Fundamental Properties of Lambda-calculus

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November 22, 2007

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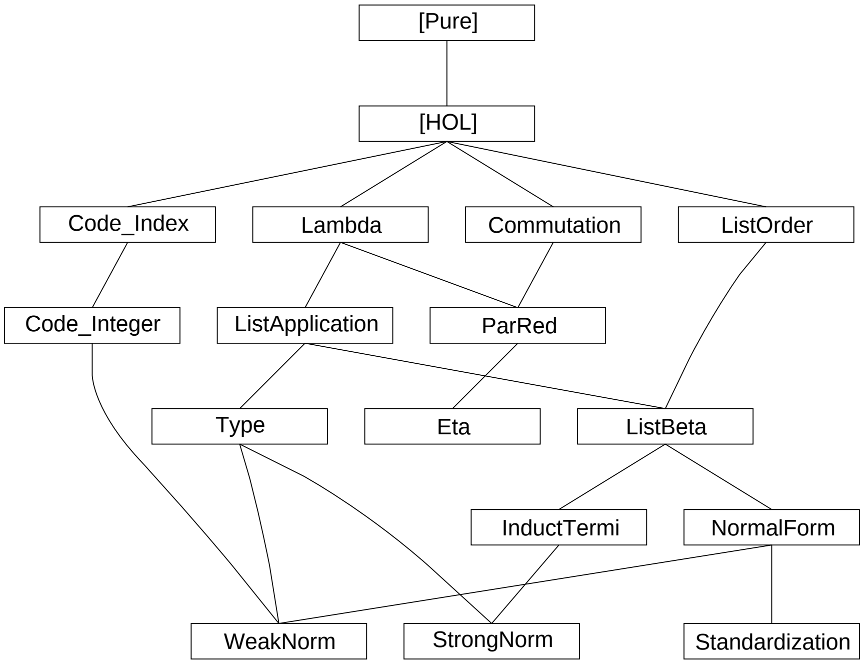
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1 Basic definitions of Lambda-calculus

theory *Lambda* **imports** *Main* **begin**

1.1 Lambda-terms in de Bruijn notation and substitution

datatype *dB* =
  Var nat
| *App dB dB* (**infixl** ∘ *200*)
| *Abs dB*

consts
subst :: [*dB*, *dB*, *nat*] => *dB* (-[-'/-] [*300*, *0*, *0*] *300*)
lift :: [*dB*, *nat*] => *dB*

primrec
lift (*Var i*) *k* = (*if i* < *k then Var i else Var* (*i* + *1*))
lift (*s* ∘ *t*) *k* = *lift s k* ∘ *lift t k*
lift (*Abs s*) *k* = *Abs* (*lift s* (*k* + *1*))

primrec
subst-Var: (*Var i*)[*s*/*k*] =
  (*if k* < *i then Var* (*i* − *1*) *else if i* = *k then s else Var i*)
subst-App: (*t* ∘ *u*)[*s*/*k*] = *t*[*s*/*k*] ∘ *u*[*s*/*k*]
subst-Abs: (*Abs t*)[*s*/*k*] = *Abs* (*t*[*lift s 0* / *k*+*1*])

declare *subst-Var* [*simp del*]

Optimized versions of *subst* and *lift*.

consts
substn :: [*dB*, *dB*, *nat*] => *dB*
liftn :: [*nat*, *dB*, *nat*] => *dB*

primrec
liftn n (*Var i*) *k* = (*if i* < *k then Var i else Var* (*i* + *n*))
liftn n (*s* ∘ *t*) *k* = *liftn n s k* ∘ *liftn n t k*
liftn n (*Abs s*) *k* = *Abs* (*liftn n s* (*k* + *1*))

primrec
substn (*Var i*) *s k* =
  (*if k* < *i then Var* (*i* − *1*) *else if i* = *k then liftn k s 0 else Var i*)
substn (*t* ∘ *u*) *s k* = *substn t s k* ∘ *substn u s k*
substn (*Abs t*) *s k* = *Abs* (*substn t s* (*k* + *1*))

