (*rule iffI*) *auto*

lemmas *BIT-eqI* [*intro*!] = *conjI* [*THEN BIT-eq-iff* [*THEN iffD2*]]

lemma *less-Bits*:
(*v BIT b* < *w BIT c*) = (*v* < *w* | *v* <= *w* & *b* = *bit.B0* & *c* = *bit.B1*)
unfolding *Bit-def* **by** (*auto split*: *bit.split*)

lemma *le-Bits*:

  (*v BIT b <= w BIT c*) = (*v < w | v <= w & (b ~= bit.B1 | c ~= bit.B0)*)
  unfolding *Bit-def* **by** (*auto split*: *bit.split*)

lemma *neB1E* [*elim!*]:
  assumes *ne*: $y \neq bit.B1$
  assumes *y*: $y = bit.B0 \Longrightarrow P$
  shows *P*
  apply (*rule y*)
  apply (*cases y rule*: *bit.exhaust*, *simp*)
  apply (*simp add*: *ne*)
  done

lemma *bin-ex-rl*: *EX w b. w BIT b = bin*
  apply (*unfold Bit-def*)
  apply (*cases even bin*)
   apply (*clarsimp simp*: *even-equiv-def*)
   apply (*auto simp*: *odd-equiv-def split*: *bit.split*)
  done

lemma *bin-exhaust*:
  assumes *Q*: $\bigwedge x\ b.\ bin = x\ BIT\ b \Longrightarrow Q$
  shows *Q*
  apply (*insert bin-ex-rl* [*of bin*])
  apply (*erule exE*)+
  apply (*rule Q*)
  apply *force*
  done

lemma *bin-rl-char*: (*bin-rl w* = (*r*, *l*)) = (*r BIT l* = *w*)
  apply (*unfold bin-rl-def*)
  apply *safe*
   apply (*cases w rule*: *bin-exhaust*)
   apply *auto*
  done

lemmas *bin-rl-simps* [*THEN bin-rl-char* [*THEN iffD2*], *standard*, *simp*] =
  Pls-0-eq Min-1-eq refl

lemma *bin-abs-lem*:
  bin = (*w BIT b*) ==> ~ *bin* = *Numeral.Min* --> ~ *bin* = *Numeral.Pls* -->
    nat (*abs w*) < *nat* (*abs bin*)
  apply (*clarsimp simp add*: *bin-rl-char*)
  apply (*unfold Pls-def Min-def Bit-def*)
  apply (*cases b*)
   apply (*clarsimp*, *arith*)
  apply (*clarsimp*, *arith*)
  done

lemma *bin-induct*:

```
  assumes PPls: P Numeral.Pls
    and PMin: P Numeral.Min
    and PBit: !!bin bit. P bin ==> P (bin BIT bit)
  shows P bin
  apply (rule-tac P=P and a=bin and f1=nat o abs
                in wf-measure [THEN wf-induct])
  apply (simp add: measure-def inv-image-def)
  apply (case-tac x rule: bin-exhaust)
  apply (frule bin-abs-lem)
  apply (auto simp add : PPls PMin PBit)
  done

lemma no-no [simp]: number-of (number-of i) = i
  unfolding number-of-eq by simp

lemma Bit-B0:
  k BIT bit.B0 = k + k
   by (unfold Bit-def) simp

lemma Bit-B1:
  k BIT bit.B1 = k + k + 1
   by (unfold Bit-def) simp

lemma Bit-B0-2t: k BIT bit.B0 = 2 * k
  by (rule trans, rule Bit-B0) simp

lemma Bit-B1-2t: k BIT bit.B1 = 2 * k + 1
  by (rule trans, rule Bit-B1) simp

lemma B-mod-2':
  X = 2 ==> (w BIT bit.B1) mod X = 1 & (w BIT bit.B0) mod X = 0
  apply (simp (no-asm) only: Bit-B0 Bit-B1)
  apply (simp add: z1pmod2)
  done

lemmas B1-mod-2 [simp] = B-mod-2' [OF refl, THEN conjunct1, standard]
lemmas B0-mod-2 [simp] = B-mod-2' [OF refl, THEN conjunct2, standard]

lemma axxbyy:
  a + m + m = b + n + n ==> (a = 0 | a = 1) ==> (b = 0 | b = 1) ==>
   a = b & m = (n :: int)
  apply auto
   apply (drule-tac f=%n. n mod 2 in arg-cong)
   apply (clarsimp simp: z1pmod2)
  apply (drule-tac f=%n. n mod 2 in arg-cong)
  apply (clarsimp simp: z1pmod2)
  done

lemma axxmod2:
```

(1 + x + x) mod 2 = (1 :: int) & *(0 + x + x) mod 2 = (0 :: int)*
by *simp (rule z1pmod2)*

lemma *axxdiv2*:
(1 + x + x) div 2 = (x :: int) & *(0 + x + x) div 2 = (x :: int)*
by *simp (rule z1pdiv2)*

lemmas *iszero-minus = trans [THEN trans,*
OF iszero-def neg-equal-0-iff-equal iszero-def [symmetric], standard]

lemmas *zadd-diff-inverse = trans [OF diff-add-cancel [symmetric] add-commute,*
standard]

lemmas *add-diff-cancel2 = add-commute [THEN diff-eq-eq [THEN iffD2], standard]*

lemma *zmod-uminus*: $-$ *((a :: int) mod b) mod b =* $-$*a mod b*
by *(simp add : zmod-zminus1-eq-if)*

lemma *zmod-zsub-distrib*: *((a::int)* $-$ *b) mod c = (a mod c* $-$ *b mod c) mod c*
apply *(unfold diff-int-def)*
apply *(rule trans [OF - zmod-zadd1-eq [symmetric]])*
apply *(simp add: zmod-uminus zmod-zadd1-eq [symmetric])*
done

lemma *zmod-zsub-right-eq*: *((a::int)* $-$ *b) mod c = (a* $-$ *b mod c) mod c*
apply *(unfold diff-int-def)*
apply *(rule trans [OF - zmod-zadd-right-eq [symmetric]])*
apply *(simp add : zmod-uminus zmod-zadd-right-eq [symmetric])*
done

lemma *zmod-zsub-left-eq*: *((a::int)* $-$ *b) mod c = (a mod c* $-$ *b) mod c*
by *(rule zmod-zadd-left-eq [***where** *b =* $-$ *b, simplified diff-int-def [symmetric]])*

lemma *zmod-zsub-self [simp]*:
((b :: int) $-$ *a) mod a = b mod a*
by *(simp add: zmod-zsub-right-eq)*

lemma *zmod-zmult1-eq-rev*:
b $*$ *a mod c = b mod c* $*$ *a mod (c::int)*
apply *(simp add: mult-commute)*
apply *(subst zmod-zmult1-eq)*
apply *simp*
done

lemmas *rdmods [symmetric] = zmod-uminus [symmetric]*
zmod-zsub-left-eq zmod-zsub-right-eq zmod-zadd-left-eq
zmod-zadd-right-eq zmod-zmult1-eq zmod-zmult1-eq-rev

lemma *mod-plus-right*:
((a + x) mod m = (b + x) mod m) = (a mod m = b mod (m :: nat))
apply (*induct x*)
apply (*simp-all add*: *mod-Suc*)
apply *arith*
done

lemma *nat-minus-mod*: (n − n mod m) mod m = (0 :: nat)
by (*induct n*) (*simp-all add* : *mod-Suc*)

lemmas *nat-minus-mod-plus-right* = *trans* [*OF nat-minus-mod mod-0* [*symmetric*],
THEN mod-plus-right [*THEN iffD2*], *standard*, *simplified*]

lemmas *push-mods′* = *zmod-zadd1-eq* [*standard*]
zmod-zmult-distrib [*standard*] *zmod-zsub-distrib* [*standard*]
zmod-uminus [*symmetric*, *standard*]

lemmas *push-mods* = *push-mods′* [*THEN eq-reflection*, *standard*]
lemmas *pull-mods* = *push-mods* [*symmetric*] *rdmods* [*THEN eq-reflection*, *standard*]
lemmas *mod-simps* =
zmod-zmult-self1 [*THEN eq-reflection*] *zmod-zmult-self2* [*THEN eq-reflection*]
mod-mod-trivial [*THEN eq-reflection*]

lemma *nat-mod-eq*:
!!b. b < n ==> a mod n = b mod n ==> a mod n = (b :: nat)
by (*induct a*) *auto*

lemmas *nat-mod-eq′* = *refl* [*THEN* [2] *nat-mod-eq*]

lemma *nat-mod-lem*:
(0 :: nat) < n ==> b < n = (b mod n = b)
apply *safe*
apply (*erule nat-mod-eq′*)
apply (*erule subst*)
apply (*erule mod-less-divisor*)
done

lemma *mod-nat-add*:
(x :: nat) < z ==> y < z ==>
(x + y) mod z = (if x + y < z then x + y else x + y − z)
apply (*rule nat-mod-eq*)
apply *auto*
apply (*rule trans*)
apply (*rule le-mod-geq*)
apply *simp*
apply (*rule nat-mod-eq′*)
apply *arith*
done

```
lemma mod-nat-sub:
  (x :: nat) < z ==> (x − y) mod z = x − y
  by (rule nat-mod-eq') arith

lemma int-mod-lem:
  (0 :: int) < n ==> (0 <= b & b < n) = (b mod n = b)
  apply safe
    apply (erule (1) mod-pos-pos-trivial)
   apply (erule-tac [!] subst)
   apply auto
  done

lemma int-mod-eq:
  (0 :: int) <= b ==> b < n ==> a mod n = b mod n ==> a mod n = b
  by clarsimp (rule mod-pos-pos-trivial)

lemmas int-mod-eq' = refl [THEN [3] int-mod-eq]

lemma int-mod-le: 0 <= a ==> 0 < (n :: int) ==> a mod n <= a
  apply (cases a < n)
   apply (auto dest: mod-pos-pos-trivial pos-mod-bound [where a=a])
  done

lemma int-mod-le': 0 <= b − n ==> 0 < (n :: int) ==> b mod n <= b − n
  by (rule int-mod-le [where a = b − n and n = n, simplified])

lemma int-mod-ge: a < n ==> 0 < (n :: int) ==> a <= a mod n
  apply (cases 0 <= a)
   apply (drule (1) mod-pos-pos-trivial)
   apply simp
  apply (rule order-trans [OF - pos-mod-sign])
   apply simp
  apply assumption
  done

lemma int-mod-ge': b < 0 ==> 0 < (n :: int) ==> b + n <= b mod n
  by (rule int-mod-ge [where a = b + n and n = n, simplified])

lemma mod-add-if-z:
  (x :: int) < z ==> y < z ==> 0 <= y ==> 0 <= x ==> 0 <= z ==>
   (x + y) mod z = (if x + y < z then x + y else x + y − z)
  by (auto intro: int-mod-eq)

lemma mod-sub-if-z:
  (x :: int) < z ==> y < z ==> 0 <= y ==> 0 <= x ==> 0 <= z ==>
   (x − y) mod z = (if y <= x then x − y else x − y + z)
  by (auto intro: int-mod-eq)
```

lemmas *zmde* = *zmod-zdiv-equality* [*THEN diff-eq-eq* [*THEN iffD2*], *symmetric*]
lemmas *mcl* = *mult-cancel-left* [*THEN iffD1*, *THEN make-pos-rule*]

lemma *zdiv-mult-self*: *m* ~= (*0* :: *int*) ==> (*a* + *m* ∗ *n*) *div m* = *a div m* + *n*
  apply (*rule mcl*)
   prefer *2*
   apply (*erule asm-rl*)
  apply (*simp add*: *zmde ring-distribs*)
  apply (*simp add*: *push-mods*)
  done

lemma *eqne*: *equiv A r* ==> *X* : *A* // *r* ==> *X* ~= {}
  unfolding *equiv-def refl-def quotient-def Image-def* **by** *auto*

lemmas *Rep-Integ-ne* = *Integ.Rep-Integ*
  [*THEN equiv-intrel* [*THEN eqne*, *simplified Integ-def* [*symmetric*]], *standard*]

lemmas *riq* = *Integ.Rep-Integ* [*simplified Integ-def*]
lemmas *intrel-refl* = *refl* [*THEN equiv-intrel-iff* [*THEN iffD1*], *standard*]
lemmas *Rep-Integ-equiv* = *quotient-eq-iff*
  [*OF equiv-intrel riq riq*, *simplified Integ.Rep-Integ-inject*, *standard*]
lemmas *Rep-Integ-same* =
  Rep-Integ-equiv [*THEN intrel-refl* [*THEN rev-iffD2*], *standard*]

lemma *RI-int*: (*a*, *0*) : *Rep-Integ* (*int a*)
  unfolding *int-def* **by** *auto*

lemmas *RI-intrel* [*simp*] = *UNIV-I* [*THEN quotientI*,
  THEN Integ.Abs-Integ-inverse [*simplified Integ-def*], *standard*]

lemma *RI-minus*: (*a*, *b*) : *Rep-Integ x* ==> (*b*, *a*) : *Rep-Integ* (− *x*)
  apply (*rule-tac z=x* **in** *eq-Abs-Integ*)
  apply (*clarsimp simp*: *minus*)
  done

lemma *RI-add*:
  (*a*, *b*) : *Rep-Integ x* ==> (*c*, *d*) : *Rep-Integ y* ==>
   (*a* + *c*, *b* + *d*) : *Rep-Integ* (*x* + *y*)
  apply (*rule-tac z=x* **in** *eq-Abs-Integ*)
  apply (*rule-tac z=y* **in** *eq-Abs-Integ*)
  apply (*clarsimp simp*: *add*)
  done

lemma *mem-same*: *a* : *S* ==> *a* = *b* ==> *b* : *S*
  by *fast*

```
lemma RI-eq-diff': (a, b) : Rep-Integ (int a − int b)
  apply (unfold diff-def)
  apply (rule mem-same)
   apply (rule RI-minus RI-add RI-int)+
  apply simp
  done

lemma RI-eq-diff: ((a, b) : Rep-Integ x) = (int a − int b = x)
  apply safe
   apply (rule Rep-Integ-same)
    prefer 2
    apply (erule asm-rl)
   apply (rule RI-eq-diff')+
  done

lemma mod-power-lem:
  a > 1 ==> a ^ n mod a ^ m = (if m <= n then 0 else (a :: int) ^ n)
  apply clarsimp
  apply safe
   apply (simp add: zdvd-iff-zmod-eq-0 [symmetric])
   apply (drule le-iff-add [THEN iffD1])
   apply (force simp: zpower-zadd-distrib)
  apply (rule mod-pos-pos-trivial)
   apply (simp add: zero-le-power)
  apply (rule power-strict-increasing)
   apply auto
  done

lemma min-pm [simp]: min a b + (a − b) = (a :: nat)
  by arith

lemmas min-pm1 [simp] = trans [OF add-commute min-pm]

lemma rev-min-pm [simp]: min b a + (a − b) = (a::nat)
  by simp

lemmas rev-min-pm1 [simp] = trans [OF add-commute rev-min-pm]

lemma pl-pl-rels:
  a + b = c + d ==>
   a >= c & b <= d | a <= c & b >= (d :: nat)
  apply (cut-tac n=a and m=c in nat-le-linear)
  apply (safe dest!: le-iff-add [THEN iffD1])
        apply arith+
  done

lemmas pl-pl-rels' = add-commute [THEN [2] trans, THEN pl-pl-rels]

lemma minus-eq: (m − k = m) = (k = 0 | m = (0 :: nat))
```

```
  by arith

lemma pl-pl-mm: (a :: nat) + b = c + d ==> a - c = d - b
  by arith

lemmas pl-pl-mm' = add-commute [THEN [2] trans, THEN pl-pl-mm]

lemma min-minus [simp] : min m (m - k) = (m - k :: nat)
  by arith

lemmas min-minus' [simp] = trans [OF min-max.inf-commute min-minus]

lemma nat-no-eq-iff:
  (number-of b :: int) >= 0 ==> (number-of c :: int) >= 0 ==>
   (number-of b = (number-of c :: nat)) = (b = c)
  apply (unfold nat-number-of-def)
  apply safe
  apply (drule (2) eq-nat-nat-iff [THEN iffD1])
  apply (simp add: number-of-eq)
  done

lemmas dme = box-equals [OF div-mod-equality add-0-right add-0-right]
lemmas dtle = xtr3 [OF dme [symmetric] le-add1]
lemmas th2 = order-trans [OF order-refl [THEN [2] mult-le-mono] dtle]

lemma td-gal:
  0 < c ==> (a >= b * c) = (a div c >= (b :: nat))
  apply safe
   apply (erule (1) xtr4 [OF div-le-mono div-mult-self-is-m])
  apply (erule th2)
  done

lemmas td-gal-lt = td-gal [simplified le-def, simplified]

lemma div-mult-le: (a :: nat) div b * b <= a
  apply (cases b)
   prefer 2
   apply (rule order-refl [THEN th2])
  apply auto
  done

lemmas sdl = split-div-lemma [THEN iffD1, symmetric]

lemma given-quot: f > (0 :: nat) ==> (f * l + (f - 1)) div f = l
  by (rule sdl, assumption) (simp (no-asm))

lemma given-quot-alt: f > (0 :: nat) ==> (l * f + f - Suc 0) div f = l
  apply (frule given-quot)
  apply (rule trans)
```

prefer *2*
apply (*erule asm-rl*)
apply (*rule-tac f*=%*n*. *n div f* **in** *arg-cong*)
apply (*simp add* : *mult-ac*)
done

lemma *diff-mod-le*: (*a*::*nat*) < *d* ==> *b dvd d* ==> *a* − *a mod b* <= *d* − *b*
apply (*unfold dvd-def*)
apply *clarify*
apply (*case-tac k*)
apply *clarsimp*
apply *clarify*
apply (*cases b* > *0*)
apply (*drule mult-commute* [*THEN xtr1*])
apply (*frule* (*1*) *td-gal-lt* [*THEN iffD1*])
apply (*clarsimp simp*: *le-simps*)
apply (*rule mult-div-cancel* [*THEN* [*2*] *xtr4*])
apply (*rule mult-mono*)
apply *auto*
done

lemma *less-le-mult'*:
$w * c < b * c$ ==> $0 \leq c$ ==> $(w + 1) * c \leq b * (c::int)$
apply (*rule mult-right-mono*)
apply (*rule zless-imp-add1-zle*)
apply (*erule* (*1*) *mult-right-less-imp-less*)
apply *assumption*
done

lemmas *less-le-mult* = *less-le-mult'* [*simplified left-distrib*, *simplified*]

lemmas *less-le-mult-minus* = *iffD2* [*OF le-diff-eq less-le-mult*, *simplified left-diff-distrib*, *standard*]

lemma *lrlem'*:
assumes *d*: $(i::nat) \leq j \vee m < j'$
assumes *R1*: $i * k \leq j * k \Longrightarrow R$
assumes *R2*: $Suc\ m * k' \leq j' * k' \Longrightarrow R$
shows *R* **using** *d*
apply *safe*
apply (*rule R1*, *erule mult-le-mono1*)
apply (*rule R2*, *erule Suc-le-eq* [*THEN iffD2* [*THEN mult-le-mono1*]])
done

lemma *lrlem*: (*0*::*nat*) < *sc* ==>
(*sc* − *n* + (*n* + *lb* ∗ *n*) <= *m* ∗ *n*) = (*sc* + *lb* ∗ *n* <= *m* ∗ *n*)
apply *safe*
apply *arith*
apply (*case-tac sc* >= *n*)

```
   apply arith
  apply (insert linorder-le-less-linear [of m lb])
  apply (erule-tac k=n and k'=n in lrlem')
   apply arith
  apply simp
  done
```

```
lemma gen-minus: 0 < n ==> f n = f (Suc (n - 1))
  by auto
```

```
lemma mpl-lem: j <= (i :: nat) ==> k < j ==> i - j + k < i
  apply (induct i, clarsimp)
  apply (cases j, clarsimp)
  apply arith
  done
```

```
lemma nonneg-mod-div:
  0 <= a ==> 0 <= b ==> 0 <= (a mod b :: int) & 0 <= a div b
  apply (cases b = 0, clarsimp)
  apply (auto intro: pos-imp-zdiv-nonneg-iff [THEN iffD2])
  done
```

end

4 BinGeneral: Basic Definitions for Binary Integers

```
theory BinGeneral imports Num-Lemmas
```

begin

4.1 Recursion combinator for binary integers

```
lemma brlem: (bin = Numeral.Min) = (- bin + Numeral.pred 0 = 0)
  unfolding Min-def pred-def by arith
```

```
function
  bin-rec' :: int * 'a * 'a * (int => bit => 'a => 'a) => 'a
  where
  bin-rec' (bin, f1, f2, f3) = (if bin = Numeral.Pls then f1
    else if bin = Numeral.Min then f2
    else case bin-rl bin of (w, b) => f3 w b (bin-rec' (w, f1, f2, f3)))
  by pat-completeness auto
```

```
termination
  apply (relation measure (nat o abs o fst))
   apply simp
```

apply (*simp add*: *Pls-def brlem*)
apply (*clarsimp simp*: *bin-rl-char pred-def*)
apply (*frule thin-rl* [*THEN refl* [*THEN bin-abs-lem* [*rule-format*]]])
  apply (*unfold Pls-def Min-def number-of-eq*)
  prefer *2*
  apply (*erule asm-rl*)
 apply *auto*
done

constdefs
bin-rec :: $'a$ => $'a$ => (*int* => *bit* => $'a$ => $'a$) => *int* => $'a$
bin-rec f1 f2 f3 bin == *bin-rec'* (*bin*, *f1*, *f2*, *f3*)

lemma *bin-rec-PM*:
f = *bin-rec f1 f2 f3* ==> *f Numeral.Pls* = *f1* & *f Numeral.Min* = *f2*
apply *safe*
 apply (*unfold bin-rec-def*)
 apply (*auto intro*: *bin-rec'.simps* [*THEN trans*])
done

lemmas *bin-rec-Pls* = *refl* [*THEN bin-rec-PM*, *THEN conjunct1*, *standard*]
lemmas *bin-rec-Min* = *refl* [*THEN bin-rec-PM*, *THEN conjunct2*, *standard*]

lemma *bin-rec-Bit*:
f = *bin-rec f1 f2 f3* ==> *f3 Numeral.Pls bit.B0 f1* = *f1* ==>
  f3 Numeral.Min bit.B1 f2 = *f2* ==> *f* (*w BIT b*) = *f3 w b* (*f w*)
apply *clarify*
apply (*unfold bin-rec-def*)
apply (*rule bin-rec'.simps* [*THEN trans*])
apply *auto*
    apply (*unfold Pls-def Min-def Bit-def*)
    apply (*cases b*, *auto*)+
done

lemmas *bin-rec-simps* = *refl* [*THEN bin-rec-Bit*] *bin-rec-Pls bin-rec-Min*

4.2 Destructors for binary integers

consts
— corresponding operations analysing bins
bin-last :: *int* => *bit*
bin-rest :: *int* => *int*
bin-sign :: *int* => *int*
bin-nth :: *int* => *nat* => *bool*

primrec
Z : *bin-nth w 0* = (*bin-last w* = *bit.B1*)
Suc : *bin-nth w* (*Suc n*) = *bin-nth* (*bin-rest w*) *n*

defs
bin-rest-def : *bin-rest w* == *fst* (*bin-rl w*)
bin-last-def : *bin-last w* == *snd* (*bin-rl w*)
bin-sign-def : *bin-sign* == *bin-rec Numeral.Pls Numeral.Min* (%*w b s. s*)

lemma *bin-rl*: *bin-rl w* = (*bin-rest w*, *bin-last w*)
unfolding *bin-rest-def bin-last-def* **by** *auto*

lemmas *bin-rl-simp* [*simp*] = *iffD1* [*OF bin-rl-char bin-rl*]

lemma *bin-rest-simps* [*simp*]:
bin-rest Numeral.Pls = *Numeral.Pls*
bin-rest Numeral.Min = *Numeral.Min*
bin-rest (*w BIT b*) = *w*
unfolding *bin-rest-def* **by** *auto*

lemma *bin-last-simps* [*simp*]:
bin-last Numeral.Pls = *bit.B0*
bin-last Numeral.Min = *bit.B1*
bin-last (*w BIT b*) = *b*
unfolding *bin-last-def* **by** *auto*

lemma *bin-sign-simps* [*simp*]:
bin-sign Numeral.Pls = *Numeral.Pls*
bin-sign Numeral.Min = *Numeral.Min*
bin-sign (*w BIT b*) = *bin-sign w*
unfolding *bin-sign-def* **by** (*auto simp*: *bin-rec-simps*)

lemma *bin-r-l-extras* [*simp*]:
bin-last 0 = *bit.B0*
bin-last (− *1*) = *bit.B1*
bin-last −*1* = *bit.B1*
bin-last 1 = *bit.B1*
bin-rest 1 = *0*
bin-rest 0 = *0*
bin-rest (− *1*) = − *1*
bin-rest −*1* = −*1*
apply (*unfold number-of-Min*)
apply (*unfold Pls-def* [*symmetric*] *Min-def* [*symmetric*])
apply (*unfold numeral-1-eq-1* [*symmetric*])
apply (*auto simp*: *number-of-eq*)
done

lemma *bin-last-mod*:
bin-last w = (*if w mod 2* = *0 then bit.B0 else bit.B1*)
apply (*case-tac w rule*: *bin-exhaust*)
apply (*case-tac b*)
apply *auto*
done

```
lemma bin-rest-div:
  bin-rest w = w div 2
  apply (case-tac w rule: bin-exhaust)
  apply (rule trans)
   apply clarsimp
   apply (rule refl)
  apply (drule trans)
   apply (rule Bit-def)
  apply (simp add: z1pdiv2 split: bit.split)
  done

lemma Bit-div2 [simp]: (w BIT b) div 2 = w
  unfolding bin-rest-div [symmetric] by auto

lemma bin-nth-lem [rule-format]:
  ALL y. bin-nth x = bin-nth y --> x = y
  apply (induct x rule: bin-induct)
    apply safe
    apply (erule rev-mp)
    apply (induct-tac y rule: bin-induct)
      apply safe
      apply (drule-tac x=0 in fun-cong, force)
     apply (erule notE, rule ext,
            drule-tac x=Suc x in fun-cong, force)
    apply (drule-tac x=0 in fun-cong, force)
   apply (erule rev-mp)
   apply (induct-tac y rule: bin-induct)
     apply safe
     apply (drule-tac x=0 in fun-cong, force)
    apply (erule notE, rule ext,
           drule-tac x=Suc x in fun-cong, force)
   apply (drule-tac x=0 in fun-cong, force)
  apply (case-tac y rule: bin-exhaust)
  apply clarify
  apply (erule allE)
  apply (erule impE)
   prefer 2
   apply (erule BIT-eqI)
   apply (drule-tac x=0 in fun-cong, force)
  apply (rule ext)
  apply (drule-tac x=Suc ?x in fun-cong, force)
  done

lemma bin-nth-eq-iff: (bin-nth x = bin-nth y) = (x = y)
  by (auto elim: bin-nth-lem)

lemmas bin-eqI = ext [THEN bin-nth-eq-iff [THEN iffD1], standard]
```

lemma *bin-nth-Pls* [*simp*]: ~ *bin-nth Numeral.Pls n*
  by (*induct n*) *auto*

lemma *bin-nth-Min* [*simp*]: *bin-nth Numeral.Min n*
  by (*induct n*) *auto*

lemma *bin-nth-0-BIT*: *bin-nth* (*w BIT b*) *0* = (*b* = *bit.B1*)
  by *auto*

lemma *bin-nth-Suc-BIT*: *bin-nth* (*w BIT b*) (*Suc n*) = *bin-nth w n*
  by *auto*

lemma *bin-nth-minus* [*simp*]: *0* < *n* ==> *bin-nth* (*w BIT b*) *n* = *bin-nth w* (*n* − *1*)
  by (*cases n*) *auto*

lemmas *bin-nth-0* = *bin-nth.simps*(*1*)
lemmas *bin-nth-Suc* = *bin-nth.simps*(*2*)

lemmas *bin-nth-simps* =
  bin-nth-0 bin-nth-Suc bin-nth-Pls bin-nth-Min bin-nth-minus

lemma *bin-sign-rest* [*simp*]:
  bin-sign (*bin-rest w*) = (*bin-sign w*)
  by (*case-tac w rule*: *bin-exhaust*) *auto*

4.3 Truncating binary integers

consts
  bintrunc :: *nat* => *int* => *int*
primrec
  Z : *bintrunc 0 bin* = *Numeral.Pls*
  Suc : *bintrunc* (*Suc n*) *bin* = *bintrunc n* (*bin-rest bin*) *BIT* (*bin-last bin*)

consts
  sbintrunc :: *nat* => *int* => *int*
primrec
  Z : *sbintrunc 0 bin* =
    (*case bin-last bin of bit.B1* => *Numeral.Min* | *bit.B0* => *Numeral.Pls*)
  Suc : *sbintrunc* (*Suc n*) *bin* = *sbintrunc n* (*bin-rest bin*) *BIT* (*bin-last bin*)

lemma *sign-bintr*:
  !!*w*. *bin-sign* (*bintrunc n w*) = *Numeral.Pls*
  by (*induct n*) *auto*

lemma *bintrunc-mod2p*:
  !!*w*. *bintrunc n w* = (*w mod 2* ˆ *n* :: *int*)
  apply (*induct n*, *clarsimp*)
  apply (*simp add*: *bin-last-mod bin-rest-div Bit-def zmod-zmult2-eq*

```
            cong: number-of-False-cong)
  done

lemma sbintrunc-mod2p:
  !!w. sbintrunc n w = ((w + 2 ^ n) mod 2 ^ (Suc n) − 2 ^ n :: int)
  apply (induct n)
   apply clarsimp
   apply (subst zmod-zadd-left-eq)
   apply (simp add: bin-last-mod)
   apply (simp add: number-of-eq)
  apply clarsimp
  apply (simp add: bin-last-mod bin-rest-div Bit-def
            cong: number-of-False-cong)
  apply (clarsimp simp: zmod-zmult-zmult1 [symmetric]
        zmod-zdiv-equality [THEN diff-eq-eq [THEN iffD2 [THEN sym]]])
  apply (rule trans [symmetric, OF - emep1])
    apply auto
  apply (auto simp: even-def)
  done
```

4.4 Simplifications for (s)bintrunc

```
lemma bit-bool:
  (b = (b' = bit.B1)) = (b' = (if b then bit.B1 else bit.B0))
  by (cases b') auto

lemmas bit-bool1 [simp] = refl [THEN bit-bool [THEN iffD1], symmetric]

lemma bin-sign-lem:
  !!bin. (bin-sign (sbintrunc n bin) = Numeral.Min) = bin-nth bin n
  apply (induct n)
   apply (case-tac bin rule: bin-exhaust, case-tac b, auto)+
  done

lemma nth-bintr:
  !!w m. bin-nth (bintrunc m w) n = (n < m & bin-nth w n)
  apply (induct n)
   apply (case-tac m, auto)[1]
  apply (case-tac m, auto)[1]
  done

lemma nth-sbintr:
  !!w m. bin-nth (sbintrunc m w) n =
        (if n < m then bin-nth w n else bin-nth w m)
  apply (induct n)
   apply (case-tac m, simp-all split: bit.splits)[1]
  apply (case-tac m, simp-all split: bit.splits)[1]
  done
```

lemma *bin-nth-Bit*:
bin-nth (*w BIT b*) *n* = (*n* = *0* & *b* = *bit.B1* | (*EX m*. *n* = *Suc m* & *bin-nth w m*))
by (*cases n*) *auto*

lemma *bintrunc-bintrunc-l*:
n <= *m* ==> (*bintrunc m* (*bintrunc n w*) = *bintrunc n w*)
by (*rule bin-eqI*) (*auto simp add* : *nth-bintr*)

lemma *sbintrunc-sbintrunc-l*:
n <= *m* ==> (*sbintrunc m* (*sbintrunc n w*) = *sbintrunc n w*)
by (*rule bin-eqI*) (*auto simp*: *nth-sbintr min-def*)

lemma *bintrunc-bintrunc-ge*:
n <= *m* ==> (*bintrunc n* (*bintrunc m w*) = *bintrunc n w*)
by (*rule bin-eqI*) (*auto simp*: *nth-bintr*)

lemma *bintrunc-bintrunc-min* [*simp*]:
bintrunc m (*bintrunc n w*) = *bintrunc* (*min m n*) *w*
apply (*unfold min-def*)
apply (*rule bin-eqI*)
apply (*auto simp*: *nth-bintr*)
done

lemma *sbintrunc-sbintrunc-min* [*simp*]:
sbintrunc m (*sbintrunc n w*) = *sbintrunc* (*min m n*) *w*
apply (*unfold min-def*)
apply (*rule bin-eqI*)
apply (*auto simp*: *nth-sbintr*)
done

lemmas *bintrunc-Pls* =
bintrunc.Suc [**where** *bin*=*Numeral.Pls*, *simplified bin-last-simps bin-rest-simps*, *standard*]

lemmas *bintrunc-Min* [*simp*] =
bintrunc.Suc [**where** *bin*=*Numeral.Min*, *simplified bin-last-simps bin-rest-simps*, *standard*]

lemmas *bintrunc-BIT* [*simp*] =
bintrunc.Suc [**where** *bin*=*w BIT b*, *simplified bin-last-simps bin-rest-simps*, *standard*]

lemmas *bintrunc-Sucs* = *bintrunc-Pls bintrunc-Min bintrunc-BIT*

lemmas *sbintrunc-Suc-Pls* =
sbintrunc.Suc [**where** *bin*=*Numeral.Pls*, *simplified bin-last-simps bin-rest-simps*, *standard*]

lemmas *sbintrunc-Suc-Min* =
sbintrunc.Suc [**where** *bin=Numeral.Min*, *simplified bin-last-simps bin-rest-simps*, *standard*]

lemmas *sbintrunc-Suc-BIT* [*simp*] =
sbintrunc.Suc [**where** *bin=w BIT b*, *simplified bin-last-simps bin-rest-simps*, *standard*]

lemmas *sbintrunc-Sucs* = *sbintrunc-Suc-Pls sbintrunc-Suc-Min sbintrunc-Suc-BIT*

lemmas *sbintrunc-Pls* =
sbintrunc.Z [**where** *bin=Numeral.Pls*,
simplified bin-last-simps bin-rest-simps bit.simps, *standard*]

lemmas *sbintrunc-Min* =
sbintrunc.Z [**where** *bin=Numeral.Min*,
simplified bin-last-simps bin-rest-simps bit.simps, *standard*]

lemmas *sbintrunc-0-BIT-B0* [*simp*] =
sbintrunc.Z [**where** *bin=w BIT bit.B0*,
simplified bin-last-simps bin-rest-simps bit.simps, *standard*]

lemmas *sbintrunc-0-BIT-B1* [*simp*] =
sbintrunc.Z [**where** *bin=w BIT bit.B1*,
simplified bin-last-simps bin-rest-simps bit.simps, *standard*]

lemmas *sbintrunc-0-simps* =
sbintrunc-Pls sbintrunc-Min sbintrunc-0-BIT-B0 sbintrunc-0-BIT-B1

lemmas *bintrunc-simps* = *bintrunc.Z bintrunc-Sucs*
lemmas *sbintrunc-simps* = *sbintrunc-0-simps sbintrunc-Sucs*

lemma *bintrunc-minus*:
0 < *n* ==> *bintrunc* (*Suc* (*n* − *1*)) *w* = *bintrunc n w*
by *auto*

lemma *sbintrunc-minus*:
0 < *n* ==> *sbintrunc* (*Suc* (*n* − *1*)) *w* = *sbintrunc n w*
by *auto*

lemmas *bintrunc-minus-simps* =
bintrunc-Sucs [*THEN* [*2*] *bintrunc-minus* [*symmetric*, *THEN trans*], *standard*]
lemmas *sbintrunc-minus-simps* =
sbintrunc-Sucs [*THEN* [*2*] *sbintrunc-minus* [*symmetric*, *THEN trans*], *standard*]

lemma *bintrunc-n-Pls* [*simp*]:
bintrunc n Numeral.Pls = *Numeral.Pls*
by (*induct n*) *auto*

lemma *sbintrunc-n-PM* [*simp*]:
sbintrunc n Numeral.Pls = Numeral.Pls
sbintrunc n Numeral.Min = Numeral.Min
by (*induct n*) *auto*

lemmas *thobini1 = arg-cong* [**where** *f = %w. w BIT b*, *standard*]

lemmas *bintrunc-BIT-I = trans* [*OF bintrunc-BIT thobini1*]
lemmas *bintrunc-Min-I = trans* [*OF bintrunc-Min thobini1*]

lemmas *bmsts = bintrunc-minus-simps* [*THEN thobini1* [*THEN* [*2*] *trans*], *standard*]
lemmas *bintrunc-Pls-minus-I = bmsts*(*1*)
lemmas *bintrunc-Min-minus-I = bmsts*(*2*)
lemmas *bintrunc-BIT-minus-I = bmsts*(*3*)

lemma *bintrunc-0-Min*: *bintrunc 0 Numeral.Min = Numeral.Pls*
by *auto*
lemma *bintrunc-0-BIT*: *bintrunc 0* (*w BIT b*) *= Numeral.Pls*
by *auto*

lemma *bintrunc-Suc-lem*:
bintrunc (*Suc n*) *x = y ==> m = Suc n ==> bintrunc m x = y*
by *auto*

lemmas *bintrunc-Suc-Ialts =*
bintrunc-Min-I bintrunc-BIT-I [*THEN bintrunc-Suc-lem*, *standard*]

lemmas *sbintrunc-BIT-I = trans* [*OF sbintrunc-Suc-BIT thobini1*]

lemmas *sbintrunc-Suc-Is =*
sbintrunc-Sucs [*THEN thobini1* [*THEN* [*2*] *trans*], *standard*]

lemmas *sbintrunc-Suc-minus-Is =*
sbintrunc-minus-simps [*THEN thobini1* [*THEN* [*2*] *trans*], *standard*]

lemma *sbintrunc-Suc-lem*:
sbintrunc (*Suc n*) *x = y ==> m = Suc n ==> sbintrunc m x = y*
by *auto*

lemmas *sbintrunc-Suc-Ialts =*
sbintrunc-Suc-Is [*THEN sbintrunc-Suc-lem*, *standard*]

lemma *sbintrunc-bintrunc-lt*:
m > n ==> sbintrunc n (*bintrunc m w*) *= sbintrunc n w*
by (*rule bin-eqI*) (*auto simp*: *nth-sbintr nth-bintr*)

lemma *bintrunc-sbintrunc-le*:
m <= Suc n ==> bintrunc m (*sbintrunc n w*) *= bintrunc m w*

apply (*rule bin-eqI*)
apply (*auto simp*: *nth-sbintr nth-bintr*)
 apply (*subgoal-tac x=n*, *safe*, *arith+*)[*1*]
apply (*subgoal-tac x=n*, *safe*, *arith+*)[*1*]
done

lemmas *bintrunc-sbintrunc* [*simp*] = *order-refl* [*THEN bintrunc-sbintrunc-le*]
lemmas *sbintrunc-bintrunc* [*simp*] = *lessI* [*THEN sbintrunc-bintrunc-lt*]
lemmas *bintrunc-bintrunc* [*simp*] = *order-refl* [*THEN bintrunc-bintrunc-l*]
lemmas *sbintrunc-sbintrunc* [*simp*] = *order-refl* [*THEN sbintrunc-sbintrunc-l*]

lemma *bintrunc-sbintrunc′* [*simp*]:
0 < *n* ⟹ *bintrunc n* (*sbintrunc* (*n* − *1*) *w*) = *bintrunc n w*
by (*cases n*) (*auto simp del*: *bintrunc.Suc*)

lemma *sbintrunc-bintrunc′* [*simp*]:
0 < *n* ⟹ *sbintrunc* (*n* − *1*) (*bintrunc n w*) = *sbintrunc* (*n* − *1*) *w*
by (*cases n*) (*auto simp del*: *bintrunc.Suc*)

lemma *bin-sbin-eq-iff*:
bintrunc (*Suc n*) *x* = *bintrunc* (*Suc n*) *y* <−>
 sbintrunc n x = *sbintrunc n y*
apply (*rule iffI*)
 apply (*rule box-equals* [*OF - sbintrunc-bintrunc sbintrunc-bintrunc*])
 apply *simp*
apply (*rule box-equals* [*OF - bintrunc-sbintrunc bintrunc-sbintrunc*])
apply *simp*
done

lemma *bin-sbin-eq-iff′*:
0 < *n* ⟹ *bintrunc n x* = *bintrunc n y* <−>
 sbintrunc (*n* − *1*) *x* = *sbintrunc* (*n* − *1*) *y*
by (*cases n*) (*simp-all add*: *bin-sbin-eq-iff del*: *bintrunc.Suc*)

lemmas *bintrunc-sbintruncS0* [*simp*] = *bintrunc-sbintrunc′* [*unfolded One-nat-def*]
lemmas *sbintrunc-bintruncS0* [*simp*] = *sbintrunc-bintrunc′* [*unfolded One-nat-def*]

lemmas *bintrunc-bintrunc-l′* = *le-add1* [*THEN bintrunc-bintrunc-l*]
lemmas *sbintrunc-sbintrunc-l′* = *le-add1* [*THEN sbintrunc-sbintrunc-l*]

lemmas *nat-non0-gr* =
trans [*OF iszero-def* [*THEN Not-eq-iff* [*THEN iffD2*]] *refl*, *standard*]

lemmas *bintrunc-pred-simps* [*simp*] =
bintrunc-minus-simps [*of number-of bin*, *simplified nobm1*, *standard*]

lemmas *sbintrunc-pred-simps* [*simp*] =

sbintrunc-minus-simps [*of number-of bin*, *simplified nobm1*, *standard*]

lemma *no-bintr-alt*:
number-of (*bintrunc n w*) = *w mod 2 ^ n*
by (*simp add*: *number-of-eq bintrunc-mod2p*)

lemma *no-bintr-alt1*: *bintrunc n* = (%*w*. *w mod 2 ^ n* :: *int*)
by (*rule ext*) (*rule bintrunc-mod2p*)

lemma *range-bintrunc*: *range* (*bintrunc n*) = {*i*. *0* <= *i* & *i* < *2 ^ n*}
apply (*unfold no-bintr-alt1*)
apply (*auto simp add*: *image-iff*)
apply (*rule exI*)
apply (*auto intro*: *int-mod-lem* [*THEN iffD1*, *symmetric*])
done

lemma *no-bintr*:
number-of (*bintrunc n w*) = (*number-of w mod 2 ^ n* :: *int*)
by (*simp add* : *bintrunc-mod2p number-of-eq*)

lemma *no-sbintr-alt2*:
sbintrunc n = (%*w*. (*w* + *2 ^ n*) *mod 2 ^ Suc n* − *2 ^ n* :: *int*)
by (*rule ext*) (*simp add* : *sbintrunc-mod2p*)

lemma *no-sbintr*:
number-of (*sbintrunc n w*) =
((*number-of w* + *2 ^ n*) *mod 2 ^ Suc n* − *2 ^ n* :: *int*)
by (*simp add* : *no-sbintr-alt2 number-of-eq*)

lemma *range-sbintrunc*:
range (*sbintrunc n*) = {*i*. − (*2 ^ n*) <= *i* & *i* < *2 ^ n*}
apply (*unfold no-sbintr-alt2*)
apply (*auto simp add*: *image-iff eq-diff-eq*)
apply (*rule exI*)
apply (*auto intro*: *int-mod-lem* [*THEN iffD1*, *symmetric*])
done

lemma *sb-inc-lem*:
(*a*::*int*) + *2^k* < *0* ⟹ *a* + *2^k* + *2^*(*Suc k*) <= (*a* + *2^k*) *mod 2^*(*Suc k*)
apply (*erule int-mod-ge'* [**where** *n* = *2 ^* (*Suc k*) **and** *b* = *a* + *2 ^ k*, *simplified zless2p*])
apply (*rule TrueI*)
done

lemma *sb-inc-lem'*:
(*a*::*int*) < − (*2^k*) ⟹ *a* + *2^k* + *2^*(*Suc k*) <= (*a* + *2^k*) *mod 2^*(*Suc k*)
by (*rule iffD1* [*OF less-diff-eq*, *THEN sb-inc-lem*, *simplified OrderedGroup.diff-0*])

lemma *sbintrunc-inc*:

$x < -(2\hat{}n) ==> x + 2\hat{}(Suc\ n) <= sbintrunc\ n\ x$
unfolding *no-sbintr-alt2* **by** (*drule sb-inc-lem'*) *simp*

lemma *sb-dec-lem*:
$(0::int) <= -(2\hat{}k) + a ==> (a + 2\hat{}k)\ mod\ (2 * 2\ \hat{}\ k) <= -(2\ \hat{}\ k) + a$
by (*rule int-mod-le'* [**where** $n = 2\ \hat{}\ (Suc\ k)$ **and** $b = a + 2\ \hat{}\ k$,
simplified zless2p, *OF* - *TrueI*, *simplified*])

lemma *sb-dec-lem'*:
$(2::int)\ \hat{}\ k <= a ==> (a + 2\ \hat{}\ k)\ mod\ (2 * 2\ \hat{}\ k) <= -(2\ \hat{}\ k) + a$
by (*rule iffD1* [*OF diff-le-eq'*, *THEN sb-dec-lem*, *simplified*])

lemma *sbintrunc-dec*:
$x >= (2\ \hat{}\ n) ==> x - 2\ \hat{}\ (Suc\ n) >= sbintrunc\ n\ x$
unfolding *no-sbintr-alt2* **by** (*drule sb-dec-lem'*) *simp*

lemmas *zmod-uminus'* = *zmod-uminus* [**where** *b*=*c*, *standard*]
lemmas *zpower-zmod'* = *zpower-zmod* [**where** *m*=*c* **and** *y*=*k*, *standard*]

lemmas *brdmod1s'* [*symmetric*] =
zmod-zadd-left-eq zmod-zadd-right-eq
zmod-zsub-left-eq zmod-zsub-right-eq
zmod-zmult1-eq zmod-zmult1-eq-rev

lemmas *brdmods'* [*symmetric*] =
zpower-zmod' [*symmetric*]
trans [*OF zmod-zadd-left-eq zmod-zadd-right-eq*]
trans [*OF zmod-zsub-left-eq zmod-zsub-right-eq*]
trans [*OF zmod-zmult1-eq zmod-zmult1-eq-rev*]
zmod-uminus' [*symmetric*]
zmod-zadd-left-eq [**where** $b = 1$]
zmod-zsub-left-eq [**where** $b = 1$]

lemmas *bintr-arith1s* =
brdmod1s' [**where** $c=2\hat{}n$, *folded pred-def succ-def bintrunc-mod2p*, *standard*]
lemmas *bintr-ariths* =
brdmods' [**where** $c=2\hat{}n$, *folded pred-def succ-def bintrunc-mod2p*, *standard*]

lemmas *m2pths* = *pos-mod-sign pos-mod-bound* [*OF zless2p*, *standard*]

lemma *bintr-ge0*: $(0 :: int) <= number\text{-}of\ (bintrunc\ n\ w)$
by (*simp add* : *no-bintr m2pths*)

lemma *bintr-lt2p*: $number\text{-}of\ (bintrunc\ n\ w) < (2\ \hat{}\ n :: int)$
by (*simp add* : *no-bintr m2pths*)

lemma *bintr-Min*:
$number\text{-}of\ (bintrunc\ n\ Numeral.Min) = (2\ \hat{}\ n :: int) - 1$
by (*simp add* : *no-bintr m1mod2k*)

lemma *sbintr-ge*: (− (*2* ˆ *n*) :: *int*) <= *number-of* (*sbintrunc n w*)
  by (*simp add* : *no-sbintr m2pths*)

lemma *sbintr-lt*: *number-of* (*sbintrunc n w*) < (*2* ˆ *n* :: *int*)
  by (*simp add* : *no-sbintr m2pths*)

lemma *bintrunc-Suc*:
  bintrunc (*Suc n*) *bin* = *bintrunc n* (*bin-rest bin*) *BIT bin-last bin*
  by (*case-tac bin rule*: *bin-exhaust*) *auto*

lemma *sign-Pls-ge-0*:
  (*bin-sign bin* = *Numeral.Pls*) = (*number-of bin* >= (*0* :: *int*))
  by (*induct bin rule*: *bin-induct*) *auto*

lemma *sign-Min-lt-0*:
  (*bin-sign bin* = *Numeral.Min*) = (*number-of bin* < (*0* :: *int*))
  by (*induct bin rule*: *bin-induct*) *auto*

lemmas *sign-Min-neg* = *trans* [*OF sign-Min-lt-0 neg-def* [*symmetric*]]

lemma *bin-rest-trunc*:
  !!*bin*. (*bin-rest* (*bintrunc n bin*)) = *bintrunc* (*n* − *1*) (*bin-rest bin*)
  by (*induct n*) *auto*

lemma *bin-rest-power-trunc* [*rule-format*] :
  (*bin-rest* ˆ *k*) (*bintrunc n bin*) =
    bintrunc (*n* − *k*) ((*bin-rest* ˆ *k*) *bin*)
  by (*induct k*) (*auto simp*: *bin-rest-trunc*)

lemma *bin-rest-trunc-i*:
  bintrunc n (*bin-rest bin*) = *bin-rest* (*bintrunc* (*Suc n*) *bin*)
  by *auto*

lemma *bin-rest-strunc*:
  !!*bin*. *bin-rest* (*sbintrunc* (*Suc n*) *bin*) = *sbintrunc n* (*bin-rest bin*)
  by (*induct n*) *auto*

lemma *bintrunc-rest* [*simp*]:
  !!*bin*. *bintrunc n* (*bin-rest* (*bintrunc n bin*)) = *bin-rest* (*bintrunc n bin*)
  apply (*induct n*, *simp*)
  apply (*case-tac bin rule*: *bin-exhaust*)
  apply (*auto simp*: *bintrunc-bintrunc-l*)
  done

lemma *sbintrunc-rest* [*simp*]:
  !!*bin*. *sbintrunc n* (*bin-rest* (*sbintrunc n bin*)) = *bin-rest* (*sbintrunc n bin*)
  apply (*induct n*, *simp*)
  apply (*case-tac bin rule*: *bin-exhaust*)

apply (*auto simp*: *bintrunc-bintrunc-l split*: *bit.splits*)
done

lemma *bintrunc-rest'*:
bintrunc n o bin-rest o bintrunc n = bin-rest o bintrunc n
by (*rule ext*) *auto*

lemma *sbintrunc-rest'* :
sbintrunc n o bin-rest o sbintrunc n = bin-rest o sbintrunc n
by (*rule ext*) *auto*

lemma *rco-lem*:
f o g o f = g o f ==> f o (g o f) ^ n = g ^ n o f
apply (*rule ext*)
apply (*induct-tac n*)
apply (*simp-all* (*no-asm*))
apply (*drule fun-cong*)
apply (*unfold o-def*)
apply (*erule trans*)
apply *simp*
done

lemma *rco-alt*: (*f o g*) ^ *n o f = f o* (*g o f*) ^ *n*
apply (*rule ext*)
apply (*induct n*)
apply (*simp-all add*: *o-def*)
done

lemmas *rco-bintr = bintrunc-rest'*
[*THEN rco-lem* [*THEN fun-cong*], *unfolded o-def*]
lemmas *rco-sbintr = sbintrunc-rest'*
[*THEN rco-lem* [*THEN fun-cong*], *unfolded o-def*]

4.5 Splitting and concatenation

consts
bin-split :: *nat => int => int * int*
primrec
Z : *bin-split 0 w =* (*w*, *Numeral.Pls*)
Suc : *bin-split* (*Suc n*) *w =* (*let* (*w1*, *w2*) *= bin-split n* (*bin-rest w*)
in (*w1*, *w2 BIT bin-last w*))

consts
bin-cat :: *int => nat => int => int*
primrec
Z : *bin-cat w 0 v = w*
Suc : *bin-cat w* (*Suc n*) *v = bin-cat w n* (*bin-rest v*) *BIT bin-last v*

4.6 Miscellaneous lemmas

lemmas *funpow-minus-simp* =
  trans [*OF gen-minus* [**where** *f* = *power f*] *funpow-Suc*, *standard*]

lemmas *funpow-pred-simp* [*simp*] =
  funpow-minus-simp [*of number-of bin*, *simplified nobm1*, *standard*]

lemmas *replicate-minus-simp* =
  trans [*OF gen-minus* [**where** *f* = %*n*. *replicate n x*] *replicate.replicate-Suc*,
    standard]

lemmas *replicate-pred-simp* [*simp*] =
  replicate-minus-simp [*of number-of bin*, *simplified nobm1*, *standard*]

lemmas *power-Suc-no* [*simp*] = *power-Suc* [*of number-of a*, *standard*]

lemmas *power-minus-simp* =
  trans [*OF gen-minus* [**where** *f* = *power f*] *power-Suc*, *standard*]

lemmas *power-pred-simp* =
  power-minus-simp [*of number-of bin*, *simplified nobm1*, *standard*]
lemmas *power-pred-simp-no* [*simp*] = *power-pred-simp* [**where** *f*= *number-of f*, *standard*]

lemma *list-exhaust-size-gt0*:
  assumes *y*: $\bigwedge a$ *list*. *y* = *a* # *list* $\Longrightarrow$ *P*
  shows *0* < *length y* $\Longrightarrow$ *P*
  apply (*cases y*, *simp*)
  apply (*rule y*)
  apply *fastsimp*
  done

lemma *list-exhaust-size-eq0*:
  assumes *y*: *y* = [] $\Longrightarrow$ *P*
  shows *length y* = *0* $\Longrightarrow$ *P*
  apply (*cases y*)
   apply (*rule y*, *simp*)
  apply *simp*
  done

lemma *size-Cons-lem-eq*:
  y = *xa* # *list* ==> *size y* = *Suc k* ==> *size list* = *k*
  by *auto*

lemma *size-Cons-lem-eq-bin*:
  y = *xa* # *list* ==> *size y* = *number-of* (*Numeral.succ k*) ==>
    size list = *number-of k*
  by (*auto simp*: *pred-def succ-def split add* : *split-if-asm*)

lemmas *ls-splits* =
  prod.split split-split prod.split-asm split-split-asm split-if-asm

lemma *not-B1-is-B0*: $y \neq bit.B1 \Longrightarrow y = bit.B0$
  by (*cases y*) *auto*

lemma *B1-ass-B0*:
  assumes *y*: $y = bit.B0 \Longrightarrow y = bit.B1$
  shows $y = bit.B1$
  apply (*rule classical*)
  apply (*drule not-B1-is-B0*)
  apply (*erule y*)
  done

— simplifications for specific word lengths
lemmas *n2s-ths* [*THEN eq-reflection*] = *add-2-eq-Suc add-2-eq-Suc'*

lemmas *s2n-ths* = *n2s-ths* [*symmetric*]

end

5 BitSyntax: Syntactic class for bitwise operations

theory *BitSyntax*
imports *Main*
begin

class *bit* = *type* +
  fixes *bitNOT* :: $'a \Rightarrow 'a$
    and *bitAND* :: $'a \Rightarrow 'a \Rightarrow 'a$
    and *bitOR* :: $'a \Rightarrow 'a \Rightarrow 'a$
    and *bitXOR* :: $'a \Rightarrow 'a \Rightarrow 'a$

We want the bitwise operations to bind slightly weaker than + and −, but ~~ to bind slightly stronger than ∗.

notation
  bitNOT (*NOT* - [*70*] *71*) **and**
  bitAND (**infixr** *AND 64*) **and**
  bitOR (**infixr** *OR 59*) **and**
  bitXOR (**infixr** *XOR 59*)

Testing and shifting operations.

consts
  test-bit :: $'a::bit \Rightarrow nat \Rightarrow bool$ (**infixl** !! *100*)
  lsb :: $'a::bit \Rightarrow bool$

msb :: *'a::bit* ⇒ *bool*
set-bit :: *'a::bit* ⇒ *nat* ⇒ *bool* ⇒ *'a*
set-bits :: (*nat* ⇒ *bool*) ⇒ *'a::bit* (**binder** *BITS 10*)
shiftl :: *'a::bit* ⇒ *nat* ⇒ *'a* (**infixl** << *55*)
shiftr :: *'a::bit* ⇒ *nat* ⇒ *'a* (**infixl** >> *55*)

5.1 Bitwise operations on type *bit*

instance *bit* :: *bit*
NOT-bit-def: *NOT x* ≡ *case x of bit.B0* ⇒ *bit.B1* | *bit.B1* ⇒ *bit.B0*
AND-bit-def: *x AND y* ≡ *case x of bit.B0* ⇒ *bit.B0* | *bit.B1* ⇒ *y*
OR-bit-def: *x OR y* :: *bit* ≡ *NOT* (*NOT x AND NOT y*)
XOR-bit-def: *x XOR y* :: *bit* ≡ (*x AND NOT y*) *OR* (*NOT x AND y*)
..

lemma *bit-simps* [*simp*]:
NOT bit.B0 = *bit.B1*
NOT bit.B1 = *bit.B0*
bit.B0 AND y = *bit.B0*
bit.B1 AND y = *y*
bit.B0 OR y = *y*
bit.B1 OR y = *bit.B1*
bit.B0 XOR y = *y*
bit.B1 XOR y = *NOT y*
by (*simp-all add*: *NOT-bit-def AND-bit-def OR-bit-def*
XOR-bit-def split: *bit.split*)

lemma *bit-extra-simps* [*simp*]:
x AND bit.B0 = *bit.B0*
x AND bit.B1 = *x*
x OR bit.B1 = *bit.B1*
x OR bit.B0 = *x*
x XOR bit.B1 = *NOT x*
x XOR bit.B0 = *x*
by (*cases x*, *auto*)+

lemma *bit-ops-comm*:
(*x::bit*) *AND y* = *y AND x*
(*x::bit*) *OR y* = *y OR x*
(*x::bit*) *XOR y* = *y XOR x*
by (*cases y*, *auto*)+

lemma *bit-ops-same* [*simp*]:
(*x::bit*) *AND x* = *x*
(*x::bit*) *OR x* = *x*
(*x::bit*) *XOR x* = *bit.B0*
by (*cases x*, *auto*)+

lemma *bit-not-not* [*simp*]: *NOT* (*NOT* (*x::bit*)) = *x*

by (*cases x*) *auto*

end

6 BinOperations: Bitwise Operations on Binary Integers

theory *BinOperations* **imports** *BinGeneral BitSyntax*

begin

6.1 Logical operations

bit-wise logical operations on the int type

instance *int* :: *bit*
  int-not-def: *bitNOT* $\equiv$ *bin-rec Numeral.Min Numeral.Pls*
    ($\lambda w\ b\ s.\ s$ *BIT* (*NOT* b))
  int-and-def: *bitAND* $\equiv$ *bin-rec* ($\lambda x.$ *Numeral.Pls*) ($\lambda y.\ y$)
    ($\lambda w\ b\ s\ y.\ s$ (*bin-rest* y) *BIT* (b *AND* *bin-last* y))
  int-or-def: *bitOR* $\equiv$ *bin-rec* ($\lambda x.\ x$) ($\lambda y.$ *Numeral.Min*)
    ($\lambda w\ b\ s\ y.\ s$ (*bin-rest* y) *BIT* (b *OR* *bin-last* y))
  int-xor-def: *bitXOR* $\equiv$ *bin-rec* ($\lambda x.\ x$) *bitNOT*
    ($\lambda w\ b\ s\ y.\ s$ (*bin-rest* y) *BIT* (b *XOR* *bin-last* y))
  ..

lemma *int-not-simps* [*simp*]:
  NOT Numeral.Pls = *Numeral.Min*
  NOT Numeral.Min = *Numeral.Pls*
  NOT (w *BIT* b) = (*NOT* w) *BIT* (*NOT* b)
  by (*unfold int-not-def*) (*auto intro*: *bin-rec-simps*)

lemma *int-xor-Pls* [*simp*]:
  Numeral.Pls XOR $x = x$
  unfolding *int-xor-def* **by** (*simp add*: *bin-rec-PM*)

lemma *int-xor-Min* [*simp*]:
  Numeral.Min XOR x = *NOT* x
  unfolding *int-xor-def* **by** (*simp add*: *bin-rec-PM*)

lemma *int-xor-Bits* [*simp*]:
  (x *BIT* b) *XOR* (y *BIT* c) = (x *XOR* y) *BIT* (b *XOR* c)
  apply (*unfold int-xor-def*)
  apply (*rule bin-rec-simps* (*1*) [*THEN fun-cong*, *THEN trans*])
    apply (*rule ext*, *simp*)
   prefer *2*
   apply *simp*
  apply (*rule ext*)

apply (*simp add*: *int-not-simps* [*symmetric*])
done

lemma *int-xor-x-simps'*:
w XOR (*Numeral.Pls BIT bit.B0*) = *w*
w XOR (*Numeral.Min BIT bit.B1*) = *NOT w*
apply (*induct w rule*: *bin-induct*)
apply *simp-all*[*4*]
apply (*unfold int-xor-Bits*)
apply *clarsimp+*
done

lemmas *int-xor-extra-simps* [*simp*] = *int-xor-x-simps'* [*simplified arith-simps*]

lemma *int-or-Pls* [*simp*]:
Numeral.Pls OR x = *x*
by (*unfold int-or-def*) (*simp add*: *bin-rec-PM*)

lemma *int-or-Min* [*simp*]:
Numeral.Min OR x = *Numeral.Min*
by (*unfold int-or-def*) (*simp add*: *bin-rec-PM*)

lemma *int-or-Bits* [*simp*]:
(*x BIT b*) *OR* (*y BIT c*) = (*x OR y*) *BIT* (*b OR c*)
unfolding *int-or-def* **by** (*simp add*: *bin-rec-simps*)

lemma *int-or-x-simps'*:
w OR (*Numeral.Pls BIT bit.B0*) = *w*
w OR (*Numeral.Min BIT bit.B1*) = *Numeral.Min*
apply (*induct w rule*: *bin-induct*)
apply *simp-all*[*4*]
apply (*unfold int-or-Bits*)
apply *clarsimp+*
done

lemmas *int-or-extra-simps* [*simp*] = *int-or-x-simps'* [*simplified arith-simps*]

lemma *int-and-Pls* [*simp*]:
Numeral.Pls AND x = *Numeral.Pls*
unfolding *int-and-def* **by** (*simp add*: *bin-rec-PM*)

lemma *int-and-Min* [*simp*]:
Numeral.Min AND x = *x*
unfolding *int-and-def* **by** (*simp add*: *bin-rec-PM*)

lemma *int-and-Bits* [*simp*]:
(*x BIT b*) *AND* (*y BIT c*) = (*x AND y*) *BIT* (*b AND c*)
unfolding *int-and-def* **by** (*simp add*: *bin-rec-simps*)

lemma *int-and-x-simps'*:
  w AND (Numeral.Pls BIT bit.B0) = Numeral.Pls
  w AND (Numeral.Min BIT bit.B1) = w
  apply (*induct w rule*: *bin-induct*)
    apply *simp-all*[*4*]
  apply (*unfold int-and-Bits*)
  apply *clarsimp*+
  done

lemmas *int-and-extra-simps* [*simp*] = *int-and-x-simps'* [*simplified arith-simps*]

lemma *bin-ops-comm*:
  shows
  int-and-comm: !!*y*::*int*. *x AND y = y AND x* **and**
  int-or-comm: !!*y*::*int*. *x OR y = y OR x* **and**
  int-xor-comm: !!*y*::*int*. *x XOR y = y XOR x*
  apply (*induct x rule*: *bin-induct*)
    apply *simp-all*[*6*]
  apply (*case-tac y rule*: *bin-exhaust*, *simp add*: *bit-ops-comm*)+
  done

lemma *bin-ops-same* [*simp*]:
  (*x*::*int*) *AND x = x*
  (*x*::*int*) *OR x = x*
  (*x*::*int*) *XOR x = Numeral.Pls*
  by (*induct x rule*: *bin-induct*) *auto*

lemma *int-not-not* [*simp*]: *NOT* (*NOT* (*x*::*int*)) = *x*
  by (*induct x rule*: *bin-induct*) *auto*

lemmas *bin-log-esimps* =
  int-and-extra-simps int-or-extra-simps int-xor-extra-simps
  int-and-Pls int-and-Min int-or-Pls int-or-Min int-xor-Pls int-xor-Min

lemma *bbw-ao-absorb*:
  !!*y*::*int*. *x AND* (*y OR x*) = *x* & *x OR* (*y AND x*) = *x*
  apply (*induct x rule*: *bin-induct*)
    apply *auto*
  apply (*case-tac* [!] *y rule*: *bin-exhaust*)
  apply *auto*
  apply (*case-tac* [!] *bit*)
    apply *auto*
  done

lemma *bbw-ao-absorbs-other*:

```
  x AND (x OR y) = x ∧ (y AND x) OR x = (x::int)
  (y OR x) AND x = x ∧ x OR (x AND y) = (x::int)
  (x OR y) AND x = x ∧ (x AND y) OR x = (x::int)
  apply (auto simp: bbw-ao-absorb int-or-comm)
      apply (subst int-or-comm)
    apply (simp add: bbw-ao-absorb)
   apply (subst int-and-comm)
   apply (subst int-or-comm)
   apply (simp add: bbw-ao-absorb)
  apply (subst int-and-comm)
  apply (simp add: bbw-ao-absorb)
  done

lemmas bbw-ao-absorbs [simp] = bbw-ao-absorb bbw-ao-absorbs-other

lemma int-xor-not:
  !!y::int. (NOT x) XOR y = NOT (x XOR y) &
        x XOR (NOT y) = NOT (x XOR y)
  apply (induct x rule: bin-induct)
    apply auto
   apply (case-tac y rule: bin-exhaust, auto,
          case-tac b, auto)+
  done

lemma bbw-assocs':
  !!y z::int. (x AND y) AND z = x AND (y AND z) &
          (x OR y) OR z = x OR (y OR z) &
          (x XOR y) XOR z = x XOR (y XOR z)
  apply (induct x rule: bin-induct)
    apply (auto simp: int-xor-not)
    apply (case-tac [!] y rule: bin-exhaust)
    apply (case-tac [!] z rule: bin-exhaust)
    apply (case-tac [!] bit)
       apply (case-tac [!] b)
             apply auto
  done

lemma int-and-assoc:
  (x AND y) AND (z::int) = x AND (y AND z)
  by (simp add: bbw-assocs')

lemma int-or-assoc:
  (x OR y) OR (z::int) = x OR (y OR z)
  by (simp add: bbw-assocs')

lemma int-xor-assoc:
  (x XOR y) XOR (z::int) = x XOR (y XOR z)
  by (simp add: bbw-assocs')
```

lemmas *bbw-assocs* = *int-and-assoc int-or-assoc int-xor-assoc*

lemma *bbw-lcs* [*simp*]:
  (*y*::*int*) *AND* (*x AND z*) = *x AND* (*y AND z*)
  (*y*::*int*) *OR* (*x OR z*) = *x OR* (*y OR z*)
  (*y*::*int*) *XOR* (*x XOR z*) = *x XOR* (*y XOR z*)
  apply (*auto simp*: *bbw-assocs* [*symmetric*])
  apply (*auto simp*: *bin-ops-comm*)
  done

lemma *bbw-not-dist*:
  !!*y*::*int*. *NOT* (*x OR y*) = (*NOT x*) *AND* (*NOT y*)
  !!*y*::*int*. *NOT* (*x AND y*) = (*NOT x*) *OR* (*NOT y*)
  apply (*induct x rule*: *bin-induct*)
      apply *auto*
   apply (*case-tac* [!] *y rule*: *bin-exhaust*)
   apply (*case-tac* [!] *bit*, *auto*)
  done

lemma *bbw-oa-dist*:
  !!*y z*::*int*. (*x AND y*) *OR z* =
      (*x OR z*) *AND* (*y OR z*)
  apply (*induct x rule*: *bin-induct*)
    apply *auto*
  apply (*case-tac y rule*: *bin-exhaust*)
  apply (*case-tac z rule*: *bin-exhaust*)
  apply (*case-tac ba*, *auto*)
  done

lemma *bbw-ao-dist*:
  !!*y z*::*int*. (*x OR y*) *AND z* =
      (*x AND z*) *OR* (*y AND z*)
   apply (*induct x rule*: *bin-induct*)
    apply *auto*
  apply (*case-tac y rule*: *bin-exhaust*)
  apply (*case-tac z rule*: *bin-exhaust*)
  apply (*case-tac ba*, *auto*)
  done

lemma *plus-and-or* [*rule-format*]:
  ALL y::*int*. (*x AND y*) + (*x OR y*) = *x* + *y*
  apply (*induct x rule*: *bin-induct*)
    apply *clarsimp*
   apply *clarsimp*
  apply *clarsimp*
  apply (*case-tac y rule*: *bin-exhaust*)
  apply *clarsimp*

```
  apply (unfold Bit-def)
  apply clarsimp
  apply (erule-tac x = x in allE)
  apply (simp split: bit.split)
  done

lemma le-int-or:
  !!x.  bin-sign y = Numeral.Pls ==> x <= x OR y
  apply (induct y rule: bin-induct)
    apply clarsimp
   apply clarsimp
  apply (case-tac x rule: bin-exhaust)
  apply (case-tac b)
   apply (case-tac [!] bit)
     apply (auto simp: less-eq-numeral-code)
  done

lemmas int-and-le =
  xtr3 [OF bbw-ao-absorbs (2) [THEN conjunct2, symmetric] le-int-or]

lemma bin-nth-ops:
  !!x y. bin-nth (x AND y) n = (bin-nth x n & bin-nth y n)
  !!x y. bin-nth (x OR y) n = (bin-nth x n | bin-nth y n)
  !!x y. bin-nth (x XOR y) n = (bin-nth x n ~= bin-nth y n)
  !!x. bin-nth (NOT x) n = (~ bin-nth x n)
  apply (induct n)
         apply safe
                         apply (case-tac [!] x rule: bin-exhaust)
                         apply simp-all
                      apply (case-tac [!] y rule: bin-exhaust)
                      apply simp-all
        apply (auto dest: not-B1-is-B0 intro: B1-ass-B0)
  done



lemma bin-add-not: x + NOT x = Numeral.Min
  apply (induct x rule: bin-induct)
    apply clarsimp
   apply clarsimp
  apply (case-tac bit, auto)
  done


lemma bin-trunc-ao:
  !!x y. (bintrunc n x) AND (bintrunc n y) = bintrunc n (x AND y)
  !!x y. (bintrunc n x) OR (bintrunc n y) = bintrunc n (x OR y)
  apply (induct n)
      apply auto
```

```
    apply (case-tac [!] x rule: bin-exhaust)
    apply (case-tac [!] y rule: bin-exhaust)
    apply auto
  done
```

lemma *bin-trunc-xor*:

```
  !!x y. bintrunc n (bintrunc n x XOR bintrunc n y) =
         bintrunc n (x XOR y)
  apply (induct n)
   apply auto
   apply (case-tac [!] x rule: bin-exhaust)
   apply (case-tac [!] y rule: bin-exhaust)
   apply auto
  done
```

lemma *bin-trunc-not*:

```
  !!x. bintrunc n (NOT (bintrunc n x)) = bintrunc n (NOT x)
  apply (induct n)
   apply auto
   apply (case-tac [!] x rule: bin-exhaust)
   apply auto
  done
```

lemma *bintr-bintr-i*:

```
  x = bintrunc n y ==> bintrunc n x = bintrunc n y
  by auto
```

lemmas *bin-trunc-and* = *bin-trunc-ao*(*1*) [*THEN bintr-bintr-i*]
lemmas *bin-trunc-or* = *bin-trunc-ao*(*2*) [*THEN bintr-bintr-i*]

6.2 Setting and clearing bits

consts

```
  bin-sc :: nat => bit => int => int
```

primrec

```
  Z : bin-sc 0 b w = bin-rest w BIT b
  Suc :
    bin-sc (Suc n) b w = bin-sc n b (bin-rest w) BIT bin-last w
```

lemma *bin-nth-sc* [*simp*]:

```
  !!w. bin-nth (bin-sc n b w) n = (b = bit.B1)
  by (induct n) auto
```

lemma *bin-sc-sc-same* [*simp*]:

```
  !!w. bin-sc n c (bin-sc n b w) = bin-sc n c w
```

by (*induct n*) *auto*

lemma *bin-sc-sc-diff*:
!!*w m*. *m* ~= *n* ==>
bin-sc m c (*bin-sc n b w*) = *bin-sc n b* (*bin-sc m c w*)
apply (*induct n*)
apply (*case-tac* [!] *m*)
apply *auto*
done

lemma *bin-nth-sc-gen*:
!!*w m*. *bin-nth* (*bin-sc n b w*) *m* = (*if m* = *n then b* = *bit.B1 else bin-nth w m*)
by (*induct n*) (*case-tac* [!] *m*, *auto*)

lemma *bin-sc-nth* [*simp*]:
!!*w*. (*bin-sc n* (*If* (*bin-nth w n*) *bit.B1 bit.B0*) *w*) = *w*
by (*induct n*) *auto*

lemma *bin-sign-sc* [*simp*]:
!!*w*. *bin-sign* (*bin-sc n b w*) = *bin-sign w*
by (*induct n*) *auto*

lemma *bin-sc-bintr* [*simp*]:
!!*w m*. *bintrunc m* (*bin-sc n x* (*bintrunc m* (*w*))) = *bintrunc m* (*bin-sc n x w*)
apply (*induct n*)
apply (*case-tac* [!] *w rule*: *bin-exhaust*)
apply (*case-tac* [!] *m*, *auto*)
done

lemma *bin-clr-le*:
!!*w*. *bin-sc n bit.B0 w* <= *w*
apply (*induct n*)
apply (*case-tac* [!] *w rule*: *bin-exhaust*)
apply *auto*
apply (*unfold Bit-def*)
apply (*simp-all split*: *bit.split*)
done

lemma *bin-set-ge*:
!!*w*. *bin-sc n bit.B1 w* >= *w*
apply (*induct n*)
apply (*case-tac* [!] *w rule*: *bin-exhaust*)
apply *auto*
apply (*unfold Bit-def*)
apply (*simp-all split*: *bit.split*)
done

lemma *bintr-bin-clr-le*:
!!*w m*. *bintrunc n* (*bin-sc m bit.B0 w*) <= *bintrunc n w*

apply (*induct n*)
 apply *simp*
apply (*case-tac w rule*: *bin-exhaust*)
apply (*case-tac m*)
 apply *auto*
 apply (*unfold Bit-def*)
 apply (*simp-all split*: *bit.split*)
done

lemma *bintr-bin-set-ge*:
!!*w m. bintrunc n* (*bin-sc m bit.B1 w*) >= *bintrunc n w*
apply (*induct n*)
 apply *simp*
apply (*case-tac w rule*: *bin-exhaust*)
apply (*case-tac m*)
 apply *auto*
 apply (*unfold Bit-def*)
 apply (*simp-all split*: *bit.split*)
done

lemma *bin-sc-FP* [*simp*]: *bin-sc n bit.B0 Numeral.Pls* = *Numeral.Pls*
by (*induct n*) *auto*

lemma *bin-sc-TM* [*simp*]: *bin-sc n bit.B1 Numeral.Min* = *Numeral.Min*
by (*induct n*) *auto*

lemmas *bin-sc-simps* = *bin-sc.Z bin-sc.Suc bin-sc-TM bin-sc-FP*

lemma *bin-sc-minus*:
0 < *n* ==> *bin-sc* (*Suc* (*n* − *1*)) *b w* = *bin-sc n b w*
by *auto*

lemmas *bin-sc-Suc-minus* =
trans [*OF bin-sc-minus* [*symmetric*] *bin-sc.Suc*, *standard*]

lemmas *bin-sc-Suc-pred* [*simp*] =
bin-sc-Suc-minus [*of number-of bin*, *simplified nobm1*, *standard*]

6.3 Operations on lists of booleans

consts
bin-to-bl :: *nat* => *int* => *bool list*
bin-to-bl-aux :: *nat* => *int* => *bool list* => *bool list*
bl-to-bin :: *bool list* => *int*
bl-to-bin-aux :: *int* => *bool list* => *int*

bl-of-nth :: *nat* => (*nat* => *bool*) => *bool list*

primrec

Nil : *bl-to-bin-aux w* [] = *w*
Cons : *bl-to-bin-aux w* (*b* # *bs*) =
bl-to-bin-aux (*w BIT* (*if b then bit.B1 else bit.B0*)) *bs*

primrec
Z : *bin-to-bl-aux 0 w bl* = *bl*
Suc : *bin-to-bl-aux* (*Suc n*) *w bl* =
bin-to-bl-aux n (*bin-rest w*) ((*bin-last w* = *bit.B1*) # *bl*)

defs
bin-to-bl-def : *bin-to-bl n w* == *bin-to-bl-aux n w* []
bl-to-bin-def : *bl-to-bin bs* == *bl-to-bin-aux Numeral.Pls bs*

primrec
Suc : *bl-of-nth* (*Suc n*) *f* = *f n* # *bl-of-nth n f*
Z : *bl-of-nth 0 f* = []

consts
takefill :: *'a* => *nat* => *'a list* => *'a list*
app2 :: (*'a* => *'b* => *'c*) => *'a list* => *'b list* => *'c list*

— takefill - like take but if argument list too short,
— extends result to get requested length
primrec
Z : *takefill fill 0 xs* = []
Suc : *takefill fill* (*Suc n*) *xs* = (
case xs of [] => *fill* # *takefill fill n xs*
| *y* # *ys* => *y* # *takefill fill n ys*)

defs
app2-def : *app2 f as bs* == *map* (*split f*) (*zip as bs*)

6.4 Splitting and concatenation

— rcat and rsplit
consts
bin-rcat :: *nat* => *int list* => *int*
bin-rsplit-aux :: *nat* * *int list* * *nat* * *int* => *int list*
bin-rsplit :: *nat* => (*nat* * *int*) => *int list*
bin-rsplitl-aux :: *nat* * *int list* * *nat* * *int* => *int list*
bin-rsplitl :: *nat* => (*nat* * *int*) => *int list*

recdef *bin-rsplit-aux measure* (*fst o snd o snd*)
bin-rsplit-aux (*n*, *bs*, (*m*, *c*)) =
(*if m* = *0* | *n* = *0 then bs else*
let (*a*, *b*) = *bin-split n c*
in bin-rsplit-aux (*n*, *b* # *bs*, (*m* − *n*, *a*)))

recdef *bin-rsplitl-aux measure* (*fst o snd o snd*)

```
bin-rsplitl-aux (n, bs, (m, c)) =
  (if m = 0 | n = 0 then bs else
    let (a, b) = bin-split (min m n) c
    in bin-rsplitl-aux (n, b # bs, (m - n, a)))
```

defs

```
bin-rcat-def : bin-rcat n bs == foldl (%u v. bin-cat u n v) Numeral.Pls bs
bin-rsplit-def : bin-rsplit n w == bin-rsplit-aux (n, [], w)
bin-rsplitl-def : bin-rsplitl n w == bin-rsplitl-aux (n, [], w)
```

declare *bin-rsplit-aux.simps* [*simp del*]
declare *bin-rsplitl-aux.simps* [*simp del*]

lemma *bin-sign-cat*:

```
!!y. bin-sign (bin-cat x n y) = bin-sign x
by (induct n) auto
```

lemma *bin-cat-Suc-Bit*:

```
bin-cat w (Suc n) (v BIT b) = bin-cat w n v BIT b
by auto
```

lemma *bin-nth-cat*:

```
!!n y. bin-nth (bin-cat x k y) n =
  (if n < k then bin-nth y n else bin-nth x (n - k))
apply (induct k)
 apply clarsimp
apply (case-tac n, auto)
done
```

lemma *bin-nth-split*:

```
!!b c. bin-split n c = (a, b) ==>
  (ALL k. bin-nth a k = bin-nth c (n + k)) &
  (ALL k. bin-nth b k = (k < n & bin-nth c k))
apply (induct n)
 apply clarsimp
apply (clarsimp simp: Let-def split: ls-splits)
apply (case-tac k)
apply auto
done
```

lemma *bin-cat-assoc*:

```
!!z. bin-cat (bin-cat x m y) n z = bin-cat x (m + n) (bin-cat y n z)
by (induct n) auto
```

lemma *bin-cat-assoc-sym*: !!*z m*.

```
bin-cat x m (bin-cat y n z) = bin-cat (bin-cat x (m - n) y) (min m n) z
apply (induct n, clarsimp)
```

apply (*case-tac m*, *auto*)
done

lemma *bin-cat-Pls* [*simp*]:
!!*w*. *bin-cat Numeral.Pls n w* = *bintrunc n w*
by (*induct n*) *auto*

lemma *bintr-cat1*:
!!*b*. *bintrunc* (*k* + *n*) (*bin-cat a n b*) = *bin-cat* (*bintrunc k a*) *n b*
by (*induct n*) *auto*

lemma *bintr-cat*: *bintrunc m* (*bin-cat a n b*) =
bin-cat (*bintrunc* (*m* − *n*) *a*) *n* (*bintrunc* (*min m n*) *b*)
by (*rule bin-eqI*) (*auto simp*: *bin-nth-cat nth-bintr*)

lemma *bintr-cat-same* [*simp*]:
bintrunc n (*bin-cat a n b*) = *bintrunc n b*
by (*auto simp add* : *bintr-cat*)

lemma *cat-bintr* [*simp*]:
!!*b*. *bin-cat a n* (*bintrunc n b*) = *bin-cat a n b*
by (*induct n*) *auto*

lemma *split-bintrunc*:
!!*b c*. *bin-split n c* = (*a*, *b*) ==> *b* = *bintrunc n c*
by (*induct n*) (*auto simp*: *Let-def split*: *ls-splits*)

lemma *bin-cat-split*:
!!*v w*. *bin-split n w* = (*u*, *v*) ==> *w* = *bin-cat u n v*
by (*induct n*) (*auto simp*: *Let-def split*: *ls-splits*)

lemma *bin-split-cat*:
!!*w*. *bin-split n* (*bin-cat v n w*) = (*v*, *bintrunc n w*)
by (*induct n*) *auto*

lemma *bin-split-Pls* [*simp*]:
bin-split n Numeral.Pls = (*Numeral.Pls*, *Numeral.Pls*)
by (*induct n*) (*auto simp*: *Let-def split*: *ls-splits*)

lemma *bin-split-Min* [*simp*]:
bin-split n Numeral.Min = (*Numeral.Min*, *bintrunc n Numeral.Min*)
by (*induct n*) (*auto simp*: *Let-def split*: *ls-splits*)

lemma *bin-split-trunc*:
!!*m b c*. *bin-split* (*min m n*) *c* = (*a*, *b*) ==>
bin-split n (*bintrunc m c*) = (*bintrunc* (*m* − *n*) *a*, *b*)
apply (*induct n*, *clarsimp*)
apply (*simp add*: *bin-rest-trunc Let-def split*: *ls-splits*)
apply (*case-tac m*)

```
  apply (auto simp: Let-def split: ls-splits)
 done

lemma bin-split-trunc1:
 !!m b c. bin-split n c = (a, b) ==>
   bin-split n (bintrunc m c) = (bintrunc (m - n) a, bintrunc m b)
 apply (induct n, clarsimp)
 apply (simp add: bin-rest-trunc Let-def split: ls-splits)
 apply (case-tac m)
  apply (auto simp: Let-def split: ls-splits)
 done

lemma bin-cat-num:
 !!b. bin-cat a n b = a * 2 ^ n + bintrunc n b
 apply (induct n, clarsimp)
 apply (simp add: Bit-def cong: number-of-False-cong)
 done

lemma bin-split-num:
 !!b. bin-split n b = (b div 2 ^ n, b mod 2 ^ n)
 apply (induct n, clarsimp)
 apply (simp add: bin-rest-div zdiv-zmult2-eq)
 apply (case-tac b rule: bin-exhaust)
 apply simp
 apply (simp add: Bit-def zmod-zmult-zmult1 p1mod22k
           split: bit.split
           cong: number-of-False-cong)
 done
```

6.5 Miscellaneous lemmas

```
lemma nth-2p-bin:
 !!m. bin-nth (2 ^ n) m = (m = n)
 apply (induct n)
  apply clarsimp
  apply safe
    apply (case-tac m)
     apply (auto simp: trans [OF numeral-1-eq-1 [symmetric] number-of-eq])
  apply (case-tac m)
   apply (auto simp: Bit-B0-2t [symmetric])
 done

lemma ex-eq-or:
 (EX m. n = Suc m & (m = k | P m)) = (n = Suc k | (EX m. n = Suc m & P m))
 by auto
```

end

7 BinBoolList: Bool lists and integers

theory *BinBoolList* **imports** *BinOperations* **begin**

7.1 Arithmetic in terms of bool lists

consts
rbl-succ :: *bool list => bool list*
rbl-pred :: *bool list => bool list*
rbl-add :: *bool list => bool list => bool list*
rbl-mult :: *bool list => bool list => bool list*

primrec
Nil: *rbl-succ Nil = Nil*
Cons: *rbl-succ (x # xs) = (if x then False # rbl-succ xs else True # xs)*

primrec
Nil : *rbl-pred Nil = Nil*
Cons : *rbl-pred (x # xs) = (if x then False # xs else True # rbl-pred xs)*

primrec
Nil : *rbl-add Nil x = Nil*
Cons : *rbl-add (y # ys) x = (let ws = rbl-add ys (tl x) in*
(y ~= hd x) # (if hd x & y then rbl-succ ws else ws))

primrec
Nil : *rbl-mult Nil x = Nil*
Cons : *rbl-mult (y # ys) x = (let ws = False # rbl-mult ys x in*
if y then rbl-add ws x else ws)

lemma *tl-take*: *tl (take n l) = take (n − 1) (tl l)*
apply (*cases n*, *clarsimp*)
apply (*cases l*, *auto*)
done

lemma *take-butlast* [*rule-format*] :
ALL n. n < length l −−> take n (butlast l) = take n l
apply (*induct l*, *clarsimp*)
apply *clarsimp*
apply (*case-tac n*)
apply *auto*
done

lemma *butlast-take* [*rule-format*] :
ALL n. n <= length l −−> butlast (take n l) = take (n − 1) l

apply (*induct l*, *clarsimp*)
apply *clarsimp*
apply (*case-tac n*)
 apply *safe*
 apply *simp-all*
apply (*case-tac nat*)
 apply *auto*
done

lemma *butlast-drop* [*rule-format*] :
ALL n. butlast (*drop n l*) = *drop n* (*butlast l*)
apply (*induct l*)
 apply *clarsimp*
apply *clarsimp*
apply *safe*
 apply ((*case-tac n*, *auto*)[*1*])+
done

lemma *butlast-power*:
(*butlast* ^ *n*) *bl* = *take* (*length bl* − *n*) *bl*
by (*induct n*) (*auto simp*: *butlast-take*)

lemma *bin-to-bl-aux-Pls-minus-simp*:
0 < *n* ==> *bin-to-bl-aux n Numeral.Pls bl* =
 bin-to-bl-aux (*n* − *1*) *Numeral.Pls* (*False* # *bl*)
by (*cases n*) *auto*

lemma *bin-to-bl-aux-Min-minus-simp*:
0 < *n* ==> *bin-to-bl-aux n Numeral.Min bl* =
 bin-to-bl-aux (*n* − *1*) *Numeral.Min* (*True* # *bl*)
by (*cases n*) *auto*

lemma *bin-to-bl-aux-Bit-minus-simp*:
0 < *n* ==> *bin-to-bl-aux n* (*w BIT b*) *bl* =
 bin-to-bl-aux (*n* − *1*) *w* ((*b* = *bit.B1*) # *bl*)
by (*cases n*) *auto*

declare *bin-to-bl-aux-Pls-minus-simp* [*simp*]
bin-to-bl-aux-Min-minus-simp [*simp*]
bin-to-bl-aux-Bit-minus-simp [*simp*]

lemma *bl-to-bin-aux-append* [*rule-format*] :
ALL w. bl-to-bin-aux w (*bs* @ *cs*) = *bl-to-bin-aux* (*bl-to-bin-aux w bs*) *cs*
by (*induct bs*) *auto*

lemma *bin-to-bl-aux-append* [*rule-format*] :
ALL w bs. bin-to-bl-aux n w bs @ *cs* = *bin-to-bl-aux n w* (*bs* @ *cs*)

by (*induct n*) *auto*

lemma *bl-to-bin-append*:
bl-to-bin (*bs* @ *cs*) = *bl-to-bin-aux* (*bl-to-bin bs*) *cs*
unfolding *bl-to-bin-def* **by** (*rule bl-to-bin-aux-append*)

lemma *bin-to-bl-aux-alt*:
bin-to-bl-aux n w bs = *bin-to-bl n w* @ *bs*
unfolding *bin-to-bl-def* **by** (*simp add* : *bin-to-bl-aux-append*)

lemma *bin-to-bl-0*: *bin-to-bl 0 bs* = []
unfolding *bin-to-bl-def* **by** *auto*

lemma *size-bin-to-bl-aux* [*rule-format*] :
ALL w bs. size (*bin-to-bl-aux n w bs*) = *n* + *length bs*
by (*induct n*) *auto*

lemma *size-bin-to-bl*: *size* (*bin-to-bl n w*) = *n*
unfolding *bin-to-bl-def* **by** (*simp add* : *size-bin-to-bl-aux*)

lemma *bin-bl-bin′* [*rule-format*] :
ALL w bs. bl-to-bin (*bin-to-bl-aux n w bs*) =
bl-to-bin-aux (*bintrunc n w*) *bs*
by (*induct n*) (*auto simp add* : *bl-to-bin-def*)

lemma *bin-bl-bin*: *bl-to-bin* (*bin-to-bl n w*) = *bintrunc n w*
unfolding *bin-to-bl-def bin-bl-bin′* **by** *auto*

lemma *bl-bin-bl′* [*rule-format*] :
ALL w n. bin-to-bl (*n* + *length bs*) (*bl-to-bin-aux w bs*) =
bin-to-bl-aux n w bs
apply (*induct bs*)
apply *auto*
apply (*simp-all only* : *add-Suc* [*symmetric*])
apply (*auto simp add* : *bin-to-bl-def*)
done

lemma *bl-bin-bl*: *bin-to-bl* (*length bs*) (*bl-to-bin bs*) = *bs*
unfolding *bl-to-bin-def*
apply (*rule box-equals*)
apply (*rule bl-bin-bl′*)
prefer *2*
apply (*rule bin-to-bl-aux.Z*)
apply *simp*
done

declare
bin-to-bl-0 [*simp*]
size-bin-to-bl [*simp*]

bin-bl-bin [*simp*]
bl-bin-bl [*simp*]

lemma *bl-to-bin-inj*:
bl-to-bin bs = bl-to-bin cs ==> length bs = length cs ==> bs = cs
apply (*rule-tac box-equals*)
defer
apply (*rule bl-bin-bl*)
apply (*rule bl-bin-bl*)
apply *simp*
done

lemma *bl-to-bin-False*: *bl-to-bin* (*False # bl*) = *bl-to-bin bl*
unfolding *bl-to-bin-def* **by** *auto*

lemma *bl-to-bin-Nil*: *bl-to-bin* [] = *Numeral.Pls*
unfolding *bl-to-bin-def* **by** *auto*

lemma *bin-to-bl-Pls-aux* [*rule-format*] :
ALL bl. bin-to-bl-aux n Numeral.Pls bl = replicate n False @ bl
by (*induct n*) (*auto simp*: *replicate-app-Cons-same*)

lemma *bin-to-bl-Pls*: *bin-to-bl n Numeral.Pls = replicate n False*
unfolding *bin-to-bl-def* **by** (*simp add* : *bin-to-bl-Pls-aux*)

lemma *bin-to-bl-Min-aux* [*rule-format*] :
ALL bl. bin-to-bl-aux n Numeral.Min bl = replicate n True @ bl
by (*induct n*) (*auto simp*: *replicate-app-Cons-same*)

lemma *bin-to-bl-Min*: *bin-to-bl n Numeral.Min = replicate n True*
unfolding *bin-to-bl-def* **by** (*simp add* : *bin-to-bl-Min-aux*)

lemma *bl-to-bin-rep-F*:
bl-to-bin (*replicate n False @ bl*) = *bl-to-bin bl*
apply (*simp add*: *bin-to-bl-Pls-aux* [*symmetric*] *bin-bl-bin'*)
apply (*simp add*: *bl-to-bin-def*)
done

lemma *bin-to-bl-trunc*:
n <= m ==> bin-to-bl n (*bintrunc m w*) = *bin-to-bl n w*
by (*auto intro*: *bl-to-bin-inj*)