1.2 Beta-reduction

inductive *beta* :: [*dB*, *dB*] => *bool* (**infixl** $\rightarrow_\beta$ *50*)
where
  beta [*simp*, *intro*!]: *Abs s* ∘ *t* $\rightarrow_\beta$ *s*[*t*/*0*]
| *appL* [*simp*, *intro*!]: *s* $\rightarrow_\beta$ *t* ==> *s* ∘ *u* $\rightarrow_\beta$ *t* ∘ *u*

| *appR* [*simp*, *intro*!]: $s \to_\beta t$ ==> $u \circ s \to_\beta u \circ t$
| *abs* [*simp*, *intro*!]: $s \to_\beta t$ ==> *Abs* $s \to_\beta$ *Abs* t

abbreviation
beta-reds :: [*dB*, *dB*] => *bool* (**infixl** −>> *50*) **where**
s −>> t == *beta*ˆ∗∗ s t

notation (*latex*)
beta-reds (**infixl** $\to_\beta{}^*$ *50*)

inductive-cases *beta-cases* [*elim*!]:
Var $i \to_\beta t$
Abs $r \to_\beta s$
$s \circ t \to_\beta u$

declare *if-not-P* [*simp*] *not-less-eq* [*simp*]
— don't add *r-into-rtrancl*[*intro*!]

1.3 Congruence rules

lemma *rtrancl-beta-Abs* [*intro*!]:
$s \to_\beta{}^* s'$ ==> *Abs* $s \to_\beta{}^*$ *Abs* s'
by (*induct set*: *rtranclp*) (*blast intro*: *rtranclp.rtrancl-into-rtrancl*)+

lemma *rtrancl-beta-AppL*:
$s \to_\beta{}^* s'$ ==> $s \circ t \to_\beta{}^* s' \circ t$
by (*induct set*: *rtranclp*) (*blast intro*: *rtranclp.rtrancl-into-rtrancl*)+

lemma *rtrancl-beta-AppR*:
$t \to_\beta{}^* t'$ ==> $s \circ t \to_\beta{}^* s \circ t'$
by (*induct set*: *rtranclp*) (*blast intro*: *rtranclp.rtrancl-into-rtrancl*)+

lemma *rtrancl-beta-App* [*intro*]:
[| $s \to_\beta{}^* s'$; $t \to_\beta{}^* t'$ |] ==> $s \circ t \to_\beta{}^* s' \circ t'$
by (*blast intro*!: *rtrancl-beta-AppL rtrancl-beta-AppR intro*: *rtranclp-trans*)

1.4 Substitution-lemmas

lemma *subst-eq* [*simp*]: (*Var* k)[u/k] = u
by (*simp add*: *subst-Var*)

lemma *subst-gt* [*simp*]: $i < j$ ==> (*Var* j)[u/i] = *Var* (j − *1*)
by (*simp add*: *subst-Var*)

lemma *subst-lt* [*simp*]: $j < i$ ==> (*Var* j)[u/i] = *Var* j
by (*simp add*: *subst-Var*)

lemma *lift-lift*:
$i < k + 1 \Longrightarrow$ *lift* (*lift* t i) (*Suc* k) = *lift* (*lift* t k) i
by (*induct t arbitrary*: i k) *auto*

lemma *lift-subst* [*simp*]:
  $j < i + 1 \Longrightarrow lift\ (t[s/j])\ i = (lift\ t\ (i + 1))\ [lift\ s\ i\ /\ j]$
 by (*induct t arbitrary*: *i j s*)
  (*simp-all add*: *diff-Suc subst-Var lift-lift split*: *nat.split*)

lemma *lift-subst-lt*:
  $i < j + 1 \Longrightarrow lift\ (t[s/j])\ i = (lift\ t\ i)\ [lift\ s\ i\ /\ j + 1]$
 by (*induct t arbitrary*: *i j s*) (*simp-all add*: *subst-Var lift-lift*)

lemma *subst-lift* [*simp*]:
  $(lift\ t\ k)[s/k] = t$
 by (*induct t arbitrary*: *k s*) *simp-all*

lemma *subst-subst*:
  $i < j + 1 \Longrightarrow t[lift\ v\ i\ /\ Suc\ j][u[v/j]/i] = t[u/i][v/j]$
 by (*induct t arbitrary*: *i j u v*)
  (*simp-all add*: *diff-Suc subst-Var lift-lift* [*symmetric*] *lift-subst-lt*
   split: *nat.split*)