declare
bin-to-bl-trunc [*simp*]
bl-to-bin-False [*simp*]
bl-to-bin-Nil [*simp*]

lemma *bin-to-bl-aux-bintr* [*rule-format*] :
ALL m bin bl. bin-to-bl-aux n (*bintrunc m bin*) *bl* =

```
    replicate (n - m) False @ bin-to-bl-aux (min n m) bin bl
  apply (induct-tac n)
   apply clarsimp
  apply clarsimp
  apply (case-tac m)
   apply (clarsimp simp: bin-to-bl-Pls-aux)
   apply (erule thin-rl)
   apply (induct-tac n)
    apply auto
  done

lemmas bin-to-bl-bintr =
  bin-to-bl-aux-bintr [where bl = [], folded bin-to-bl-def]

lemma bl-to-bin-rep-False: bl-to-bin (replicate n False) = Numeral.Pls
  by (induct n) auto

lemma len-bin-to-bl-aux [rule-format] :
  ALL w bs. length (bin-to-bl-aux n w bs) = n + length bs
  by (induct n) auto

lemma len-bin-to-bl [simp]: length (bin-to-bl n w) = n
  unfolding bin-to-bl-def len-bin-to-bl-aux by auto

lemma sign-bl-bin' [rule-format] :
  ALL w. bin-sign (bl-to-bin-aux w bs) = bin-sign w
  by (induct bs) auto

lemma sign-bl-bin: bin-sign (bl-to-bin bs) = Numeral.Pls
  unfolding bl-to-bin-def by (simp add : sign-bl-bin')

lemma bl-sbin-sign-aux [rule-format] :
  ALL w bs. hd (bin-to-bl-aux (Suc n) w bs) =
    (bin-sign (sbintrunc n w) = Numeral.Min)
  apply (induct n)
   apply clarsimp
   apply (case-tac w rule: bin-exhaust)
   apply (simp split add : bit.split)
  apply clarsimp
  done

lemma bl-sbin-sign:
  hd (bin-to-bl (Suc n) w) = (bin-sign (sbintrunc n w) = Numeral.Min)
  unfolding bin-to-bl-def by (rule bl-sbin-sign-aux)

lemma bin-nth-of-bl-aux [rule-format] :
  ALL w. bin-nth (bl-to-bin-aux w bl) n =
    (n < size bl & rev bl ! n | n >= length bl & bin-nth w (n - size bl))
  apply (induct-tac bl)
```

```
  apply clarsimp
 apply clarsimp
 apply (cut-tac x=n and y=size list in linorder-less-linear)
 apply (erule disjE, simp add: nth-append)+
 apply (simp add: nth-append)
 done

lemma bin-nth-of-bl: bin-nth (bl-to-bin bl) n = (n < length bl & rev bl ! n)
 unfolding bl-to-bin-def by (simp add : bin-nth-of-bl-aux)

lemma bin-nth-bl [rule-format] : ALL m w. n < m -->
   bin-nth w n = nth (rev (bin-to-bl m w)) n
 apply (induct n)
  apply clarsimp
  apply (case-tac m, clarsimp)
  apply (clarsimp simp: bin-to-bl-def)
  apply (simp add: bin-to-bl-aux-alt)
 apply clarsimp
 apply (case-tac m, clarsimp)
 apply (clarsimp simp: bin-to-bl-def)
 apply (simp add: bin-to-bl-aux-alt)
 done

lemma nth-rev [rule-format] :
 n < length xs --> rev xs ! n = xs ! (length xs - 1 - n)
 apply (induct-tac xs)
  apply simp
 apply (clarsimp simp add : nth-append nth.simps split add : nat.split)
 apply (rule-tac f = %n. list ! n in arg-cong)
 apply arith
 done

lemmas nth-rev-alt = nth-rev [where xs = rev ys, simplified, standard]

lemma nth-bin-to-bl-aux [rule-format] :
 ALL w n bl. n < m + length bl --> (bin-to-bl-aux m w bl) ! n =
   (if n < m then bin-nth w (m - 1 - n) else bl ! (n - m))
 apply (induct-tac m)
  apply clarsimp
 apply clarsimp
 apply (case-tac w rule: bin-exhaust)
 apply clarsimp
 apply (case-tac na - n)
  apply arith
 apply simp
 apply (rule-tac f = %n. bl ! n in arg-cong)
 apply arith
 done
```

```
lemma nth-bin-to-bl: n < m ==> (bin-to-bl m w) ! n = bin-nth w (m - Suc n)
  unfolding bin-to-bl-def by (simp add : nth-bin-to-bl-aux)

lemma bl-to-bin-lt2p-aux [rule-format] :
  ALL w. bl-to-bin-aux w bs < (w + 1) * (2 ^ length bs)
  apply (induct bs)
   apply clarsimp
  apply clarsimp
  apply safe
   apply (erule allE, erule xtr8 [rotated],
         simp add: Bit-def ring-simps cong add : number-of-False-cong)+
  done

lemma bl-to-bin-lt2p: bl-to-bin bs < (2 ^ length bs)
  apply (unfold bl-to-bin-def)
  apply (rule xtr1)
   prefer 2
   apply (rule bl-to-bin-lt2p-aux)
  apply simp
  done

lemma bl-to-bin-ge2p-aux [rule-format] :
  ALL w. bl-to-bin-aux w bs >= w * (2 ^ length bs)
  apply (induct bs)
   apply clarsimp
  apply clarsimp
  apply safe
   apply (erule allE, erule less-eq-less.order-trans [rotated],
         simp add: Bit-def ring-simps cong add : number-of-False-cong)+
  done

lemma bl-to-bin-ge0: bl-to-bin bs >= 0
  apply (unfold bl-to-bin-def)
  apply (rule xtr4)
   apply (rule bl-to-bin-ge2p-aux)
  apply simp
  done

lemma butlast-rest-bin:
  butlast (bin-to-bl n w) = bin-to-bl (n - 1) (bin-rest w)
  apply (unfold bin-to-bl-def)
  apply (cases w rule: bin-exhaust)
  apply (cases n, clarsimp)
  apply clarsimp
  apply (auto simp add: bin-to-bl-aux-alt)
  done

lemmas butlast-bin-rest = butlast-rest-bin
  [where w=bl-to-bin bl and n=length bl, simplified, standard]
```

```
lemma butlast-rest-bl2bin-aux [rule-format] :
  ALL w. bl ~= [] -->
    bl-to-bin-aux w (butlast bl) = bin-rest (bl-to-bin-aux w bl)
  by (induct bl) auto

lemma butlast-rest-bl2bin:
  bl-to-bin (butlast bl) = bin-rest (bl-to-bin bl)
  apply (unfold bl-to-bin-def)
  apply (cases bl)
   apply (auto simp add: butlast-rest-bl2bin-aux)
  done

lemma trunc-bl2bin-aux [rule-format] :
  ALL w. bintrunc m (bl-to-bin-aux w bl) =
    bl-to-bin-aux (bintrunc (m - length bl) w) (drop (length bl - m) bl)
  apply (induct-tac bl)
   apply clarsimp
  apply clarsimp
  apply safe
   apply (case-tac m - size list)
    apply (simp add : diff-is-0-eq [THEN iffD1, THEN Suc-diff-le])
   apply simp
   apply (rule-tac f = %nat. bl-to-bin-aux (bintrunc nat w BIT bit.B1) list
                in arg-cong)
   apply simp
  apply (case-tac m - size list)
   apply (simp add: diff-is-0-eq [THEN iffD1, THEN Suc-diff-le])
  apply simp
  apply (rule-tac f = %nat. bl-to-bin-aux (bintrunc nat w BIT bit.B0) list
               in arg-cong)
  apply simp
  done

lemma trunc-bl2bin:
  bintrunc m (bl-to-bin bl) = bl-to-bin (drop (length bl - m) bl)
  unfolding bl-to-bin-def by (simp add : trunc-bl2bin-aux)

lemmas trunc-bl2bin-len [simp] =
  trunc-bl2bin [of length bl bl, simplified, standard]

lemma bl2bin-drop:
  bl-to-bin (drop k bl) = bintrunc (length bl - k) (bl-to-bin bl)
  apply (rule trans)
   prefer 2
   apply (rule trunc-bl2bin [symmetric])
  apply (cases k <= length bl)
   apply auto
  done
```

lemma *nth-rest-power-bin* [*rule-format*] :
  ALL n. bin-nth ((*bin-rest* ^ *k*) *w*) *n* = *bin-nth w* (*n* + *k*)
  apply (*induct k*, *clarsimp*)
  apply *clarsimp*
  apply (*simp only*: *bin-nth.Suc* [*symmetric*] *add-Suc*)
  done

lemma *take-rest-power-bin*:
  m <= *n* ==> *take m* (*bin-to-bl n w*) = *bin-to-bl m* ((*bin-rest* ^ (*n* − *m*)) *w*)
  apply (*rule nth-equalityI*)
   apply *simp*
  apply (*clarsimp simp add*: *nth-bin-to-bl nth-rest-power-bin*)
  done

lemma *hd-butlast*: *size xs* > *1* ==> *hd* (*butlast xs*) = *hd xs*
  by (*cases xs*) *auto*

lemma *last-bin-last'* [*rule-format*] :
  ALL w. size xs > *0* −−> *last xs* = (*bin-last* (*bl-to-bin-aux w xs*) = *bit.B1*)
  by (*induct xs*) *auto*

lemma *last-bin-last*:
  size xs > *0* ==> *last xs* = (*bin-last* (*bl-to-bin xs*) = *bit.B1*)
  unfolding *bl-to-bin-def* **by** (*erule last-bin-last'*)

lemma *bin-last-last*:
  bin-last w = (*if last* (*bin-to-bl* (*Suc n*) *w*) *then bit.B1 else bit.B0*)
  apply (*unfold bin-to-bl-def*)
  apply *simp*
  apply (*auto simp add*: *bin-to-bl-aux-alt*)
  done

lemma *app2-Nil* [*simp*]: *app2 f* [] *ys* = []
  unfolding *app2-def* **by** *auto*

lemma *app2-Cons* [*simp*]:
  app2 f (*x* # *xs*) (*y* # *ys*) = *f x y* # *app2 f xs ys*
  unfolding *app2-def* **by** *auto*

lemma *bl-xor-aux-bin* [*rule-format*] : *ALL v w bs cs.*
    app2 (%*x y. x* ~= *y*) (*bin-to-bl-aux n v bs*) (*bin-to-bl-aux n w cs*) =
    bin-to-bl-aux n (*v XOR w*) (*app2* (%*x y. x* ~= *y*) *bs cs*)
  apply (*induct-tac n*)
   apply *safe*
   apply *simp*
  apply (*case-tac v rule*: *bin-exhaust*)

```
  apply (case-tac w rule: bin-exhaust)
  apply clarsimp
  apply (case-tac b)
  apply (case-tac ba, safe, simp-all)+
  done

lemma bl-or-aux-bin [rule-format] : ALL v w bs cs.
    app2 (op | ) (bin-to-bl-aux n v bs) (bin-to-bl-aux n w cs) =
    bin-to-bl-aux n (v OR w) (app2 (op | ) bs cs)
  apply (induct-tac n)
   apply safe
   apply simp
  apply (case-tac v rule: bin-exhaust)
  apply (case-tac w rule: bin-exhaust)
  apply clarsimp
  apply (case-tac b)
  apply (case-tac ba, safe, simp-all)+
  done

lemma bl-and-aux-bin [rule-format] : ALL v w bs cs.
    app2 (op & ) (bin-to-bl-aux n v bs) (bin-to-bl-aux n w cs) =
    bin-to-bl-aux n (v AND w) (app2 (op & ) bs cs)
  apply (induct-tac n)
   apply safe
   apply simp
  apply (case-tac v rule: bin-exhaust)
  apply (case-tac w rule: bin-exhaust)
  apply clarsimp
  apply (case-tac b)
  apply (case-tac ba, safe, simp-all)+
  done

lemma bl-not-aux-bin [rule-format] :
  ALL w cs. map Not (bin-to-bl-aux n w cs) =
    bin-to-bl-aux n (NOT w) (map Not cs)
  apply (induct n)
   apply clarsimp
  apply clarsimp
  apply (case-tac w rule: bin-exhaust)
  apply (case-tac b)
   apply auto
  done

lemmas bl-not-bin = bl-not-aux-bin
  [where cs = [], unfolded bin-to-bl-def [symmetric] map.simps]

lemmas bl-and-bin = bl-and-aux-bin [where bs=[] and cs=[],
                                   unfolded app2-Nil, folded bin-to-bl-def]
```

```
lemmas bl-or-bin = bl-or-aux-bin [where bs=[] and cs=[],
                                  unfolded app2-Nil, folded bin-to-bl-def]

lemmas bl-xor-bin = bl-xor-aux-bin [where bs=[] and cs=[],
                                    unfolded app2-Nil, folded bin-to-bl-def]

lemma drop-bin2bl-aux [rule-format] :
  ALL m bin bs. drop m (bin-to-bl-aux n bin bs) =
    bin-to-bl-aux (n - m) bin (drop (m - n) bs)
  apply (induct n, clarsimp)
  apply clarsimp
  apply (case-tac bin rule: bin-exhaust)
  apply (case-tac m <= n, simp)
  apply (case-tac m - n, simp)
  apply simp
  apply (rule-tac f = %nat. drop nat bs in arg-cong)
  apply simp
  done

lemma drop-bin2bl: drop m (bin-to-bl n bin) = bin-to-bl (n - m) bin
  unfolding bin-to-bl-def by (simp add : drop-bin2bl-aux)

lemma take-bin2bl-lem1 [rule-format] :
  ALL w bs. take m (bin-to-bl-aux m w bs) = bin-to-bl m w
  apply (induct m, clarsimp)
  apply clarsimp
  apply (simp add: bin-to-bl-aux-alt)
  apply (simp add: bin-to-bl-def)
  apply (simp add: bin-to-bl-aux-alt)
  done

lemma take-bin2bl-lem [rule-format] :
  ALL w bs. take m (bin-to-bl-aux (m + n) w bs) =
    take m (bin-to-bl (m + n) w)
  apply (induct n)
   apply clarify
   apply (simp-all (no-asm) add: bin-to-bl-def take-bin2bl-lem1)
  apply simp
  done

lemma bin-split-take [rule-format] :
  ALL b c. bin-split n c = (a, b) -->
    bin-to-bl m a = take m (bin-to-bl (m + n) c)
  apply (induct n)
   apply clarsimp
  apply (clarsimp simp: Let-def split: ls-splits)
  apply (simp add: bin-to-bl-def)
  apply (simp add: take-bin2bl-lem)
  done
```

lemma *bin-split-take1*:
  $k = m + n ==> bin\text{-}split\ n\ c = (a,\ b) ==>$
    $bin\text{-}to\text{-}bl\ m\ a = take\ m\ (bin\text{-}to\text{-}bl\ k\ c)$
  by (*auto elim*: *bin-split-take*)

lemma *nth-takefill* [*rule-format*] : *ALL m l. m < n -->*
    takefill fill n l ! m = (if m < length l then l ! m else fill)
  apply (*induct n*, *clarsimp*)
  apply *clarsimp*
  apply (*case-tac m*)
   apply (*simp split*: *list.split*)
  apply *clarsimp*
  apply (*erule allE*)+
  apply (*erule* (*1*) *impE*)
  apply (*simp split*: *list.split*)
  done

lemma *takefill-alt* [*rule-format*] :
  ALL l. takefill fill n l = take n l @ replicate (n − length l) fill
  by (*induct n*) (*auto split*: *list.split*)

lemma *takefill-replicate* [*simp*]:
  takefill fill n (replicate m fill) = replicate n fill
  by (*simp add* : *takefill-alt replicate-add* [*symmetric*])

lemma *takefill-le′* [*rule-format*] :
  ALL l n. n = m + k --> takefill x m (takefill x n l) = takefill x m l
  by (*induct m*) (*auto split*: *list.split*)

lemma *length-takefill* [*simp*]: *length (takefill fill n l) = n*
  by (*simp add* : *takefill-alt*)

lemma *take-takefill′*:
  !!w n. n = k + m ==> take k (takefill fill n w) = takefill fill k w
  by (*induct k*) (*auto split add* : *list.split*)

lemma *drop-takefill*:
  !!w. drop k (takefill fill (m + k) w) = takefill fill m (drop k w)
  by (*induct k*) (*auto split add* : *list.split*)

lemma *takefill-le* [*simp*]:
  $m \le n \Longrightarrow$ *takefill x m (takefill x n l) = takefill x m l*
  by (*auto simp*: *le-iff-add takefill-le′*)

lemma *take-takefill* [*simp*]:
  $m \le n \Longrightarrow$ *take m (takefill fill n w) = takefill fill m w*
  by (*auto simp*: *le-iff-add take-takefill′*)

lemma *takefill-append*:
takefill fill (*m* + *length xs*) (*xs* @ *w*) = *xs* @ (*takefill fill m w*)
by (*induct xs*) *auto*

lemma *takefill-same'*:
l = *length xs* ==> *takefill fill l xs* = *xs*
by *clarify* (*induct xs*, *auto*)

lemmas *takefill-same* [*simp*] = *takefill-same'* [*OF refl*]

lemma *takefill-bintrunc*:
takefill False n bl = *rev* (*bin-to-bl n* (*bl-to-bin* (*rev bl*)))
apply (*rule nth-equalityI*)
apply *simp*
apply (*clarsimp simp*: *nth-takefill nth-rev nth-bin-to-bl bin-nth-of-bl*)
done

lemma *bl-bin-bl-rtf*:
bin-to-bl n (*bl-to-bin bl*) = *rev* (*takefill False n* (*rev bl*))
by (*simp add* : *takefill-bintrunc*)

lemmas *bl-bin-bl-rep-drop* =
bl-bin-bl-rtf [*simplified takefill-alt*,
simplified, *simplified rev-take*, *simplified*]

lemma *tf-rev*:
n + *k* = *m* + *length bl* ==> *takefill x m* (*rev* (*takefill y n bl*)) =
rev (*takefill y m* (*rev* (*takefill x k* (*rev bl*))))
apply (*rule nth-equalityI*)
apply (*auto simp add*: *nth-takefill nth-rev*)
apply (*rule-tac f* = %*n*. *bl* ! *n* **in** *arg-cong*)
apply *arith*
done

lemma *takefill-minus*:
0 < *n* ==> *takefill fill* (*Suc* (*n* − *1*)) *w* = *takefill fill n w*
by *auto*

lemmas *takefill-Suc-cases* =
list.cases [*THEN takefill.Suc* [*THEN trans*], *standard*]

lemmas *takefill-Suc-Nil* = *takefill-Suc-cases* (*1*)
lemmas *takefill-Suc-Cons* = *takefill-Suc-cases* (*2*)

lemmas *takefill-minus-simps* = *takefill-Suc-cases* [*THEN* [*2*]
takefill-minus [*symmetric*, *THEN trans*], *standard*]

lemmas *takefill-pred-simps* [*simp*] =
takefill-minus-simps [**where** *n*=*number-of bin*, *simplified nobm1*, *standard*]

```
lemma bl-to-bin-aux-cat:
  !!nv v. bl-to-bin-aux (bin-cat w nv v) bs =
    bin-cat w (nv + length bs) (bl-to-bin-aux v bs)
  apply (induct bs)
   apply simp
  apply (simp add: bin-cat-Suc-Bit [symmetric] del: bin-cat.simps)
  done

lemma bin-to-bl-aux-cat:
  !!w bs. bin-to-bl-aux (nv + nw) (bin-cat v nw w) bs =
    bin-to-bl-aux nv v (bin-to-bl-aux nw w bs)
  by (induct nw) auto

lemmas bl-to-bin-aux-alt =
  bl-to-bin-aux-cat [where nv = 0 and v = Numeral.Pls,
    simplified bl-to-bin-def [symmetric], simplified]

lemmas bin-to-bl-cat =
  bin-to-bl-aux-cat [where bs = [], folded bin-to-bl-def]

lemmas bl-to-bin-aux-app-cat =
  trans [OF bl-to-bin-aux-append bl-to-bin-aux-alt]

lemmas bin-to-bl-aux-cat-app =
  trans [OF bin-to-bl-aux-cat bin-to-bl-aux-alt]

lemmas bl-to-bin-app-cat = bl-to-bin-aux-app-cat
  [where w = Numeral.Pls, folded bl-to-bin-def]

lemmas bin-to-bl-cat-app = bin-to-bl-aux-cat-app
  [where bs = [], folded bin-to-bl-def]


lemma bl-to-bin-app-cat-alt:
  bin-cat (bl-to-bin cs) n w = bl-to-bin (cs @ bin-to-bl n w)
  by (simp add : bl-to-bin-app-cat)

lemma mask-lem: (bl-to-bin (True # replicate n False)) =
    Numeral.succ (bl-to-bin (replicate n True))
  apply (unfold bl-to-bin-def)
  apply (induct n)
   apply simp
  apply (simp only: Suc-eq-add-numeral-1 replicate-add
                  append-Cons [symmetric] bl-to-bin-aux-append)
  apply simp
  done
```

lemma *length-bl-of-nth* [*simp*]: *length* (*bl-of-nth n f*) = *n*
  by (*induct n*) *auto*

lemma *nth-bl-of-nth* [*simp*]:
  m < *n* ⟹ *rev* (*bl-of-nth n f*) ! *m* = *f m*
  apply (*induct n*)
   apply *simp*
  apply (*clarsimp simp add* : *nth-append*)
  apply (*rule-tac f* = *f* **in** *arg-cong*)
  apply *simp*
  done

lemma *bl-of-nth-inj*:
  (!!*k*. *k* < *n* ==> *f k* = *g k*) ==> *bl-of-nth n f* = *bl-of-nth n g*
  by (*induct n*) *auto*

lemma *bl-of-nth-nth-le* [*rule-format*] : *ALL xs*.
    length xs >= *n* −−> *bl-of-nth n* (*nth* (*rev xs*)) = *drop* (*length xs* − *n*) *xs*
  apply (*induct n*, *clarsimp*)
  apply *clarsimp*
  apply (*rule trans* [*OF* - *hd-Cons-tl*])
   apply (*frule Suc-le-lessD*)
   apply (*simp add*: *nth-rev trans* [*OF drop-Suc drop-tl*, *symmetric*])
   apply (*subst hd-drop-conv-nth*)
     apply *force*
    apply *simp-all*
  apply (*rule-tac f* = %*n*. *drop n xs* **in** *arg-cong*)
  apply *simp*
  done

lemmas *bl-of-nth-nth* [*simp*] = *order-refl* [*THEN bl-of-nth-nth-le*, *simplified*]

lemma *size-rbl-pred*: *length* (*rbl-pred bl*) = *length bl*
  by (*induct bl*) *auto*

lemma *size-rbl-succ*: *length* (*rbl-succ bl*) = *length bl*
  by (*induct bl*) *auto*

lemma *size-rbl-add*:
  !!*cl*. *length* (*rbl-add bl cl*) = *length bl*
  by (*induct bl*) (*auto simp*: *Let-def size-rbl-succ*)

lemma *size-rbl-mult*:
  !!*cl*. *length* (*rbl-mult bl cl*) = *length bl*
  by (*induct bl*) (*auto simp add* : *Let-def size-rbl-add*)

lemmas *rbl-sizes* [*simp*] =

size-rbl-pred size-rbl-succ size-rbl-add size-rbl-mult

lemmas *rbl-Nils* =
rbl-pred.Nil rbl-succ.Nil rbl-add.Nil rbl-mult.Nil

lemma *rbl-pred*:
!!*bin*. *rbl-pred* (*rev* (*bin-to-bl n bin*)) = *rev* (*bin-to-bl n* (*Numeral.pred bin*))
apply (*induct n*, *simp*)
apply (*unfold bin-to-bl-def*)
apply *clarsimp*
apply (*case-tac bin rule*: *bin-exhaust*)
apply (*case-tac b*)
apply (*clarsimp simp*: *bin-to-bl-aux-alt*)+
done

lemma *rbl-succ*:
!!*bin*. *rbl-succ* (*rev* (*bin-to-bl n bin*)) = *rev* (*bin-to-bl n* (*Numeral.succ bin*))
apply (*induct n*, *simp*)
apply (*unfold bin-to-bl-def*)
apply *clarsimp*
apply (*case-tac bin rule*: *bin-exhaust*)
apply (*case-tac b*)
apply (*clarsimp simp*: *bin-to-bl-aux-alt*)+
done

lemma *rbl-add*:
!!*bina binb*. *rbl-add* (*rev* (*bin-to-bl n bina*)) (*rev* (*bin-to-bl n binb*)) =
rev (*bin-to-bl n* (*bina* + *binb*))
apply (*induct n*, *simp*)
apply (*unfold bin-to-bl-def*)
apply *clarsimp*
apply (*case-tac bina rule*: *bin-exhaust*)
apply (*case-tac binb rule*: *bin-exhaust*)
apply (*case-tac b*)
apply (*case-tac* [!] *ba*)
apply (*auto simp*: *rbl-succ succ-def bin-to-bl-aux-alt Let-def add-ac*)
done

lemma *rbl-add-app2*:
!!*blb*. *length blb* >= *length bla* ==>
rbl-add bla (*blb* @ *blc*) = *rbl-add bla blb*
apply (*induct bla*, *simp*)
apply *clarsimp*
apply (*case-tac blb*, *clarsimp*)
apply (*clarsimp simp*: *Let-def*)
done

lemma *rbl-add-take2*:
!!*blb*. *length blb* >= *length bla* ==>

rbl-add bla (*take* (*length bla*) *blb*) = *rbl-add bla blb*
apply (*induct bla*, *simp*)
apply *clarsimp*
apply (*case-tac blb*, *clarsimp*)
apply (*clarsimp simp*: *Let-def*)
done

lemma *rbl-add-long*:
m >= *n* ==> *rbl-add* (*rev* (*bin-to-bl n bina*)) (*rev* (*bin-to-bl m binb*)) =
rev (*bin-to-bl n* (*bina* + *binb*))
apply (*rule box-equals* [*OF* - *rbl-add-take2 rbl-add*])
apply (*rule-tac f* = *rbl-add* (*rev* (*bin-to-bl n bina*)) **in** *arg-cong*)
apply (*rule rev-swap* [*THEN iffD1*])
apply (*simp add*: *rev-take drop-bin2bl*)
apply *simp*
done

lemma *rbl-mult-app2*:
!!*blb*. *length blb* >= *length bla* ==>
rbl-mult bla (*blb* @ *blc*) = *rbl-mult bla blb*
apply (*induct bla*, *simp*)
apply *clarsimp*
apply (*case-tac blb*, *clarsimp*)
apply (*clarsimp simp*: *Let-def rbl-add-app2*)
done

lemma *rbl-mult-take2*:
length blb >= *length bla* ==>
rbl-mult bla (*take* (*length bla*) *blb*) = *rbl-mult bla blb*
apply (*rule trans*)
apply (*rule rbl-mult-app2* [*symmetric*])
apply *simp*
apply (*rule-tac f* = *rbl-mult bla* **in** *arg-cong*)
apply (*rule append-take-drop-id*)
done

lemma *rbl-mult-gt1*:
m >= *length bl* ==> *rbl-mult bl* (*rev* (*bin-to-bl m binb*)) =
rbl-mult bl (*rev* (*bin-to-bl* (*length bl*) *binb*))
apply (*rule trans*)
apply (*rule rbl-mult-take2* [*symmetric*])
apply *simp-all*
apply (*rule-tac f* = *rbl-mult bl* **in** *arg-cong*)
apply (*rule rev-swap* [*THEN iffD1*])
apply (*simp add*: *rev-take drop-bin2bl*)
done

lemma *rbl-mult-gt*:
m > *n* ==> *rbl-mult* (*rev* (*bin-to-bl n bina*)) (*rev* (*bin-to-bl m binb*)) =

```
    rbl-mult (rev (bin-to-bl n bina)) (rev (bin-to-bl n binb))
  by (auto intro: trans [OF rbl-mult-gt1])

lemmas rbl-mult-Suc = lessI [THEN rbl-mult-gt]

lemma rbbl-Cons:
  b # rev (bin-to-bl n x) = rev (bin-to-bl (Suc n) (x BIT If b bit.B1 bit.B0))
  apply (unfold bin-to-bl-def)
  apply simp
  apply (simp add: bin-to-bl-aux-alt)
  done

lemma rbl-mult: !!bina binb.
    rbl-mult (rev (bin-to-bl n bina)) (rev (bin-to-bl n binb)) =
    rev (bin-to-bl n (bina * binb))
  apply (induct n)
   apply simp
  apply (unfold bin-to-bl-def)
  apply clarsimp
  apply (case-tac bina rule: bin-exhaust)
  apply (case-tac binb rule: bin-exhaust)
  apply (case-tac b)
   apply (case-tac [!] ba)
     apply (auto simp: bin-to-bl-aux-alt Let-def)
     apply (auto simp: rbbl-Cons rbl-mult-Suc rbl-add)
  done

lemma rbl-add-split:
  P (rbl-add (y # ys) (x # xs)) =
    (ALL ws. length ws = length ys --> ws = rbl-add ys xs -->
    (y --> ((x --> P (False # rbl-succ ws)) & (~ x --> P (True # ws))))
&
    (~ y --> P (x # ws)))
  apply (auto simp add: Let-def)
   apply (case-tac [!] y)
     apply auto
  done

lemma rbl-mult-split:
  P (rbl-mult (y # ys) xs) =
    (ALL ws. length ws = Suc (length ys) --> ws = False # rbl-mult ys xs -->

    (y --> P (rbl-add ws xs)) & (~ y --> P ws))
  by (clarsimp simp add : Let-def)

lemma and-len: xs = ys ==> xs = ys & length xs = length ys
  by auto

lemma size-if: size (if p then xs else ys) = (if p then size xs else size ys)
```

by *auto*

lemma *tl-if*: tl (*if p then xs else ys*) = (*if p then tl xs else tl ys*)
by *auto*

lemma *hd-if*: hd (*if p then xs else ys*) = (*if p then hd xs else hd ys*)
by *auto*

lemma *if-Not-x*: (*if p then* ~ *x else x*) = (p = (~ x))
by *auto*

lemma *if-x-Not*: (*if p then x else* ~ *x*) = (p = x)
by *auto*

lemma *if-same-and*: (*If p x y* & *If p u v*) = (*if p then x* & *u else y* & *v*)
by *auto*

lemma *if-same-eq*: (*If p x y* = (*If p u v*)) = (*if p then x* = (u) *else y* = (v))
by *auto*

lemma *if-same-eq-not*:
(*If p x y* = (~ *If p u v*)) = (*if p then x* = (~u) *else y* = (~v))
by *auto*

lemma *if-Cons*: (*if p then x* # *xs else y* # *ys*) = *If p x y* # *If p xs ys*
by *auto*

lemma *if-single*:
(*if xc then* [*xab*] *else* [*an*]) = [*if xc then xab else an*]
by *auto*

lemma *if-bool-simps*:
If p True y = (p | y) & *If p False y* = (~p & y) &
If p y True = (p --> y) & *If p y False* = (p & y)
by *auto*

lemmas *if-simps* = *if-x-Not if-Not-x if-cancel if-True if-False if-bool-simps*

lemmas *seqr* = *eq-reflection* [**where** x = *size w*, *standard*]

lemmas *tl-Nil* = *tl.simps* (*1*)
lemmas *tl-Cons* = *tl.simps* (*2*)

7.2 Repeated splitting or concatenation

lemma *sclem*:
size (*concat* (*map* (*bin-to-bl n*) *xs*)) = *length xs* $*$ n
by (*induct xs*) *auto*

lemma *bin-cat-foldl-lem* [*rule-format*] :
  ALL x. foldl (*%u. bin-cat u n*) *x xs* =
    bin-cat x (*size xs* ∗ *n*) (*foldl* (*%u. bin-cat u n*) *y xs*)
  apply (*induct xs*)
   apply *simp*
  apply *clarify*
  apply (*simp* (*no-asm*))
  apply (*frule asm-rl*)
  apply (*drule spec*)
  apply (*erule trans*)
  apply (*drule-tac x* = *bin-cat y n a* **in** *spec*)
  apply (*simp add* : *bin-cat-assoc-sym min-def*)
  done

lemma *bin-rcat-bl*:
  (*bin-rcat n wl*) = *bl-to-bin* (*concat* (*map* (*bin-to-bl n*) *wl*))
  apply (*unfold bin-rcat-def*)
  apply (*rule sym*)
  apply (*induct wl*)
   apply (*auto simp add* : *bl-to-bin-append*)
  apply (*simp add* : *bl-to-bin-aux-alt sclem*)
  apply (*simp add* : *bin-cat-foldl-lem* [*symmetric*])
  done

lemmas *bin-rsplit-aux-simps* = *bin-rsplit-aux.simps bin-rsplitl-aux.simps*
lemmas *rsplit-aux-simps* = *bin-rsplit-aux-simps*

lemmas *th-if-simp1* = *split-if* [**where** *P* = *op* = *l*,
  THEN iffD1, *THEN conjunct1*, *THEN mp*, *standard*]
lemmas *th-if-simp2* = *split-if* [**where** *P* = *op* = *l*,
  THEN iffD1, *THEN conjunct2*, *THEN mp*, *standard*]

lemmas *rsplit-aux-simp1s* = *rsplit-aux-simps* [*THEN th-if-simp1*]

lemmas *rsplit-aux-simp2ls* = *rsplit-aux-simps* [*THEN th-if-simp2*]

lemmas *bin-rsplit-aux-simp2s* [*simp*] = *rsplit-aux-simp2ls* [*unfolded Let-def*]
lemmas *rbscl* = *bin-rsplit-aux-simp2s* (*2*)

lemmas *rsplit-aux-0-simps* [*simp*] =
  rsplit-aux-simp1s [*OF disjI1*] *rsplit-aux-simp1s* [*OF disjI2*]

lemma *bin-rsplit-aux-append*:
  bin-rsplit-aux (*n*, *bs* @ *cs*, *m*, *c*) = *bin-rsplit-aux* (*n*, *bs*, *m*, *c*) @ *cs*
  apply (*rule-tac u*=*n* **and** *v*=*bs* **and** *w*=*m* **and** *x*=*c* **in** *bin-rsplit-aux.induct*)
  apply (*subst bin-rsplit-aux.simps*)
  apply (*subst bin-rsplit-aux.simps*)
  apply (*clarsimp split*: *ls-splits*)

done

lemma *bin-rsplitl-aux-append*:
bin-rsplitl-aux (*n*, *bs* @ *cs*, *m*, *c*) = *bin-rsplitl-aux* (*n*, *bs*, *m*, *c*) @ *cs*
apply (*rule-tac u*=*n* **and** *v*=*bs* **and** *w*=*m* **and** *x*=*c* **in** *bin-rsplitl-aux.induct*)
apply (*subst bin-rsplitl-aux.simps*)
apply (*subst bin-rsplitl-aux.simps*)
apply (*clarsimp split*: *ls-splits*)
done

lemmas *rsplit-aux-apps* [**where** *bs* = []] =
bin-rsplit-aux-append bin-rsplitl-aux-append

lemmas *rsplit-def-auxs* = *bin-rsplit-def bin-rsplitl-def*

lemmas *rsplit-aux-alts* = *rsplit-aux-apps*
[*unfolded append-Nil rsplit-def-auxs* [*symmetric*]]

lemma *bin-split-minus*: *0* < *n* ==> *bin-split* (*Suc* (*n* − *1*)) *w* = *bin-split n w*
by *auto*

lemmas *bin-split-minus-simp* =
bin-split.Suc [*THEN* [*2*] *bin-split-minus* [*symmetric*, *THEN trans*], *standard*]

lemma *bin-split-pred-simp* [*simp*]:
(*0*::*nat*) < *number-of bin* ⟹
bin-split (*number-of bin*) *w* =
(*let* (*w1*, *w2*) = *bin-split* (*number-of* (*Numeral.pred bin*)) (*bin-rest w*)
in (*w1*, *w2 BIT bin-last w*))
by (*simp only*: *nobm1 bin-split-minus-simp*)

declare *bin-split-pred-simp* [*simp*]

lemma *bin-rsplit-aux-simp-alt*:
bin-rsplit-aux (*n*, *bs*, *m*, *c*) =
(*if m* = *0* ∨ *n* = *0*
then bs
else let (*a*, *b*) = *bin-split n c in bin-rsplit n* (*m* − *n*, *a*) @ *b* # *bs*)
apply (*rule trans*)
apply (*subst bin-rsplit-aux.simps*, *rule refl*)
apply (*simp add*: *rsplit-aux-alts*)
done

lemmas *bin-rsplit-simp-alt* =
trans [*OF bin-rsplit-def* [*THEN meta-eq-to-obj-eq*]
bin-rsplit-aux-simp-alt, *standard*]

lemmas *bthrs* = *bin-rsplit-simp-alt* [*THEN* [*2*] *trans*]

lemma *bin-rsplit-size-sign'* [*rule-format*] :
n > *0* ==> (*ALL nw w*. *rev sw* = *bin-rsplit n* (*nw*, *w*) -->
(*ALL v*: *set sw*. *bintrunc n v* = *v*))
apply (*induct sw*)
apply *clarsimp*
apply *clarsimp*
apply (*drule bthrs*)
apply (*simp* (*no-asm-use*) *add*: *Let-def split*: *ls-splits*)
apply *clarify*
apply (*erule impE*, *rule exI*, *erule exI*)
apply (*drule split-bintrunc*)
apply *simp*
done

lemmas *bin-rsplit-size-sign* = *bin-rsplit-size-sign'* [*OF asm-rl*
rev-rev-ident [*THEN trans*] *set-rev* [*THEN equalityD2* [*THEN subsetD*]],
standard]

lemma *bin-nth-rsplit* [*rule-format*] :
n > *0* ==> *m* < *n* ==> (*ALL w k nw*. *rev sw* = *bin-rsplit n* (*nw*, *w*) -->
k < *size sw* --> *bin-nth* (*sw* ! *k*) *m* = *bin-nth w* (*k* * *n* + *m*))
apply (*induct sw*)
apply *clarsimp*
apply *clarsimp*
apply (*drule bthrs*)
apply (*simp* (*no-asm-use*) *add*: *Let-def split*: *ls-splits*)
apply *clarify*
apply (*erule allE*, *erule impE*, *erule exI*)
apply (*case-tac k*)
apply *clarsimp*
prefer *2*
apply *clarsimp*
apply (*erule allE*)
apply (*erule* (*1*) *impE*)
apply (*drule bin-nth-split*, *erule conjE*, *erule allE*,
erule trans, *simp add* : *add-ac*)+
done

lemma *bin-rsplit-all*:
0 < *nw* ==> *nw* <= *n* ==> *bin-rsplit n* (*nw*, *w*) = [*bintrunc n w*]
unfolding *bin-rsplit-def*
by (*clarsimp dest*!: *split-bintrunc simp*: *rsplit-aux-simp2ls split*: *ls-splits*)

lemma *bin-rsplit-l* [*rule-format*] :
ALL bin. *bin-rsplitl n* (*m*, *bin*) = *bin-rsplit n* (*m*, *bintrunc m bin*)
apply (*rule-tac a* = *m* **in** *wf-less-than* [*THEN wf-induct*])
apply (*simp* (*no-asm*) *add* : *bin-rsplitl-def bin-rsplit-def*)
apply (*rule allI*)
apply (*subst bin-rsplitl-aux.simps*)

apply (*subst bin-rsplit-aux.simps*)
apply (*clarsimp simp*: *rsplit-aux-alts Let-def split*: *ls-splits*)
apply (*drule bin-split-trunc*)
apply (*drule sym* [*THEN trans*], *assumption*)
apply *fast*
done

lemma *bin-rsplit-rcat* [*rule-format*] :
n > 0 --> bin-rsplit n (*n * size ws*, *bin-rcat n ws*) = *map* (*bintrunc n*) *ws*
apply (*unfold bin-rsplit-def bin-rcat-def*)
apply (*rule-tac xs = ws* **in** *rev-induct*)
apply *clarsimp*
apply *clarsimp*
apply (*clarsimp simp add*: *bin-split-cat rsplit-aux-alts*)
done

lemma *bin-rsplit-aux-len-le* [*rule-format*] :
ALL ws m. n ≠ 0 --> ws = bin-rsplit-aux (*n*, *bs*, *nw*, *w*) *-->*
(*length ws <= m*) = (*nw + length bs * n <= m * n*)
apply (*rule-tac u=n* **and** *v=bs* **and** *w=nw* **and** *x=w* **in** *bin-rsplit-aux.induct*)
apply (*subst bin-rsplit-aux.simps*)
apply (*simp add:lrlem Let-def split*: *ls-splits*)
done

lemma *bin-rsplit-len-le*:
n ≠ 0 --> ws = bin-rsplit n (*nw*, *w*) *-->* (*length ws <= m*) = (*nw <= m **
n)
unfolding *bin-rsplit-def* **by** (*clarsimp simp add* : *bin-rsplit-aux-len-le*)

lemma *bin-rsplit-aux-len* [*rule-format*] :
n≠0 --> length (*bin-rsplit-aux* (*n*, *cs*, *nw*, *w*)) =
(*nw + n − 1*) *div n + length cs*
apply (*rule-tac u=n* **and** *v=cs* **and** *w=nw* **and** *x=w* **in** *bin-rsplit-aux.induct*)
apply (*subst bin-rsplit-aux.simps*)
apply (*clarsimp simp*: *Let-def split*: *ls-splits*)
apply (*erule thin-rl*)
apply (*case-tac m <= n*)
prefer *2*
apply (*simp add*: *div-add-self2* [*symmetric*])
apply (*case-tac m*, *clarsimp*)
apply (*simp add*: *div-add-self2*)
done

lemma *bin-rsplit-len*:
n≠0 ==> length (*bin-rsplit n* (*nw*, *w*)) = (*nw + n − 1*) *div n*
unfolding *bin-rsplit-def* **by** (*clarsimp simp add* : *bin-rsplit-aux-len*)

lemma *bin-rsplit-aux-len-indep* [*rule-format*] :
n≠0 ==> (*ALL v bs. length bs = length cs -->*

```
    length (bin-rsplit-aux (n, bs, nw, v)) =
    length (bin-rsplit-aux (n, cs, nw, w)))
  apply (rule-tac u=n and v=cs and w=nw and x=w in bin-rsplit-aux.induct)
  apply clarsimp
  apply (simp (no-asm-simp) add: bin-rsplit-aux-simp-alt Let-def
                         split: ls-splits)
  apply clarify
  apply (erule allE)+
  apply (erule impE)
   apply (fast elim!: sym)
  apply (simp (no-asm-use) add: rsplit-aux-alts)
  apply (erule impE)
  apply (rule-tac x=ba # bs in exI)
  apply auto
  done

lemma bin-rsplit-len-indep:
  n≠0 ==> length (bin-rsplit n (nw, v)) = length (bin-rsplit n (nw, w))
  apply (unfold bin-rsplit-def)
  apply (erule bin-rsplit-aux-len-indep)
  apply (rule refl)
  done

end
```

8 TdThs: Type Definition Theorems

theory *TdThs* **imports** *Main* **begin**

9 More lemmas about normal type definitions

```
lemma
  tdD1: type-definition Rep Abs A ⟹ ∀ x. Rep x ∈ A and
  tdD2: type-definition Rep Abs A ⟹ ∀ x. Abs (Rep x) = x and
  tdD3: type-definition Rep Abs A ⟹ ∀ y. y ∈ A ⟶ Rep (Abs y) = y
  by (auto simp: type-definition-def)

lemma td-nat-int:
  type-definition int nat (Collect (op <= 0))
  unfolding type-definition-def by auto

context type-definition
begin

lemmas Rep' [iff] = Rep [simplified]
```

declare *Rep-inverse* [*simp*] *Rep-inject* [*simp*]

lemma *Abs-eqD*: *Abs x = Abs y ==> x ∈ A ==> y ∈ A ==> x = y*
  by (*simp add*: *Abs-inject*)

lemma *Abs-inverse'*:
  r : A ==> Abs r = a ==> Rep a = r
  by (*safe elim*!: *Abs-inverse*)

lemma *Rep-comp-inverse*:
  Rep o f = g ==> Abs o g = f
  using *Rep-inverse* **by** (*auto intro*: *ext*)

lemma *Rep-eqD* [*elim*!]: *Rep x = Rep y ==> x = y*
  by *simp*

lemma *Rep-inverse'*: *Rep a = r ==> Abs r = a*
  by (*safe intro*!: *Rep-inverse*)

lemma *comp-Abs-inverse*:
  f o Abs = g ==> g o Rep = f
  using *Rep-inverse* **by** (*auto intro*: *ext*)

lemma *set-Rep*:
  A = range Rep
proof (*rule set-ext*)
  fix *x*
  show (*x ∈ A*) = (*x ∈ range Rep*)
    by (*auto dest*: *Abs-inverse* [*of x*, *symmetric*])
qed

lemma *set-Rep-Abs*: *A = range* (*Rep o Abs*)
proof (*rule set-ext*)
  fix *x*
  show (*x ∈ A*) = (*x ∈ range* (*Rep o Abs*))
    by (*auto dest*: *Abs-inverse* [*of x*, *symmetric*])
qed

lemma *Abs-inj-on*: *inj-on Abs A*
  unfolding *inj-on-def*
  by (*auto dest*: *Abs-inject* [*THEN iffD1*])

lemma *image*: *Abs ‘ A = UNIV*
  by (*auto intro*!: *image-eqI*)

lemmas *td-thm = type-definition-axioms*

lemma *fns1*:
  Rep o fa = fr o Rep | fa o Abs = Abs o fr ==> Abs o fr o Rep = fa

by (*auto dest*: *Rep-comp-inverse elim*: *comp-Abs-inverse simp*: *o-assoc*)

lemmas *fns1a* = *disjI1* [*THEN fns1*]
lemmas *fns1b* = *disjI2* [*THEN fns1*]

lemma *fns4*:
Rep o fa o Abs = *fr* ==>
Rep o fa = *fr o Rep* & *fa o Abs* = *Abs o fr*
by (*auto intro*!: *ext*)

end

interpretation *nat-int*: *type-definition* [*int nat Collect* (*op* <= *0*)]
by (*rule td-nat-int*)

— resetting to the default nat induct and cases rules
declare *Nat.induct* [*case-names 0 Suc*, *induct type*]
declare *Nat.exhaust* [*case-names 0 Suc*, *cases type*]

9.1 Extended form of type definition predicate

lemma *td-conds*:
norm o norm = *norm* ==> (*fr o norm* = *norm o fr*) =
(*norm o fr o norm* = *fr o norm* & *norm o fr o norm* = *norm o fr*)
apply *safe*
apply (*simp-all add*: *o-assoc* [*symmetric*])
apply (*simp-all add*: *o-assoc*)
done

lemma *fn-comm-power*:
fa o tr = *tr o fr* ==> *fa* ^ *n o tr* = *tr o fr* ^ *n*
apply (*rule ext*)
apply (*induct n*)
apply (*auto dest*: *fun-cong*)
done

lemmas *fn-comm-power'* =
ext [*THEN fn-comm-power*, *THEN fun-cong*, *unfolded o-def*, *standard*]

locale *td-ext* = *type-definition* +
fixes *norm*
assumes *eq-norm*: $\bigwedge$*x*. *Rep* (*Abs x*) = *norm x*
begin

lemma *Abs-norm* [*simp*]:
Abs (*norm x*) = *Abs x*
using *eq-norm* [*of x*] **by** (*auto elim*: *Rep-inverse'*)

lemma *td-th*:
g o Abs = f ==> f (Rep x) = g x
by (*drule comp-Abs-inverse* [*symmetric*]) *simp*

lemma *eq-norm'*: *Rep o Abs = norm*
by (*auto simp*: *eq-norm intro*!: *ext*)

lemma *norm-Rep* [*simp*]: *norm* (*Rep x*) = *Rep x*
by (*auto simp*: *eq-norm' intro*: *td-th*)

lemmas *td = td-thm*

lemma *set-iff-norm*: *w : A <-> w = norm w*
by (*auto simp*: *set-Rep-Abs eq-norm' eq-norm* [*symmetric*])

lemma *inverse-norm*:
(*Abs n = w*) = (*Rep w = norm n*)
apply (*rule iffI*)
 apply (*clarsimp simp add*: *eq-norm*)
apply (*simp add*: *eq-norm'* [*symmetric*])
done

lemma *norm-eq-iff*:
(*norm x = norm y*) = (*Abs x = Abs y*)
by (*simp add*: *eq-norm'* [*symmetric*])

lemma *norm-comps*:
Abs o norm = Abs
norm o Rep = Rep
norm o norm = norm
by (*auto simp*: *eq-norm'* [*symmetric*] *o-def*)

lemmas *norm-norm* [*simp*] = *norm-comps*

lemma *fns5*:
Rep o fa o Abs = fr ==>
fr o norm = fr & norm o fr = fr
by (*fold eq-norm'*) (*auto intro*!: *ext*)

lemma *fns2*:
Abs o fr o Rep = fa ==>
 (*norm o fr o norm = fr o norm*) = (*Rep o fa = fr o Rep*)
apply (*fold eq-norm'*)
apply *safe*
 prefer *2*
 apply (*simp add*: *o-assoc*)
apply (*rule ext*)
apply (*drule-tac x=Rep x* **in** *fun-cong*)

```
  apply auto
  done

lemma fns3:
  Abs o fr o Rep = fa ==>
   (norm o fr o norm = norm o fr) = (fa o Abs = Abs o fr)
  apply (fold eq-norm')
  apply safe
   prefer 2
   apply (simp add: o-assoc [symmetric])
  apply (rule ext)
  apply (drule fun-cong)
  apply simp
  done

lemma fns:
  fr o norm = norm o fr ==>
    (fa o Abs = Abs o fr) = (Rep o fa = fr o Rep)
  apply safe
   apply (frule fns1b)
   prefer 2
   apply (frule fns1a)
   apply (rule fns3 [THEN iffD1])
     prefer 3
     apply (rule fns2 [THEN iffD1])
       apply (simp-all add: o-assoc [symmetric])
   apply (simp-all add: o-assoc)
  done

lemma range-norm:
  range (Rep o Abs) = A
  by (simp add: set-Rep-Abs)

end

lemmas td-ext-def' =
  td-ext-def [unfolded type-definition-def td-ext-axioms-def]

end
```

10 WordDefinition: Definition of Word Type

```
theory WordDefinition imports Size BinBoolList TdThs begin

typedef (open word) 'a word
  = {(0::int) ..< 2^len-of TYPE('a::len0)} by auto
```

instance *word* :: (*len0*) *number* ..
instance *word* :: (*type*) *minus* ..
instance *word* :: (*type*) *plus* ..
instance *word* :: (*type*) *one* ..
instance *word* :: (*type*) *zero* ..
instance *word* :: (*type*) *times* ..
instance *word* :: (*type*) *Divides.div* ..
instance *word* :: (*type*) *power* ..
instance *word* :: (*type*) *ord* ..
instance *word* :: (*type*) *size* ..
instance *word* :: (*type*) *inverse* ..
instance *word* :: (*type*) *bit* ..

10.1 Type conversions and casting

constdefs
— representation of words using unsigned or signed bins, only difference in these is the type class
word-of-int :: *int* => *'a* :: *len0 word*
word-of-int w == *Abs-word* (*bintrunc* (*len-of TYPE* (*'a*)) *w*)

— uint and sint cast a word to an integer, uint treats the word as unsigned, sint treats the most-significant-bit as a sign bit
uint :: *'a* :: *len0 word* => *int*
uint w == *Rep-word w*
sint :: *'a* :: *len word* => *int*
sint-uint: *sint w* == *sbintrunc* (*len-of TYPE* (*'a*) − *1*) (*uint w*)
unat :: *'a* :: *len0 word* => *nat*
unat w == *nat* (*uint w*)

— the sets of integers representing the words
uints :: *nat* => *int set*
uints n == *range* (*bintrunc n*)
sints :: *nat* => *int set*
sints n == *range* (*sbintrunc* (*n* − *1*))
unats :: *nat* => *nat set*
unats n == {*i*. *i* < *2* ˆ *n*}
norm-sint :: *nat* => *int* => *int*
norm-sint n w == (*w* + *2* ˆ (*n* − *1*)) *mod 2* ˆ *n* − *2* ˆ (*n* − *1*)

— cast a word to a different length
scast :: *'a* :: *len word* => *'b* :: *len word*
scast w == *word-of-int* (*sint w*)
ucast :: *'a* :: *len0 word* => *'b* :: *len0 word*
ucast w == *word-of-int* (*uint w*)

— whether a cast (or other) function is to a longer or shorter length
source-size :: (*'a* :: *len0 word* => *'b*) => *nat*
source-size c == *let arb* = *arbitrary* ; *x* = *c arb in size arb*

target-size :: (*'a* => *'b* :: *len0 word*) => *nat*
target-size c == *size* (*c arbitrary*)
is-up :: (*'a* :: *len0 word* => *'b* :: *len0 word*) => *bool*
is-up c == *source-size c* <= *target-size c*
is-down :: (*'a* :: *len0 word* => *'b* :: *len0 word*) => *bool*
is-down c == *target-size c* <= *source-size c*

constdefs
of-bl :: *bool list* => *'a* :: *len0 word*
of-bl bl == *word-of-int* (*bl-to-bin bl*)
to-bl :: *'a* :: *len0 word* => *bool list*
to-bl w ==
bin-to-bl (*len-of TYPE* (*'a*)) (*uint w*)

word-reverse :: *'a* :: *len0 word* => *'a word*
word-reverse w == *of-bl* (*rev* (*to-bl w*))

defs (**overloaded**)
word-size: *size* (*w* :: *'a* :: *len0 word*) == *len-of TYPE*(*'a*)
word-number-of-def: *number-of w* == *word-of-int w*

constdefs
word-int-case :: (*int* => *'b*) => (*'a* :: *len0 word*) => *'b*
word-int-case f w == *f* (*uint w*)

syntax
of-int :: *int* => *'a*
translations
case x of of-int y => *b* == *word-int-case* (%*y*. *b*) *x*

10.2 Arithmetic operations

defs (**overloaded**)
word-1-wi: (*1* :: (*'a* :: *len0*) *word*) == *word-of-int 1*
word-0-wi: (*0* :: (*'a* :: *len0*) *word*) == *word-of-int 0*

word-le-def: *a* <= *b* == *uint a* <= *uint b*
word-less-def: *x* < *y* == *x* <= *y* & *x* ~= (*y* :: *'a* :: *len0 word*)

constdefs
word-succ :: *'a* :: *len0 word* => *'a word*
word-succ a == *word-of-int* (*Numeral.succ* (*uint a*))

word-pred :: *'a* :: *len0 word* => *'a word*
word-pred a == *word-of-int* (*Numeral.pred* (*uint a*))

udvd :: *'a::len word* => *'a::len word* => *bool* (**infixl** *udvd 50*)
a udvd b == *EX n>=0*. *uint b* = *n* * *uint a*

word-sle :: 'a :: len word => 'a word => bool ((-/ <=s -) [50, 51] 50)
a <=s b == sint a <= sint b

word-sless :: 'a :: len word => 'a word => bool ((-/ <s -) [50, 51] 50)
(x <s y) == (x <=s y & x ~= y)

consts
word-power :: 'a :: len0 word => nat => 'a word
primrec
word-power a 0 = 1
word-power a (Suc n) = a * word-power a n

defs (overloaded)
word-pow: power == word-power
word-add-def: a + b == word-of-int (uint a + uint b)
word-sub-wi: a − b == word-of-int (uint a − uint b)
word-minus-def: − a == word-of-int (− uint a)
word-mult-def: a * b == word-of-int (uint a * uint b)
word-div-def: a div b == word-of-int (uint a div uint b)
word-mod-def: a mod b == word-of-int (uint a mod uint b)

10.3 Bit-wise operations

defs (overloaded)
word-and-def:
(a::'a::len0 word) AND b == word-of-int (uint a AND uint b)

word-or-def:
(a::'a::len0 word) OR b == word-of-int (uint a OR uint b)

word-xor-def:
(a::'a::len0 word) XOR b == word-of-int (uint a XOR uint b)

word-not-def:
NOT (a::'a::len0 word) == word-of-int (NOT (uint a))

word-test-bit-def:
test-bit (a::'a::len0 word) == bin-nth (uint a)

word-set-bit-def:
set-bit (a::'a::len0 word) n x ==
 word-of-int (bin-sc n (If x bit.B1 bit.B0) (uint a))

word-set-bits-def:
(BITS n. f n)::'a::len0 word == of-bl (bl-of-nth (len-of TYPE ('a)) f)

word-lsb-def:
lsb (a::'a::len0 word) == bin-last (uint a) = bit.B1

word-msb-def:
$msb\ (a::'a::len\ word) == bin\text{-}sign\ (sint\ a) = Numeral.Min$

constdefs
$setBit :: 'a :: len0\ word => nat => 'a\ word$
$setBit\ w\ n == set\text{-}bit\ w\ n\ True$

$clearBit :: 'a :: len0\ word => nat => 'a\ word$
$clearBit\ w\ n == set\text{-}bit\ w\ n\ False$

10.4 Shift operations

constdefs
$shiftl1 :: 'a :: len0\ word => 'a\ word$
$shiftl1\ w == word\text{-}of\text{-}int\ (uint\ w\ BIT\ bit.B0)$

— shift right as unsigned or as signed, ie logical or arithmetic
$shiftr1 :: 'a :: len0\ word => 'a\ word$
$shiftr1\ w == word\text{-}of\text{-}int\ (bin\text{-}rest\ (uint\ w))$

$sshiftr1 :: 'a :: len\ word => 'a\ word$
$sshiftr1\ w == word\text{-}of\text{-}int\ (bin\text{-}rest\ (sint\ w))$

$bshiftr1 :: bool => 'a :: len\ word => 'a\ word$
$bshiftr1\ b\ w == of\text{-}bl\ (b\ \#\ butlast\ (to\text{-}bl\ w))$

$sshiftr :: 'a :: len\ word => nat => 'a\ word$ (**infixl** $>>>\ 55$)
$w >>> n == (sshiftr1\ \hat{}\ n)\ w$

$mask :: nat => 'a::len\ word$
$mask\ n == (1 << n) - 1$

$revcast :: 'a :: len0\ word => 'b :: len0\ word$
$revcast\ w == of\text{-}bl\ (takefill\ False\ (len\text{-}of\ TYPE('b))\ (to\text{-}bl\ w))$

$slice1 :: nat => 'a :: len0\ word => 'b :: len0\ word$
$slice1\ n\ w == of\text{-}bl\ (takefill\ False\ n\ (to\text{-}bl\ w))$

$slice :: nat => 'a :: len0\ word => 'b :: len0\ word$
$slice\ n\ w == slice1\ (size\ w - n)\ w$

defs (**overloaded**)
shiftl-def: $(w::'a::len0\ word) << n == (shiftl1\ \hat{}\ n)\ w$
shiftr-def: $(w::'a::len0\ word) >> n == (shiftr1\ \hat{}\ n)\ w$

10.5 Rotation

constdefs

rotater1 :: *'a list => 'a list*
rotater1 ys ==
  case ys of [] => [] | *x # xs => last ys # butlast ys*

rotater :: *nat => 'a list => 'a list*
rotater n == rotater1 ^ n

word-rotr :: *nat => 'a :: len0 word => 'a :: len0 word*
word-rotr n w == of-bl (rotater n (to-bl w))

word-rotl :: *nat => 'a :: len0 word => 'a :: len0 word*
word-rotl n w == of-bl (rotate n (to-bl w))

word-roti :: *int => 'a :: len0 word => 'a :: len0 word*
word-roti i w == if i >= 0 then word-rotr (nat i) w
  else word-rotl (nat (− i)) w

10.6 Split and cat operations

constdefs
word-cat :: *'a :: len0 word => 'b :: len0 word => 'c :: len0 word*
word-cat a b == word-of-int (bin-cat (uint a) (len-of TYPE ('b)) (uint b))

word-split :: *'a :: len0 word => ('b :: len0 word) ∗ ('c :: len0 word)*
word-split a ==
 case bin-split (len-of TYPE ('c)) (uint a) of
  (u, v) => (word-of-int u, word-of-int v)

word-rcat :: *'a :: len0 word list => 'b :: len0 word*
word-rcat ws ==
word-of-int (bin-rcat (len-of TYPE ('a)) (map uint ws))

word-rsplit :: *'a :: len0 word => 'b :: len word list*
word-rsplit w ==
map word-of-int (bin-rsplit (len-of TYPE ('b)) (len-of TYPE ('a), uint w))

constdefs
— Largest representable machine integer.
max-word :: *'a::len word*
max-word ≡ *word-of-int (2^len-of TYPE('a) − 1)*

consts
of-bool :: *bool* ⇒ *'a::len word*
primrec
of-bool False = 0
of-bool True = 1

lemmas *of-nth-def* = *word-set-bits-def*

lemmas *word-size-gt-0* [*iff*] =
xtr1 [*OF word-size* [*THEN meta-eq-to-obj-eq*] *len-gt-0*, *standard*]
lemmas *lens-gt-0* = *word-size-gt-0 len-gt-0*
lemmas *lens-not-0* [*iff*] = *lens-gt-0* [*THEN gr-implies-not0*, *standard*]

lemma *uints-num*: *uints n* = $\{i.\ 0 \leq i \wedge i < 2\ \hat{}\ n\}$
by (*simp add*: *uints-def range-bintrunc*)

lemma *sints-num*: *sints n* = $\{i.\ -(2\ \hat{}\ (n - 1)) \leq i \wedge i < 2\ \hat{}\ (n - 1)\}$
by (*simp add*: *sints-def range-sbintrunc*)

lemmas *atLeastLessThan-alt* = *atLeastLessThan-def* [*unfolded*
atLeast-def lessThan-def Collect-conj-eq [*symmetric*]]

lemma *mod-in-reps*: *m* > *0* ==> *y mod m* : {*0*::*int* ..< *m*}
unfolding *atLeastLessThan-alt* **by** *auto*

lemma
Rep-word-0:*0* <= *Rep-word x* **and**
Rep-word-lt: *Rep-word* (*x*::'*a*::*len0 word*) < *2* ^ *len-of TYPE*('*a*)
by (*auto simp*: *Rep-word* [*simplified*])

lemma *Rep-word-mod-same*:
Rep-word x mod 2 ^ *len-of TYPE*('*a*) = *Rep-word* (*x*::'*a*::*len0 word*)
by (*simp add*: *int-mod-eq Rep-word-lt Rep-word-0*)

lemma *td-ext-uint*:
td-ext (*uint* :: '*a word* => *int*) *word-of-int* (*uints* (*len-of TYPE*('*a*::*len0*)))
(%*w*::*int*. *w mod 2* ^ *len-of TYPE*('*a*))
apply (*unfold td-ext-def*')
apply (*simp add*: *uints-num uint-def word-of-int-def bintrunc-mod2p*)
apply (*simp add*: *Rep-word-mod-same Rep-word-0 Rep-word-lt*
word.Rep-word-inverse word.Abs-word-inverse int-mod-lem)
done

lemmas *int-word-uint* = *td-ext-uint* [*THEN td-ext.eq-norm*, *standard*]

interpretation *word-uint*:
td-ext [*uint*::'*a*::*len0 word* ⇒ *int*
word-of-int
uints (*len-of TYPE*('*a*::*len0*))
λ*w*. *w mod 2* ^ *len-of TYPE*('*a*::*len0*)]
by (*rule td-ext-uint*)

lemmas *td-uint* = *word-uint.td-thm*

lemmas *td-ext-ubin* = *td-ext-uint* [*simplified len-gt-0 no-bintr-alt1* [*symmetric*]]

interpretation *word-ubin*:
  td-ext [*uint*::$'$*a*::*len0 word* $\Rightarrow$ *int*
      word-of-int
      uints (*len-of TYPE*($'$*a*::*len0*))
      bintrunc (*len-of TYPE*($'$*a*::*len0*))]
  by (*rule td-ext-ubin*)

lemma *sint-sbintrunc*$'$:
  sint (*word-of-int bin* :: $'$*a word*) =
    (*sbintrunc* (*len-of TYPE* ($'$*a* :: *len*) $-$ *1*) *bin*)
  unfolding *sint-uint*
  by (*auto simp*: *word-ubin.eq-norm sbintrunc-bintrunc-lt*)

lemma *uint-sint*:
  uint w = *bintrunc* (*len-of TYPE*($'$*a*)) (*sint* (*w* :: $'$*a* :: *len word*))
  unfolding *sint-uint* **by** (*auto simp*: *bintrunc-sbintrunc-le*)

lemma *bintr-uint*$'$:
  n >= *size w* ==> *bintrunc n* (*uint w*) = *uint w*
  apply (*unfold word-size*)
  apply (*subst word-ubin.norm-Rep* [*symmetric*])
  apply (*simp only*: *bintrunc-bintrunc-min word-size min-def*)
  apply *simp*
  done

lemma *wi-bintr*$'$:
  wb = *word-of-int bin* ==> *n* >= *size wb* ==>
    word-of-int (*bintrunc n bin*) = *wb*
  unfolding *word-size*
  by (*clarsimp simp add* : *word-ubin.norm-eq-iff* [*symmetric*] *min-def*)

lemmas *bintr-uint* = *bintr-uint*$'$ [*unfolded word-size*]
lemmas *wi-bintr* = *wi-bintr*$'$ [*unfolded word-size*]

lemma *td-ext-sbin*:
  td-ext (*sint* :: $'$*a word* => *int*) *word-of-int* (*sints* (*len-of TYPE*($'$*a*::*len*)))
    (*sbintrunc* (*len-of TYPE*($'$*a*) $-$ *1*))
  apply (*unfold td-ext-def*$'$ *sint-uint*)
  apply (*simp add* : *word-ubin.eq-norm*)
  apply (*cases len-of TYPE*($'$*a*))
   apply (*auto simp add* : *sints-def*)
  apply (*rule sym* [*THEN trans*])
  apply (*rule word-ubin.Abs-norm*)
  apply (*simp only*: *bintrunc-sbintrunc*)
  apply (*drule sym*)
  apply *simp*
  done

lemmas *td-ext-sint* = *td-ext-sbin*
[*simplified len-gt-0 no-sbintr-alt2 Suc-pred′* [*symmetric*]]

interpretation *word-sint*:
td-ext [*sint* ::′*a*::*len word* => *int*
word-of-int
sints (*len-of TYPE*(′*a*::*len*))
%*w*. (*w* + *2*ˆ(*len-of TYPE*(′*a*::*len*) − *1*)) *mod 2*ˆ*len-of TYPE*(′*a*::*len*) −
2 ˆ (*len-of TYPE*(′*a*::*len*) − *1*)]
by (*rule td-ext-sint*)

interpretation *word-sbin*:
td-ext [*sint* ::′*a*::*len word* => *int*
word-of-int
sints (*len-of TYPE*(′*a*::*len*))
sbintrunc (*len-of TYPE*(′*a*::*len*) − *1*)]
by (*rule td-ext-sbin*)

lemmas *int-word-sint* = *td-ext-sint* [*THEN td-ext.eq-norm*, *standard*]

lemmas *td-sint* = *word-sint.td*

lemma *word-number-of-alt*: *number-of b* == *word-of-int* (*number-of b*)
unfolding *word-number-of-def* **by** (*simp add*: *number-of-eq*)

lemma *word-no-wi*: *number-of* = *word-of-int*
by (*auto simp*: *word-number-of-def intro*: *ext*)

lemma *to-bl-def′*:
(*to-bl* :: ′*a* :: *len0 word* => *bool list*) =
bin-to-bl (*len-of TYPE*(′*a*)) *o uint*
by (*auto simp*: *to-bl-def intro*: *ext*)

lemmas *word-reverse-no-def* [*simp*] = *word-reverse-def* [*of number-of w*, *standard*]

lemmas *uints-mod* = *uints-def* [*unfolded no-bintr-alt1*]

lemma *uint-bintrunc*: *uint* (*number-of bin* :: ′*a word*) =
number-of (*bintrunc* (*len-of TYPE* (′*a* :: *len0*)) *bin*)
unfolding *word-number-of-def number-of-eq*
by (*auto intro*: *word-ubin.eq-norm*)

lemma *sint-sbintrunc*: *sint* (*number-of bin* :: ′*a word*) =
number-of (*sbintrunc* (*len-of TYPE* (′*a* :: *len*) − *1*) *bin*)
unfolding *word-number-of-def number-of-eq*
by (*subst word-sbin.eq-norm*) *simp*

lemma *unat-bintrunc*:
  unat (*number-of bin* :: *'a* :: *len0 word*) =
    number-of (*bintrunc* (*len-of TYPE*(*'a*)) *bin*)
  unfolding *unat-def nat-number-of-def*
  by (*simp only*: *uint-bintrunc*)

declare
  uint-bintrunc [*simp*]
  sint-sbintrunc [*simp*]
  unat-bintrunc [*simp*]

lemma *size-0-eq*: *size* (*w* :: *'a* :: *len0 word*) = *0* ==> *v* = *w*
  apply (*unfold word-size*)
  apply (*rule word-uint.Rep-eqD*)
  apply (*rule box-equals*)
    defer
    apply (*rule word-ubin.norm-Rep*)+
  apply *simp*
  done

lemmas *uint-lem* = *word-uint.Rep* [*unfolded uints-num mem-Collect-eq*]
lemmas *sint-lem* = *word-sint.Rep* [*unfolded sints-num mem-Collect-eq*]
lemmas *uint-ge-0* [*iff*] = *uint-lem* [*THEN conjunct1*, *standard*]
lemmas *uint-lt2p* [*iff*] = *uint-lem* [*THEN conjunct2*, *standard*]
lemmas *sint-ge* = *sint-lem* [*THEN conjunct1*, *standard*]
lemmas *sint-lt* = *sint-lem* [*THEN conjunct2*, *standard*]

lemma *sign-uint-Pls* [*simp*]:
  bin-sign (*uint x*) = *Numeral.Pls*
  by (*simp add*: *sign-Pls-ge-0 number-of-eq*)

lemmas *uint-m2p-neg* = *iffD2* [*OF diff-less-0-iff-less uint-lt2p*, *standard*]
lemmas *uint-m2p-not-non-neg* =
  iffD2 [*OF linorder-not-le uint-m2p-neg*, *standard*]

lemma *lt2p-lem*:
  len-of TYPE(*'a*) <= *n* ==> *uint* (*w* :: *'a* :: *len0 word*) < *2* ^ *n*
  by (*rule xtr8* [*OF* - *uint-lt2p*]) *simp*

lemmas *uint-le-0-iff* [*simp*] =
  uint-ge-0 [*THEN leD*, *THEN linorder-antisym-conv1*, *standard*]

lemma *uint-nat*: *uint w* == *int* (*unat w*)
  unfolding *unat-def* **by** *auto*

lemma *uint-number-of*:
  uint (*number-of b* :: *'a* :: *len0 word*) = *number-of b mod 2* ^ *len-of TYPE*(*'a*)
  unfolding *word-number-of-alt*

by (*simp only*: *int-word-uint*)

lemma *unat-number-of*:
bin-sign b = Numeral.Pls ==>
unat (number-of b::'a::len0 word) = number-of b mod 2 ^ len-of TYPE ('a)
apply (*unfold unat-def*)
apply (*clarsimp simp only*: *uint-number-of*)
apply (*rule nat-mod-distrib* [*THEN trans*])
apply (*erule sign-Pls-ge-0* [*THEN iffD1*])
apply (*simp-all add*: *nat-power-eq*)
done

lemma *sint-number-of*: *sint (number-of b :: 'a :: len word) = (number-of b +*
2 ^ (len-of TYPE('a) − 1)) mod 2 ^ len-of TYPE('a) −
2 ^ (len-of TYPE('a) − 1)
unfolding *word-number-of-alt* **by** (*rule int-word-sint*)

lemma *word-of-int-bin* [*simp*] :
(*word-of-int (number-of bin) :: 'a :: len0 word) = (number-of bin)*
unfolding *word-number-of-alt* **by** *auto*

lemma *word-int-case-wi*:
word-int-case f (word-of-int i :: 'b word) =
f (i mod 2 ^ len-of TYPE('b::len0))
unfolding *word-int-case-def* **by** (*simp add*: *word-uint.eq-norm*)

lemma *word-int-split*:
P (word-int-case f x) =
(ALL i. x = (word-of-int i :: 'b :: len0 word) &
0 <= i & i < 2 ^ len-of TYPE('b) −−> P (f i))
unfolding *word-int-case-def*
by (*auto simp*: *word-uint.eq-norm int-mod-eq'*)

lemma *word-int-split-asm*:
P (word-int-case f x) =
(~ (EX n. x = (word-of-int n :: 'b::len0 word) &
0 <= n & n < 2 ^ len-of TYPE('b::len0) & ~ P (f n)))
unfolding *word-int-case-def*
by (*auto simp*: *word-uint.eq-norm int-mod-eq'*)

lemmas *uint-range' =*
word-uint.Rep [*unfolded uints-num mem-Collect-eq*, *standard*]
lemmas *sint-range' = word-sint.Rep* [*unfolded One-nat-def*
sints-num mem-Collect-eq, *standard*]

lemma *uint-range-size*: *0 <= uint w & uint w < 2 ^ size w*
unfolding *word-size* **by** (*rule uint-range'*)

lemma *sint-range-size*:

$- (2 \hat{} (size\ w - Suc\ 0)) <= sint\ w\ \&\ sint\ w < 2 \hat{} (size\ w - Suc\ 0)$
unfolding *word-size* **by** (*rule sint-range'*)

lemmas *sint-above-size* = *sint-range-size*
[*THEN conjunct2*, *THEN* [*2*] *xtr8*, *folded One-nat-def*, *standard*]

lemmas *sint-below-size* = *sint-range-size*
[*THEN conjunct1*, *THEN* [*2*] *order-trans*, *folded One-nat-def*, *standard*]

lemma *test-bit-eq-iff*: (*test-bit* (*u*::'*a*::*len0 word*) = *test-bit v*) = (*u* = *v*)
unfolding *word-test-bit-def* **by** (*simp add*: *bin-nth-eq-iff*)

lemma *test-bit-size* [*rule-format*] : (*w*::'*a*::*len0 word*) !! *n* −−> *n* < *size w*
apply (*unfold word-test-bit-def*)
apply (*subst word-ubin.norm-Rep* [*symmetric*])
apply (*simp only*: *nth-bintr word-size*)
apply *fast*
done

lemma *word-eqI* [*rule-format*] :
fixes *u* :: '*a*::*len0 word*
shows (*ALL n*. *n* < *size u* −−> *u* !! *n* = *v* !! *n*) ==> *u* = *v*
apply (*rule test-bit-eq-iff* [*THEN iffD1*])
apply (*rule ext*)
apply (*erule allE*)
apply (*erule impCE*)
prefer *2*
apply *assumption*
apply (*auto dest*!: *test-bit-size simp add*: *word-size*)
done

lemmas *word-eqD* = *test-bit-eq-iff* [*THEN iffD2*, *THEN fun-cong*, *standard*]

lemma *test-bit-bin'*: *w* !! *n* = (*n* < *size w* & *bin-nth* (*uint w*) *n*)
unfolding *word-test-bit-def word-size*
by (*simp add*: *nth-bintr* [*symmetric*])

lemmas *test-bit-bin* = *test-bit-bin'* [*unfolded word-size*]

lemma *bin-nth-uint-imp'*: *bin-nth* (*uint w*) *n* −−> *n* < *size w*
apply (*unfold word-size*)
apply (*rule impI*)
apply (*rule nth-bintr* [*THEN iffD1*, *THEN conjunct1*])
apply (*subst word-ubin.norm-Rep*)
apply *assumption*
done

lemma *bin-nth-sint'*:
n >= *size w* −−> *bin-nth* (*sint w*) *n* = *bin-nth* (*sint w*) (*size w* − *1*)

apply (*rule impI*)
apply (*subst word-sbin.norm-Rep* [*symmetric*])
apply (*simp add* : *nth-sbintr word-size*)
apply *auto*
done

lemmas *bin-nth-uint-imp* = *bin-nth-uint-imp'* [*rule-format*, *unfolded word-size*]
lemmas *bin-nth-sint* = *bin-nth-sint'* [*rule-format*, *unfolded word-size*]

lemma *td-bl*:
type-definition (*to-bl* :: *'a*::*len0 word* => *bool list*)
of-bl
{*bl*. *length bl* = *len-of TYPE*(*'a*)}
apply (*unfold type-definition-def of-bl-def to-bl-def*)
apply (*simp add*: *word-ubin.eq-norm*)
apply *safe*
apply (*drule sym*)
apply *simp*
done

interpretation *word-bl*:
type-definition [*to-bl* :: *'a*::*len0 word* => *bool list*
of-bl
{*bl*. *length bl* = *len-of TYPE*(*'a*::*len0*)}]
by (*rule td-bl*)

lemma *word-size-bl*: *size w* == *size* (*to-bl w*)
unfolding *word-size* **by** *auto*

lemma *to-bl-use-of-bl*:
(*to-bl w* = *bl*) = (*w* = *of-bl bl* ∧ *length bl* = *length* (*to-bl w*))
by (*fastsimp elim*!: *word-bl.Abs-inverse* [*simplified*])

lemma *to-bl-word-rev*: *to-bl* (*word-reverse w*) = *rev* (*to-bl w*)
unfolding *word-reverse-def* **by** (*simp add*: *word-bl.Abs-inverse*)

lemma *word-rev-rev* [*simp*] : *word-reverse* (*word-reverse w*) = *w*
unfolding *word-reverse-def* **by** (*simp add* : *word-bl.Abs-inverse*)

lemma *word-rev-gal*: *word-reverse w* = *u* ==> *word-reverse u* = *w*
by *auto*

lemmas *word-rev-gal'* = *sym* [*THEN word-rev-gal*, *symmetric*, *standard*]

lemmas *length-bl-gt-0* [*iff*] = *xtr1* [*OF word-bl.Rep' len-gt-0*, *standard*]
lemmas *bl-not-Nil* [*iff*] =
length-bl-gt-0 [*THEN length-greater-0-conv* [*THEN iffD1*], *standard*]
lemmas *length-bl-neq-0* [*iff*] = *length-bl-gt-0* [*THEN gr-implies-not0*]

lemma *hd-bl-sign-sint*: *hd* (*to-bl* *w*) = (*bin-sign* (*sint* *w*) = *Numeral.Min*)
  apply (*unfold to-bl-def sint-uint*)
  apply (*rule trans* [*OF* - *bl-sbin-sign*])
  apply *simp*
  done

lemma *of-bl-drop′*:
  lend = *length bl* − *len-of TYPE* (′*a* :: *len0*) ==>
    of-bl (*drop lend bl*) = (*of-bl bl* :: ′*a word*)
  apply (*unfold of-bl-def*)
  apply (*clarsimp simp add* : *trunc-bl2bin* [*symmetric*])
  done

lemmas *of-bl-no* = *of-bl-def* [*folded word-number-of-def*]

lemma *test-bit-of-bl*:
  (*of-bl bl*::′*a*::*len0 word*) !! *n* = (*rev bl* ! *n* ∧ *n* < *len-of TYPE*(′*a*) ∧ *n* < *length bl*)
  apply (*unfold of-bl-def word-test-bit-def*)
  apply (*auto simp add*: *word-size word-ubin.eq-norm nth-bintr bin-nth-of-bl*)
  done

lemma *no-of-bl*:
  (*number-of bin* ::′*a*::*len0 word*) = *of-bl* (*bin-to-bl* (*len-of TYPE* (′*a*)) *bin*)
  unfolding *word-size of-bl-no* **by** (*simp add* : *word-number-of-def*)

lemma *uint-bl*: *to-bl w* == *bin-to-bl* (*size w*) (*uint w*)
  unfolding *word-size to-bl-def* **by** *auto*

lemma *to-bl-bin*: *bl-to-bin* (*to-bl w*) = *uint w*
  unfolding *uint-bl* **by** (*simp add* : *word-size*)

lemma *to-bl-of-bin*:
  to-bl (*word-of-int bin*::′*a*::*len0 word*) = *bin-to-bl* (*len-of TYPE*(′*a*)) *bin*
  unfolding *uint-bl* **by** (*clarsimp simp add*: *word-ubin.eq-norm word-size*)

lemmas *to-bl-no-bin* [*simp*] = *to-bl-of-bin* [*folded word-number-of-def*]

lemma *to-bl-to-bin* [*simp*] : *bl-to-bin* (*to-bl w*) = *uint w*
  unfolding *uint-bl* **by** (*simp add* : *word-size*)

lemmas *uint-bl-bin* [*simp*] = *trans* [*OF bin-bl-bin word-ubin.norm-Rep*, *standard*]

lemmas *num-AB-u* [*simp*] = *word-uint.Rep-inverse*
  [*unfolded o-def word-number-of-def* [*symmetric*], *standard*]
lemmas *num-AB-s* [*simp*] = *word-sint.Rep-inverse*
  [*unfolded o-def word-number-of-def* [*symmetric*], *standard*]

lemma *uints-unats*: *uints n* = *int* ' *unats n*
  apply (*unfold unats-def uints-num*)
  apply *safe*
  apply (*rule-tac image-eqI*)
  apply (*erule-tac nat-0-le* [*symmetric*])
  apply *auto*
  apply (*erule-tac nat-less-iff* [*THEN iffD2*])
  apply (*rule-tac* [*2*] *zless-nat-eq-int-zless* [*THEN iffD1*])
  apply (*auto simp add* : *nat-power-eq int-power*)
  done

lemma *unats-uints*: *unats n* = *nat* ' *uints n*
  by (*auto simp add* : *uints-unats image-iff*)

lemmas *bintr-num* = *word-ubin.norm-eq-iff*
  [*symmetric*, *folded word-number-of-def*, *standard*]
lemmas *sbintr-num* = *word-sbin.norm-eq-iff*
  [*symmetric*, *folded word-number-of-def*, *standard*]

lemmas *num-of-bintr* = *word-ubin.Abs-norm* [*folded word-number-of-def*, *standard*]
lemmas *num-of-sbintr* = *word-sbin.Abs-norm* [*folded word-number-of-def*, *standard*]

lemma *num-of-bintr'*:
  bintrunc (*len-of TYPE*(*'a* :: *len0*)) *a* = *b* ==>
    number-of a = (*number-of b* :: *'a word*)
  apply *safe*
  apply (*rule-tac num-of-bintr* [*symmetric*])
  done

lemma *num-of-sbintr'*:
  sbintrunc (*len-of TYPE*(*'a* :: *len*) − *1*) *a* = *b* ==>
    number-of a = (*number-of b* :: *'a word*)
  apply *safe*
  apply (*rule-tac num-of-sbintr* [*symmetric*])
  done

lemmas *num-abs-bintr* = *sym* [*THEN trans*,
  OF num-of-bintr word-number-of-def [*THEN meta-eq-to-obj-eq*], *standard*]
lemmas *num-abs-sbintr* = *sym* [*THEN trans*,
  OF num-of-sbintr word-number-of-def [*THEN meta-eq-to-obj-eq*], *standard*]

lemma *ucast-id*: *ucast w* = *w*

unfolding *ucast-def* **by** *auto*

lemma *scast-id*: *scast w* = *w*
unfolding *scast-def* **by** *auto*

lemma *ucast-bl*: *ucast w* == *of-bl* (*to-bl w*)
unfolding *ucast-def of-bl-def uint-bl*
by (*auto simp add* : *word-size*)

lemma *nth-ucast*:
(*ucast w*::′*a*::*len0 word*) !! *n* = (*w* !! *n* & *n* < *len-of TYPE*(′*a*))
apply (*unfold ucast-def test-bit-bin*)
apply (*simp add*: *word-ubin.eq-norm nth-bintr word-size*)
apply (*fast elim*!: *bin-nth-uint-imp*)
done

lemma *ucast-bintr* [*simp*]:
ucast (*number-of w* ::′*a*::*len0 word*) =
number-of (*bintrunc* (*len-of TYPE*(′*a*)) *w*)
unfolding *ucast-def* **by** *simp*

lemma *scast-sbintr* [*simp*]:
scast (*number-of w* ::′*a*::*len word*) =
number-of (*sbintrunc* (*len-of TYPE*(′*a*) − *Suc 0*) *w*)
unfolding *scast-def* **by** *simp*

lemmas *source-size* = *source-size-def* [*unfolded Let-def word-size*]
lemmas *target-size* = *target-size-def* [*unfolded Let-def word-size*]
lemmas *is-down* = *is-down-def* [*unfolded source-size target-size*]
lemmas *is-up* = *is-up-def* [*unfolded source-size target-size*]

lemmas *is-up-down* =
trans [*OF is-up* [*THEN meta-eq-to-obj-eq*]
is-down [*THEN meta-eq-to-obj-eq*, *symmetric*],
standard]

lemma *down-cast-same*′: *uc* = *ucast* ==> *is-down uc* ==> *uc* = *scast*
apply (*unfold is-down*)
apply *safe*
apply (*rule ext*)
apply (*unfold ucast-def scast-def uint-sint*)
apply (*rule word-ubin.norm-eq-iff* [*THEN iffD1*])
apply *simp*
done

lemma *word-rev-tf*′:
r = *to-bl* (*of-bl bl*) ==> *r* = *rev* (*takefill False* (*length r*) (*rev bl*))

```
  unfolding of-bl-def uint-bl
  by (clarsimp simp add: bl-bin-bl-rtf word-ubin.eq-norm word-size)

lemmas word-rev-tf = refl [THEN word-rev-tf', unfolded word-bl.Rep', standard]

lemmas word-rep-drop = word-rev-tf [simplified takefill-alt,
  simplified, simplified rev-take, simplified]

lemma to-bl-ucast:
  to-bl (ucast (w::'b::len0 word) ::'a::len0 word) =
   replicate (len-of TYPE('a) - len-of TYPE('b)) False @
   drop (len-of TYPE('b) - len-of TYPE('a)) (to-bl w)
  apply (unfold ucast-bl)
  apply (rule trans)
   apply (rule word-rep-drop)
  apply simp
  done

lemma ucast-up-app':
  uc = ucast ==> source-size uc + n = target-size uc ==>
    to-bl (uc w) = replicate n False @ (to-bl w)
  apply (auto simp add : source-size target-size to-bl-ucast)
  apply (rule-tac f = %n. replicate n False in arg-cong)
  apply simp
  done

lemma ucast-down-drop':
  uc = ucast ==> source-size uc = target-size uc + n ==>
    to-bl (uc w) = drop n (to-bl w)
  by (auto simp add : source-size target-size to-bl-ucast)

lemma scast-down-drop':
  sc = scast ==> source-size sc = target-size sc + n ==>
    to-bl (sc w) = drop n (to-bl w)
  apply (subgoal-tac sc = ucast)
   apply safe
   apply simp
   apply (erule refl [THEN ucast-down-drop'])
  apply (rule refl [THEN down-cast-same', symmetric])
  apply (simp add : source-size target-size is-down)
  done

lemma sint-up-scast':
  sc = scast ==> is-up sc ==> sint (sc w) = sint w
  apply (unfold is-up)
  apply safe
  apply (simp add: scast-def word-sbin.eq-norm)
  apply (rule box-equals)
    prefer 3
```

apply (*rule word-sbin.norm-Rep*)
apply (*rule sbintrunc-sbintrunc-l*)
defer
apply (*subst word-sbin.norm-Rep*)
apply (*rule refl*)
apply *simp*
done

lemma *uint-up-ucast'*:
uc = *ucast* ==> *is-up uc* ==> *uint* (*uc w*) = *uint w*
apply (*unfold is-up*)
apply *safe*
apply (*rule bin-eqI*)
apply (*fold word-test-bit-def*)
apply (*auto simp add*: *nth-ucast*)
apply (*auto simp add*: *test-bit-bin*)
done

lemmas *down-cast-same* = *refl* [*THEN down-cast-same'*]
lemmas *ucast-up-app* = *refl* [*THEN ucast-up-app'*]
lemmas *ucast-down-drop* = *refl* [*THEN ucast-down-drop'*]
lemmas *scast-down-drop* = *refl* [*THEN scast-down-drop'*]
lemmas *uint-up-ucast* = *refl* [*THEN uint-up-ucast'*]
lemmas *sint-up-scast* = *refl* [*THEN sint-up-scast'*]

lemma *ucast-up-ucast'*: *uc* = *ucast* ==> *is-up uc* ==> *ucast* (*uc w*) = *ucast w*
apply (*simp* (*no-asm*) *add*: *ucast-def*)
apply (*clarsimp simp add*: *uint-up-ucast*)
done

lemma *scast-up-scast'*: *sc* = *scast* ==> *is-up sc* ==> *scast* (*sc w*) = *scast w*
apply (*simp* (*no-asm*) *add*: *scast-def*)
apply (*clarsimp simp add*: *sint-up-scast*)
done

lemma *ucast-of-bl-up'*:
w = *of-bl bl* ==> *size bl* <= *size w* ==> *ucast w* = *of-bl bl*
by (*auto simp add* : *nth-ucast word-size test-bit-of-bl intro*!: *word-eqI*)

lemmas *ucast-up-ucast* = *refl* [*THEN ucast-up-ucast'*]
lemmas *scast-up-scast* = *refl* [*THEN scast-up-scast'*]
lemmas *ucast-of-bl-up* = *refl* [*THEN ucast-of-bl-up'*]

lemmas *ucast-up-ucast-id* = *trans* [*OF ucast-up-ucast ucast-id*]
lemmas *scast-up-scast-id* = *trans* [*OF scast-up-scast scast-id*]

lemmas *isduu* = *is-up-down* [**where** *c* = *ucast*, *THEN iffD2*]
lemmas *isdus* = *is-up-down* [**where** *c* = *scast*, *THEN iffD2*]
lemmas *ucast-down-ucast-id* = *isduu* [*THEN ucast-up-ucast-id*]

lemmas *scast-down-scast-id* = *isdus* [*THEN ucast-up-ucast-id*]

lemma *up-ucast-surj*:
is-up (*ucast* :: *'b*::*len0 word* => *'a*::*len0 word*) ==>
surj (*ucast* :: *'a word* => *'b word*)
by (*rule surjI*, *erule ucast-up-ucast-id*)

lemma *up-scast-surj*:
is-up (*scast* :: *'b*::*len word* => *'a*::*len word*) ==>
surj (*scast* :: *'a word* => *'b word*)
by (*rule surjI*, *erule scast-up-scast-id*)

lemma *down-scast-inj*:
is-down (*scast* :: *'b*::*len word* => *'a*::*len word*) ==>
inj-on (*ucast* :: *'a word* => *'b word*) *A*
by (*rule inj-on-inverseI*, *erule scast-down-scast-id*)

lemma *down-ucast-inj*:
is-down (*ucast* :: *'b*::*len0 word* => *'a*::*len0 word*) ==>
inj-on (*ucast* :: *'a word* => *'b word*) *A*
by (*rule inj-on-inverseI*, *erule ucast-down-ucast-id*)

lemma *of-bl-append-same*: *of-bl* (*X* @ *to-bl w*) = *w*
by (*rule word-bl.Rep-eqD*) (*simp add*: *word-rep-drop*)

lemma *ucast-down-no'*:
uc = *ucast* ==> *is-down uc* ==> *uc* (*number-of bin*) = *number-of bin*
apply (*unfold word-number-of-def is-down*)
apply (*clarsimp simp add*: *ucast-def word-ubin.eq-norm*)
apply (*rule word-ubin.norm-eq-iff* [*THEN iffD1*])
apply (*erule bintrunc-bintrunc-ge*)
done

lemmas *ucast-down-no* = *ucast-down-no'* [*OF refl*]

lemma *ucast-down-bl'*: *uc* = *ucast* ==> *is-down uc* ==> *uc* (*of-bl bl*) = *of-bl bl*
unfolding *of-bl-no* **by** *clarify* (*erule ucast-down-no*)

lemmas *ucast-down-bl* = *ucast-down-bl'* [*OF refl*]

lemmas *slice-def'* = *slice-def* [*unfolded word-size*]
lemmas *test-bit-def'* = *word-test-bit-def* [*THEN meta-eq-to-obj-eq*, *THEN fun-cong*]

lemmas *word-log-defs* = *word-and-def word-or-def word-xor-def word-not-def*
lemmas *word-log-bin-defs* = *word-log-defs*

end

11 WordArith: Word Arithmetic

theory *WordArith* **imports** *WordDefinition* **begin**

lemma *word-less-alt*: $(a < b) = (uint\ a < uint\ b)$
  unfolding *word-less-def word-le-def*
  by (*auto simp del*: *word-uint.Rep-inject*
      simp: *word-uint.Rep-inject* [*symmetric*])