1.5 Equivalence proof for optimized substitution

lemma *liftn-0* [*simp*]: $liftn\ 0\ t\ k = t$
 by (*induct t arbitrary*: *k*) (*simp-all add*: *subst-Var*)

lemma *liftn-lift* [*simp*]: $liftn\ (Suc\ n)\ t\ k = lift\ (liftn\ n\ t\ k)\ k$
 by (*induct t arbitrary*: *k*) (*simp-all add*: *subst-Var*)

lemma *substn-subst-n* [*simp*]: $substn\ t\ s\ n = t[liftn\ n\ s\ 0\ /\ n]$
 by (*induct t arbitrary*: *n*) (*simp-all add*: *subst-Var*)

theorem *substn-subst-0*: $substn\ t\ s\ 0 = t[s/0]$
 by *simp*

1.6 Preservation theorems

Not used in Church-Rosser proof, but in Strong Normalization.

theorem *subst-preserves-beta* [*simp*]:
  $r \to_\beta s ==> r[t/i] \to_\beta s[t/i]$
 by (*induct arbitrary*: *t i set*: *beta*) (*simp-all add*: *subst-subst* [*symmetric*])

theorem *subst-preserves-beta'*: $r \to_\beta{}^* s ==> r[t/i] \to_\beta{}^* s[t/i]$
 apply (*induct set*: *rtranclp*)
  apply (*rule rtranclp.rtrancl-refl*)
 apply (*erule rtranclp.rtrancl-into-rtrancl*)
 apply (*erule subst-preserves-beta*)
 done

theorem *lift-preserves-beta* [*simp*]:
  $r \to_\beta s$ ==> *lift r i* $\to_\beta$ *lift s i*
 by (*induct arbitrary*: *i set*: *beta*) *auto*

theorem *lift-preserves-beta'*: $r \to_\beta{}^* s$ ==> *lift r i* $\to_\beta{}^*$ *lift s i*
 apply (*induct set*: *rtranclp*)
  apply (*rule rtranclp.rtrancl-refl*)
 apply (*erule rtranclp.rtrancl-into-rtrancl*)
 apply (*erule lift-preserves-beta*)
 done

theorem *subst-preserves-beta2* [*simp*]: $r \to_\beta s$ ==> $t[r/i] \to_\beta{}^* t[s/i]$
 apply (*induct t arbitrary*: *r s i*)
   apply (*simp add*: *subst-Var r-into-rtranclp*)
  apply (*simp add*: *rtrancl-beta-App*)
 apply (*simp add*: *rtrancl-beta-Abs*)
 done

theorem *subst-preserves-beta2'*: $r \to_\beta{}^* s$ ==> $t[r/i] \to_\beta{}^* t[s/i]$
 apply (*induct set*: *rtranclp*)
  apply (*rule rtranclp.rtrancl-refl*)
 apply (*erule rtranclp-trans*)
 apply (*erule subst-preserves-beta2*)
 done

end

2 Abstract commutation and confluence notions

theory *Commutation* **imports** *Main* **begin**

2.1 Basic definitions

definition
 square :: ['a => 'a => bool, 'a => 'a => bool, 'a => 'a => bool, 'a => 'a => bool] => bool **where**
 square R S T U =
  ($\forall x\ y.\ R\ x\ y$ --> ($\forall z.\ S\ x\ z$ --> ($\exists u.\ T\ y\ u \wedge U\ z\ u$)))

definition
 commute :: ['a => 'a => bool, 'a => 'a => bool] => bool **where**
 commute R S = *square R S S R*

definition
 diamond :: ('a => 'a => bool) => bool **where**
 diamond R = *commute R R*

definition

Church-Rosser :: ($'a$ => $'a$ => *bool*) => *bool* **where**
Church-Rosser R =
($\forall x\ y.$ (*sup R* (R^−−*1*)) ^** $x\ y$ −−> ($\exists z.$ R^** $x\ z \wedge R$^** $y\ z$))

abbreviation
confluent :: ($'a$ => $'a$ => *bool*) => *bool* **where**
confluent R == *diamond* (R^**)