lemma *signed-linorder*: *linorder word-sle word-sless*
  apply *unfold-locales*
    apply (*unfold word-sle-def word-sless-def*)
  by *auto*

interpretation *signed*: *linorder* [*word-sle word-sless*]
  by (*rule signed-linorder*)

lemmas *word-arith-wis* [*THEN meta-eq-to-obj-eq*] =
  word-add-def word-mult-def word-minus-def
  word-succ-def word-pred-def word-0-wi word-1-wi

lemma *udvdI*:
  $0 \leq n \Longrightarrow uint\ b = n * uint\ a \Longrightarrow a\ udvd\ b$
  by (*auto simp*: *udvd-def*)

lemmas *word-div-no* [*simp*] =
  word-div-def [*of number-of a number-of b*, *standard*]

lemmas *word-mod-no* [*simp*] =
  word-mod-def [*of number-of a number-of b*, *standard*]

lemmas *word-less-no* [*simp*] =
  word-less-def [*of number-of a number-of b*, *standard*]

lemmas *word-le-no* [*simp*] =
  word-le-def [*of number-of a number-of b*, *standard*]

lemmas *word-sless-no* [*simp*] =
  word-sless-def [*of number-of a number-of b*, *standard*]

lemmas *word-sle-no* [*simp*] =
  word-sle-def [*of number-of a number-of b*, *standard*]

lemmas *word-0-wi-Pls* = *word-0-wi* [*folded Pls-def*]
lemmas *word-0-no* = *word-0-wi-Pls* [*folded word-no-wi*]

lemma *int-one-bin*: $(1 :: int) == (Numeral.Pls\ BIT\ bit.B1)$

unfolding *Pls-def Bit-def* **by** *auto*

lemma *word-1-no*:
$(1 :: 'a :: len0\ word) == number\text{-}of\ (Numeral.Pls\ BIT\ bit.B1)$
unfolding *word-1-wi word-number-of-def int-one-bin* **by** *auto*

lemma *word-m1-wi*: $-1 == word\text{-}of\text{-}int\ -1$
by (*rule word-number-of-alt*)

lemma *word-m1-wi-Min*: $-1 = word\text{-}of\text{-}int\ Numeral.Min$
by (*simp add*: *word-m1-wi number-of-eq*)

lemma *word-0-bl*: $of\text{-}bl\ [] = 0$
unfolding *word-0-wi of-bl-def* **by** (*simp add* : *Pls-def*)

lemma *word-1-bl*: $of\text{-}bl\ [True] = 1$
unfolding *word-1-wi of-bl-def*
by (*simp add* : *bl-to-bin-def Bit-def Pls-def*)

lemma *uint-0* [*simp*] : $(uint\ 0 = 0)$
unfolding *word-0-wi*
by (*simp add*: *word-ubin.eq-norm Pls-def* [*symmetric*])

lemma *of-bl-0* [*simp*] : $of\text{-}bl\ (replicate\ n\ False) = 0$
by (*simp add* : *word-0-wi of-bl-def bl-to-bin-rep-False Pls-def*)

lemma *to-bl-0*:
$to\text{-}bl\ (0::'a::len0\ word) = replicate\ (len\text{-}of\ TYPE('a))\ False$
unfolding *uint-bl*
by (*simp add* : *word-size bin-to-bl-Pls Pls-def* [*symmetric*])

lemma *uint-0-iff*: $(uint\ x = 0) = (x = 0)$
by (*auto intro*!: *word-uint.Rep-eqD*)

lemma *unat-0-iff*: $(unat\ x = 0) = (x = 0)$
unfolding *unat-def* **by** (*auto simp add* : *nat-eq-iff uint-0-iff*)

lemma *unat-0* [*simp*]: $unat\ 0 = 0$
unfolding *unat-def* **by** *auto*

lemma *size-0-same'*: $size\ w = 0 ==> w = (v :: 'a :: len0\ word)$
apply (*unfold word-size*)
apply (*rule box-equals*)
  defer
  apply (*rule word-uint.Rep-inverse*)+
apply (*rule word-ubin.norm-eq-iff* [*THEN iffD1*])
apply *simp*
done

lemmas *size-0-same* = *size-0-same′* [*folded word-size*]

lemmas *unat-eq-0* = *unat-0-iff*
lemmas *unat-eq-zero* = *unat-0-iff*

lemma *unat-gt-0*: (*0* < *unat x*) = (*x* ~= *0*)
by (*auto simp*: *unat-0-iff* [*symmetric*])

lemma *ucast-0* [*simp*] : *ucast 0* = *0*
unfolding *ucast-def*
by *simp* (*simp add*: *word-0-wi*)

lemma *sint-0* [*simp*] : *sint 0* = *0*
unfolding *sint-uint*
by (*simp add*: *Pls-def* [*symmetric*])

lemma *scast-0* [*simp*] : *scast 0* = *0*
apply (*unfold scast-def*)
apply *simp*
apply (*simp add*: *word-0-wi*)
done

lemma *sint-n1* [*simp*] : *sint* −*1* = −*1*
apply (*unfold word-m1-wi-Min*)
apply (*simp add*: *word-sbin.eq-norm*)
apply (*unfold Min-def number-of-eq*)
apply *simp*
done

lemma *scast-n1* [*simp*] : *scast* −*1* = −*1*
  apply (*unfold scast-def sint-n1*)
  apply (*unfold word-number-of-alt*)
  apply (*rule refl*)
  done

lemma *uint-1* [*simp*] : *uint* (*1* :: ′*a* :: *len word*) = *1*
  unfolding *word-1-wi*
  by (*simp add*: *word-ubin.eq-norm int-one-bin bintrunc-minus-simps*)

lemma *unat-1* [*simp*] : *unat* (*1* :: ′*a* :: *len word*) = *1*
  by (*unfold unat-def uint-1*) *auto*

lemma *ucast-1* [*simp*] : *ucast* (*1* :: ′*a* :: *len word*) = *1*
  unfolding *ucast-def word-1-wi*
  by (*simp add*: *word-ubin.eq-norm int-one-bin bintrunc-minus-simps*)

lemmas *arths* =

bintr-ariths [*THEN word-ubin.norm-eq-iff* [*THEN iffD1*],
folded word-ubin.eq-norm, *standard*]

lemma *wi-homs*:
shows
wi-hom-add: *word-of-int a* + *word-of-int b* = *word-of-int* (*a* + *b*) **and**
wi-hom-mult: *word-of-int a* ∗ *word-of-int b* = *word-of-int* (*a* ∗ *b*) **and**
wi-hom-neg: − *word-of-int a* = *word-of-int* (− *a*) **and**
wi-hom-succ: *word-succ* (*word-of-int a*) = *word-of-int* (*Numeral.succ a*) **and**
wi-hom-pred: *word-pred* (*word-of-int a*) = *word-of-int* (*Numeral.pred a*)
by (*auto simp*: *word-arith-wis arths*)

lemmas *wi-hom-syms* = *wi-homs* [*symmetric*]

lemma *word-sub-def*: *a* − *b* == *a* + − (*b* :: *'a* :: *len0 word*)
unfolding *word-sub-wi diff-def*
by (*simp only* : *word-uint.Rep-inverse wi-hom-syms*)

lemmas *word-diff-minus* = *word-sub-def* [*THEN meta-eq-to-obj-eq*, *standard*]

lemma *word-of-int-sub-hom*:
(*word-of-int a*) − *word-of-int b* = *word-of-int* (*a* − *b*)
unfolding *word-sub-def diff-def* **by** (*simp only* : *wi-homs*)

lemmas *new-word-of-int-homs* =
word-of-int-sub-hom wi-homs word-0-wi word-1-wi

lemmas *new-word-of-int-hom-syms* = *new-word-of-int-homs* [*symmetric*, *standard*]

lemmas *word-of-int-hom-syms* =
new-word-of-int-hom-syms [*unfolded succ-def pred-def*]

lemmas *word-of-int-homs* =
new-word-of-int-homs [*unfolded succ-def pred-def*]

lemmas *word-of-int-add-hom* = *word-of-int-homs* (*2*)
lemmas *word-of-int-mult-hom* = *word-of-int-homs* (*3*)
lemmas *word-of-int-minus-hom* = *word-of-int-homs* (*4*)
lemmas *word-of-int-succ-hom* = *word-of-int-homs* (*5*)
lemmas *word-of-int-pred-hom* = *word-of-int-homs* (*6*)
lemmas *word-of-int-0-hom* = *word-of-int-homs* (*7*)
lemmas *word-of-int-1-hom* = *word-of-int-homs* (*8*)

lemmas *word-arith-alts* =
word-sub-wi [*unfolded succ-def pred-def*, *THEN meta-eq-to-obj-eq*, *standard*]
word-arith-wis [*unfolded succ-def pred-def*, *standard*]

lemmas *word-sub-alt* = *word-arith-alts* (*1*)
lemmas *word-add-alt* = *word-arith-alts* (*2*)
lemmas *word-mult-alt* = *word-arith-alts* (*3*)
lemmas *word-minus-alt* = *word-arith-alts* (*4*)
lemmas *word-succ-alt* = *word-arith-alts* (*5*)
lemmas *word-pred-alt* = *word-arith-alts* (*6*)
lemmas *word-0-alt* = *word-arith-alts* (*7*)
lemmas *word-1-alt* = *word-arith-alts* (*8*)

11.1 Transferring goals from words to ints

lemma *word-ths*:
shows
word-succ-p1: *word-succ a* = *a* + *1* **and**
word-pred-m1: *word-pred a* = *a* − *1* **and**
word-pred-succ: *word-pred* (*word-succ a*) = *a* **and**
word-succ-pred: *word-succ* (*word-pred a*) = *a* **and**
word-mult-succ: *word-succ a* ∗ *b* = *b* + *a* ∗ *b*
by (*rule word-uint.Abs-cases* [*of b*],
rule word-uint.Abs-cases [*of a*],
simp add: *pred-def succ-def add-commute mult-commute*
ring-distribs new-word-of-int-homs)+

lemmas *uint-cong* = *arg-cong* [**where** *f* = *uint*]

lemmas *uint-word-ariths* =
word-arith-alts [*THEN trans* [*OF uint-cong int-word-uint*], *standard*]

lemmas *uint-word-arith-bintrs* = *uint-word-ariths* [*folded bintrunc-mod2p*]

lemmas *sint-word-ariths* = *uint-word-arith-bintrs*
[*THEN uint-sint* [*symmetric*, *THEN trans*],
unfolded uint-sint bintr-arith1s bintr-ariths
len-gt-0 [*THEN bin-sbin-eq-iff'*] *word-sbin.norm-Rep*, *standard*]

lemmas *uint-div-alt* = *word-div-def*
[*THEN meta-eq-to-obj-eq* [*THEN trans* [*OF uint-cong int-word-uint*]], *standard*]
lemmas *uint-mod-alt* = *word-mod-def*
[*THEN meta-eq-to-obj-eq* [*THEN trans* [*OF uint-cong int-word-uint*]], *standard*]

lemma *word-pred-0-n1*: *word-pred 0* = *word-of-int* −*1*
unfolding *word-pred-def number-of-eq*
by (*simp add* : *pred-def word-no-wi*)

lemma *word-pred-0-Min*: *word-pred 0* = *word-of-int Numeral.Min*
by (*simp add*: *word-pred-0-n1 number-of-eq*)

lemma *word-m1-Min*: − *1* = *word-of-int Numeral.Min*

unfolding *Min-def* **by** (*simp only*: *word-of-int-hom-syms*)

lemma *succ-pred-no* [*simp*]:
word-succ (*number-of bin*) = *number-of* (*Numeral.succ bin*) &
word-pred (*number-of bin*) = *number-of* (*Numeral.pred bin*)
unfolding *word-number-of-def* **by** (*simp add* : *new-word-of-int-homs*)

lemma *word-sp-01* [*simp*] :
word-succ -1 = *0* & *word-succ 0* = *1* & *word-pred 0* = -1 & *word-pred 1* = *0*
by (*unfold word-0-no word-1-no*) *auto*

lemma *word-of-int-Ex*:
$\exists\, y.\ x = $ *word-of-int y*
by (*rule-tac x=uint x* **in** *exI*) *simp*

lemma *word-arith-eqs*:
fixes *a* :: *'a::len0 word*
fixes *b* :: *'a::len0 word*
shows
word-add-0: $0 + a = a$ **and**
word-add-0-right: $a + 0 = a$ **and**
word-mult-1: $1 * a = a$ **and**
word-mult-1-right: $a * 1 = a$ **and**
word-add-commute: $a + b = b + a$ **and**
word-add-assoc: $a + b + c = a + (b + c)$ **and**
word-add-left-commute: $a + (b + c) = b + (a + c)$ **and**
word-mult-commute: $a * b = b * a$ **and**
word-mult-assoc: $a * b * c = a * (b * c)$ **and**
word-mult-left-commute: $a * (b * c) = b * (a * c)$ **and**
word-left-distrib: $(a + b) * c = a * c + b * c$ **and**
word-right-distrib: $a * (b + c) = a * b + a * c$ **and**
word-left-minus: $- a + a = 0$ **and**
word-diff-0-right: $a - 0 = a$ **and**
word-diff-self: $a - a = 0$
using *word-of-int-Ex* [*of a*]
word-of-int-Ex [*of b*]
word-of-int-Ex [*of c*]
by (*auto simp*: *word-of-int-hom-syms* [*symmetric*]
zadd-0-right add-commute add-assoc add-left-commute
mult-commute mult-assoc mult-left-commute
plus-times.left-distrib plus-times.right-distrib)

lemmas *word-add-ac* = *word-add-commute word-add-assoc word-add-left-commute*
lemmas *word-mult-ac* = *word-mult-commute word-mult-assoc word-mult-left-commute*

lemmas *word-plus-ac0* = *word-add-0 word-add-0-right word-add-ac*
lemmas *word-times-ac1* = *word-mult-1 word-mult-1-right word-mult-ac*

11.2 Order on fixed-length words

lemma *word-order-trans*: $x <= y ==> y <= z ==> x <= (z :: 'a :: len0\ word)$
unfolding *word-le-def* **by** *auto*

lemma *word-order-refl*: $z <= (z :: 'a :: len0\ word)$
unfolding *word-le-def* **by** *auto*

lemma *word-order-antisym*: $x <= y ==> y <= x ==> x = (y :: 'a :: len0\ word)$
unfolding *word-le-def* **by** (*auto intro*!: *word-uint.Rep-eqD*)

lemma *word-order-linear*:
$y <= x \mid x <= (y :: 'a :: len0\ word)$
unfolding *word-le-def* **by** *auto*

lemma *word-zero-le* [*simp*] :
$0 <= (y :: 'a :: len0\ word)$
unfolding *word-le-def* **by** *auto*

instance *word* :: (*len0*) *semigroup-add*
by *intro-classes* (*simp add*: *word-add-assoc*)

instance *word* :: (*len0*) *linorder*
by *intro-classes* (*auto simp*: *word-less-def word-le-def*)

instance *word* :: (*len0*) *ring*
by *intro-classes*
(*auto simp*: *word-arith-eqs word-diff-minus*
word-diff-self [*unfolded word-diff-minus*])

lemma *word-m1-ge* [*simp*] : *word-pred 0* $>= y$
unfolding *word-le-def*
by (*simp only* : *word-pred-0-n1 word-uint.eq-norm m1mod2k*) *auto*

lemmas *word-n1-ge* [*simp*] = *word-m1-ge* [*simplified word-sp-01*]

lemmas *word-not-simps* [*simp*] =
word-zero-le [*THEN leD*] *word-m1-ge* [*THEN leD*] *word-n1-ge* [*THEN leD*]

lemma *word-gt-0*: $0 < y = (0\ \tilde{}= (y :: 'a :: len0\ word))$
unfolding *word-less-def* **by** *auto*

lemmas *word-gt-0-no* [*simp*] = *word-gt-0* [*of number-of y*, *standard*]

lemma *word-sless-alt*: $(a <s\ b) == (sint\ a < sint\ b)$
unfolding *word-sle-def word-sless-def*
by (*auto simp add* : *less-eq-less.less-le*)

lemma *word-le-nat-alt*: $(a <= b) = (unat\ a <= unat\ b)$
unfolding *unat-def word-le-def*

by (*rule nat-le-eq-zle* [*symmetric*]) *simp*

lemma *word-less-nat-alt*: $(a < b) = (unat\ a < unat\ b)$
unfolding *unat-def word-less-alt*
by (*rule nat-less-eq-zless* [*symmetric*]) *simp*

lemma *wi-less*:
$(word\text{-}of\text{-}int\ n < (word\text{-}of\text{-}int\ m :: 'a :: len0\ word)) =$
$(n\ mod\ 2\ \hat{}\ len\text{-}of\ TYPE('a) < m\ mod\ 2\ \hat{}\ len\text{-}of\ TYPE('a))$
unfolding *word-less-alt* **by** (*simp add*: *word-uint.eq-norm*)

lemma *wi-le*:
$(word\text{-}of\text{-}int\ n <= (word\text{-}of\text{-}int\ m :: 'a :: len0\ word)) =$
$(n\ mod\ 2\ \hat{}\ len\text{-}of\ TYPE('a) <= m\ mod\ 2\ \hat{}\ len\text{-}of\ TYPE('a))$
unfolding *word-le-def* **by** (*simp add*: *word-uint.eq-norm*)

lemma *udvd-nat-alt*: $a\ udvd\ b = (EX\ n{>=}0.\ unat\ b = n * unat\ a)$
apply (*unfold udvd-def*)
apply *safe*
apply (*simp add*: *unat-def nat-mult-distrib*)
apply (*simp add*: *uint-nat int-mult*)
apply (*rule exI*)
apply *safe*
prefer *2*
apply (*erule notE*)
apply (*rule refl*)
apply *force*
done

lemma *udvd-iff-dvd*: $x\ udvd\ y <-> unat\ x\ dvd\ unat\ y$
unfolding *dvd-def udvd-nat-alt* **by** *force*

lemmas *unat-mono* = *word-less-nat-alt* [*THEN iffD1*, *standard*]

lemma *word-zero-neq-one*: $0 < len\text{-}of\ TYPE\ ('a :: len0) ==> (0 :: 'a\ word)\ \tilde{}= 1$
unfolding *word-arith-wis*
by (*auto simp add*: *word-ubin.norm-eq-iff* [*symmetric*] *gr0-conv-Suc*)

lemmas *lenw1-zero-neq-one* = *len-gt-0* [*THEN word-zero-neq-one*]

lemma *no-no* [*simp*] : *number-of* (*number-of b*) = *number-of b*
by (*simp add*: *number-of-eq*)

lemma *unat-minus-one*: $x\ \tilde{}= 0 ==> unat\ (x - 1) = unat\ x - 1$
apply (*unfold unat-def*)
apply (*simp only*: *int-word-uint word-arith-alts rdmods*)
apply (*subgoal-tac uint x >= 1*)
prefer *2*

```
   apply (drule contrapos-nn)
    apply (erule word-uint.Rep-inverse' [symmetric])
   apply (insert uint-ge-0 [of x])[1]
   apply arith
  apply (rule box-equals)
    apply (rule nat-diff-distrib)
     prefer 2
     apply assumption
    apply simp
   apply (subst mod-pos-pos-trivial)
     apply arith
    apply (insert uint-lt2p [of x])[1]
    apply arith
   apply (rule refl)
  apply simp
  done

lemma measure-unat: p ~= 0 ==> unat (p - 1) < unat p
  by (simp add: unat-minus-one) (simp add: unat-0-iff [symmetric])

lemmas uint-add-ge0 [simp] =
  add-nonneg-nonneg [OF uint-ge-0 uint-ge-0, standard]
lemmas uint-mult-ge0 [simp] =
  mult-nonneg-nonneg [OF uint-ge-0 uint-ge-0, standard]

lemma uint-sub-lt2p [simp]:
  uint (x :: 'a :: len0 word) - uint (y :: 'b :: len0 word) <
    2 ^ len-of TYPE('a)
  using uint-ge-0 [of y] uint-lt2p [of x] by arith
```

11.3 Conditions for the addition (etc) of two words to overflow

```
lemma uint-add-lem:
  (uint x + uint y < 2 ^ len-of TYPE('a)) =
    (uint (x + y :: 'a :: len0 word) = uint x + uint y)
  by (unfold uint-word-ariths) (auto intro!: trans [OF - int-mod-lem])

lemma uint-mult-lem:
  (uint x * uint y < 2 ^ len-of TYPE('a)) =
    (uint (x * y :: 'a :: len0 word) = uint x * uint y)
  by (unfold uint-word-ariths) (auto intro!: trans [OF - int-mod-lem])

lemma uint-sub-lem:
  (uint x >= uint y) = (uint (x - y) = uint x - uint y)
  by (unfold uint-word-ariths) (auto intro!: trans [OF - int-mod-lem])

lemma uint-add-le: uint (x + y) <= uint x + uint y
  unfolding uint-word-ariths by (auto simp: mod-add-if-z)
```

lemma *uint-sub-ge*: *uint* $(x - y) >= $ *uint* $x -$ *uint* y
  unfolding *uint-word-ariths* **by** (*auto simp*: *mod-sub-if-z*)

lemmas *uint-sub-if'* =
  trans [*OF uint-word-ariths*(*1*) *mod-sub-if-z*, *simplified*, *standard*]
lemmas *uint-plus-if'* =
  trans [*OF uint-word-ariths*(*2*) *mod-add-if-z*, *simplified*, *standard*]

11.4 Definition of uint_arith

lemma *word-of-int-inverse*:
  word-of-int $r = a$ ==> *0* <= r ==> $r <$ *2* ^ *len-of TYPE*('a) ==>
   uint (a::'a::*len0 word*) = r
  apply (*erule word-uint.Abs-inverse'* [*rotated*])
  apply (*simp add*: *uints-num*)
  done

lemma *uint-split*:
  fixes x::'a::*len0 word*
  shows P (*uint* x) =
        (*ALL* i. *word-of-int* $i = x$ & *0* <= i & $i <$ *2*^*len-of TYPE*('a) --> P i)
  apply (*fold word-int-case-def*)
  apply (*auto dest*!: *word-of-int-inverse simp*: *int-word-uint int-mod-eq'*
        split: *word-int-split*)
  done

lemma *uint-split-asm*:
  fixes x::'a::*len0 word*
  shows P (*uint* x) =
        (~(*EX* i. *word-of-int* $i = x$ & *0* <= i & $i <$ *2*^*len-of TYPE*('a) & ~ P i))
  by (*auto dest*!: *word-of-int-inverse*
        simp: *int-word-uint int-mod-eq'*
        split: *uint-split*)

lemmas *uint-splits* = *uint-split uint-split-asm*

lemmas *uint-arith-simps* =
  word-le-def word-less-alt
  word-uint.Rep-inject [*symmetric*]
  uint-sub-if' uint-plus-if'

lemma *power-False-cong*: *False* ==> a ^ b = c ^ d
  by *auto*

ML ⟪
fun uint-arith-ss-of ss =

```
  ss addsimps @{thms uint-arith-simps}
     delsimps @{thms word-uint.Rep-inject}
     addsplits @{thms split-if-asm}
     addcongs @{thms power-False-cong}

fun uint-arith-tacs ctxt =
  let fun arith-tac' n t = arith-tac ctxt n t handle COOPER => Seq.empty
  in
    [ CLASET' clarify-tac 1,
      SIMPSET' (full-simp-tac o uint-arith-ss-of) 1,
      ALLGOALS (full-simp-tac (HOL-ss addsplits @{thms uint-splits}
                               addcongs @{thms power-False-cong})),
      rewrite-goals-tac @{thms word-size},
      ALLGOALS (fn n => REPEAT (resolve-tac [allI, impI] n) THEN
                        REPEAT (etac conjE n) THEN
                        REPEAT (dtac @{thm word-of-int-inverse} n
                                THEN atac n
                                THEN atac n)),
      TRYALL arith-tac' ]
  end

fun uint-arith-tac ctxt = SELECT-GOAL (EVERY (uint-arith-tacs ctxt))
⟫
```

method-setup *uint-arith* =
Method.ctxt-args (*fn ctxt* => *Method.SIMPLE-METHOD* (*uint-arith-tac ctxt 1*))

solving word arithmetic via integers and arith

11.5 More on overflows and monotonicity

lemma *no-plus-overflow-uint-size*:
((*x* :: *'a* :: *len0 word*) <= *x* + *y*) = (*uint x* + *uint y* < *2* ^ *size x*)
unfolding *word-size* **by** *uint-arith*

lemmas *no-olen-add* = *no-plus-overflow-uint-size* [*unfolded word-size*]

lemma *no-ulen-sub*: ((*x* :: *'a* :: *len0 word*) >= *x* − *y*) = (*uint y* <= *uint x*)
by *uint-arith*

lemma *no-olen-add'*:
fixes *x* :: *'a*::*len0 word*
shows ($x \leq y + x$) = (*uint y* + *uint x* < *2* ^ *len-of TYPE*(*'a*))
by (*simp add*: *word-add-ac add-ac no-olen-add*)

lemmas *olen-add-eqv* = *trans* [*OF no-olen-add no-olen-add'* [*symmetric*], *standard*]

lemmas *uint-plus-simple-iff* = *trans* [*OF no-olen-add uint-add-lem*, *standard*]

lemmas *uint-plus-simple* = *uint-plus-simple-iff* [*THEN iffD1*, *standard*]
lemmas *uint-minus-simple-iff* = *trans* [*OF no-ulen-sub uint-sub-lem*, *standard*]
lemmas *uint-minus-simple-alt* = *uint-sub-lem* [*folded word-le-def*]
lemmas *word-sub-le-iff* = *no-ulen-sub* [*folded word-le-def*]
lemmas *word-sub-le* = *word-sub-le-iff* [*THEN iffD2*, *standard*]

lemma *word-less-sub1*:
  $(x :: 'a :: len\ word)\ \tilde{}{=}\ 0 ==> (1 < x) = (0 < x - 1)$
  by *uint-arith*

lemma *word-le-sub1*:
  $(x :: 'a :: len\ word)\ \tilde{}{=}\ 0 ==> (1 <= x) = (0 <= x - 1)$
  by *uint-arith*

lemma *sub-wrap-lt*:
  $((x :: 'a :: len0\ word) < x - z) = (x < z)$
  by *uint-arith*

lemma *sub-wrap*:
  $((x :: 'a :: len0\ word) <= x - z) = (z = 0 \mid x < z)$
  by *uint-arith*

lemma *plus-minus-not-NULL-ab*:
  $(x :: 'a :: len0\ word) <= ab - c ==> c <= ab ==> c\ \tilde{}{=}\ 0 ==> x + c\ \tilde{}{=}\ 0$
  by *uint-arith*

lemma *plus-minus-no-overflow-ab*:
  $(x :: 'a :: len0\ word) <= ab - c ==> c <= ab ==> x <= x + c$
  by *uint-arith*

lemma *le-minus'*:
  $(a :: 'a :: len0\ word) + c <= b ==> a <= a + c ==> c <= b - a$
  by *uint-arith*

lemma *le-plus'*:
  $(a :: 'a :: len0\ word) <= b ==> c <= b - a ==> a + c <= b$
  by *uint-arith*

lemmas *le-plus* = *le-plus'* [*rotated*]

lemmas *le-minus* = *leD* [*THEN thin-rl*, *THEN le-minus'*, *standard*]

lemma *word-plus-mono-right*:
  $(y :: 'a :: len0\ word) <= z ==> x <= x + z ==> x + y <= x + z$
  by *uint-arith*

lemma *word-less-minus-cancel*:
  $y - x < z - x ==> x <= z ==> (y :: 'a :: len0\ word) < z$
  by *uint-arith*

lemma *word-less-minus-mono-left*:
$(y :: 'a :: len0\ word) < z ==> x <= y ==> y - x < z - x$
by *uint-arith*

lemma *word-less-minus-mono*:
$a < c ==> d < b ==> a - b < a ==> c - d < c$
$==> a - b < c - (d::'a::len\ word)$
by *uint-arith*

lemma *word-le-minus-cancel*:
$y - x <= z - x ==> x <= z ==> (y :: 'a :: len0\ word) <= z$
by *uint-arith*

lemma *word-le-minus-mono-left*:
$(y :: 'a :: len0\ word) <= z ==> x <= y ==> y - x <= z - x$
by *uint-arith*

lemma *word-le-minus-mono*:
$a <= c ==> d <= b ==> a - b <= a ==> c - d <= c$
$==> a - b <= c - (d::'a::len\ word)$
by *uint-arith*

lemma *plus-le-left-cancel-wrap*:
$(x :: 'a :: len0\ word) + y' < x ==> x + y < x ==> (x + y' < x + y) = (y' < y)$
by *uint-arith*

lemma *plus-le-left-cancel-nowrap*:
$(x :: 'a :: len0\ word) <= x + y' ==> x <= x + y ==>$
$(x + y' < x + y) = (y' < y)$
by *uint-arith*

lemma *word-plus-mono-right2*:
$(a :: 'a :: len0\ word) <= a + b ==> c <= b ==> a <= a + c$
by *uint-arith*

lemma *word-less-add-right*:
$(x :: 'a :: len0\ word) < y - z ==> z <= y ==> x + z < y$
by *uint-arith*

lemma *word-less-sub-right*:
$(x :: 'a :: len0\ word) < y + z ==> y <= x ==> x - y < z$
by *uint-arith*

lemma *word-le-plus-either*:
$(x :: 'a :: len0\ word) <= y \mid x <= z ==> y <= y + z ==> x <= y + z$
by *uint-arith*

lemma *word-less-nowrapI*:
  $(x :: 'a :: len0\ word) < z - k ==> k <= z ==> 0 < k ==> x < x + k$
  by *uint-arith*

lemma *inc-le*: $(i :: 'a :: len\ word) < m ==> i + 1 <= m$
  by *uint-arith*

lemma *inc-i*:
  $(1 :: 'a :: len\ word) <= i ==> i < m ==> 1 <= (i + 1)\ \&\ i + 1 <= m$
  by *uint-arith*

lemma *udvd-incr-lem*:
  $up < uq ==> up = ua + n * uint\ K ==>$
    $uq = ua + n' * uint\ K ==> up + uint\ K <= uq$
  apply *clarsimp*
  apply (*drule less-le-mult*)
  apply *safe*
  done

lemma *udvd-incr'*:
  $p < q ==> uint\ p = ua + n * uint\ K ==>$
    $uint\ q = ua + n' * uint\ K ==> p + K <= q$
  apply (*unfold word-less-alt word-le-def*)
  apply (*drule* (*2*) *udvd-incr-lem*)
  apply (*erule uint-add-le* [*THEN order-trans*])
  done

lemma *udvd-decr'*:
  $p < q ==> uint\ p = ua + n * uint\ K ==>$
    $uint\ q = ua + n' * uint\ K ==> p <= q - K$
  apply (*unfold word-less-alt word-le-def*)
  apply (*drule* (*2*) *udvd-incr-lem*)
  apply (*drule le-diff-eq* [*THEN iffD2*])
  apply (*erule order-trans*)
  apply (*rule uint-sub-ge*)
  done

lemmas *udvd-incr-lem0* = *udvd-incr-lem* [**where** *ua=0*, *simplified*]
lemmas *udvd-incr0* = *udvd-incr'* [**where** *ua=0*, *simplified*]
lemmas *udvd-decr0* = *udvd-decr'* [**where** *ua=0*, *simplified*]

lemma *udvd-minus-le'*:
  $xy < k ==> z\ udvd\ xy ==> z\ udvd\ k ==> xy <= k - z$
  apply (*unfold udvd-def*)
  apply *clarify*
  apply (*erule* (*2*) *udvd-decr0*)
  done

lemma *udvd-incr2-K*:

p < a + s ==> a <= a + s ==> K udvd s ==> K udvd p − a ==> a <= p ==>
0 < K ==> p <= p + K & p + K <= a + s
apply (unfold udvd-def)
apply clarify
apply (simp add: uint-arith-simps split: split-if-asm)
prefer 2
apply (insert uint-range′ [of s])[1]
apply arith
apply (drule add-commute [THEN xtr1])
apply (simp add: diff-less-eq [symmetric])
apply (drule less-le-mult)
apply arith
apply simp
done

lemma word-succ-rbl:
to-bl w = bl ==> to-bl (word-succ w) = (rev (rbl-succ (rev bl)))
apply (unfold word-succ-def)
apply clarify
apply (simp add: to-bl-of-bin)
apply (simp add: to-bl-def rbl-succ)
done

lemma word-pred-rbl:
to-bl w = bl ==> to-bl (word-pred w) = (rev (rbl-pred (rev bl)))
apply (unfold word-pred-def)
apply clarify
apply (simp add: to-bl-of-bin)
apply (simp add: to-bl-def rbl-pred)
done

lemma word-add-rbl:
to-bl v = vbl ==> to-bl w = wbl ==>
to-bl (v + w) = (rev (rbl-add (rev vbl) (rev wbl)))
apply (unfold word-add-def)
apply clarify
apply (simp add: to-bl-of-bin)
apply (simp add: to-bl-def rbl-add)
done

lemma word-mult-rbl:
to-bl v = vbl ==> to-bl w = wbl ==>
to-bl (v * w) = (rev (rbl-mult (rev vbl) (rev wbl)))
apply (unfold word-mult-def)
apply clarify
apply (simp add: to-bl-of-bin)
apply (simp add: to-bl-def rbl-mult)

done

lemma *rtb-rbl-ariths*:
rev (*to-bl* *w*) = *ys* $\Longrightarrow$ *rev* (*to-bl* (*word-succ* *w*)) = *rbl-succ* *ys*

rev (*to-bl* *w*) = *ys* $\Longrightarrow$ *rev* (*to-bl* (*word-pred* *w*)) = *rbl-pred* *ys*

[| *rev* (*to-bl* *v*) = *ys*; *rev* (*to-bl* *w*) = *xs* |]
==> *rev* (*to-bl* (*v* $*$ *w*)) = *rbl-mult* *ys* *xs*

[| *rev* (*to-bl* *v*) = *ys*; *rev* (*to-bl* *w*) = *xs* |]
==> *rev* (*to-bl* (*v* + *w*)) = *rbl-add* *ys* *xs*
by (*auto simp*: *rev-swap* [*symmetric*] *word-succ-rbl*
word-pred-rbl word-mult-rbl word-add-rbl)

11.6 Arithmetic type class instantiations

instance *word* :: (*len0*) *comm-monoid-add* **..**

instance *word* :: (*len0*) *comm-monoid-mult*
apply (*intro-classes*)
apply (*simp add*: *word-mult-commute*)
apply (*simp add*: *word-mult-1*)
done

instance *word* :: (*len0*) *comm-semiring*
by (*intro-classes*) (*simp add* : *word-left-distrib*)

instance *word* :: (*len0*) *ab-group-add* **..**

instance *word* :: (*len0*) *comm-ring* **..**

instance *word* :: (*len*) *comm-semiring-1*
by (*intro-classes*) (*simp add*: *lenw1-zero-neq-one*)

instance *word* :: (*len*) *comm-ring-1* **..**

instance *word* :: (*len0*) *comm-semiring-0* **..**

instance *word* :: (*len0*) *order* **..**

instance *word* :: (*len*) *recpower*
by (*intro-classes*) (*simp-all add*: *word-pow*)

lemma *zero-bintrunc*:
iszero (*number-of* *x* :: *'a* :: *len word*) =
(*bintrunc* (*len-of* *TYPE*(*'a*)) *x* = *Numeral.Pls*)
apply (*unfold iszero-def word-0-wi word-no-wi*)

```
  apply (rule word-ubin.norm-eq-iff [symmetric, THEN trans])
  apply (simp add : Pls-def [symmetric])
  done

lemmas word-le-0-iff [simp] =
  word-zero-le [THEN leD, THEN linorder-antisym-conv1]

lemma word-of-nat: of-nat n = word-of-int (int n)
  by (induct n) (auto simp add : word-of-int-hom-syms)

lemma word-of-int: of-int = word-of-int
  apply (rule ext)
  apply (unfold of-int-def)
  apply (rule contentsI)
  apply safe
  apply (simp-all add: word-of-nat word-of-int-homs)
   defer
   apply (rule Rep-Integ-ne [THEN nonemptyE])
   apply (rule bexI)
    prefer 2
    apply assumption
   apply (auto simp add: RI-eq-diff)
  done

lemma word-of-int-nat:
  0 <= x ==> word-of-int x = of-nat (nat x)
  by (simp add: of-nat-nat word-of-int)

lemma word-number-of-eq:
  number-of w = (of-int w :: 'a :: len word)
  unfolding word-number-of-def word-of-int by auto

instance word :: (len) number-ring
  by (intro-classes) (simp add : word-number-of-eq)

lemma iszero-word-no [simp] :
  iszero (number-of bin :: 'a :: len word) =
    iszero (number-of (bintrunc (len-of TYPE('a)) bin) :: int)
  apply (simp add: zero-bintrunc number-of-is-id)
  apply (unfold iszero-def Pls-def)
  apply (rule refl)
  done
```

11.7 Word and nat

```
lemma td-ext-unat':
  n = len-of TYPE ('a :: len) ==>
    td-ext (unat :: 'a word => nat) of-nat
    (unats n) (%i. i mod 2 ^ n)
```

```
  apply (unfold td-ext-def' unat-def word-of-nat unats-uints)
  apply (auto intro!: imageI simp add : word-of-int-hom-syms)
  apply (erule word-uint.Abs-inverse [THEN arg-cong])
  apply (simp add: int-word-uint nat-mod-distrib nat-power-eq)
  done

lemmas td-ext-unat = refl [THEN td-ext-unat']
lemmas unat-of-nat = td-ext-unat [THEN td-ext.eq-norm, standard]

interpretation word-unat:
  td-ext [unat::'a::len word => nat
          of-nat
          unats (len-of TYPE('a::len))
          %i. i mod 2 ^ len-of TYPE('a::len)]
  by (rule td-ext-unat)

lemmas td-unat = word-unat.td-thm

lemmas unat-lt2p [iff] = word-unat.Rep [unfolded unats-def mem-Collect-eq]

lemma unat-le: y <= unat (z :: 'a :: len word) ==> y : unats (len-of TYPE
('a))
  apply (unfold unats-def)
  apply clarsimp
  apply (rule xtrans, rule unat-lt2p, assumption)
  done

lemma word-nchotomy:
  ALL w. EX n. (w :: 'a :: len word) = of-nat n & n < 2 ^ len-of TYPE ('a)
  apply (rule allI)
  apply (rule word-unat.Abs-cases)
  apply (unfold unats-def)
  apply auto
  done

lemma of-nat-eq:
  fixes w :: 'a::len word
  shows (of-nat n = w) = (∃ q. n = unat w + q * 2 ^ len-of TYPE('a))
  apply (rule trans)
   apply (rule word-unat.inverse-norm)
  apply (rule iffI)
   apply (rule mod-eqD)
   apply simp
  apply clarsimp
  done

lemma of-nat-eq-size:
  (of-nat n = w) = (EX q. n = unat w + q * 2 ^ size w)
  unfolding word-size by (rule of-nat-eq)
```

lemma of-nat-0:
(of-nat m = (0::'a::len word)) = (∃ q. m = q * 2 ^ len-of TYPE('a))
by (simp add: of-nat-eq)

lemmas of-nat-2p = mult-1 [symmetric, THEN iffD2 [OF of-nat-0 exI]]

lemma of-nat-gt-0: of-nat k ~= 0 ==> 0 < k
by (cases k) auto

lemma of-nat-neq-0:
0 < k ==> k < 2 ^ len-of TYPE ('a :: len) ==> of-nat k ~= (0 :: 'a word)
by (clarsimp simp add : of-nat-0)

lemma Abs-fnat-hom-add:
of-nat a + of-nat b = of-nat (a + b)
by simp

lemma Abs-fnat-hom-mult:
of-nat a * of-nat b = (of-nat (a * b) :: 'a :: len word)
by (simp add: word-of-nat word-of-int-mult-hom zmult-int)

lemma Abs-fnat-hom-Suc:
word-succ (of-nat a) = of-nat (Suc a)
by (simp add: word-of-nat word-of-int-succ-hom add-ac)

lemma Abs-fnat-hom-0: (0::'a::len word) = of-nat 0
by (simp add: word-of-nat word-0-wi)

lemma Abs-fnat-hom-1: (1::'a::len word) = of-nat (Suc 0)
by (simp add: word-of-nat word-1-wi)

lemmas Abs-fnat-homs =
Abs-fnat-hom-add Abs-fnat-hom-mult Abs-fnat-hom-Suc
Abs-fnat-hom-0 Abs-fnat-hom-1

lemma word-arith-nat-add:
a + b = of-nat (unat a + unat b)
by simp

lemma word-arith-nat-mult:
a * b = of-nat (unat a * unat b)
by (simp add: Abs-fnat-hom-mult [symmetric])

lemma word-arith-nat-Suc:
word-succ a = of-nat (Suc (unat a))
by (subst Abs-fnat-hom-Suc [symmetric]) simp

lemma word-arith-nat-div:

a div b = *of-nat* (*unat a div unat b*)
by (*simp add*: *word-div-def word-of-nat zdiv-int uint-nat*)

lemma *word-arith-nat-mod*:
a mod b = *of-nat* (*unat a mod unat b*)
by (*simp add*: *word-mod-def word-of-nat zmod-int uint-nat*)

lemmas *word-arith-nat-defs* =
word-arith-nat-add word-arith-nat-mult
word-arith-nat-Suc Abs-fnat-hom-0
Abs-fnat-hom-1 word-arith-nat-div
word-arith-nat-mod

lemmas *unat-cong* = *arg-cong* [**where** *f* = *unat*]

lemmas *unat-word-ariths* = *word-arith-nat-defs*
[*THEN trans* [*OF unat-cong unat-of-nat*], *standard*]

lemmas *word-sub-less-iff* = *word-sub-le-iff*
[*simplified linorder-not-less* [*symmetric*], *simplified*]

lemma *unat-add-lem*:
(*unat x* + *unat y* < *2* ^ *len-of TYPE*(*'a*)) =
(*unat* (*x* + *y* :: *'a* :: *len word*) = *unat x* + *unat y*)
unfolding *unat-word-ariths*
by (*auto intro*!: *trans* [*OF* - *nat-mod-lem*])

lemma *unat-mult-lem*:
(*unat x* ∗ *unat y* < *2* ^ *len-of TYPE*(*'a*)) =
(*unat* (*x* ∗ *y* :: *'a* :: *len word*) = *unat x* ∗ *unat y*)
unfolding *unat-word-ariths*
by (*auto intro*!: *trans* [*OF* - *nat-mod-lem*])

lemmas *unat-plus-if'* =
trans [*OF unat-word-ariths*(*1*) *mod-nat-add*, *simplified*, *standard*]

lemma *le-no-overflow*:
x <= *b* ==> *a* <= *a* + *b* ==> *x* <= *a* + (*b* :: *'a* :: *len0 word*)
apply (*erule order-trans*)
apply (*erule olen-add-eqv* [*THEN iffD1*])
done

lemmas *un-ui-le* = *trans*
[*OF word-le-nat-alt* [*symmetric*]
word-le-def [*THEN meta-eq-to-obj-eq*],
standard]

lemma *unat-sub-if-size*:
unat (*x* − *y*) = (*if unat y* <= *unat x*

then unat x − unat y
else unat x + 2 ˆ size x − unat y)
apply (*unfold word-size*)
apply (*simp add*: *un-ui-le*)
apply (*auto simp add*: *unat-def uint-sub-if'*)
apply (*rule nat-diff-distrib*)
prefer *3*
apply (*simp add*: *group-simps*)
apply (*rule nat-diff-distrib* [*THEN trans*])
prefer *3*
apply (*subst nat-add-distrib*)
prefer *3*
apply (*simp add*: *nat-power-eq*)
apply *auto*
apply *uint-arith*
done

lemmas *unat-sub-if'* = *unat-sub-if-size* [*unfolded word-size*]

lemma *unat-div*: *unat* ((*x* :: *'a* :: *len word*) *div y*) = *unat x div unat y*
apply (*simp add* : *unat-word-ariths*)
apply (*rule unat-lt2p* [*THEN xtr7*, *THEN nat-mod-eq'*])
apply (*rule div-le-dividend*)
done

lemma *unat-mod*: *unat* ((*x* :: *'a* :: *len word*) *mod y*) = *unat x mod unat y*
apply (*clarsimp simp add* : *unat-word-ariths*)
apply (*cases unat y*)
prefer *2*
apply (*rule unat-lt2p* [*THEN xtr7*, *THEN nat-mod-eq'*])
apply (*rule mod-le-divisor*)
apply *auto*
done

lemma *uint-div*: *uint* ((*x* :: *'a* :: *len word*) *div y*) = *uint x div uint y*
unfolding *uint-nat* **by** (*simp add* : *unat-div zdiv-int*)

lemma *uint-mod*: *uint* ((*x* :: *'a* :: *len word*) *mod y*) = *uint x mod uint y*
unfolding *uint-nat* **by** (*simp add* : *unat-mod zmod-int*)

11.8 Definition of unat_arith tactic

lemma *unat-split*:
fixes *x*::*'a*::*len word*
shows *P* (*unat x*) =
(*ALL n. of-nat n* = *x* & *n* < *2ˆlen-of TYPE*(*'a*) −−> *P n*)
by (*auto simp*: *unat-of-nat*)

lemma *unat-split-asm*:

```
  fixes x::'a::len word
  shows P (unat x) =
        (~(EX n. of-nat n = x & n < 2^len-of TYPE('a) & ~ P n))
  by (auto simp: unat-of-nat)

lemmas of-nat-inverse =
  word-unat.Abs-inverse' [rotated, unfolded unats-def, simplified]

lemmas unat-splits = unat-split unat-split-asm

lemmas unat-arith-simps =
  word-le-nat-alt word-less-nat-alt
  word-unat.Rep-inject [symmetric]
  unat-sub-if' unat-plus-if' unat-div unat-mod


ML ⟪
fun unat-arith-ss-of ss =
  ss addsimps @{thms unat-arith-simps}
     delsimps @{thms word-unat.Rep-inject}
     addsplits @{thms split-if-asm}
     addcongs @{thms power-False-cong}

fun unat-arith-tacs ctxt =
  let fun arith-tac' n t = arith-tac ctxt n t handle COOPER => Seq.empty
  in
    [ CLASET' clarify-tac 1,
      SIMPSET' (full-simp-tac o unat-arith-ss-of) 1,
      ALLGOALS (full-simp-tac (HOL-ss addsplits @{thms unat-splits}
                                addcongs @{thms power-False-cong})),
      rewrite-goals-tac @{thms word-size},
      ALLGOALS (fn n => REPEAT (resolve-tac [allI, impI] n) THEN
                        REPEAT (etac conjE n) THEN
                        REPEAT (dtac @{thm of-nat-inverse} n THEN atac n)),
      TRYALL arith-tac' ]
  end

fun unat-arith-tac ctxt = SELECT-GOAL (EVERY (unat-arith-tacs ctxt))
⟫

method-setup unat-arith =
  Method.ctxt-args (fn ctxt => Method.SIMPLE-METHOD (unat-arith-tac ctxt
1))
  solving word arithmetic via natural numbers and arith

lemma no-plus-overflow-unat-size:
  ((x :: 'a :: len word) <= x + y) = (unat x + unat y < 2 ^ size x)
  unfolding word-size by unat-arith
```

lemma *unat-sub*: $b <= a ==> unat\ (a - b) = unat\ a - unat\ (b :: 'a :: len\ word)$
  by *unat-arith*

lemmas *no-olen-add-nat* = *no-plus-overflow-unat-size* [*unfolded word-size*]

lemmas *unat-plus-simple* = *trans* [*OF no-olen-add-nat unat-add-lem*, *standard*]

lemma *word-div-mult*:
  $(0 :: 'a :: len\ word) < y ==> unat\ x * unat\ y < 2\ \hat{}\ len\text{-}of\ TYPE('a) ==>$
    $x * y\ div\ y = x$
  apply *unat-arith*
  apply *clarsimp*
  apply (*subst unat-mult-lem* [*THEN iffD1*])
  apply *auto*
  done

lemma *div-lt'*: $(i :: 'a :: len\ word) <= k\ div\ x ==>$
    $unat\ i * unat\ x < 2\ \hat{}\ len\text{-}of\ TYPE('a)$
  apply *unat-arith*
  apply *clarsimp*
  apply (*drule mult-le-mono1*)
  apply (*erule order-le-less-trans*)
  apply (*rule xtr7* [*OF unat-lt2p div-mult-le*])
  done

lemmas *div-lt''* = *order-less-imp-le* [*THEN div-lt'*]

lemma *div-lt-mult*: $(i :: 'a :: len\ word) < k\ div\ x ==> 0 < x ==> i * x < k$
  apply (*frule div-lt''* [*THEN unat-mult-lem* [*THEN iffD1*]])
  apply (*simp add*: *unat-arith-simps*)
  apply (*drule* (*1*) *mult-less-mono1*)
  apply (*erule order-less-le-trans*)
  apply (*rule div-mult-le*)
  done

lemma *div-le-mult*:
  $(i :: 'a :: len\ word) <= k\ div\ x ==> 0 < x ==> i * x <= k$
  apply (*frule div-lt'* [*THEN unat-mult-lem* [*THEN iffD1*]])
  apply (*simp add*: *unat-arith-simps*)
  apply (*drule mult-le-mono1*)
  apply (*erule order-trans*)
  apply (*rule div-mult-le*)
  done

lemma *div-lt-uint'*:
  $(i :: 'a :: len\ word) <= k\ div\ x ==> uint\ i * uint\ x < 2\ \hat{}\ len\text{-}of\ TYPE('a)$
  apply (*unfold uint-nat*)
  apply (*drule div-lt'*)
  apply (*simp add*: *zmult-int zless-nat-eq-int-zless* [*symmetric*]

```
                    nat-power-eq)
  done

lemmas div-lt-uint'' = order-less-imp-le [THEN div-lt-uint']

lemma word-le-exists':
  (x :: 'a :: len0 word) <= y ==>
    (EX z. y = x + z & uint x + uint z < 2 ^ len-of TYPE('a))
  apply (rule exI)
  apply (rule conjI)
  apply (rule zadd-diff-inverse)
  apply uint-arith
  done

lemmas plus-minus-not-NULL = order-less-imp-le [THEN plus-minus-not-NULL-ab]

lemmas plus-minus-no-overflow =
  order-less-imp-le [THEN plus-minus-no-overflow-ab]

lemmas mcs = word-less-minus-cancel word-less-minus-mono-left
  word-le-minus-cancel word-le-minus-mono-left

lemmas word-l-diffs = mcs [where y = w + x, unfolded add-diff-cancel, standard]
lemmas word-diff-ls = mcs [where z = w + x, unfolded add-diff-cancel, standard]
lemmas word-plus-mcs = word-diff-ls
  [where y = v + x, unfolded add-diff-cancel, standard]

lemmas le-unat-uoi = unat-le [THEN word-unat.Abs-inverse]

lemmas thd = refl [THEN [2] split-div-lemma [THEN iffD2], THEN conjunct1]

lemma thd1:
  a div b * b ≤ (a::nat)
  using gt-or-eq-0 [of b]
  apply (rule disjE)
   apply (erule xtr4 [OF thd mult-commute])
  apply clarsimp
  done

lemmas uno-simps [THEN le-unat-uoi, standard] =
  mod-le-divisor div-le-dividend thd1

lemma word-mod-div-equality:
  (n div b) * b + (n mod b) = (n :: 'a :: len word)
  apply (unfold word-less-nat-alt word-arith-nat-defs)
  apply (cut-tac y=unat b in gt-or-eq-0)
  apply (erule disjE)
   apply (simp add: mod-div-equality uno-simps)
  apply simp
```

```
  done

lemma word-div-mult-le: a div b * b <= (a::'a::len word)
  apply (unfold word-le-nat-alt word-arith-nat-defs)
  apply (cut-tac y=unat b in gt-or-eq-0)
  apply (erule disjE)
   apply (simp add: div-mult-le uno-simps)
  apply simp
  done

lemma word-mod-less-divisor: 0 < n ==> m mod n < (n :: 'a :: len word)
  apply (simp only: word-less-nat-alt word-arith-nat-defs)
  apply (clarsimp simp add : uno-simps)
  done

lemma word-of-int-power-hom:
  word-of-int a ^ n = (word-of-int (a ^ n) :: 'a :: len word)
  by (induct n) (simp-all add : word-of-int-hom-syms power-Suc)

lemma word-arith-power-alt:
  a ^ n = (word-of-int (uint a ^ n) :: 'a :: len word)
  by (simp add : word-of-int-power-hom [symmetric])

lemma of-bl-length-less:
  length x = k ==> k < len-of TYPE('a) ==> (of-bl x :: 'a :: len word) < 2 ^ k
  apply (unfold of-bl-no [unfolded word-number-of-def]
              word-less-alt word-number-of-alt)
  apply safe
  apply (simp (no-asm) add: word-of-int-power-hom word-uint.eq-norm
                     del: word-of-int-bin)
  apply (simp add: mod-pos-pos-trivial)
  apply (subst mod-pos-pos-trivial)
    apply (rule bl-to-bin-ge0)
   apply (rule order-less-trans)
    apply (rule bl-to-bin-lt2p)
   apply simp
  apply (rule bl-to-bin-lt2p)
  done
```

11.9 Cardinality, finiteness of set of words

```
lemmas card-lessThan' = card-lessThan [unfolded lessThan-def]

lemmas card-eq = word-unat.Abs-inj-on [THEN card-image,
  unfolded word-unat.image, unfolded unats-def, standard]

lemmas card-word = trans [OF card-eq card-lessThan', standard]

lemma finite-word-UNIV: finite (UNIV :: 'a :: len word set)
```

apply (*rule contrapos-np*)
 prefer *2*
 apply (*erule card-infinite*)
apply (*simp add*: *card-word*)
done

lemma *card-word-size*:
  card (*UNIV* :: *'a* :: *len word set*) = (*2* ^ *size* (*x* :: *'a word*))
unfolding *word-size* **by** (*rule card-word*)

end

12 WordBitwise: Bitwise Operations on Words

theory *WordBitwise* **imports** *WordArith* **begin**

lemmas *bin-log-bintrs* = *bin-trunc-not bin-trunc-xor bin-trunc-and bin-trunc-or*

lemmas *wils1* = *bin-log-bintrs* [*THEN word-ubin.norm-eq-iff* [*THEN iffD1*],
  folded word-ubin.eq-norm, *THEN eq-reflection*, *standard*]

lemmas *word-log-binary-defs* =
  word-and-def word-or-def word-xor-def

lemmas *word-no-log-defs* [*simp*] =
  word-not-def [**where** *a*=*number-of a*,
        unfolded word-no-wi wils1, *folded word-no-wi*, *standard*]
  word-log-binary-defs [**where** *a*=*number-of a* **and** *b*=*number-of b*,
        unfolded word-no-wi wils1, *folded word-no-wi*, *standard*]

lemmas *word-wi-log-defs* = *word-no-log-defs* [*unfolded word-no-wi*]

lemma *uint-or*: *uint* (*x OR y*) = (*uint x*) *OR* (*uint y*)
  by (*simp add*: *word-or-def word-no-wi* [*symmetric*] *number-of-is-id*
        bin-trunc-ao(*2*) [*symmetric*])

lemma *uint-and*: *uint* (*x AND y*) = (*uint x*) *AND* (*uint y*)
  by (*simp add*: *word-and-def number-of-is-id word-no-wi* [*symmetric*]
        bin-trunc-ao(*1*) [*symmetric*])

lemma *word-ops-nth-size*:
  n < *size* (*x*::*'a*::*len0 word*) ==>
    (*x OR y*) !! *n* = (*x* !! *n* | *y* !! *n*) &

$(x\ AND\ y)\ !!\ n = (x\ !!\ n\ \&\ y\ !!\ n)\ \&$
$(x\ XOR\ y)\ !!\ n = (x\ !!\ n\ {}^{\sim}{=}\ y\ !!\ n)\ \&$
$(NOT\ x)\ !!\ n = ({}^{\sim}\ x\ !!\ n)$
unfolding *word-size word-no-wi word-test-bit-def word-log-defs*
by (*clarsimp simp add* : *word-ubin.eq-norm nth-bintr bin-nth-ops*)

lemma *word-ao-nth*:
fixes x :: *'a::len0 word*
shows $(x\ OR\ y)\ !!\ n = (x\ !!\ n\ |\ y\ !!\ n)\ \&$
$(x\ AND\ y)\ !!\ n = (x\ !!\ n\ \&\ y\ !!\ n)$
apply (*cases* $n <$ *size* x)
apply (*drule-tac* $y = y$ **in** *word-ops-nth-size*)
apply *simp*
apply (*simp add* : *test-bit-bin word-size*)
done

lemmas *bwsimps* =
word-of-int-homs(*2*)
word-0-wi-Pls
word-m1-wi-Min
word-wi-log-defs

lemma *word-bw-assocs*:
fixes x :: *'a::len0 word*
shows
$(x\ AND\ y)\ AND\ z = x\ AND\ y\ AND\ z$
$(x\ OR\ y)\ OR\ z = x\ OR\ y\ OR\ z$
$(x\ XOR\ y)\ XOR\ z = x\ XOR\ y\ XOR\ z$
using *word-of-int-Ex* [**where** $x{=}x$]
word-of-int-Ex [**where** $x{=}y$]
word-of-int-Ex [**where** $x{=}z$]
by (*auto simp*: *bwsimps bbw-assocs*)

lemma *word-bw-comms*:
fixes x :: *'a::len0 word*
shows
$x\ AND\ y = y\ AND\ x$
$x\ OR\ y = y\ OR\ x$
$x\ XOR\ y = y\ XOR\ x$
using *word-of-int-Ex* [**where** $x{=}x$]
word-of-int-Ex [**where** $x{=}y$]
by (*auto simp*: *bwsimps bin-ops-comm*)

lemma *word-bw-lcs*:
fixes x :: *'a::len0 word*
shows
$y\ AND\ x\ AND\ z = x\ AND\ y\ AND\ z$

$y\ OR\ x\ OR\ z = x\ OR\ y\ OR\ z$
$y\ XOR\ x\ XOR\ z = x\ XOR\ y\ XOR\ z$
using *word-of-int-Ex* [**where** $x=x$]
word-of-int-Ex [**where** $x=y$]
word-of-int-Ex [**where** $x=z$]
by (*auto simp*: *bwsimps*)

lemma *word-log-esimps* [*simp*]:
fixes x :: $'a$::*len0 word*
shows
$x\ AND\ 0 = 0$
$x\ AND\ -1 = x$
$x\ OR\ 0 = x$
$x\ OR\ -1 = -1$
$x\ XOR\ 0 = x$
$x\ XOR\ -1 = NOT\ x$
$0\ AND\ x = 0$
$-1\ AND\ x = x$
$0\ OR\ x = x$
$-1\ OR\ x = -1$
$0\ XOR\ x = x$
$-1\ XOR\ x = NOT\ x$
using *word-of-int-Ex* [**where** $x=x$]
by (*auto simp*: *bwsimps*)

lemma *word-not-dist*:
fixes x :: $'a$::*len0 word*
shows
$NOT\ (x\ OR\ y) = NOT\ x\ AND\ NOT\ y$
$NOT\ (x\ AND\ y) = NOT\ x\ OR\ NOT\ y$
using *word-of-int-Ex* [**where** $x=x$]
word-of-int-Ex [**where** $x=y$]
by (*auto simp*: *bwsimps bbw-not-dist*)

lemma *word-bw-same*:
fixes x :: $'a$::*len0 word*
shows
$x\ AND\ x = x$
$x\ OR\ x = x$
$x\ XOR\ x = 0$
using *word-of-int-Ex* [**where** $x=x$]
by (*auto simp*: *bwsimps*)

lemma *word-ao-absorbs* [*simp*]:
fixes x :: $'a$::*len0 word*
shows
$x\ AND\ (y\ OR\ x) = x$
$x\ OR\ y\ AND\ x = x$
$x\ AND\ (x\ OR\ y) = x$

y AND x OR x = x
(y OR x) AND x = x
x OR x AND y = x
(x OR y) AND x = x
x AND y OR x = x
using *word-of-int-Ex* [**where** *x=x*]
word-of-int-Ex [**where** *x=y*]
by (*auto simp*: *bwsimps*)

lemma *word-not-not* [*simp*]:
NOT NOT (x::'a::len0 word) = x
using *word-of-int-Ex* [**where** *x=x*]
by (*auto simp*: *bwsimps*)

lemma *word-ao-dist*:
fixes *x :: 'a::len0 word*
shows *(x OR y) AND z = x AND z OR y AND z*
using *word-of-int-Ex* [**where** *x=x*]
word-of-int-Ex [**where** *x=y*]
word-of-int-Ex [**where** *x=z*]
by (*auto simp*: *bwsimps bbw-ao-dist simp del*: *bin-ops-comm*)

lemma *word-oa-dist*:
fixes *x :: 'a::len0 word*
shows *x AND y OR z = (x OR z) AND (y OR z)*
using *word-of-int-Ex* [**where** *x=x*]
word-of-int-Ex [**where** *x=y*]
word-of-int-Ex [**where** *x=z*]
by (*auto simp*: *bwsimps bbw-oa-dist simp del*: *bin-ops-comm*)

lemma *word-add-not* [*simp*]:
fixes *x :: 'a::len0 word*
shows *x + NOT x = −1*
using *word-of-int-Ex* [**where** *x=x*]
by (*auto simp*: *bwsimps bin-add-not*)

lemma *word-plus-and-or* [*simp*]:
fixes *x :: 'a::len0 word*
shows *(x AND y) + (x OR y) = x + y*
using *word-of-int-Ex* [**where** *x=x*]
word-of-int-Ex [**where** *x=y*]
by (*auto simp*: *bwsimps plus-and-or*)

lemma *leoa*:
fixes *x :: 'a::len0 word*
shows *(w = (x OR y)) ==> (y = (w AND y))* **by** *auto*
lemma *leao*:
fixes *x' :: 'a::len0 word*
shows *(w' = (x' AND y')) ==> (x' = (x' OR w'))* **by** *auto*

lemmas *word-ao-equiv* = *leao* [*COMP leoa* [*COMP iffI*]]

lemma *le-word-or2*: *x* <= *x OR* (*y*::′*a*::*len0 word*)
  unfolding *word-le-def uint-or*
  by (*auto intro*: *le-int-or*)

lemmas *le-word-or1* = *xtr3* [*OF word-bw-comms* (*2*) *le-word-or2*, *standard*]
lemmas *word-and-le1* =
  xtr3 [*OF word-ao-absorbs* (*4*) [*symmetric*] *le-word-or2*, *standard*]
lemmas *word-and-le2* =
  xtr3 [*OF word-ao-absorbs* (*8*) [*symmetric*] *le-word-or2*, *standard*]

lemma *bl-word-not*: *to-bl* (*NOT w*) = *map Not* (*to-bl w*)
  unfolding *to-bl-def word-log-defs*
  by (*simp add*: *bl-not-bin number-of-is-id word-no-wi* [*symmetric*])

lemma *bl-word-xor*: *to-bl* (*v XOR w*) = *app2 op* ~= (*to-bl v*) (*to-bl w*)
  unfolding *to-bl-def word-log-defs bl-xor-bin*
  by (*simp add*: *number-of-is-id word-no-wi* [*symmetric*])

lemma *bl-word-or*: *to-bl* (*v OR w*) = *app2 op* | (*to-bl v*) (*to-bl w*)
  unfolding *to-bl-def word-log-defs*
  by (*simp add*: *bl-or-bin number-of-is-id word-no-wi* [*symmetric*])

lemma *bl-word-and*: *to-bl* (*v AND w*) = *app2 op* & (*to-bl v*) (*to-bl w*)
  unfolding *to-bl-def word-log-defs*
  by (*simp add*: *bl-and-bin number-of-is-id word-no-wi* [*symmetric*])

lemma *word-lsb-alt*: *lsb* (*w*::′*a*::*len0 word*) = *test-bit w 0*
  by (*auto simp*: *word-test-bit-def word-lsb-def*)

lemma *word-lsb-1-0*: *lsb* (*1*::′*a*::*len word*) & ~ *lsb* (*0*::′*b*::*len0 word*)
  unfolding *word-lsb-def word-1-no word-0-no* **by** *auto*

lemma *word-lsb-last*: *lsb* (*w*::′*a*::*len word*) = *last* (*to-bl w*)
  apply (*unfold word-lsb-def uint-bl bin-to-bl-def*)
  apply (*rule-tac bin*=*uint w* **in** *bin-exhaust*)
  apply (*cases size w*)
   apply *auto*
   apply (*auto simp add*: *bin-to-bl-aux-alt*)
  done

lemma *word-lsb-int*: *lsb w* = (*uint w mod 2* = *1*)
  unfolding *word-lsb-def bin-last-mod* **by** *auto*

lemma *word-msb-sint*: *msb w* = (*sint w* < *0*)
  unfolding *word-msb-def*
  by (*simp add* : *sign-Min-lt-0 number-of-is-id*)

lemma *word-msb-no'*:
  w = *number-of bin* ==> *msb* (*w*::'*a*::*len word*) = *bin-nth bin* (*size w* − *1*)
  unfolding *word-msb-def word-number-of-def*
  by (*clarsimp simp add*: *word-sbin.eq-norm word-size bin-sign-lem*)

lemmas *word-msb-no* = *refl* [*THEN word-msb-no'*, *unfolded word-size*]

lemma *word-msb-nth'*: *msb* (*w*::'*a*::*len word*) = *bin-nth* (*uint w*) (*size w* − *1*)
  apply (*unfold word-size*)
  apply (*rule trans* [*OF* - *word-msb-no*])
  apply (*simp add* : *word-number-of-def*)
  done

lemmas *word-msb-nth* = *word-msb-nth'* [*unfolded word-size*]

lemma *word-msb-alt*: *msb* (*w*::'*a*::*len word*) = *hd* (*to-bl w*)
  apply (*unfold word-msb-nth uint-bl*)
  apply (*subst hd-conv-nth*)
  apply (*rule length-greater-0-conv* [*THEN iffD1*])
   apply *simp*
  apply (*simp add* : *nth-bin-to-bl word-size*)
  done

lemma *word-set-nth*:
  set-bit w n (*test-bit w n*) = (*w*::'*a*::*len0 word*)
  unfolding *word-test-bit-def word-set-bit-def* **by** *auto*

lemma *bin-nth-uint'*:
  bin-nth (*uint w*) *n* = (*rev* (*bin-to-bl* (*size w*) (*uint w*)) ! *n* & *n* < *size w*)
  apply (*unfold word-size*)
  apply (*safe elim*!: *bin-nth-uint-imp*)
   apply (*frule bin-nth-uint-imp*)
   apply (*fast dest*!: *bin-nth-bl*)+
  done

lemmas *bin-nth-uint* = *bin-nth-uint'* [*unfolded word-size*]

lemma *test-bit-bl*: *w* !! *n* = (*rev* (*to-bl w*) ! *n* & *n* < *size w*)
  unfolding *to-bl-def word-test-bit-def word-size*
  by (*rule bin-nth-uint*)

lemma *to-bl-nth*: *n* < *size w* ==> *to-bl w* ! *n* = *w* !! (*size w* − *Suc n*)
  apply (*unfold test-bit-bl*)
  apply *clarsimp*
  apply (*rule trans*)
   apply (*rule nth-rev-alt*)
   apply (*auto simp add*: *word-size*)
  done

```
lemma test-bit-set:
  fixes w :: 'a::len0 word
  shows (set-bit w n x) !! n = (n < size w & x)
  unfolding word-size word-test-bit-def word-set-bit-def
  by (clarsimp simp add : word-ubin.eq-norm nth-bintr)

lemma test-bit-set-gen:
  fixes w :: 'a::len0 word
  shows test-bit (set-bit w n x) m =
        (if m = n then n < size w & x else test-bit w m)
  apply (unfold word-size word-test-bit-def word-set-bit-def)
  apply (clarsimp simp add: word-ubin.eq-norm nth-bintr bin-nth-sc-gen)
  apply (auto elim!: test-bit-size [unfolded word-size]
              simp add: word-test-bit-def [symmetric])
  done

lemma of-bl-rep-False: of-bl (replicate n False @ bs) = of-bl bs
  unfolding of-bl-def bl-to-bin-rep-F by auto

lemma msb-nth':
  fixes w :: 'a::len word
  shows msb w = w !! (size w - 1)
  unfolding word-msb-nth' word-test-bit-def by simp

lemmas msb-nth = msb-nth' [unfolded word-size]

lemmas msb0 = len-gt-0 [THEN diff-Suc-less, THEN
  word-ops-nth-size [unfolded word-size], standard]
lemmas msb1 = msb0 [where i = 0]
lemmas word-ops-msb = msb1 [unfolded msb-nth [symmetric, unfolded One-nat-def]]

lemmas lsb0 = len-gt-0 [THEN word-ops-nth-size [unfolded word-size], standard]
lemmas word-ops-lsb = lsb0 [unfolded word-lsb-alt]

lemma td-ext-nth':
  n = size (w::'a::len0 word) ==> ofn = set-bits ==> [w, ofn g] = l ==>
    td-ext test-bit ofn {f. ALL i. f i --> i < n} (%h i. h i & i < n)
  apply (unfold word-size td-ext-def')
  apply safe
     apply (rule-tac [3] ext)
     apply (rule-tac [4] ext)
     apply (unfold word-size of-nth-def test-bit-bl)
     apply safe
       defer
       apply (clarsimp simp: word-bl.Abs-inverse)+
  apply (rule word-bl.Rep-inverse')
  apply (rule sym [THEN trans])
  apply (rule bl-of-nth-nth)
```

apply *simp*
apply (*rule bl-of-nth-inj*)
apply (*clarsimp simp add* : *test-bit-bl word-size*)
done

lemmas *td-ext-nth* = *td-ext-nth'* [*OF refl refl refl*, *unfolded word-size*]

interpretation *test-bit*:
td-ext [*op* !! :: *'a*::*len0 word* => *nat* => *bool*
set-bits
$\{f.\ \forall i.\ f\ i \longrightarrow i < \mathit{len\text{-}of}\ \mathit{TYPE}('a::\mathit{len0})\}$
$(\lambda h\ i.\ h\ i \wedge i < \mathit{len\text{-}of}\ \mathit{TYPE}('a::\mathit{len0}))]$
by (*rule td-ext-nth*)

declare *test-bit.Rep'* [*simp del*]
declare *test-bit.Rep'* [*rule del*]

lemmas *td-nth* = *test-bit.td-thm*

lemma *word-set-set-same*:
fixes *w* :: *'a*::*len0 word*
shows *set-bit* (*set-bit w n x*) *n y* = *set-bit w n y*
by (*rule word-eqI*) (*simp add* : *test-bit-set-gen word-size*)

lemma *word-set-set-diff*:
fixes *w* :: *'a*::*len0 word*
assumes *m* ~= *n*
shows *set-bit* (*set-bit w m x*) *n y* = *set-bit* (*set-bit w n y*) *m x*
by (*rule word-eqI*) (*clarsimp simp add* : *test-bit-set-gen word-size prems*)

lemma *test-bit-no'*:
fixes *w* :: *'a*::*len0 word*
shows *w* = *number-of bin* ==> *test-bit w n* = (*n* < *size w* & *bin-nth bin n*)
unfolding *word-test-bit-def word-number-of-def word-size*
by (*simp add* : *nth-bintr* [*symmetric*] *word-ubin.eq-norm*)

lemmas *test-bit-no* =
refl [*THEN test-bit-no'*, *unfolded word-size*, *THEN eq-reflection*, *standard*]

lemma *nth-0*: ~ (*0*::*'a*::*len0 word*) !! *n*
unfolding *test-bit-no word-0-no* **by** *auto*

lemma *nth-sint*:
fixes *w* :: *'a*::*len word*
defines $l \equiv$ *len-of TYPE* (*'a*)
shows *bin-nth* (*sint w*) *n* = (*if n* < *l* − *1 then w* !! *n else w* !! (*l* − *1*))
unfolding *sint-uint l-def*
by (*clarsimp simp add*: *nth-sbintr word-test-bit-def* [*symmetric*])

lemma *word-lsb-no*:
  lsb (*number-of bin* :: *'a* :: *len word*) = (*bin-last bin* = *bit.B1*)
  unfolding *word-lsb-alt test-bit-no* **by** *auto*

lemma *word-set-no*:
  set-bit (*number-of bin*::*'a*::*len0 word*) *n b* =
    number-of (*bin-sc n* (*if b then bit.B1 else bit.B0*) *bin*)
  apply (*unfold word-set-bit-def word-number-of-def* [*symmetric*])
  apply (*rule word-eqI*)
  apply (*clarsimp simp*: *word-size bin-nth-sc-gen number-of-is-id*
            test-bit-no nth-bintr)
  done

lemmas *setBit-no* = *setBit-def* [*THEN trans* [*OF meta-eq-to-obj-eq word-set-no*],
  simplified if-simps, *THEN eq-reflection*, *standard*]
lemmas *clearBit-no* = *clearBit-def* [*THEN trans* [*OF meta-eq-to-obj-eq word-set-no*],
  simplified if-simps, *THEN eq-reflection*, *standard*]

lemma *to-bl-n1*:
  to-bl (*−1*::*'a*::*len0 word*) = *replicate* (*len-of TYPE* (*'a*)) *True*
  apply (*rule word-bl.Abs-inverse'*)
   apply *simp*
  apply (*rule word-eqI*)
  apply (*clarsimp simp add*: *word-size test-bit-no*)
  apply (*auto simp add*: *word-bl.Abs-inverse test-bit-bl word-size*)
  done

lemma *word-msb-n1*: *msb* (*−1*::*'a*::*len word*)
  unfolding *word-msb-alt word-msb-alt to-bl-n1* **by** *simp*

declare *word-set-set-same* [*simp*] *word-set-nth* [*simp*]
  test-bit-no [*simp*] *word-set-no* [*simp*] *nth-0* [*simp*]
  setBit-no [*simp*] *clearBit-no* [*simp*]
  word-lsb-no [*simp*] *word-msb-no* [*simp*] *word-msb-n1* [*simp*] *word-lsb-1-0* [*simp*]

lemma *word-set-nth-iff*:
  (*set-bit w n b* = *w*) = (*w* !! *n* = *b* | *n* >= *size* (*w*::*'a*::*len0 word*))
  apply (*rule iffI*)
   apply (*rule disjCI*)
   apply (*drule word-eqD*)
   apply (*erule sym* [*THEN trans*])
   apply (*simp add*: *test-bit-set*)
  apply (*erule disjE*)
   apply *clarsimp*
  apply (*rule word-eqI*)
  apply (*clarsimp simp add* : *test-bit-set-gen*)
  apply (*drule test-bit-size*)
  apply *force*
  done

```
lemma test-bit-2p':
  w = word-of-int (2 ^ n) ==>
    w !! m = (m = n & m < size (w :: 'a :: len word))
  unfolding word-test-bit-def word-size
  by (auto simp add: word-ubin.eq-norm nth-bintr nth-2p-bin)

lemmas test-bit-2p = refl [THEN test-bit-2p', unfolded word-size]

lemmas nth-w2p = test-bit-2p [unfolded of-int-number-of-eq
  word-of-int [symmetric] of-int-power]

lemma uint-2p:
  (0::'a::len word) < 2 ^ n ==> uint (2 ^ n::'a::len word) = 2 ^ n
  apply (unfold word-arith-power-alt)
  apply (case-tac len-of TYPE ('a))
   apply clarsimp
  apply (case-tac nat)
   apply clarsimp
   apply (case-tac n)
    apply (clarsimp simp add : word-1-wi [symmetric])
   apply (clarsimp simp add : word-0-wi [symmetric])
  apply (drule word-gt-0 [THEN iffD1])
  apply (safe intro!: word-eqI bin-nth-lem ext)
     apply (auto simp add: test-bit-2p nth-2p-bin word-test-bit-def [symmetric])
  done

lemma word-of-int-2p: (word-of-int (2 ^ n) :: 'a :: len word) = 2 ^ n
  apply (unfold word-arith-power-alt)
  apply (case-tac len-of TYPE ('a))
   apply clarsimp
  apply (case-tac nat)
   apply (rule word-ubin.norm-eq-iff [THEN iffD1])
   apply (rule box-equals)
     apply (rule-tac [2] bintr-ariths (1))+
   apply (clarsimp simp add : number-of-is-id)
  apply simp
  done

lemma bang-is-le: x !! m ==> 2 ^ m <= (x :: 'a :: len word)
  apply (rule xtr3)
  apply (rule-tac [2] y = x in le-word-or2)
  apply (rule word-eqI)
  apply (auto simp add: word-ao-nth nth-w2p word-size)
  done

lemma word-clr-le:
  fixes w :: 'a::len0 word
  shows w >= set-bit w n False
```

apply (*unfold word-set-bit-def word-le-def word-ubin.eq-norm*)
apply *simp*
apply (*rule order-trans*)
apply (*rule bintr-bin-clr-le*)
apply *simp*
done

lemma *word-set-ge*:
fixes *w* :: *'a::len word*
shows *w* <= *set-bit w n True*
apply (*unfold word-set-bit-def word-le-def word-ubin.eq-norm*)
apply *simp*
apply (*rule order-trans* [*OF - bintr-bin-set-ge*])
apply *simp*
done

end

13 WordShift: Shifting, Rotating, and Splitting Words

theory *WordShift* **imports** *WordBitwise* **begin**

13.1 Bit shifting

lemma *shiftl1-number* [*simp*] :
shiftl1 (*number-of w*) = *number-of* (*w BIT bit.B0*)
apply (*unfold shiftl1-def word-number-of-def*)
apply (*simp add*: *word-ubin.norm-eq-iff* [*symmetric*] *word-ubin.eq-norm*)
apply (*subst refl* [*THEN bintrunc-BIT-I*, *symmetric*])
apply (*subst bintrunc-bintrunc-min*)
apply *simp*
done

lemma *shiftl1-0* [*simp*] : *shiftl1 0* = *0*
unfolding *word-0-no shiftl1-number* **by** *auto*

lemmas *shiftl1-def-u* = *shiftl1-def* [*folded word-number-of-def*]

lemma *shiftl1-def-s*: *shiftl1 w* = *number-of* (*sint w BIT bit.B0*)
by (*rule trans* [*OF - shiftl1-number*]) *simp*

lemma *shiftr1-0* [*simp*] : *shiftr1 0* = *0*
unfolding *shiftr1-def*
by *simp* (*simp add*: *word-0-wi*)

lemma *sshiftr1-0* [*simp*] : *sshiftr1 0* = *0*
  apply (*unfold sshiftr1-def*)
  apply *simp*
  apply (*simp add* : *word-0-wi*)
  done

lemma *sshiftr1-n1* [*simp*] : *sshiftr1* −*1* = −*1*
  unfolding *sshiftr1-def* **by** *auto*

lemma *shiftl-0* [*simp*] : (*0*::'*a*::*len0 word*) << *n* = *0*
  unfolding *shiftl-def* **by** (*induct n*) *auto*

lemma *shiftr-0* [*simp*] : (*0*::'*a*::*len0 word*) >> *n* = *0*
  unfolding *shiftr-def* **by** (*induct n*) *auto*

lemma *sshiftr-0* [*simp*] : *0* >>> *n* = *0*
  unfolding *sshiftr-def* **by** (*induct n*) *auto*

lemma *sshiftr-n1* [*simp*] : −*1* >>> *n* = −*1*
  unfolding *sshiftr-def* **by** (*induct n*) *auto*

lemma *nth-shiftl1*: *shiftl1 w* !! *n* = (*n* < *size w* & *n* > *0* & *w* !! (*n* − *1*))
  apply (*unfold shiftl1-def word-test-bit-def*)
  apply (*simp add*: *nth-bintr word-ubin.eq-norm word-size*)
  apply (*cases n*)
   apply *auto*
  done

lemma *nth-shiftl'* [*rule-format*]:
  ALL n. ((*w*::'*a*::*len0 word*) << *m*) !! *n* = (*n* < *size w* & *n* >= *m* & *w* !! (*n* − *m*))
  apply (*unfold shiftl-def*)
  apply (*induct m*)
   apply (*force elim*!: *test-bit-size*)
  apply (*clarsimp simp add* : *nth-shiftl1 word-size*)
  apply *arith*
  done

lemmas *nth-shiftl* = *nth-shiftl'* [*unfolded word-size*]

lemma *nth-shiftr1*: *shiftr1 w* !! *n* = *w* !! *Suc n*
  apply (*unfold shiftr1-def word-test-bit-def*)
  apply (*simp add*: *nth-bintr word-ubin.eq-norm*)
  apply *safe*
  apply (*drule bin-nth.Suc* [*THEN iffD2*, *THEN bin-nth-uint-imp*])
  apply *simp*
  done

lemma *nth-shiftr*:

$\bigwedge n.\ ((w{::}'a{::}len0\ word) >> m)\ !!\ n = w\ !!\ (n + m)$
 apply (*unfold shiftr-def*)
 apply (*induct m*)
  apply (*auto simp add* : *nth-shiftr1*)
 done

lemma *uint-shiftr1*: *uint* (*shiftr1 w*) = *bin-rest* (*uint w*)
 apply (*unfold shiftr1-def word-ubin.eq-norm bin-rest-trunc-i*)
 apply (*subst bintr-uint* [*symmetric*, *OF order-refl*])
 apply (*simp only* : *bintrunc-bintrunc-l*)
 apply *simp*
 done

lemma *nth-sshiftr1*:
 sshiftr1 w !! *n* = (*if n* = *size w* − *1 then w* !! *n else w* !! *Suc n*)
 apply (*unfold sshiftr1-def word-test-bit-def*)
 apply (*simp add*: *nth-bintr word-ubin.eq-norm*
   bin-nth.Suc [*symmetric*] *word-size*
  del: *bin-nth.simps*)
 apply (*simp add*: *nth-bintr uint-sint del* : *bin-nth.simps*)
 apply (*auto simp add*: *bin-nth-sint*)
 done

lemma *nth-sshiftr* [*rule-format*] :
 ALL n. sshiftr w m !! *n* = (*n* < *size w* &
  (*if n* + *m* >= *size w then w* !! (*size w* − *1*) *else w* !! (*n* + *m*)))
 apply (*unfold sshiftr-def*)
 apply (*induct-tac m*)
  apply (*simp add*: *test-bit-bl*)
 apply (*clarsimp simp add*: *nth-sshiftr1 word-size*)
 apply *safe*
      apply *arith*
     apply *arith*
    apply (*erule thin-rl*)
    apply (*case-tac n*)
     apply *safe*
     apply *simp*
    apply *simp*
   apply (*erule thin-rl*)
   apply (*case-tac n*)
    apply *safe*
    apply *simp*
   apply *simp*
  apply *arith+*
 done

lemma *shiftr1-div-2*: *uint* (*shiftr1 w*) = *uint w div 2*

apply (*unfold shiftr1-def bin-rest-div*)
apply (*rule word-uint.Abs-inverse*)
apply (*simp add*: *uints-num pos-imp-zdiv-nonneg-iff*)
apply (*rule xtr7*)
 prefer *2*
 apply (*rule zdiv-le-dividend*)
  apply *auto*
done

lemma *sshiftr1-div-2*: *sint* (*sshiftr1 w*) = *sint w div 2*
apply (*unfold sshiftr1-def bin-rest-div* [*symmetric*])
apply (*simp add*: *word-sbin.eq-norm*)
apply (*rule trans*)
 defer
 apply (*subst word-sbin.norm-Rep* [*symmetric*])
 apply (*rule refl*)
apply (*subst word-sbin.norm-Rep* [*symmetric*])
apply (*unfold One-nat-def*)
apply (*rule sbintrunc-rest*)
done

lemma *shiftr-div-2n*: *uint* (*shiftr w n*) = *uint w div 2 ^ n*
apply (*unfold shiftr-def*)
apply (*induct n*)
 apply *simp*
apply (*simp add*: *shiftr1-div-2 mult-commute*
  zdiv-zmult2-eq [*symmetric*])
done

lemma *sshiftr-div-2n*: *sint* (*sshiftr w n*) = *sint w div 2 ^ n*
apply (*unfold sshiftr-def*)
apply (*induct n*)
 apply *simp*
apply (*simp add*: *sshiftr1-div-2 mult-commute*
  zdiv-zmult2-eq [*symmetric*])
done

13.1.1 shift functions in terms of lists of bools

lemmas *bshiftr1-no-bin* [*simp*] =
bshiftr1-def [**where** *w=number-of w*, *unfolded to-bl-no-bin*, *standard*]

lemma *bshiftr1-bl*: *to-bl* (*bshiftr1 b w*) = *b* # *butlast* (*to-bl w*)
unfolding *bshiftr1-def* **by** (*rule word-bl.Abs-inverse*) *simp*

lemma *shiftl1-of-bl*: *shiftl1* (*of-bl bl*) = *of-bl* (*bl* @ [*False*])
unfolding *uint-bl of-bl-no*
by (*simp add*: *bl-to-bin-aux-append bl-to-bin-def*)

lemma *shiftl1-bl*: *shiftl1* (w::$'a$::*len0 word*) = *of-bl* (*to-bl* w @ [*False*])
proof −
  have *shiftl1* w = *shiftl1* (*of-bl* (*to-bl* w)) **by** *simp*
  also have ... = *of-bl* (*to-bl* w @ [*False*]) **by** (*rule shiftl1-of-bl*)
  finally show *?thesis* **.**
qed

lemma *bl-shiftl1*:
  to-bl (*shiftl1* (w :: $'a$ :: *len word*)) = *tl* (*to-bl* w) @ [*False*]
  apply (*simp add*: *shiftl1-bl word-rep-drop drop-Suc drop-Cons'*)
  apply (*fast intro*!: *Suc-leI*)
  done

lemma *shiftr1-bl*: *shiftr1* w = *of-bl* (*butlast* (*to-bl* w))
  apply (*unfold shiftr1-def uint-bl of-bl-def*)
  apply (*simp add*: *butlast-rest-bin word-size*)
  apply (*simp add*: *bin-rest-trunc* [*symmetric*, *unfolded One-nat-def*])
  done

lemma *bl-shiftr1*:
  to-bl (*shiftr1* (w :: $'a$ :: *len word*)) = *False* # *butlast* (*to-bl* w)
  unfolding *shiftr1-bl*
  by (*simp add* : *word-rep-drop len-gt-0* [*THEN Suc-leI*])

lemma *shiftl1-rev*:
  shiftl1 (w :: $'a$ :: *len word*) = *word-reverse* (*shiftr1* (*word-reverse* w))
  apply (*unfold word-reverse-def*)
  apply (*rule word-bl.Rep-inverse'* [*symmetric*])
  apply (*simp add*: *bl-shiftl1 bl-shiftr1 word-bl.Abs-inverse*)
  apply (*cases to-bl* w)
   apply *auto*
  done

lemma *shiftl-rev*:
  shiftl (w :: $'a$ :: *len word*) n = *word-reverse* (*shiftr* (*word-reverse* w) n)
  apply (*unfold shiftl-def shiftr-def*)
  apply (*induct* n)
   apply (*auto simp add* : *shiftl1-rev*)
  done

lemmas *rev-shiftl* =
  shiftl-rev [**where** w = *word-reverse* w, *simplified*, *standard*]

lemmas *shiftr-rev* = *rev-shiftl* [*THEN word-rev-gal'*, *standard*]
lemmas *rev-shiftr* = *shiftl-rev* [*THEN word-rev-gal'*, *standard*]

lemma *bl-sshiftr1*:

to-bl (*sshiftr1* (*w* :: *'a* :: *len word*)) = *hd* (*to-bl w*) # *butlast* (*to-bl w*)
apply (*unfold sshiftr1-def uint-bl word-size*)
apply (*simp add*: *butlast-rest-bin word-ubin.eq-norm*)
apply (*simp add*: *sint-uint*)
apply (*rule nth-equalityI*)
 apply *clarsimp*
apply *clarsimp*
apply (*case-tac i*)
 apply (*simp-all add*: *hd-conv-nth length-0-conv* [*symmetric*]
 nth-bin-to-bl bin-nth.Suc [*symmetric*]
 nth-sbintr
 del: *bin-nth.Suc*)
 apply *force*
apply (*rule impI*)
apply (*rule-tac f* = *bin-nth* (*uint w*) **in** *arg-cong*)
apply *simp*
done

lemma *drop-shiftr*:
drop n (*to-bl* ((*w* :: *'a* :: *len word*) >> *n*)) = *take* (*size w* − *n*) (*to-bl w*)
apply (*unfold shiftr-def*)
apply (*induct n*)
 prefer *2*
 apply (*simp add*: *drop-Suc bl-shiftr1 butlast-drop* [*symmetric*])
 apply (*rule butlast-take* [*THEN trans*])
apply (*auto simp*: *word-size*)
done

lemma *drop-sshiftr*:
drop n (*to-bl* ((*w* :: *'a* :: *len word*) >>> *n*)) = *take* (*size w* − *n*) (*to-bl w*)
apply (*unfold sshiftr-def*)
apply (*induct n*)
 prefer *2*
 apply (*simp add*: *drop-Suc bl-sshiftr1 butlast-drop* [*symmetric*])
 apply (*rule butlast-take* [*THEN trans*])
apply (*auto simp*: *word-size*)
done

lemma *take-shiftr* [*rule-format*] :
n <= *size* (*w* :: *'a* :: *len word*) −−> *take n* (*to-bl* (*w* >> *n*)) =
 replicate n False
apply (*unfold shiftr-def*)
apply (*induct n*)
 prefer *2*
 apply (*simp add*: *bl-shiftr1*)
 apply (*rule impI*)
 apply (*rule take-butlast* [*THEN trans*])
apply (*auto simp*: *word-size*)
done

```
lemma take-sshiftr' [rule-format] :
  n <= size (w :: 'a :: len word) --> hd (to-bl (w >>> n)) = hd (to-bl w) &
    take n (to-bl (w >>> n)) = replicate n (hd (to-bl w))
  apply (unfold sshiftr-def)
  apply (induct n)
   prefer 2
   apply (simp add: bl-sshiftr1)
   apply (rule impI)
   apply (rule take-butlast [THEN trans])
  apply (auto simp: word-size)
  done

lemmas hd-sshiftr = take-sshiftr' [THEN conjunct1, standard]
lemmas take-sshiftr = take-sshiftr' [THEN conjunct2, standard]

lemma atd-lem: take n xs = t ==> drop n xs = d ==> xs = t @ d
  by (auto intro: append-take-drop-id [symmetric])

lemmas bl-shiftr = atd-lem [OF take-shiftr drop-shiftr]
lemmas bl-sshiftr = atd-lem [OF take-sshiftr drop-sshiftr]

lemma shiftl-of-bl: of-bl bl << n = of-bl (bl @ replicate n False)
  unfolding shiftl-def
  by (induct n) (auto simp: shiftl1-of-bl replicate-app-Cons-same)

lemma shiftl-bl:
  (w::'a::len0 word) << (n::nat) = of-bl (to-bl w @ replicate n False)
proof -
  have w << n = of-bl (to-bl w) << n by simp
  also have ... = of-bl (to-bl w @ replicate n False) by (rule shiftl-of-bl)
  finally show ?thesis .
qed

lemmas shiftl-number [simp] = shiftl-def [where w=number-of w, standard]

lemma bl-shiftl:
  to-bl (w << n) = drop n (to-bl w) @ replicate (min (size w) n) False
  by (simp add: shiftl-bl word-rep-drop word-size min-def)

lemma shiftl-zero-size:
  fixes x :: 'a::len0 word
  shows size x <= n ==> x << n = 0
  apply (unfold word-size)
  apply (rule word-eqI)
  apply (clarsimp simp add: shiftl-bl word-size test-bit-of-bl nth-append)
  done
```

lemma *shiftl1-2t*: *shiftl1* (*w* :: *'a* :: *len word*) = *2* ∗ *w*
  apply (*simp add*: *shiftl1-def-u*)
  apply (*simp only*: *double-number-of-BIT* [*symmetric*])
  apply *simp*
  done

lemma *shiftl1-p*: *shiftl1* (*w* :: *'a* :: *len word*) = *w* + *w*
  apply (*simp add*: *shiftl1-def-u*)
  apply (*simp only*: *double-number-of-BIT* [*symmetric*])
  apply *simp*
  done

lemma *shiftl-t2n*: *shiftl* (*w* :: *'a* :: *len word*) *n* = *2* ˆ *n* ∗ *w*
  unfolding *shiftl-def*
  by (*induct n*) (*auto simp*: *shiftl1-2t power-Suc*)

lemma *shiftr1-bintr* [*simp*]:
  (*shiftr1* (*number-of w*) :: *'a* :: *len0 word*) =
    number-of (*bin-rest* (*bintrunc* (*len-of TYPE* (*'a*)) *w*))
  unfolding *shiftr1-def word-number-of-def*
  by (*simp add* : *word-ubin.eq-norm*)

lemma *sshiftr1-sbintr* [*simp*] :
  (*sshiftr1* (*number-of w*) :: *'a* :: *len word*) =
    number-of (*bin-rest* (*sbintrunc* (*len-of TYPE* (*'a*) − *1*) *w*))
  unfolding *sshiftr1-def word-number-of-def*
  by (*simp add* : *word-sbin.eq-norm*)

lemma *shiftr-no'*:
  w = *number-of bin* ==>
  (*w*::*'a*::*len0 word*) >> *n* = *number-of* ((*bin-rest* ˆ *n*) (*bintrunc* (*size w*) *bin*))
  apply *clarsimp*
  apply (*rule word-eqI*)
  apply (*auto simp*: *nth-shiftr nth-rest-power-bin nth-bintr word-size*)
  done

lemma *sshiftr-no'*:
  w = *number-of bin* ==> *w* >>> *n* = *number-of* ((*bin-rest* ˆ *n*)
    (*sbintrunc* (*size w* − *1*) *bin*))
  apply *clarsimp*
  apply (*rule word-eqI*)
  apply (*auto simp*: *nth-sshiftr nth-rest-power-bin nth-sbintr word-size*)
   apply (*subgoal-tac na* + *n* = *len-of TYPE*(*'a*) − *Suc 0*, *simp*, *simp*)+
  done

lemmas *sshiftr-no* [*simp*] =
  sshiftr-no' [**where** *w* = *number-of w*, *OF refl*, *unfolded word-size*, *standard*]

```
lemmas shiftr-no [simp] =
  shiftr-no' [where w = number-of w, OF refl, unfolded word-size, standard]

lemma shiftr1-bl-of':
  us = shiftr1 (of-bl bl) ==> length bl <= size us ==>
    us = of-bl (butlast bl)
  by (clarsimp simp: shiftr1-def of-bl-def word-size butlast-rest-bl2bin
                   word-ubin.eq-norm trunc-bl2bin)

lemmas shiftr1-bl-of = refl [THEN shiftr1-bl-of', unfolded word-size]

lemma shiftr-bl-of' [rule-format]:
  us = of-bl bl >> n ==> length bl <= size us -->
   us = of-bl (take (length bl - n) bl)
  apply (unfold shiftr-def)
  apply hypsubst
  apply (unfold word-size)
  apply (induct n)
   apply clarsimp
  apply clarsimp
  apply (subst shiftr1-bl-of)
   apply simp
  apply (simp add: butlast-take)
  done

lemmas shiftr-bl-of = refl [THEN shiftr-bl-of', unfolded word-size]

lemmas shiftr-bl = word-bl.Rep' [THEN eq-imp-le, THEN shiftr-bl-of,
  simplified word-size, simplified, THEN eq-reflection, standard]

lemma msb-shift': msb (w::'a::len word) <-> (w >> (size w - 1)) ~= 0
  apply (unfold shiftr-bl word-msb-alt)
  apply (simp add: word-size Suc-le-eq take-Suc)
  apply (cases hd (to-bl w))
   apply (auto simp: word-1-bl word-0-bl
                   of-bl-rep-False [where n=1 and bs=[], simplified])
  done

lemmas msb-shift = msb-shift' [unfolded word-size]

lemma align-lem-or [rule-format] :
  ALL x m. length x = n + m --> length y = n + m -->
    drop m x = replicate n False --> take m y = replicate m False -->
    app2 op | x y = take m x @ drop m y
  apply (induct-tac y)
   apply force
  apply clarsimp
  apply (case-tac x, force)
  apply (case-tac m, auto)
```

apply (*drule sym*)
apply *auto*
apply (*induct-tac list*, *auto*)
done

lemma *align-lem-and* [*rule-format*] :
ALL x m. length x = n + m --> length y = n + m -->
drop m x = replicate n False --> take m y = replicate m False -->
app2 op & x y = replicate (n + m) False
apply (*induct-tac y*)
apply *force*
apply *clarsimp*
apply (*case-tac x*, *force*)
apply (*case-tac m*, *auto*)
apply (*drule sym*)
apply *auto*
apply (*induct-tac list*, *auto*)
done

lemma *aligned-bl-add-size'*:
size x − n = m ==> n <= size x ==> drop m (to-bl x) = replicate n False ==>
take m (to-bl y) = replicate m False ==>
to-bl (x + y) = take m (to-bl x) @ drop m (to-bl y)
apply (*subgoal-tac x AND y = 0*)
prefer *2*
apply (*rule word-bl.Rep-eqD*)
apply (*simp add*: *bl-word-and to-bl-0*)
apply (*rule align-lem-and* [*THEN trans*])
apply (*simp-all add*: *word-size*)[*5*]
apply (*rule-tac f = %n. replicate n False* **in** *arg-cong*)
apply *simp*
apply (*subst word-plus-and-or* [*symmetric*])
apply (*simp add* : *bl-word-or*)
apply (*rule align-lem-or*)
apply (*simp-all add*: *word-size*)
done

lemmas *aligned-bl-add-size* = *refl* [*THEN aligned-bl-add-size'*]