2.2 Basic lemmas

square

lemma *square-sym*: *square R S T U* ==> *square S R U T*
apply (*unfold square-def*)
apply *blast*
done

lemma *square-subset*:
[| *square R S T U*; $T \leq T'$ |] ==> *square R S T' U*
apply (*unfold square-def*)
apply (*blast dest*: *predicate2D*)
done

lemma *square-reflcl*:
[| *square R S T* (R^==); $S \leq T$ |] ==> *square* (R^==) *S T* (R^==)
apply (*unfold square-def*)
apply (*blast dest*: *predicate2D*)
done

lemma *square-rtrancl*:
square R S S T ==> *square* (R^**) *S S* (T^**)
apply (*unfold square-def*)
apply (*intro strip*)
apply (*erule rtranclp-induct*)
apply *blast*
apply (*blast intro*: *rtranclp.rtrancl-into-rtrancl*)
done

lemma *square-rtrancl-reflcl-commute*:
square R S (S^**) (R^==) ==> *commute* (R^**) (S^**)
apply (*unfold commute-def*)
apply (*fastsimp dest*: *square-reflcl square-sym* [*THEN square-rtrancl*])
done

commute

lemma *commute-sym*: *commute R S* ==> *commute S R*
apply (*unfold commute-def*)
apply (*blast intro*: *square-sym*)
done

lemma *commute-rtrancl*: *commute R S* ==> *commute* (*R*ˆ**) (*S*ˆ**)
  apply (*unfold commute-def*)
  apply (*blast intro*: *square-rtrancl square-sym*)
  done

lemma *commute-Un*:
  [| *commute R T*; *commute S T* |] ==> *commute* (*sup R S*) *T*
  apply (*unfold commute-def square-def*)
  apply *blast*
  done

diamond, confluence, and union

lemma *diamond-Un*:
  [| *diamond R*; *diamond S*; *commute R S* |] ==> *diamond* (*sup R S*)
  apply (*unfold diamond-def*)
  apply (*blast intro*: *commute-Un commute-sym*)
  done

lemma *diamond-confluent*: *diamond R* ==> *confluent R*
  apply (*unfold diamond-def*)
  apply (*erule commute-rtrancl*)
  done

lemma *square-reflcl-confluent*:
  square R R (*R*ˆ==) (*R*ˆ==) ==> *confluent R*
  apply (*unfold diamond-def*)
  apply (*fast intro*: *square-rtrancl-reflcl-commute elim*: *square-subset*)
  done

lemma *confluent-Un*:
  [| *confluent R*; *confluent S*; *commute* (*R*ˆ**) (*S*ˆ**) |] ==> *confluent* (*sup R S*)
  apply (*rule rtranclp-sup-rtranclp* [*THEN subst*])
  apply (*blast dest*: *diamond-Un intro*: *diamond-confluent*)
  done

lemma *diamond-to-confluence*:
  [| *diamond R*; $T \leq R$; $R \leq T$ˆ** |] ==> *confluent T*
  apply (*force intro*: *diamond-confluent*
  dest: *rtranclp-subset* [*symmetric*])
  done

2.3 Church-Rosser

lemma *Church-Rosser-confluent*: *Church-Rosser R* = *confluent R*
  apply (*unfold square-def commute-def diamond-def Church-Rosser-def*)
  apply (*tactic* ⟪ *safe-tac HOL-cs* ⟫)
  apply (*tactic* ⟪

blast-tac (*HOL-cs addIs*
[*thm sup-ge2 RS thm rtranclp-mono RS thm predicate2D RS thm rtranclp-trans*,
thm rtranclp-converseI, *thm conversepI*,
thm sup-ge1 RS thm rtranclp-mono RS thm predicate2D]) *1* ⟫)
apply (*erule rtranclp-induct*)
apply *blast*
apply (*blast del*: *rtranclp.rtrancl-refl intro*: *rtranclp-trans*)
done