13.1.2 Mask

lemma *nth-mask'*: *m = mask n ==> test-bit m i = (i < n & i < size m)*
apply (*unfold mask-def test-bit-bl*)
apply (*simp only*: *word-1-bl* [*symmetric*] *shiftl-of-bl*)
apply (*clarsimp simp add*: *word-size*)
apply (*simp only*: *of-bl-no mask-lem number-of-succ add-diff-cancel2*)
apply (*fold of-bl-no*)
apply (*simp add*: *word-1-bl*)

apply (*rule test-bit-of-bl* [*THEN trans*, *unfolded test-bit-bl word-size*])
apply *auto*
done

lemmas *nth-mask* [*simp*] = *refl* [*THEN nth-mask'*]

lemma *mask-bl*: *mask n* = *of-bl* (*replicate n True*)
by (*auto simp add* : *test-bit-of-bl word-size intro*: *word-eqI*)

lemma *mask-bin*: *mask n* = *number-of* (*bintrunc n Numeral.Min*)
by (*auto simp add*: *nth-bintr word-size intro*: *word-eqI*)

lemma *and-mask-bintr*: *w AND mask n* = *number-of* (*bintrunc n* (*uint w*))
apply (*rule word-eqI*)
apply (*simp add*: *nth-bintr word-size word-ops-nth-size*)
apply (*auto simp add*: *test-bit-bin*)
done

lemma *and-mask-no*: *number-of i AND mask n* = *number-of* (*bintrunc n i*)
by (*auto simp add* : *nth-bintr word-size word-ops-nth-size intro*: *word-eqI*)

lemmas *and-mask-wi* = *and-mask-no* [*unfolded word-number-of-def*]

lemma *bl-and-mask*:
to-bl (*w AND mask n* :: *'a* :: *len word*) =
replicate (*len-of TYPE*(*'a*) − *n*) *False* @
drop (*len-of TYPE*(*'a*) − *n*) (*to-bl w*)
apply (*rule nth-equalityI*)
apply *simp*
apply (*clarsimp simp add*: *to-bl-nth word-size*)
apply (*simp add*: *word-size word-ops-nth-size*)
apply (*auto simp add*: *word-size test-bit-bl nth-append nth-rev*)
done

lemmas *and-mask-mod-2p* =
and-mask-bintr [*unfolded word-number-of-alt no-bintr-alt*]

lemma *and-mask-lt-2p*: *uint* (*w AND mask n*) < *2* ˆ *n*
apply (*simp add* : *and-mask-bintr no-bintr-alt*)
apply (*rule xtr8*)
prefer *2*
apply (*rule pos-mod-bound*)
apply *auto*
done

lemmas *eq-mod-iff* = *trans* [*symmetric*, *OF int-mod-lem eq-sym-conv*]

lemma *mask-eq-iff*: (*w AND mask n*) = *w* <−> *uint w* < *2* ˆ *n*
apply (*simp add*: *and-mask-bintr word-number-of-def*)

apply (*simp add*: *word-ubin.inverse-norm*)
apply (*simp add*: *eq-mod-iff bintrunc-mod2p min-def*)
apply (*fast intro*!: *lt2p-lem*)
done

lemma *and-mask-dvd*: *2 ^ n dvd uint w* = (*w AND mask n* = *0*)
apply (*simp add*: *zdvd-iff-zmod-eq-0 and-mask-mod-2p*)
apply (*simp add*: *word-uint.norm-eq-iff* [*symmetric*] *word-of-int-homs*)
apply (*subst word-uint.norm-Rep* [*symmetric*])
apply (*simp only*: *bintrunc-bintrunc-min bintrunc-mod2p* [*symmetric*] *min-def*)
apply *auto*
done

lemma *and-mask-dvd-nat*: *2 ^ n dvd unat w* = (*w AND mask n* = *0*)
apply (*unfold unat-def*)
apply (*rule trans* [*OF - and-mask-dvd*])
apply (*unfold dvd-def*)
apply *auto*
apply (*drule uint-ge-0* [*THEN nat-int.Abs-inverse′* [*simplified*], *symmetric*])
apply (*simp add* : *int-mult int-power*)
apply (*simp add* : *nat-mult-distrib nat-power-eq*)
done

lemma *word-2p-lem*:
n < *size w* ==> *w* < *2 ^ n* = (*uint* (*w* :: ′*a* :: *len word*) < *2 ^ n*)
apply (*unfold word-size word-less-alt word-number-of-alt*)
apply (*clarsimp simp add*: *word-of-int-power-hom word-uint.eq-norm*
int-mod-eq′
simp del: *word-of-int-bin*)
done

lemma *less-mask-eq*: *x* < *2 ^ n* ==> *x AND mask n* = (*x* :: ′*a* :: *len word*)
apply (*unfold word-less-alt word-number-of-alt*)
apply (*clarsimp simp add*: *and-mask-mod-2p word-of-int-power-hom*
word-uint.eq-norm
simp del: *word-of-int-bin*)
apply (*drule xtr8* [*rotated*])
apply (*rule int-mod-le*)
apply (*auto simp add* : *mod-pos-pos-trivial*)
done

lemmas *mask-eq-iff-w2p* =
trans [*OF mask-eq-iff word-2p-lem* [*symmetric*], *standard*]

lemmas *and-mask-less′* =
iffD2 [*OF word-2p-lem and-mask-lt-2p*, *simplified word-size*, *standard*]

lemma *and-mask-less-size*: *n* < *size x* ==> *x AND mask n* < *2^n*
unfolding *word-size* **by** (*erule and-mask-less′*)

lemma *word-mod-2p-is-mask'*:
  c = 2 ^ n ==> c > 0 ==> x mod c = (x :: 'a :: len word) AND mask n
  by (*clarsimp simp add*: *word-mod-def uint-2p and-mask-mod-2p*)

lemmas *word-mod-2p-is-mask = refl* [*THEN word-mod-2p-is-mask'*]

lemma *mask-eqs*:
  (*a AND mask n*) + *b AND mask n* = *a* + *b AND mask n*
  a + (*b AND mask n*) *AND mask n* = *a* + *b AND mask n*
  (*a AND mask n*) − *b AND mask n* = *a* − *b AND mask n*
  a − (*b AND mask n*) *AND mask n* = *a* − *b AND mask n*
  a ∗ (*b AND mask n*) *AND mask n* = *a* ∗ *b AND mask n*
  (*b AND mask n*) ∗ *a AND mask n* = *b* ∗ *a AND mask n*
  (*a AND mask n*) + (*b AND mask n*) *AND mask n* = *a* + *b AND mask n*
  (*a AND mask n*) − (*b AND mask n*) *AND mask n* = *a* − *b AND mask n*
  (*a AND mask n*) ∗ (*b AND mask n*) *AND mask n* = *a* ∗ *b AND mask n*
  − (*a AND mask n*) *AND mask n* = − *a AND mask n*
  word-succ (*a AND mask n*) *AND mask n* = *word-succ a AND mask n*
  word-pred (*a AND mask n*) *AND mask n* = *word-pred a AND mask n*
  using *word-of-int-Ex* [**where** *x=a*] *word-of-int-Ex* [**where** *x=b*]
  by (*auto simp*: *and-mask-wi bintr-ariths bintr-arith1s new-word-of-int-homs*)

lemma *mask-power-eq*:
  (*x AND mask n*) ^ *k AND mask n* = *x* ^ *k AND mask n*
  using *word-of-int-Ex* [**where** *x=x*]
  by (*clarsimp simp*: *and-mask-wi word-of-int-power-hom bintr-ariths*)

13.1.3 Revcast

lemmas *revcast-def' = revcast-def* [*simplified*]
lemmas *revcast-def'' = revcast-def'* [*simplified word-size*]
lemmas *revcast-no-def* [*simp*] =
  revcast-def' [**where** *w=number-of w*, *unfolded word-size*, *standard*]

lemma *to-bl-revcast*:
  to-bl (*revcast w* :: *'a* :: *len0 word*) =
   takefill False (*len-of TYPE* (*'a*)) (*to-bl w*)
  apply (*unfold revcast-def' word-size*)
  apply (*rule word-bl.Abs-inverse*)
  apply *simp*
  done

lemma *revcast-rev-ucast'*:
  cs = [*rc*, *uc*] ==> *rc* = *revcast* (*word-reverse w*) ==> *uc* = *ucast w* ==>
   rc = *word-reverse uc*
  apply (*unfold ucast-def revcast-def' Let-def word-reverse-def*)
  apply (*clarsimp simp add* : *to-bl-of-bin takefill-bintrunc*)
  apply (*simp add* : *word-bl.Abs-inverse word-size*)

```
  done

lemmas revcast-rev-ucast = revcast-rev-ucast' [OF refl refl refl]

lemmas revcast-ucast = revcast-rev-ucast
  [where w = word-reverse w, simplified word-rev-rev, standard]

lemmas ucast-revcast = revcast-rev-ucast [THEN word-rev-gal', standard]
lemmas ucast-rev-revcast = revcast-ucast [THEN word-rev-gal', standard]
```

— linking revcast and cast via shift

```
lemmas wsst-TYs = source-size target-size word-size

lemma revcast-down-uu':
  rc = revcast ==> source-size rc = target-size rc + n ==>
    rc (w :: 'a :: len word) = ucast (w >> n)
  apply (simp add: revcast-def')
  apply (rule word-bl.Rep-inverse')
  apply (rule trans, rule ucast-down-drop)
   prefer 2
   apply (rule trans, rule drop-shiftr)
   apply (auto simp: takefill-alt wsst-TYs)
  done

lemma revcast-down-us':
  rc = revcast ==> source-size rc = target-size rc + n ==>
    rc (w :: 'a :: len word) = ucast (w >>> n)
  apply (simp add: revcast-def')
  apply (rule word-bl.Rep-inverse')
  apply (rule trans, rule ucast-down-drop)
   prefer 2
   apply (rule trans, rule drop-sshiftr)
   apply (auto simp: takefill-alt wsst-TYs)
  done

lemma revcast-down-su':
  rc = revcast ==> source-size rc = target-size rc + n ==>
    rc (w :: 'a :: len word) = scast (w >> n)
  apply (simp add: revcast-def')
  apply (rule word-bl.Rep-inverse')
  apply (rule trans, rule scast-down-drop)
   prefer 2
   apply (rule trans, rule drop-shiftr)
   apply (auto simp: takefill-alt wsst-TYs)
  done

lemma revcast-down-ss':
```

rc = revcast ==> source-size rc = target-size rc + n ==>
rc (w :: 'a :: len word) = scast (w >>> n)
apply (*simp add*: *revcast-def'*)
apply (*rule word-bl.Rep-inverse'*)
apply (*rule trans*, *rule scast-down-drop*)
prefer *2*
apply (*rule trans*, *rule drop-sshiftr*)
apply (*auto simp*: *takefill-alt wsst-TYs*)
done

lemmas *revcast-down-uu = refl* [*THEN revcast-down-uu'*]
lemmas *revcast-down-us = refl* [*THEN revcast-down-us'*]
lemmas *revcast-down-su = refl* [*THEN revcast-down-su'*]
lemmas *revcast-down-ss = refl* [*THEN revcast-down-ss'*]

lemma *cast-down-rev*:
uc = ucast ==> source-size uc = target-size uc + n ==>
uc w = revcast ((w :: 'a :: len word) << n)
apply (*unfold shiftl-rev*)
apply *clarify*
apply (*simp add*: *revcast-rev-ucast*)
apply (*rule word-rev-gal'*)
apply (*rule trans* [*OF - revcast-rev-ucast*])
apply (*rule revcast-down-uu* [*symmetric*])
apply (*auto simp add*: *wsst-TYs*)
done

lemma *revcast-up'*:
rc = revcast ==> source-size rc + n = target-size rc ==>
rc w = (ucast w :: 'a :: len word) << n
apply (*simp add*: *revcast-def'*)
apply (*rule word-bl.Rep-inverse'*)
apply (*simp add*: *takefill-alt*)
apply (*rule bl-shiftl* [*THEN trans*])
apply (*subst ucast-up-app*)
apply (*auto simp add*: *wsst-TYs*)
apply (*drule sym*)
apply (*simp add*: *min-def*)
done

lemmas *revcast-up = refl* [*THEN revcast-up'*]

lemmas *rc1 = revcast-up* [*THEN*
revcast-rev-ucast [*symmetric*, *THEN trans*, *THEN word-rev-gal*, *symmetric*]]
lemmas *rc2 = revcast-down-uu* [*THEN*
revcast-rev-ucast [*symmetric*, *THEN trans*, *THEN word-rev-gal*, *symmetric*]]

lemmas *ucast-up =*
rc1 [*simplified rev-shiftr* [*symmetric*] *revcast-ucast* [*symmetric*]]

lemmas *ucast-down* =
rc2 [*simplified rev-shiftr revcast-ucast* [*symmetric*]]

13.1.4 Slices

lemmas *slice1-no-bin* [*simp*] =
slice1-def [**where** *w*=*number-of w*, *unfolded to-bl-no-bin*, *standard*]

lemmas *slice-no-bin* [*simp*] =
trans [*OF slice-def* [*THEN meta-eq-to-obj-eq*]
slice1-no-bin [*THEN meta-eq-to-obj-eq*],
unfolded word-size, *standard*]

lemma *slice1-0* [*simp*] : *slice1 n 0* = *0*
unfolding *slice1-def* **by** (*simp add* : *to-bl-0*)

lemma *slice-0* [*simp*] : *slice n 0* = *0*
unfolding *slice-def* **by** *auto*

lemma *slice-take'*: *slice n w* = *of-bl* (*take* (*size w* − *n*) (*to-bl w*))
unfolding *slice-def' slice1-def*
by (*simp add* : *takefill-alt word-size*)

lemmas *slice-take* = *slice-take'* [*unfolded word-size*]

— shiftr to a word of the same size is just slice, slice is just shiftr then ucast
lemmas *shiftr-slice* = *trans*
[*OF shiftr-bl* [*THEN meta-eq-to-obj-eq*] *slice-take* [*symmetric*], *standard*]

lemma *slice-shiftr*: *slice n w* = *ucast* (*w* >> *n*)
apply (*unfold slice-take shiftr-bl*)
apply (*rule ucast-of-bl-up* [*symmetric*])
apply (*simp add*: *word-size*)
done

lemma *nth-slice*:
(*slice n w* :: *'a* :: *len0 word*) !! *m* =
(*w* !! (*m* + *n*) & *m* < *len-of TYPE* (*'a*))
unfolding *slice-shiftr*
by (*simp add* : *nth-ucast nth-shiftr*)

lemma *slice1-down-alt'*:
sl = *slice1 n w* ==> *fs* = *size sl* ==> *fs* + *k* = *n* ==>
to-bl sl = *takefill False fs* (*drop k* (*to-bl w*))
unfolding *slice1-def word-size of-bl-def uint-bl*
by (*clarsimp simp*: *word-ubin.eq-norm bl-bin-bl-rep-drop drop-takefill*)

lemma *slice1-up-alt'*:
sl = *slice1 n w* ==> *fs* = *size sl* ==> *fs* = *n* + *k* ==>

to-bl sl = takefill False fs (replicate k False @ (to-bl w))
apply (*unfold slice1-def word-size of-bl-def uint-bl*)
apply (*clarsimp simp*: *word-ubin.eq-norm bl-bin-bl-rep-drop takefill-append* [*symmetric*])
apply (*rule-tac f = %k. takefill False (len-of TYPE('a)) (replicate k False @ bin-to-bl (len-of TYPE('b)) (uint w))* **in** *arg-cong*)
apply *arith*
done

lemmas *sd1 = slice1-down-alt'* [*OF refl refl*, *unfolded word-size*]
lemmas *su1 = slice1-up-alt'* [*OF refl refl*, *unfolded word-size*]
lemmas *slice1-down-alt = le-add-diff-inverse* [*THEN sd1*]
lemmas *slice1-up-alts* =
le-add-diff-inverse [*symmetric*, *THEN su1*]
le-add-diff-inverse2 [*symmetric*, *THEN su1*]

lemma *ucast-slice1*: *ucast w = slice1 (size w) w*
unfolding *slice1-def ucast-bl*
by (*simp add* : *takefill-same' word-size*)

lemma *ucast-slice*: *ucast w = slice 0 w*
unfolding *slice-def* **by** (*simp add* : *ucast-slice1*)

lemmas *slice-id = trans* [*OF ucast-slice* [*symmetric*] *ucast-id*]

lemma *revcast-slice1'*:
rc = revcast w ==> slice1 (size rc) w = rc
unfolding *slice1-def revcast-def'* **by** (*simp add* : *word-size*)

lemmas *revcast-slice1 = refl* [*THEN revcast-slice1'*]

lemma *slice1-tf-tf'*:
to-bl (slice1 n w :: 'a :: len0 word) =
rev (takefill False (len-of TYPE('a)) (rev (takefill False n (to-bl w))))
unfolding *slice1-def* **by** (*rule word-rev-tf*)

lemmas *slice1-tf-tf = slice1-tf-tf'*
[*THEN word-bl.Rep-inverse'*, *symmetric*, *standard*]

lemma *rev-slice1*:
n + k = len-of TYPE('a) + len-of TYPE('b) $\Longrightarrow$
slice1 n (word-reverse w :: 'b :: len0 word) =
word-reverse (slice1 k w :: 'a :: len0 word)
apply (*unfold word-reverse-def slice1-tf-tf*)
apply (*rule word-bl.Rep-inverse'*)
apply (*rule rev-swap* [*THEN iffD1*])
apply (*rule trans* [*symmetric*])
apply (*rule tf-rev*)
apply (*simp add*: *word-bl.Abs-inverse*)

```
  apply (simp add: word-bl.Abs-inverse)
  done

lemma rev-slice':
  res = slice n (word-reverse w) ==> n + k + size res = size w ==>
    res = word-reverse (slice k w)
  apply (unfold slice-def word-size)
  apply clarify
  apply (rule rev-slice1)
  apply arith
  done

lemmas rev-slice = refl [THEN rev-slice', unfolded word-size]

lemmas sym-notr =
  not-iff [THEN iffD2, THEN not-sym, THEN not-iff [THEN iffD1]]
```

— problem posed by TPHOLs referee: criterion for overflow of addition of signed integers

```
lemma sofl-test:
  (sint (x :: 'a :: len word) + sint y = sint (x + y)) =
     ((((x+y) XOR x) AND ((x+y) XOR y)) >> (size x - 1) = 0)
  apply (unfold word-size)
  apply (cases len-of TYPE('a), simp)
  apply (subst msb-shift [THEN sym-notr])
  apply (simp add: word-ops-msb)
  apply (simp add: word-msb-sint)
  apply safe
       apply simp-all
     apply (unfold sint-word-ariths)
     apply (unfold word-sbin.set-iff-norm [symmetric] sints-num)
     apply safe
        apply (insert sint-range' [where x=x])
        apply (insert sint-range' [where x=y])
        defer
        apply (simp (no-asm), arith)
       apply (simp (no-asm), arith)
      defer
      defer
      apply (simp (no-asm), arith)
     apply (simp (no-asm), arith)
    apply (rule notI [THEN notnotD],
           drule leI not-leE,
           drule sbintrunc-inc sbintrunc-dec,
           simp)+
  done
```

13.2 Split and cat

lemmas *word-split-bin′* = *word-split-def* [*THEN meta-eq-to-obj-eq*, *standard*]
lemmas *word-cat-bin′* = *word-cat-def* [*THEN meta-eq-to-obj-eq*, *standard*]

lemma *word-rsplit-no*:
  (*word-rsplit* (*number-of bin* :: ′*b* :: *len0 word*) :: ′*a word list*) =
    map number-of (*bin-rsplit* (*len-of TYPE*(′*a* :: *len*))
      (*len-of TYPE*(′*b*), *bintrunc* (*len-of TYPE*(′*b*)) *bin*))
  apply (*unfold word-rsplit-def word-no-wi*)
  apply (*simp add*: *word-ubin.eq-norm*)
  done

lemmas *word-rsplit-no-cl* [*simp*] = *word-rsplit-no*
  [*unfolded bin-rsplitl-def bin-rsplit-l* [*symmetric*]]

lemma *test-bit-cat*:
  wc = *word-cat a b* ==> *wc* !! *n* = (*n* < *size wc* &
    (*if n* < *size b then b* !! *n else a* !! (*n* − *size b*)))
  apply (*unfold word-cat-bin′ test-bit-bin*)
  apply (*auto simp add* : *word-ubin.eq-norm nth-bintr bin-nth-cat word-size*)
  apply (*erule bin-nth-uint-imp*)
  done

lemma *word-cat-bl*: *word-cat a b* = *of-bl* (*to-bl a* @ *to-bl b*)
  apply (*unfold of-bl-def to-bl-def word-cat-bin′*)
  apply (*simp add*: *bl-to-bin-app-cat*)
  done

lemma *of-bl-append*:
  (*of-bl* (*xs* @ *ys*) :: ′*a* :: *len word*) = *of-bl xs* ∗ *2*ˆ(*length ys*) + *of-bl ys*
  apply (*unfold of-bl-def*)
  apply (*simp add*: *bl-to-bin-app-cat bin-cat-num*)
  apply (*simp add*: *word-of-int-power-hom* [*symmetric*] *new-word-of-int-hom-syms*)
  done

lemma *of-bl-False* [*simp*]:
  of-bl (*False*#*xs*) = *of-bl xs*
  by (*rule word-eqI*)
    (*auto simp add*: *test-bit-of-bl nth-append*)

lemma *of-bl-True*:
  (*of-bl* (*True*#*xs*)::′*a*::*len word*) = *2*ˆ*length xs* + *of-bl xs*
  by (*subst of-bl-append* [**where** *xs*=[*True*], *simplified*])
    (*simp add*: *word-1-bl*)

lemma *of-bl-Cons*:
  of-bl (*x*#*xs*) = *of-bool x* ∗ *2*ˆ*length xs* + *of-bl xs*
  by (*cases x*) (*simp-all add*: *of-bl-True*)

lemma *split-uint-lem*: *bin-split n* (*uint* (*w* :: *'a* :: *len0 word*)) = (*a*, *b*) ==>
  a = *bintrunc* (*len-of TYPE*(*'a*) − *n*) *a* & *b* = *bintrunc* (*len-of TYPE*(*'a*)) *b*
  apply (*frule word-ubin.norm-Rep* [*THEN ssubst*])
  apply (*drule bin-split-trunc1*)
  apply (*drule sym* [*THEN trans*])
  apply *assumption*
  apply *safe*
  done

lemma *word-split-bl'*:
  std = *size c* − *size b* ==> (*word-split c* = (*a*, *b*)) ==>
    (*a* = *of-bl* (*take std* (*to-bl c*)) & *b* = *of-bl* (*drop std* (*to-bl c*)))
  apply (*unfold word-split-bin'*)
  apply *safe*
   defer
   apply (*clarsimp split*: *prod.splits*)
   apply (*drule word-ubin.norm-Rep* [*THEN ssubst*])
   apply (*drule split-bintrunc*)
   apply (*simp add* : *of-bl-def bl2bin-drop word-size*
      word-ubin.norm-eq-iff [*symmetric*] *min-def del* : *word-ubin.norm-Rep*)
  apply (*clarsimp split*: *prod.splits*)
  apply (*frule split-uint-lem* [*THEN conjunct1*])
  apply (*unfold word-size*)
  apply (*cases len-of TYPE*(*'a*) >= *len-of TYPE*(*'b*))
   defer
   apply (*simp add*: *word-0-bl word-0-wi-Pls*)
  apply (*simp add* : *of-bl-def to-bl-def*)
  apply (*subst bin-split-take1* [*symmetric*])
    prefer *2*
    apply *assumption*
   apply *simp*
  apply (*erule thin-rl*)
  apply (*erule arg-cong* [*THEN trans*])
  apply (*simp add* : *word-ubin.norm-eq-iff* [*symmetric*])
  done

lemma *word-split-bl*: *std* = *size c* − *size b* ==>
    (*a* = *of-bl* (*take std* (*to-bl c*)) & *b* = *of-bl* (*drop std* (*to-bl c*))) <−>
    word-split c = (*a*, *b*)
  apply (*rule iffI*)
   defer
   apply (*erule* (*1*) *word-split-bl'*)
  apply (*case-tac word-split c*)
  apply (*auto simp add* : *word-size*)
  apply (*frule word-split-bl'* [*rotated*])
  apply (*auto simp add* : *word-size*)
  done

lemma *word-split-bl-eq*:

(word-split (c::'a::len word) :: ('c :: len0 word * 'd :: len0 word)) =
(of-bl (take (len-of TYPE('a::len) − len-of TYPE('d::len0)) (to-bl c)),
of-bl (drop (len-of TYPE('a) − len-of TYPE('d)) (to-bl c)))
apply (rule word-split-bl [THEN iffD1])
apply (unfold word-size)
apply (rule refl conjI)+
done

— keep quantifiers for use in simplification
lemma test-bit-split':
word-split c = (a, b) −−> (ALL n m. b !! n = (n < size b & c !! n) &
a !! m = (m < size a & c !! (m + size b)))
apply (unfold word-split-bin' test-bit-bin)
apply (clarify)
apply (clarsimp simp: word-ubin.eq-norm nth-bintr word-size split: prod.splits)
apply (drule bin-nth-split)
apply safe
apply (simp-all add: add-commute)
apply (erule bin-nth-uint-imp)+
done

lemmas test-bit-split =
test-bit-split' [THEN mp, simplified all-simps, standard]

lemma test-bit-split-eq: word-split c = (a, b) <−>
((ALL n::nat. b !! n = (n < size b & c !! n)) &
(ALL m::nat. a !! m = (m < size a & c !! (m + size b))))
apply (rule-tac iffI)
apply (rule-tac conjI)
apply (erule test-bit-split [THEN conjunct1])
apply (erule test-bit-split [THEN conjunct2])
apply (case-tac word-split c)
apply (frule test-bit-split)
apply (erule trans)
apply (fastsimp intro ! : word-eqI simp add : word-size)
done

— this odd result is analogous to ucast_id, result to the length given by the result type

lemma word-cat-id: word-cat a b = b
unfolding word-cat-bin' **by** (simp add: word-ubin.inverse-norm)

— limited hom result
lemma word-cat-hom:
len-of TYPE('a::len0) <= len-of TYPE('b::len0) + len-of TYPE ('c::len0)
==>
(word-cat (word-of-int w :: 'b word) (b :: 'c word) :: 'a word) =
word-of-int (bin-cat w (size b) (uint b))

apply (*unfold word-cat-def word-size*)
apply (*clarsimp simp add* : *word-ubin.norm-eq-iff* [*symmetric*]
word-ubin.eq-norm bintr-cat min-def)
apply *arith*
done

lemma *word-cat-split-alt*:
size w <= size u + size v ==> word-split w = (u, v) ==> word-cat u v = w
apply (*rule word-eqI*)
apply (*drule test-bit-split*)
apply (*clarsimp simp add* : *test-bit-cat word-size*)
apply *safe*
apply *arith*
done

lemmas *word-cat-split-size* =
sym [*THEN* [*2*] *word-cat-split-alt* [*symmetric*], *standard*]

13.2.1 Split and slice

lemma *split-slices*:
word-split w = (u, v) ==> u = slice (size v) w & v = slice 0 w
apply (*drule test-bit-split*)
apply (*rule conjI*)
apply (*rule word-eqI*, *clarsimp simp*: *nth-slice word-size*)+
done

lemma *slice-cat1'*:
wc = word-cat a b ==> size wc >= size a + size b ==> slice (size b) wc = a
apply *safe*
apply (*rule word-eqI*)
apply (*simp add*: *nth-slice test-bit-cat word-size*)
done

lemmas *slice-cat1* = *refl* [*THEN slice-cat1'*]
lemmas *slice-cat2* = *trans* [*OF slice-id word-cat-id*]

lemma *cat-slices*:
a = slice n c ==> b = slice 0 c ==> n = size b ==>
size a + size b >= size c ==> word-cat a b = c
apply *safe*
apply (*rule word-eqI*)
apply (*simp add*: *nth-slice test-bit-cat word-size*)
apply *safe*
apply *arith*
done

lemma *word-split-cat-alt*:
w = word-cat u v ==> size u + size v <= size w ==> word-split w = (u, v)

```
  apply (case-tac word-split ?w)
  apply (rule trans, assumption)
  apply (drule test-bit-split)
  apply safe
   apply (rule word-eqI, clarsimp simp: test-bit-cat word-size)+
  done

lemmas word-cat-bl-no-bin [simp] =
  word-cat-bl [where a=number-of a
                and b=number-of b,
              unfolded to-bl-no-bin, standard]

lemmas word-split-bl-no-bin [simp] =
  word-split-bl-eq [where c=number-of c, unfolded to-bl-no-bin, standard]
```

— this odd result arises from the fact that the statement of the result implies that the decoded words are of the same type, and therefore of the same length, as the original word

```
lemma word-rsplit-same: word-rsplit w = [w]
  unfolding word-rsplit-def by (simp add : bin-rsplit-all)

lemma word-rsplit-empty-iff-size:
  (word-rsplit w = []) = (size w = 0)
  unfolding word-rsplit-def bin-rsplit-def word-size
  by (simp add: bin-rsplit-aux-simp-alt Let-def split: split-split)

lemma test-bit-rsplit:
  sw = word-rsplit w ==> m < size (hd sw :: 'a :: len word) ==>
    k < length sw ==> (rev sw ! k) !! m = (w !! (k * size (hd sw) + m))
  apply (unfold word-rsplit-def word-test-bit-def)
  apply (rule trans)
   apply (rule-tac f = %x. bin-nth x m in arg-cong)
   apply (rule nth-map [symmetric])
   apply simp
  apply (rule bin-nth-rsplit)
     apply simp-all
  apply (simp add : word-size rev-map map-compose [symmetric])
  apply (rule trans)
   defer
   apply (rule map-ident [THEN fun-cong])
  apply (rule refl [THEN map-cong])
  apply (simp add : word-ubin.eq-norm)
  apply (erule bin-rsplit-size-sign [OF len-gt-0 refl])
  done

lemma word-rcat-bl: word-rcat wl == of-bl (concat (map to-bl wl))
  unfolding word-rcat-def to-bl-def' of-bl-def
  by (clarsimp simp add : bin-rcat-bl map-compose)
```

lemma *size-rcat-lem'*:
size (*concat* (*map to-bl wl*)) = *length wl* ∗ *size* (*hd wl*)
unfolding *word-size* **by** (*induct wl*) *auto*

lemmas *size-rcat-lem* = *size-rcat-lem'* [*unfolded word-size*]

lemmas *td-gal-lt-len* = *len-gt-0* [*THEN td-gal-lt*, *standard*]

lemma *nth-rcat-lem'* [*rule-format*] :
sw = *size* (*hd wl* :: *'a* :: *len word*) ==> (*ALL n*. *n* < *size wl* ∗ *sw* −−>
rev (*concat* (*map to-bl wl*)) ! *n* =
rev (*to-bl* (*rev wl* ! (*n div sw*))) ! (*n mod sw*))
apply (*unfold word-size*)
apply (*induct wl*)
apply *clarsimp*
apply (*clarsimp simp add* : *nth-append size-rcat-lem*)
apply (*simp* (*no-asm-use*) *only*: *mult-Suc* [*symmetric*]
td-gal-lt-len less-Suc-eq-le mod-div-equality')
apply *clarsimp*
done

lemmas *nth-rcat-lem* = *refl* [*THEN nth-rcat-lem'*, *unfolded word-size*]

lemma *test-bit-rcat*:
sw = *size* (*hd wl* :: *'a* :: *len word*) ==> *rc* = *word-rcat wl* ==> *rc* !! *n* =
(*n* < *size rc* & *n div sw* < *size wl* & (*rev wl*) ! (*n div sw*) !! (*n mod sw*))
apply (*unfold word-rcat-bl word-size*)
apply (*clarsimp simp add* :
test-bit-of-bl size-rcat-lem word-size td-gal-lt-len)
apply *safe*
apply (*auto simp add* :
test-bit-bl word-size td-gal-lt-len [*THEN iffD2*, *THEN nth-rcat-lem*])
done

lemma *foldl-eq-foldr* [*rule-format*] :
ALL x. *foldl op* + *x xs* = *foldr op* + (*x* # *xs*) (*0* :: *'a* :: *comm-monoid-add*)
by (*induct xs*) (*auto simp add* : *add-assoc*)

lemmas *test-bit-cong* = *arg-cong* [**where** *f* = *test-bit*, *THEN fun-cong*]

lemmas *test-bit-rsplit-alt* =
trans [*OF nth-rev-alt* [*THEN test-bit-cong*]
test-bit-rsplit [*OF refl asm-rl diff-Suc-less*]]

— lazy way of expressing that u and v, and su and sv, have same types
lemma *word-rsplit-len-indep'*:
[*u*,*v*] = *p* ==> [*su*,*sv*] = *q* ==> *word-rsplit u* = *su* ==>
word-rsplit v = *sv* ==> *length su* = *length sv*

```
  apply (unfold word-rsplit-def)
  apply (auto simp add : bin-rsplit-len-indep)
  done

lemmas word-rsplit-len-indep = word-rsplit-len-indep' [OF refl refl refl refl]

lemma length-word-rsplit-size:
  n = len-of TYPE ('a :: len) ==>
    (length (word-rsplit w :: 'a word list) <= m) = (size w <= m * n)
  apply (unfold word-rsplit-def word-size)
  apply (clarsimp simp add : bin-rsplit-len-le)
  done

lemmas length-word-rsplit-lt-size =
  length-word-rsplit-size [unfolded Not-eq-iff linorder-not-less [symmetric]]

lemma length-word-rsplit-exp-size:
  n = len-of TYPE ('a :: len) ==>
    length (word-rsplit w :: 'a word list) = (size w + n - 1) div n
  unfolding word-rsplit-def by (clarsimp simp add : word-size bin-rsplit-len)

lemma length-word-rsplit-even-size:
  n = len-of TYPE ('a :: len) ==> size w = m * n ==>
    length (word-rsplit w :: 'a word list) = m
  by (clarsimp simp add : length-word-rsplit-exp-size given-quot-alt)

lemmas length-word-rsplit-exp-size' = refl [THEN length-word-rsplit-exp-size]


lemmas tdle = iffD2 [OF split-div-lemma refl, THEN conjunct1]
lemmas dtle = xtr4 [OF tdle mult-commute]

lemma word-rcat-rsplit: word-rcat (word-rsplit w) = w
  apply (rule word-eqI)
  apply (clarsimp simp add : test-bit-rcat word-size)
  apply (subst refl [THEN test-bit-rsplit])
    apply (simp-all add: word-size
      refl [THEN length-word-rsplit-size [simplified le-def, simplified]])
   apply safe
   apply (erule xtr7, rule len-gt-0 [THEN dtle])+
  done

lemma size-word-rsplit-rcat-size':
  word-rcat (ws :: 'a :: len word list) = frcw ==>
    size frcw = length ws * len-of TYPE ('a) ==>
    size (hd [word-rsplit frcw, ws]) = size ws
  apply (clarsimp simp add : word-size length-word-rsplit-exp-size')
  apply (fast intro: given-quot-alt)
  done
```

lemmas *size-word-rsplit-rcat-size* =
size-word-rsplit-rcat-size' [*simplified*]

lemma *msrevs*:
fixes *n*::*nat*
shows $0 < n \Longrightarrow (k * n + m)$ *div* $n = m$ *div* $n + k$
and $(k * n + m)$ *mod* $n = m$ *mod* n
by (*auto simp*: *add-commute*)

lemma *word-rsplit-rcat-size'*:
word-rcat (*ws* :: *'a* :: *len word list*) = *frcw* ==>
size frcw = *length ws* ∗ *len-of TYPE* (*'a*) ==> *word-rsplit frcw* = *ws*
apply (*frule size-word-rsplit-rcat-size*, *assumption*)
apply (*clarsimp simp add* : *word-size*)
apply (*rule nth-equalityI*, *assumption*)
apply *clarsimp*
apply (*rule word-eqI*)
apply (*rule trans*)
apply (*rule test-bit-rsplit-alt*)
apply (*clarsimp simp*: *word-size*)+
apply (*rule trans*)
apply (*rule test-bit-rcat* [*OF refl refl*])
apply (*simp add* : *word-size msrevs*)
apply (*subst nth-rev*)
apply *arith*
apply (*simp add* : *le0* [*THEN* [*2*] *xtr7*, *THEN diff-Suc-less*])
apply *safe*
apply (*simp add* : *diff-mult-distrib*)
apply (*rule mpl-lem*)
apply (*cases size ws*)
apply *simp-all*
done

lemmas *word-rsplit-rcat-size* = *refl* [*THEN word-rsplit-rcat-size'*]

13.3 Rotation

lemmas *rotater-0'* [*simp*] = *rotater-def* [**where** *n* = *0*, *simplified*]

lemmas *word-rot-defs* = *word-roti-def word-rotr-def word-rotl-def*

lemma *rotate-eq-mod*:
m mod length xs = *n mod length xs* ==> *rotate m xs* = *rotate n xs*
apply (*rule box-equals*)
defer
apply (*rule rotate-conv-mod* [*symmetric*])+
apply *simp*
done

lemmas *rotate-eqs* [*standard*] =
  trans [*OF rotate0* [*THEN fun-cong*] *id-apply*]
  rotate-rotate [*symmetric*]
  rotate-id
  rotate-conv-mod
  rotate-eq-mod

13.3.1 Rotation of list to right

lemma *rotate1-rl'*: *rotater1* (*l* @ [*a*]) = *a* # *l*
  unfolding *rotater1-def* **by** (*cases l*) *auto*

lemma *rotate1-rl* [*simp*] : *rotater1* (*rotate1 l*) = *l*
  apply (*unfold rotater1-def*)
  apply (*cases l*)
  apply (*case-tac* [*2*] *list*)
  apply *auto*
  done

lemma *rotate1-lr* [*simp*] : *rotate1* (*rotater1 l*) = *l*
  unfolding *rotater1-def* **by** (*cases l*) *auto*

lemma *rotater1-rev'*: *rotater1* (*rev xs*) = *rev* (*rotate1 xs*)
  apply (*cases xs*)
  apply (*simp add* : *rotater1-def*)
  apply (*simp add* : *rotate1-rl'*)
  done

lemma *rotater-rev'*: *rotater n* (*rev xs*) = *rev* (*rotate n xs*)
  unfolding *rotater-def* **by** (*induct n*) (*auto intro*: *rotater1-rev'*)

lemmas *rotater-rev* = *rotater-rev'* [**where** *xs* = *rev ys*, *simplified*, *standard*]

lemma *rotater-drop-take*:
  rotater n xs =
   drop (*length xs* − *n mod length xs*) *xs* @
   take (*length xs* − *n mod length xs*) *xs*
  by (*clarsimp simp add* : *rotater-rev rotate-drop-take rev-take rev-drop*)

lemma *rotater-Suc* [*simp*] :
  rotater (*Suc n*) *xs* = *rotater1* (*rotater n xs*)
  unfolding *rotater-def* **by** *auto*

lemma *rotate-inv-plus* [*rule-format*] :
  ALL k. *k* = *m* + *n* −−> *rotater k* (*rotate n xs*) = *rotater m xs* &
    rotate k (*rotater n xs*) = *rotate m xs* &
    rotater n (*rotate k xs*) = *rotate m xs* &
    rotate n (*rotater k xs*) = *rotater m xs*

unfolding *rotater-def rotate-def*
by (*induct n*) (*auto intro*: *funpow-swap1* [*THEN trans*])

lemmas *rotate-inv-rel* = *le-add-diff-inverse2* [*symmetric*, *THEN rotate-inv-plus*]

lemmas *rotate-inv-eq* = *order-refl* [*THEN rotate-inv-rel*, *simplified*]

lemmas *rotate-lr* [*simp*] = *rotate-inv-eq* [*THEN conjunct1*, *standard*]
lemmas *rotate-rl* [*simp*] =
rotate-inv-eq [*THEN conjunct2*, *THEN conjunct1*, *standard*]

lemma *rotate-gal*: (*rotater n xs* = *ys*) = (*rotate n ys* = *xs*)
by *auto*

lemma *rotate-gal'*: (*ys* = *rotater n xs*) = (*xs* = *rotate n ys*)
by *auto*

lemma *length-rotater* [*simp*]:
length (*rotater n xs*) = *length xs*
by (*simp add* : *rotater-rev*)

lemmas *rrs0* = *rotate-eqs* [*THEN restrict-to-left*,
simplified rotate-gal [*symmetric*] *rotate-gal'* [*symmetric*], *standard*]
lemmas *rrs1* = *rrs0* [*THEN refl* [*THEN rev-iffD1*]]
lemmas *rotater-eqs* = *rrs1* [*simplified length-rotater*, *standard*]
lemmas *rotater-0* = *rotater-eqs* (*1*)
lemmas *rotater-add* = *rotater-eqs* (*2*)

13.3.2 map, app2, commuting with rotate(r)

lemma *last-map*: *xs* ~= [] ==> *last* (*map f xs*) = *f* (*last xs*)
by (*induct xs*) *auto*

lemma *butlast-map*:
xs ~= [] ==> *butlast* (*map f xs*) = *map f* (*butlast xs*)
by (*induct xs*) *auto*

lemma *rotater1-map*: *rotater1* (*map f xs*) = *map f* (*rotater1 xs*)
unfolding *rotater1-def*
by (*cases xs*) (*auto simp add*: *last-map butlast-map*)

lemma *rotater-map*:
rotater n (*map f xs*) = *map f* (*rotater n xs*)
unfolding *rotater-def*
by (*induct n*) (*auto simp add* : *rotater1-map*)

lemma *but-last-zip* [*rule-format*] :
ALL ys. *length xs* = *length ys* --> *xs* ~= [] -->
last (*zip xs ys*) = (*last xs*, *last ys*) &

```
  butlast (zip xs ys) = zip (butlast xs) (butlast ys)
 apply (induct xs)
 apply auto
   apply ((case-tac ys, auto simp: neq-Nil-conv)[1])+
 done

lemma but-last-app2 [rule-format] :
 ALL ys. length xs = length ys --> xs ~= [] -->
  last (app2 f xs ys) = f (last xs) (last ys) &
  butlast (app2 f xs ys) = app2 f (butlast xs) (butlast ys)
 apply (induct xs)
 apply auto
   apply (unfold app2-def)
   apply ((case-tac ys, auto simp: neq-Nil-conv)[1])+
 done

lemma rotater1-zip:
 length xs = length ys ==>
  rotater1 (zip xs ys) = zip (rotater1 xs) (rotater1 ys)
 apply (unfold rotater1-def)
 apply (cases xs)
  apply auto
  apply ((case-tac ys, auto simp: neq-Nil-conv but-last-zip)[1])+
 done

lemma rotater1-app2:
 length xs = length ys ==>
  rotater1 (app2 f xs ys) = app2 f (rotater1 xs) (rotater1 ys)
 unfolding app2-def by (simp add: rotater1-map rotater1-zip)

lemmas lrth =
 box-equals [OF asm-rl length-rotater [symmetric]
              length-rotater [symmetric],
            THEN rotater1-app2]

lemma rotater-app2:
 length xs = length ys ==>
  rotater n (app2 f xs ys) = app2 f (rotater n xs) (rotater n ys)
 by (induct n) (auto intro!: lrth)

lemma rotate1-app2:
 length xs = length ys ==>
  rotate1 (app2 f xs ys) = app2 f (rotate1 xs) (rotate1 ys)
 apply (unfold app2-def)
 apply (cases xs)
  apply (cases ys, auto simp add : rotate1-def)+
 done

lemmas lth = box-equals [OF asm-rl length-rotate [symmetric]
```

length-rotate [*symmetric*], *THEN rotate1-app2*]

lemma *rotate-app2*:
length xs = *length ys* ==>
rotate n (*app2 f xs ys*) = *app2 f* (*rotate n xs*) (*rotate n ys*)
by (*induct n*) (*auto intro*!: *lth*)

— corresponding equalities for word rotation

lemma *to-bl-rotl*:
to-bl (*word-rotl n w*) = *rotate n* (*to-bl w*)
by (*simp add*: *word-bl.Abs-inverse′ word-rotl-def*)

lemmas *blrs0* = *rotate-eqs* [*THEN to-bl-rotl* [*THEN trans*]]

lemmas *word-rotl-eqs* =
blrs0 [*simplified word-bl.Rep′ word-bl.Rep-inject to-bl-rotl* [*symmetric*]]

lemma *to-bl-rotr*:
to-bl (*word-rotr n w*) = *rotater n* (*to-bl w*)
by (*simp add*: *word-bl.Abs-inverse′ word-rotr-def*)

lemmas *brrs0* = *rotater-eqs* [*THEN to-bl-rotr* [*THEN trans*]]

lemmas *word-rotr-eqs* =
brrs0 [*simplified word-bl.Rep′ word-bl.Rep-inject to-bl-rotr* [*symmetric*]]

declare *word-rotr-eqs* (*1*) [*simp*]
declare *word-rotl-eqs* (*1*) [*simp*]

lemma
word-rot-rl [*simp*]:
word-rotl k (*word-rotr k v*) = *v* **and**
word-rot-lr [*simp*]:
word-rotr k (*word-rotl k v*) = *v*
by (*auto simp add*: *to-bl-rotr to-bl-rotl word-bl.Rep-inject* [*symmetric*])

lemma
word-rot-gal:
(*word-rotr n v* = *w*) = (*word-rotl n w* = *v*) **and**
word-rot-gal′:
(*w* = *word-rotr n v*) = (*v* = *word-rotl n w*)
by (*auto simp*: *to-bl-rotr to-bl-rotl word-bl.Rep-inject* [*symmetric*]
dest: *sym*)

lemma *word-rotr-rev*:
word-rotr n w = *word-reverse* (*word-rotl n* (*word-reverse w*))
by (*simp add*: *word-bl.Rep-inject* [*symmetric*] *to-bl-word-rev*

```
        to-bl-rotr to-bl-rotl rotater-rev)

lemma word-roti-0 [simp]: word-roti 0 w = w
  by (unfold word-rot-defs) auto

lemmas abl-cong = arg-cong [where f = of-bl]

lemma word-roti-add:
  word-roti (m + n) w = word-roti m (word-roti n w)
proof -
  have rotater-eq-lem:
    ⋀m n xs. m = n ==> rotater m xs = rotater n xs
    by auto

  have rotate-eq-lem:
    ⋀m n xs. m = n ==> rotate m xs = rotate n xs
    by auto

  note rpts [symmetric, standard] =
    rotate-inv-plus [THEN conjunct1]
    rotate-inv-plus [THEN conjunct2, THEN conjunct1]
    rotate-inv-plus [THEN conjunct2, THEN conjunct2, THEN conjunct1]
    rotate-inv-plus [THEN conjunct2, THEN conjunct2, THEN conjunct2]

  note rrp = trans [symmetric, OF rotate-rotate rotate-eq-lem]
  note rrrp = trans [symmetric, OF rotater-add [symmetric] rotater-eq-lem]

  show ?thesis
  apply (unfold word-rot-defs)
  apply (simp only: split: split-if)
  apply (safe intro!: abl-cong)
  apply (simp-all only: to-bl-rotl [THEN word-bl.Rep-inverse']
                 to-bl-rotl
                 to-bl-rotr [THEN word-bl.Rep-inverse']
                 to-bl-rotr)
  apply (rule rrp rrrp rpts,
       simp add: nat-add-distrib [symmetric]
                nat-diff-distrib [symmetric])+
  done
qed

lemma word-roti-conv-mod': word-roti n w = word-roti (n mod int (size w)) w
  apply (unfold word-rot-defs)
  apply (cut-tac y=size w in gt-or-eq-0)
  apply (erule disjE)
   apply simp-all
  apply (safe intro!: abl-cong)
   apply (rule rotater-eqs)
   apply (simp add: word-size nat-mod-distrib)
```

apply (*simp add*: *rotater-add* [*symmetric*] *rotate-gal* [*symmetric*])
apply (*rule rotater-eqs*)
apply (*simp add*: *word-size nat-mod-distrib*)
apply (*rule int-eq-0-conv* [*THEN iffD1*])
apply (*simp only*: *zmod-int zadd-int* [*symmetric*])
apply (*simp add*: *rdmods*)
done

lemmas *word-roti-conv-mod* = *word-roti-conv-mod'* [*unfolded word-size*]

13.3.3 Word rotation commutes with bit-wise operations

locale *word-rotate*

context *word-rotate*
begin

lemmas *word-rot-defs'* = *to-bl-rotl to-bl-rotr*

lemmas *blwl-syms* [*symmetric*] = *bl-word-not bl-word-and bl-word-or bl-word-xor*

lemmas *lbl-lbl* = *trans* [*OF word-bl.Rep' word-bl.Rep'* [*symmetric*]]

lemmas *ths-app2* [*OF lbl-lbl*] = *rotate-app2 rotater-app2*

lemmas *ths-map* [**where** *xs* = *to-bl v*] = *rotate-map rotater-map*

lemmas *th1s* [*simplified word-rot-defs'* [*symmetric*]] = *ths-app2 ths-map*

lemma *word-rot-logs*:
word-rotl n (*NOT v*) = *NOT word-rotl n v*
word-rotr n (*NOT v*) = *NOT word-rotr n v*
word-rotl n (*x AND y*) = *word-rotl n x AND word-rotl n y*
word-rotr n (*x AND y*) = *word-rotr n x AND word-rotr n y*
word-rotl n (*x OR y*) = *word-rotl n x OR word-rotl n y*
word-rotr n (*x OR y*) = *word-rotr n x OR word-rotr n y*
word-rotl n (*x XOR y*) = *word-rotl n x XOR word-rotl n y*
word-rotr n (*x XOR y*) = *word-rotr n x XOR word-rotr n y*
by (*rule word-bl.Rep-eqD*,
rule word-rot-defs' [*THEN trans*],
simp only: *blwl-syms* [*symmetric*],
rule th1s [*THEN trans*],
rule refl)+
end

lemmas *word-rot-logs* = *word-rotate.word-rot-logs*

lemmas *bl-word-rotl-dt* = *trans* [*OF to-bl-rotl rotate-drop-take*,
simplified word-bl.Rep', *standard*]

```
lemmas bl-word-rotr-dt = trans [OF to-bl-rotr rotater-drop-take,
  simplified word-bl.Rep', standard]

lemma bl-word-roti-dt':
  n = nat ((- i) mod int (size (w :: 'a :: len word))) ==>
    to-bl (word-roti i w) = drop n (to-bl w) @ take n (to-bl w)
  apply (unfold word-roti-def)
  apply (simp add: bl-word-rotl-dt bl-word-rotr-dt word-size)
  apply safe
   apply (simp add: zmod-zminus1-eq-if)
   apply safe
    apply (simp add: nat-mult-distrib)
   apply (simp add: nat-diff-distrib [OF pos-mod-sign pos-mod-conj
                                      [THEN conjunct2, THEN order-less-imp-le]]
                    nat-mod-distrib)
  apply (simp add: nat-mod-distrib)
  done

lemmas bl-word-roti-dt = bl-word-roti-dt' [unfolded word-size]

lemmas word-rotl-dt = bl-word-rotl-dt
  [THEN word-bl.Rep-inverse' [symmetric], standard]
lemmas word-rotr-dt = bl-word-rotr-dt
  [THEN word-bl.Rep-inverse' [symmetric], standard]
lemmas word-roti-dt = bl-word-roti-dt
  [THEN word-bl.Rep-inverse' [symmetric], standard]

lemma word-rotx-0 [simp] : word-rotr i 0 = 0 & word-rotl i 0 = 0
  by (simp add : word-rotr-dt word-rotl-dt to-bl-0 replicate-add [symmetric])

lemma word-roti-0' [simp] : word-roti n 0 = 0
  unfolding word-roti-def by auto

lemmas word-rotr-dt-no-bin' [simp] =
  word-rotr-dt [where w=number-of w, unfolded to-bl-no-bin, standard]

lemmas word-rotl-dt-no-bin' [simp] =
  word-rotl-dt [where w=number-of w, unfolded to-bl-no-bin, standard]

declare word-roti-def [simp]

end
```

14 Boolean-Algebra: Boolean Algebras

theory *Boolean-Algebra*

imports *Main*
begin

locale *boolean* =
  fixes *conj* :: $'a \Rightarrow 'a \Rightarrow 'a$ (**infixr** $\sqcap$ *70*)
  fixes *disj* :: $'a \Rightarrow 'a \Rightarrow 'a$ (**infixr** $\sqcup$ *65*)
  fixes *compl* :: $'a \Rightarrow 'a$ ($\sim$ - [*81*] *80*)
  fixes *zero* :: $'a$ (**0**)
  fixes *one* :: $'a$ (**1**)
  assumes *conj-assoc*: $(x \sqcap y) \sqcap z = x \sqcap (y \sqcap z)$
  assumes *disj-assoc*: $(x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$
  assumes *conj-commute*: $x \sqcap y = y \sqcap x$
  assumes *disj-commute*: $x \sqcup y = y \sqcup x$
  assumes *conj-disj-distrib*: $x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$
  assumes *disj-conj-distrib*: $x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z)$
  assumes *conj-one-right* [*simp*]: $x \sqcap \mathbf{1} = x$
  assumes *disj-zero-right* [*simp*]: $x \sqcup \mathbf{0} = x$
  assumes *conj-cancel-right* [*simp*]: $x \sqcap \sim x = \mathbf{0}$
  assumes *disj-cancel-right* [*simp*]: $x \sqcup \sim x = \mathbf{1}$
begin

lemmas *disj-ac* =
  disj-assoc disj-commute
  mk-left-commute [**where** $'a = 'a$, *of disj*, *OF disj-assoc disj-commute*]

lemmas *conj-ac* =
  conj-assoc conj-commute
  mk-left-commute [**where** $'a = 'a$, *of conj*, *OF conj-assoc conj-commute*]

lemma *dual*: *boolean disj conj compl one zero*
apply (*rule boolean.intro*)
apply (*rule disj-assoc*)
apply (*rule conj-assoc*)
apply (*rule disj-commute*)
apply (*rule conj-commute*)
apply (*rule disj-conj-distrib*)
apply (*rule conj-disj-distrib*)
apply (*rule disj-zero-right*)
apply (*rule conj-one-right*)
apply (*rule disj-cancel-right*)
apply (*rule conj-cancel-right*)
done

14.1 Complement

lemma *complement-unique*:
  assumes *1*: $a \sqcap x = \mathbf{0}$
  assumes *2*: $a \sqcup x = \mathbf{1}$
  assumes *3*: $a \sqcap y = \mathbf{0}$

assumes *4*: $a \sqcup y = \mathbf{1}$
shows $x = y$
proof –
have $(a \sqcap x) \sqcup (x \sqcap y) = (a \sqcap y) \sqcup (x \sqcap y)$ **using** *1 3* **by** *simp*
hence $(x \sqcap a) \sqcup (x \sqcap y) = (y \sqcap a) \sqcup (y \sqcap x)$ **using** *conj-commute* **by** *simp*
hence $x \sqcap (a \sqcup y) = y \sqcap (a \sqcup x)$ **using** *conj-disj-distrib* **by** *simp*
hence $x \sqcap \mathbf{1} = y \sqcap \mathbf{1}$ **using** *2 4* **by** *simp*
thus $x = y$ **using** *conj-one-right* **by** *simp*
qed

lemma *compl-unique*: $[\![x \sqcap y = \mathbf{0}; x \sqcup y = \mathbf{1}]\!] \Longrightarrow \sim x = y$
by (*rule complement-unique* [*OF conj-cancel-right disj-cancel-right*])

lemma *double-compl* [*simp*]: $\sim (\sim x) = x$
proof (*rule compl-unique*)
from *conj-cancel-right* **show** $\sim x \sqcap x = \mathbf{0}$ **by** (*simp only*: *conj-commute*)
from *disj-cancel-right* **show** $\sim x \sqcup x = \mathbf{1}$ **by** (*simp only*: *disj-commute*)
qed

lemma *compl-eq-compl-iff* [*simp*]: $(\sim x = \sim y) = (x = y)$
by (*rule inj-eq* [*OF inj-on-inverseI*], *rule double-compl*)

14.2 Conjunction

lemma *conj-absorb* [*simp*]: $x \sqcap x = x$
proof –
have $x \sqcap x = (x \sqcap x) \sqcup \mathbf{0}$ **using** *disj-zero-right* **by** *simp*
also have ... $= (x \sqcap x) \sqcup (x \sqcap \sim x)$ **using** *conj-cancel-right* **by** *simp*
also have ... $= x \sqcap (x \sqcup \sim x)$ **using** *conj-disj-distrib* **by** (*simp only*:)
also have ... $= x \sqcap \mathbf{1}$ **using** *disj-cancel-right* **by** *simp*
also have ... $= x$ **using** *conj-one-right* **by** *simp*
finally show *?thesis* **.**
qed

lemma *conj-zero-right* [*simp*]: $x \sqcap \mathbf{0} = \mathbf{0}$
proof –
have $x \sqcap \mathbf{0} = x \sqcap (x \sqcap \sim x)$ **using** *conj-cancel-right* **by** *simp*
also have ... $= (x \sqcap x) \sqcap \sim x$ **using** *conj-assoc* **by** (*simp only*:)
also have ... $= x \sqcap \sim x$ **using** *conj-absorb* **by** *simp*
also have ... $= \mathbf{0}$ **using** *conj-cancel-right* **by** *simp*
finally show *?thesis* **.**
qed

lemma *compl-one* [*simp*]: $\sim \mathbf{1} = \mathbf{0}$
by (*rule compl-unique* [*OF conj-zero-right disj-zero-right*])

lemma *conj-zero-left* [*simp*]: $\mathbf{0} \sqcap x = \mathbf{0}$
by (*subst conj-commute*) (*rule conj-zero-right*)

lemma *conj-one-left* [*simp*]: $\mathbf{1} \sqcap x = x$
by (*subst conj-commute*) (*rule conj-one-right*)

lemma *conj-cancel-left* [*simp*]: $\sim x \sqcap x = \mathbf{0}$
by (*subst conj-commute*) (*rule conj-cancel-right*)

lemma *conj-left-absorb* [*simp*]: $x \sqcap (x \sqcap y) = x \sqcap y$
by (*simp only*: *conj-assoc* [*symmetric*] *conj-absorb*)

lemma *conj-disj-distrib2*:
$(y \sqcup z) \sqcap x = (y \sqcap x) \sqcup (z \sqcap x)$
by (*simp only*: *conj-commute conj-disj-distrib*)

lemmas *conj-disj-distribs* =
conj-disj-distrib conj-disj-distrib2

14.3 Disjunction

lemma *disj-absorb* [*simp*]: $x \sqcup x = x$
by (*rule boolean.conj-absorb* [*OF dual*])

lemma *disj-one-right* [*simp*]: $x \sqcup \mathbf{1} = \mathbf{1}$
by (*rule boolean.conj-zero-right* [*OF dual*])

lemma *compl-zero* [*simp*]: $\sim \mathbf{0} = \mathbf{1}$
by (*rule boolean.compl-one* [*OF dual*])

lemma *disj-zero-left* [*simp*]: $\mathbf{0} \sqcup x = x$
by (*rule boolean.conj-one-left* [*OF dual*])

lemma *disj-one-left* [*simp*]: $\mathbf{1} \sqcup x = \mathbf{1}$
by (*rule boolean.conj-zero-left* [*OF dual*])

lemma *disj-cancel-left* [*simp*]: $\sim x \sqcup x = \mathbf{1}$
by (*rule boolean.conj-cancel-left* [*OF dual*])

lemma *disj-left-absorb* [*simp*]: $x \sqcup (x \sqcup y) = x \sqcup y$
by (*rule boolean.conj-left-absorb* [*OF dual*])

lemma *disj-conj-distrib2*:
$(y \sqcap z) \sqcup x = (y \sqcup x) \sqcap (z \sqcup x)$
by (*rule boolean.conj-disj-distrib2* [*OF dual*])

lemmas *disj-conj-distribs* =
disj-conj-distrib disj-conj-distrib2

14.4 De Morgan's Laws

lemma *de-Morgan-conj* [*simp*]: $\sim (x \sqcap y) = \sim x \sqcup \sim y$
proof (*rule compl-unique*)

have $(x \sqcap y) \sqcap (\sim x \sqcup \sim y) = ((x \sqcap y) \sqcap \sim x) \sqcup ((x \sqcap y) \sqcap \sim y)$
by (*rule conj-disj-distrib*)
also have ... $= (y \sqcap (x \sqcap \sim x)) \sqcup (x \sqcap (y \sqcap \sim y))$
by (*simp only*: *conj-ac*)
finally show $(x \sqcap y) \sqcap (\sim x \sqcup \sim y) = \mathbf{0}$
by (*simp only*: *conj-cancel-right conj-zero-right disj-zero-right*)
next
have $(x \sqcap y) \sqcup (\sim x \sqcup \sim y) = (x \sqcup (\sim x \sqcup \sim y)) \sqcap (y \sqcup (\sim x \sqcup \sim y))$
by (*rule disj-conj-distrib2*)
also have ... $= (\sim y \sqcup (x \sqcup \sim x)) \sqcap (\sim x \sqcup (y \sqcup \sim y))$
by (*simp only*: *disj-ac*)
finally show $(x \sqcap y) \sqcup (\sim x \sqcup \sim y) = \mathbf{1}$
by (*simp only*: *disj-cancel-right disj-one-right conj-one-right*)
qed

lemma *de-Morgan-disj* [*simp*]: $\sim (x \sqcup y) = \sim x \sqcap \sim y$
by (*rule boolean.de-Morgan-conj* [*OF dual*])

end

14.5 Symmetric Difference

locale *boolean-xor* = *boolean* +
fixes *xor* :: *'a => 'a => 'a* (**infixr** $\oplus$ *65*)
assumes *xor-def*: $x \oplus y = (x \sqcap \sim y) \sqcup (\sim x \sqcap y)$
begin

lemma *xor-def2*:
$x \oplus y = (x \sqcup y) \sqcap (\sim x \sqcup \sim y)$
by (*simp only*: *xor-def conj-disj-distribs*
disj-ac conj-ac conj-cancel-right disj-zero-left)

lemma *xor-commute*: $x \oplus y = y \oplus x$
by (*simp only*: *xor-def conj-commute disj-commute*)

lemma *xor-assoc*: $(x \oplus y) \oplus z = x \oplus (y \oplus z)$
proof −
let *?t* $= (x \sqcap y \sqcap z) \sqcup (x \sqcap \sim y \sqcap \sim z) \sqcup$
$(\sim x \sqcap y \sqcap \sim z) \sqcup (\sim x \sqcap \sim y \sqcap z)$
have *?t* $\sqcup (z \sqcap x \sqcap \sim x) \sqcup (z \sqcap y \sqcap \sim y) =$
?t $\sqcup (x \sqcap y \sqcap \sim y) \sqcup (x \sqcap z \sqcap \sim z)$
by (*simp only*: *conj-cancel-right conj-zero-right*)
thus $(x \oplus y) \oplus z = x \oplus (y \oplus z)$
apply (*simp only*: *xor-def de-Morgan-disj de-Morgan-conj double-compl*)
apply (*simp only*: *conj-disj-distribs conj-ac disj-ac*)
done
qed

lemmas *xor-ac* =

xor-assoc xor-commute
mk-left-commute [**where** $'a = 'a$, *of xor*, *OF xor-assoc xor-commute*]

lemma *xor-zero-right* [*simp*]: $x \oplus \mathbf{0} = x$
by (*simp only*: *xor-def compl-zero conj-one-right conj-zero-right disj-zero-right*)

lemma *xor-zero-left* [*simp*]: $\mathbf{0} \oplus x = x$
by (*subst xor-commute*) (*rule xor-zero-right*)

lemma *xor-one-right* [*simp*]: $x \oplus \mathbf{1} = \sim x$
by (*simp only*: *xor-def compl-one conj-zero-right conj-one-right disj-zero-left*)

lemma *xor-one-left* [*simp*]: $\mathbf{1} \oplus x = \sim x$
by (*subst xor-commute*) (*rule xor-one-right*)

lemma *xor-self* [*simp*]: $x \oplus x = \mathbf{0}$
by (*simp only*: *xor-def conj-cancel-right conj-cancel-left disj-zero-right*)

lemma *xor-left-self* [*simp*]: $x \oplus (x \oplus y) = y$
by (*simp only*: *xor-assoc* [*symmetric*] *xor-self xor-zero-left*)

lemma *xor-compl-left*: $\sim x \oplus y = \sim (x \oplus y)$
apply (*simp only*: *xor-def de-Morgan-disj de-Morgan-conj double-compl*)
apply (*simp only*: *conj-disj-distribs*)
apply (*simp only*: *conj-cancel-right conj-cancel-left*)
apply (*simp only*: *disj-zero-left disj-zero-right*)
apply (*simp only*: *disj-ac conj-ac*)
done

lemma *xor-compl-right*: $x \oplus \sim y = \sim (x \oplus y)$
apply (*simp only*: *xor-def de-Morgan-disj de-Morgan-conj double-compl*)
apply (*simp only*: *conj-disj-distribs*)
apply (*simp only*: *conj-cancel-right conj-cancel-left*)
apply (*simp only*: *disj-zero-left disj-zero-right*)
apply (*simp only*: *disj-ac conj-ac*)
done

lemma *xor-cancel-right* [*simp*]: $x \oplus \sim x = \mathbf{1}$
by (*simp only*: *xor-compl-right xor-self compl-zero*)

lemma *xor-cancel-left* [*simp*]: $\sim x \oplus x = \mathbf{1}$
by (*subst xor-commute*) (*rule xor-cancel-right*)

lemma *conj-xor-distrib*: $x \sqcap (y \oplus z) = (x \sqcap y) \oplus (x \sqcap z)$
proof −
  have $(x \sqcap y \sqcap \sim z) \sqcup (x \sqcap \sim y \sqcap z) =$
    $(y \sqcap x \sqcap \sim x) \sqcup (z \sqcap x \sqcap \sim x) \sqcup (x \sqcap y \sqcap \sim z) \sqcup (x \sqcap \sim y \sqcap z)$
   by (*simp only*: *conj-cancel-right conj-zero-right disj-zero-left*)
  thus $x \sqcap (y \oplus z) = (x \sqcap y) \oplus (x \sqcap z)$

by (*simp* (*no-asm-use*) *only*:
xor-def de-Morgan-disj de-Morgan-conj double-compl
conj-disj-distribs conj-ac disj-ac)
qed

lemma *conj-xor-distrib2*:
$(y \oplus z) \sqcap x = (y \sqcap x) \oplus (z \sqcap x)$
proof $-$
have $x \sqcap (y \oplus z) = (x \sqcap y) \oplus (x \sqcap z)$
by (*rule conj-xor-distrib*)
thus $(y \oplus z) \sqcap x = (y \sqcap x) \oplus (z \sqcap x)$
by (*simp only*: *conj-commute*)
qed

lemmas *conj-xor-distribs* =
conj-xor-distrib conj-xor-distrib2

end

end

15 WordGenLib: Miscellaneous Library for Words

theory *WordGenLib* **imports** *WordShift Boolean-Algebra*
begin

declare *of-nat-2p* [*simp*]

lemma *word-int-cases*:
$[\![\bigwedge n.\ [\![(x :: 'a::len0\ word) = word\text{-}of\text{-}int\ n;\ 0 \leq n;\ n < 2\hat{\ }len\text{-}of\ TYPE('a)]\!] \Longrightarrow$
$P]\!]$
$\Longrightarrow P$
by (*cases x rule*: *word-uint.Abs-cases*) (*simp add*: *uints-num*)

lemma *word-nat-cases* [*cases type*: *word*]:
$[\![\bigwedge n.\ [\![(x :: 'a::len\ word) = of\text{-}nat\ n;\ n < 2\hat{\ }len\text{-}of\ TYPE('a)]\!] \Longrightarrow P]\!]$
$\Longrightarrow P$
by (*cases x rule*: *word-unat.Abs-cases*) (*simp add*: *unats-def*)

lemma *max-word-eq*:
$(max\text{-}word::'a::len\ word) = 2\hat{\ }len\text{-}of\ TYPE('a) - 1$
by (*simp add*: *max-word-def word-of-int-hom-syms word-of-int-2p*)

lemma *max-word-max* [*simp*,*intro*!]:
$n \leq max\text{-}word$
by (*cases n rule*: *word-int-cases*)
(*simp add*: *max-word-def word-le-def int-word-uint int-mod-eq'*)

lemma *word-of-int-2p-len*:
word-of-int $(2 \hat{}\ len\text{-}of\ TYPE('a)) = (0{::}'a{::}len0\ word)$
by (*subst word-uint.Abs-norm* [*symmetric*])
(*simp add*: *word-of-int-hom-syms*)

lemma *word-pow-0*:
$(2{::}'a{::}len\ word)\ \hat{}\ len\text{-}of\ TYPE('a) = 0$
proof −
have *word-of-int* $(2 \hat{}\ len\text{-}of\ TYPE('a)) = (0{::}'a\ word)$
by (*rule word-of-int-2p-len*)
thus *?thesis* **by** (*simp add*: *word-of-int-2p*)
qed

lemma *max-word-wrap*: $x + 1 = 0 \Longrightarrow x = max\text{-}word$
apply (*simp add*: *max-word-eq*)
apply *uint-arith*
apply *auto*
apply (*simp add*: *word-pow-0*)
done

lemma *max-word-minus*:
$max\text{-}word = (-1{::}'a{::}len\ word)$
proof −
have $-1 + 1 = (0{::}'a\ word)$ **by** *simp*
thus *?thesis* **by** (*rule max-word-wrap* [*symmetric*])
qed

lemma *max-word-bl* [*simp*]:
to-bl $(max\text{-}word{::}'a{::}len\ word) = replicate\ (len\text{-}of\ TYPE('a))\ True$
by (*subst max-word-minus to-bl-n1*)+ *simp*

lemma *max-test-bit* [*simp*]:
$(max\text{-}word{::}'a{::}len\ word)\ !!\ n = (n < len\text{-}of\ TYPE('a))$
by (*auto simp add*: *test-bit-bl word-size*)

lemma *word-and-max* [*simp*]:
x AND max-word = *x*
by (*rule word-eqI*) (*simp add*: *word-ops-nth-size word-size*)

lemma *word-or-max* [*simp*]:
x OR max-word = *max-word*
by (*rule word-eqI*) (*simp add*: *word-ops-nth-size word-size*)

lemma *word-ao-dist2*:
x AND (*y OR z*) = *x AND y OR x AND* $(z{::}'a{::}len0\ word)$
by (*rule word-eqI*) (*auto simp add*: *word-ops-nth-size word-size*)

lemma *word-oa-dist2*:
x OR y AND z = (*x OR y*) *AND* (*x OR* $(z{::}'a{::}len0\ word)$)

```
  by (rule word-eqI) (auto simp add: word-ops-nth-size word-size)

lemma word-and-not [simp]:
  x AND NOT x = (0::'a::len0 word)
  by (rule word-eqI) (auto simp add: word-ops-nth-size word-size)

lemma word-or-not [simp]:
  x OR NOT x = max-word
  by (rule word-eqI) (auto simp add: word-ops-nth-size word-size)

lemma word-boolean:
  boolean (op AND) (op OR) bitNOT 0 max-word
  apply (rule boolean.intro)
           apply (rule word-bw-assocs)
          apply (rule word-bw-assocs)
         apply (rule word-bw-comms)
        apply (rule word-bw-comms)
       apply (rule word-ao-dist2)
      apply (rule word-oa-dist2)
     apply (rule word-and-max)
    apply (rule word-log-esimps)
   apply (rule word-and-not)
  apply (rule word-or-not)
  done

interpretation word-bool-alg:
  boolean [op AND op OR bitNOT 0 max-word]
  by (rule word-boolean)

lemma word-xor-and-or:
  x XOR y = x AND NOT y OR NOT x AND (y::'a::len0 word)
  by (rule word-eqI) (auto simp add: word-ops-nth-size word-size)

interpretation word-bool-alg:
  boolean-xor [op AND op OR bitNOT 0 max-word op XOR]
  apply (rule boolean-xor.intro)
   apply (rule word-boolean)
  apply (rule boolean-xor-axioms.intro)
  apply (rule word-xor-and-or)
  done

lemma shiftr-0 [iff]:
  (x::'a::len0 word) >> 0 = x
  by (simp add: shiftr-bl)

lemma shiftl-0 [simp]:
  (x :: 'a :: len word) << 0 = x
  by (simp add: shiftl-t2n)
```

lemma *shiftl-1* [*simp*]:
(*1*::′*a*::*len word*) << *n* = *2*ˆ*n*
by (*simp add*: *shiftl-t2n*)

lemma *uint-lt-0* [*simp*]:
uint x < *0* = *False*
by (*simp add*: *linorder-not-less*)

lemma *shiftr1-1* [*simp*]:
shiftr1 (*1*::′*a*::*len word*) = *0*
by (*simp add*: *shiftr1-def word-0-alt*)

lemma *shiftr-1*[*simp*]:
(*1*::′*a*::*len word*) >> *n* = (*if n* = *0 then 1 else 0*)
by (*induct n*) (*auto simp*: *shiftr-def*)

lemma *word-less-1* [*simp*]:
((*x*::′*a*::*len word*) < *1*) = (*x* = *0*)
by (*simp add*: *word-less-nat-alt unat-0-iff*)

lemma *to-bl-mask*:
to-bl (*mask n* :: ′*a*::*len word*) =
replicate (*len-of TYPE*(′*a*) − *n*) *False* @
replicate (*min* (*len-of TYPE*(′*a*)) *n*) *True*
by (*simp add*: *mask-bl word-rep-drop min-def*)

lemma *map-replicate-True*:
n = *length xs* ==>
map (λ(*x*,*y*). *x* & *y*) (*zip xs* (*replicate n True*)) = *xs*
by (*induct xs arbitrary*: *n*) *auto*

lemma *map-replicate-False*:
n = *length xs* ==> *map* (λ(*x*,*y*). *x* & *y*)
(*zip xs* (*replicate n False*)) = *replicate n False*
by (*induct xs arbitrary*: *n*) *auto*

lemma *bl-and-mask*:
fixes *w* :: ′*a*::*len word*
fixes *n*
defines *n*′ ≡ *len-of TYPE*(′*a*) − *n*
shows *to-bl* (*w AND mask n*) = *replicate n*′ *False* @ *drop n*′ (*to-bl w*)
proof −
note [*simp*] = *map-replicate-True map-replicate-False*
have *to-bl* (*w AND mask n*) =
app2 op & (*to-bl w*) (*to-bl* (*mask n*::′*a*::*len word*))
by (*simp add*: *bl-word-and*)
also
have *to-bl w* = *take n*′ (*to-bl w*) @ *drop n*′ (*to-bl w*) **by** *simp*
also

have *app2 op & ... (to-bl (mask n::'a::len word))* =
replicate n' False @ drop n' (to-bl w)
unfolding *to-bl-mask n'-def app2-def*
by *(subst zip-append) auto*
finally
show *?thesis* **.**
qed

lemma *drop-rev-takefill*:
length xs $\leq$ *n ==>*
drop (n $-$ *length xs) (rev (takefill False n (rev xs)))* = *xs*
by *(simp add: takefill-alt rev-take)*

lemma *map-nth-0* [*simp*]:
map (op !! (0::'a::len0 word)) xs = *replicate (length xs) False*
by *(induct xs) auto*

lemma *uint-plus-if-size*:
uint (x + y) =
(if uint x + uint y < 2^size x then
uint x + uint y
else
uint x + uint y $-$ *2^size x)*
by *(simp add: word-arith-alts int-word-uint mod-add-if-z*
word-size)

lemma *unat-plus-if-size*:
unat (x + (y::'a::len word)) =
(if unat x + unat y < 2^size x then
unat x + unat y
else
unat x + unat y $-$ *2^size x)*
apply *(subst word-arith-nat-defs)*
apply *(subst unat-of-nat)*
apply *(simp add: mod-nat-add word-size)*
done

lemma *word-neq-0-conv* [*simp*]:
fixes *w :: 'a :: len word*
shows $(w \neq 0) = (0 < w)$
proof $-$
have $0 \leq w$ **by** *(rule word-zero-le)*
thus *?thesis* **by** *(auto simp add: word-less-def)*
qed

lemma *max-lt*:
unat (max a b div c) = *unat (max a b) div unat (c:: 'a :: len word)*
apply *(subst word-arith-nat-defs)*
apply *(subst word-unat.eq-norm)*

apply (*subst mod-if*)
apply *clarsimp*
apply (*erule notE*)
apply (*insert div-le-dividend* [*of unat* (*max a b*) *unat c*])
apply (*erule order-le-less-trans*)
apply (*insert unat-lt2p* [*of max a b*])
apply *simp*
done

lemma *uint-sub-if-size*:
uint $(x - y) =$
(*if uint* $y \leq$ *uint* x *then*
uint $x -$ *uint* y
else
uint $x -$ *uint* y + *2^size* x)
by (*simp add*: *word-arith-alts int-word-uint mod-sub-if-z*
word-size)

lemma *unat-sub-simple*:
$x \leq y$ ==> *unat* $(y - x)$ = *unat* y − *unat* x
by (*simp add*: *unat-def uint-sub-if-size word-le-def nat-diff-distrib*)

lemmas *unat-sub* = *unat-sub-simple*

lemma *word-less-sub1*:
fixes *x* :: *'a* :: *len word*
shows $x \neq 0$ ==> $1 < x = (0 < x - 1)$
by (*simp add*: *unat-sub-if-size word-less-nat-alt*)

lemma *word-le-sub1*:
fixes *x* :: *'a* :: *len word*
shows $x \neq 0$ ==> $1 \leq x = (0 \leq x - 1)$
by (*simp add*: *unat-sub-if-size order-le-less word-less-nat-alt*)

lemmas *word-less-sub1-numberof* [*simp*] =
word-less-sub1 [*of number-of w*, *standard*]
lemmas *word-le-sub1-numberof* [*simp*] =
word-le-sub1 [*of number-of w*, *standard*]

lemma *word-of-int-minus*:
word-of-int (*2^len-of TYPE('a)* − *i*) = (*word-of-int* (−*i*)::*'a*::*len word*)
proof −
have *x*: *2^len-of TYPE('a)* − *i* = −*i* + *2^len-of TYPE('a)* **by** *simp*
show *?thesis*
apply (*subst x*)
apply (*subst word-uint.Abs-norm* [*symmetric*], *subst zmod-zadd-self2*)
apply *simp*
done
qed

lemmas *word-of-int-inj* =
  word-uint.Abs-inject [*unfolded uints-num*, *simplified*]

lemma *word-le-less-eq*:
  $(x :: 'z::len\ word) \le y = (x = y \lor x < y)$
  by (*auto simp add*: *word-less-def*)

lemma *mod-plus-cong*:
  assumes *1*: $(b::int) = b'$
    and *2*: $x\ mod\ b' = x'\ mod\ b'$
    and *3*: $y\ mod\ b' = y'\ mod\ b'$
    and *4*: $x' + y' = z'$
  shows $(x + y)\ mod\ b = z'\ mod\ b'$
proof −
  from *1 2*[*symmetric*] *3*[*symmetric*] **have** $(x + y)\ mod\ b = (x'\ mod\ b' + y'\ mod\ b')\ mod\ b'$
    by (*simp add*: *zmod-zadd1-eq*[*symmetric*])
  also have $\ldots = (x' + y')\ mod\ b'$
    by (*simp add*: *zmod-zadd1-eq*[*symmetric*])
  finally show *?thesis* **by** (*simp add*: *4*)
qed

lemma *mod-minus-cong*:
  assumes *1*: $(b::int) = b'$
    and *2*: $x\ mod\ b' = x'\ mod\ b'$
    and *3*: $y\ mod\ b' = y'\ mod\ b'$
    and *4*: $x' - y' = z'$
  shows $(x - y)\ mod\ b = z'\ mod\ b'$
  using *assms*
  apply (*subst zmod-zsub-left-eq*)
  apply (*subst zmod-zsub-right-eq*)
  apply (*simp add*: *zmod-zsub-left-eq* [*symmetric*] *zmod-zsub-right-eq* [*symmetric*])
  done

lemma *word-induct-less*:
  $[\![P\ (0::'a::len\ word);\ \bigwedge n.\ [\![n < m;\ P\ n]\!] \Longrightarrow P\ (1 + n)]\!] \Longrightarrow P\ m$
  apply (*cases m*)
  apply *atomize*
  apply (*erule rev-mp*)+
  apply (*rule-tac x=m* **in** *spec*)
  apply (*induct-tac n*)
   apply *simp*
  apply *clarsimp*
  apply (*erule impE*)
   apply *clarsimp*
   apply (*erule-tac x=n* **in** *allE*)
   apply (*erule impE*)
    apply (*simp add*: *unat-arith-simps*)

```
    apply (clarsimp simp: unat-of-nat)
   apply simp
  apply (erule-tac x=of-nat na in allE)
  apply (erule impE)
   apply (simp add: unat-arith-simps)
   apply (clarsimp simp: unat-of-nat)
  apply simp
  done

lemma word-induct:
  ⟦P (0::'a::len word); ⋀n. P n ⟹ P (1 + n)⟧ ⟹ P m
  by (erule word-induct-less, simp)

lemma word-induct2 [induct type]:
  ⟦P 0; ⋀n. ⟦1 + n ≠ 0; P n⟧ ⟹ P (1 + n)⟧ ⟹ P (n::'b::len word)
  apply (rule word-induct, simp)
  apply (case-tac 1+n = 0, auto)
  done

constdefs
  word-rec :: 'a ⇒ ('b::len word ⇒ 'a ⇒ 'a) ⇒ 'b word ⇒ 'a
  word-rec forZero forSuc n ≡ nat-rec forZero (forSuc ∘ of-nat) (unat n)

lemma word-rec-0: word-rec z s 0 = z
  by (simp add: word-rec-def)

lemma word-rec-Suc:
  1 + n ≠ (0::'a::len word) ⟹ word-rec z s (1 + n) = s n (word-rec z s n)
  apply (simp add: word-rec-def unat-word-ariths)
  apply (subst nat-mod-eq')
   apply (cut-tac x=n in unat-lt2p)
   apply (drule Suc-mono)
   apply (simp add: less-Suc-eq-le)
   apply (simp only: order-less-le, simp)
   apply (erule contrapos-pn, simp)
   apply (drule arg-cong[where f=of-nat])
   apply simp
   apply (subst (asm) word-unat.Rep-Abs-A.Rep-inverse[of n])
   apply simp
  apply simp
  done

lemma word-rec-Pred:
  n ≠ 0 ⟹ word-rec z s n = s (n - 1) (word-rec z s (n - 1))
  apply (rule subst[where t=n and s=1 + (n - 1)])
   apply simp
  apply (subst word-rec-Suc)
   apply simp
  apply simp
```

done

lemma *word-rec-in*:
f (*word-rec* z (λ-. f) n) = *word-rec* (f z) (λ-. f) n
by (*induct* n) (*simp-all add*: *word-rec-0 word-rec-Suc*)

lemma *word-rec-in2*:
f n (*word-rec* z f n) = *word-rec* (f 0 z) ($f \circ op + 1$) n
by (*induct* n) (*simp-all add*: *word-rec-0 word-rec-Suc*)

lemma *word-rec-twice*:
$m \leq n \Longrightarrow$ *word-rec* z f n = *word-rec* (*word-rec* z f $(n - m)$) ($f \circ op + (n - m)$) m
apply (*erule rev-mp*)
apply (*rule-tac* $x=z$ **in** *spec*)
apply (*rule-tac* $x=f$ **in** *spec*)
apply (*induct* n)
apply (*simp add*: *word-rec-0*)
apply *clarsimp*
apply (*rule-tac* $t=1 + n - m$ **and** $s=1 + (n - m)$ **in** *subst*)
apply *simp*
apply (*case-tac* $1 + (n - m) = 0$)
apply (*simp add*: *word-rec-0*)
apply (*rule-tac* f = *word-rec* $?a$ $?b$ **in** *arg-cong*)
apply (*rule-tac* $t=m$ **and** $s=m + (1 + (n - m))$ **in** *subst*)
apply *simp*
apply (*simp* (*no-asm-use*))
apply (*simp add*: *word-rec-Suc word-rec-in2*)
apply (*erule impE*)
apply *uint-arith*
apply (*drule-tac* $x=x \circ op + 1$ **in** *spec*)
apply (*drule-tac* $x=x$ 0 xa **in** *spec*)
apply *simp*
apply (*rule-tac* $t=\lambda a.\ x\ (1 + (n - m + a))$ **and** $s=\lambda a.\ x\ (1 + (n - m) + a)$ **in** *subst*)
apply (*clarsimp simp add*: *expand-fun-eq*)
apply (*rule-tac* $t=(1 + (n - m + xb))$ **and** $s=1 + (n - m) + xb$ **in** *subst*)
apply *simp*
apply (*rule refl*)
apply (*rule refl*)
done

lemma *word-rec-id*: *word-rec* z (λ-. *id*) $n = z$
by (*induct* n) (*auto simp add*: *word-rec-0 word-rec-Suc*)

lemma *word-rec-id-eq*: $\forall\, m < n.\ f\ m = id \Longrightarrow$ *word-rec* z f $n = z$
apply (*erule rev-mp*)
apply (*induct* n)
apply (*auto simp add*: *word-rec-0 word-rec-Suc*)

apply (*drule spec*, *erule mp*)
apply *uint-arith*
apply (*drule-tac* $x{=}n$ **in** *spec*, *erule impE*)
apply *uint-arith*
apply *simp*
done

lemma *word-rec-max*:
$\forall\, m{\geq}n.\ m \neq -1 \longrightarrow f\ m = id \Longrightarrow word\text{-}rec\ z\ f\ {-1} = word\text{-}rec\ z\ f\ n$
apply (*subst word-rec-twice*[**where** $n{=}{-1}$ **and** $m{=}{-1} - n$])
apply *simp*
apply *simp*
apply (*rule word-rec-id-eq*)
apply *clarsimp*
apply (*drule spec*, *rule mp*, *erule mp*)
apply (*rule word-plus-mono-right2*[*OF - order-less-imp-le*])
prefer *2*
apply *assumption*
apply *simp*
apply (*erule contrapos-pn*)
apply *simp*
apply (*drule arg-cong*[**where** $f{=}\lambda x.\ x - n$])
apply *simp*
done

lemma *unatSuc*:
$1 + n \neq (0{::}'a{::}len\ word) \Longrightarrow unat\ (1 + n) = Suc\ (unat\ n)$
by *unat-arith*

end

16 WordMain: Main Word Library

theory *WordMain* **imports** *WordGenLib*
begin

lemmas *word-no-1* [*simp*] = *word-1-no* [*symmetric*]
lemmas *word-no-0* [*simp*] = *word-0-no* [*symmetric*]

declare *word-0-bl* [*simp*]
declare *bin-to-bl-def* [*simp*]
declare *to-bl-0* [*simp*]
declare *of-bl-True* [*simp*]

Examples

types *word32* = *32 word*
types *word8* = *8 word*
types *byte* = *word8*

for more see WordExampes.thy

end

References

[1] Jeremy Dawson. Isabelle theories for machine words. In Michael Goldsmith and Bill Roscoe, editors, *Seventh International Workshop on Automated Verification of Critical Systems (AVOCS'07)*, Electronic Notes in Theoretical Computer Science, page 15, Oxford, September 2007. Elsevier. to appear.