2.4 Newman's lemma

Proof by Stefan Berghofer

theorem *newman*:
assumes *wf*: *wfP* (R^{-1-1})
and *lc*: $\bigwedge a\ b\ c.\ R\ a\ b \Longrightarrow R\ a\ c \Longrightarrow$
$\exists\, d.\ R^{**}\ b\ d \wedge R^{**}\ c\ d$
shows $\bigwedge b\ c.\ R^{**}\ a\ b \Longrightarrow R^{**}\ a\ c \Longrightarrow$
$\exists\, d.\ R^{**}\ b\ d \wedge R^{**}\ c\ d$
using *wf*
proof *induct*
case (*less x b c*)
have *xc*: $R^{**}\ x\ c$ **by** *fact*
have *xb*: $R^{**}\ x\ b$ **by** *fact* **thus** *?case*
proof (*rule converse-rtranclpE*)
assume $x = b$
with *xc* **have** $R^{**}\ b\ c$ **by** *simp*
thus *?thesis* **by** *iprover*
next
fix y
assume *xy*: $R\ x\ y$
assume *yb*: $R^{**}\ y\ b$
from *xc* **show** *?thesis*
proof (*rule converse-rtranclpE*)
assume $x = c$
with *xb* **have** $R^{**}\ c\ b$ **by** *simp*
thus *?thesis* **by** *iprover*
next
fix y'
assume *y'c*: $R^{**}\ y'\ c$
assume *xy'*: $R\ x\ y'$
with *xy* **have** $\exists\, u.\ R^{**}\ y\ u \wedge R^{**}\ y'\ u$ **by** (*rule lc*)
then obtain u **where** *yu*: $R^{**}\ y\ u$ **and** *y'u*: $R^{**}\ y'\ u$ **by** *iprover*
from *xy* **have** $R^{-1-1}\ y\ x$ **..**
from *this* **and** *yb yu* **have** $\exists\, d.\ R^{**}\ b\ d \wedge R^{**}\ u\ d$ **by** (*rule less*)
then obtain v **where** *bv*: $R^{**}\ b\ v$ **and** *uv*: $R^{**}\ u\ v$ **by** *iprover*
from *xy'* **have** $R^{-1-1}\ y'\ x$ **..**
moreover from *y'u* **and** *uv* **have** $R^{**}\ y'\ v$ **by** (*rule rtranclp-trans*)
moreover note *y'c*
ultimately have $\exists\, d.\ R^{**}\ v\ d \wedge R^{**}\ c\ d$ **by** (*rule less*)

then obtain *w* **where** *vw*: $R^{**}\ v\ w$ **and** *cw*: $R^{**}\ c\ w$ **by** *iprover*
from *bv vw* **have** $R^{**}\ b\ w$ **by** (*rule rtranclp-trans*)
with *cw* **show** *?thesis* **by** *iprover*
qed
qed
qed

Alternative version. Partly automated by Tobias Nipkow. Takes 2 minutes (2002).

This is the maximal amount of automation possible at the moment.

theorem *newman'*:
assumes *wf*: $wfP\ (R^{-1-1})$
and *lc*: $\bigwedge a\ b\ c.\ R\ a\ b \Longrightarrow R\ a\ c \Longrightarrow$
$\exists\, d.\ R^{**}\ b\ d \wedge R^{**}\ c\ d$
shows $\bigwedge b\ c.\ R^{**}\ a\ b \Longrightarrow R^{**}\ a\ c \Longrightarrow$
$\exists\, d.\ R^{**}\ b\ d \wedge R^{**}\ c\ d$
using *wf*
proof *induct*
case (*less x b c*)
note $IH = \langle\bigwedge y\ b\ c.\ [\![R^{-1-1}\ y\ x;\ R^{**}\ y\ b;\ R^{**}\ y\ c]\!]$
$\Longrightarrow \exists\, d.\ R^{**}\ b\ d \wedge R^{**}\ c\ d\rangle$
have *xc*: $R^{**}\ x\ c$ **by** *fact*
have *xb*: $R^{**}\ x\ b$ **by** *fact*
thus *?case*
proof (*rule converse-rtranclpE*)
assume $x = b$
with *xc* **have** $R^{**}\ b\ c$ **by** *simp*
thus *?thesis* **by** *iprover*
next
fix *y*
assume *xy*: $R\ x\ y$
assume *yb*: $R^{**}\ y\ b$
from *xc* **show** *?thesis*
proof (*rule converse-rtranclpE*)
assume $x = c$
with *xb* **have** $R^{**}\ c\ b$ **by** *simp*
thus *?thesis* **by** *iprover*
next
fix y'
assume $y'c$: $R^{**}\ y'\ c$
assume xy': $R\ x\ y'$
with *xy* **obtain** *u* **where** *u*: $R^{**}\ y\ u\ R^{**}\ y'\ u$
by (*blast dest*: *lc*)
from *yb u* $y'c$ **show** *?thesis*
by (*blast del*: *rtranclp.rtrancl-refl*
intro: *rtranclp-trans*
dest: *IH* [*OF conversepI*, *OF xy*] *IH* [*OF conversepI*, *OF xy'*])
qed

qed
qed

end

3 Parallel reduction and a complete developments

theory *ParRed* **imports** *Lambda Commutation* **begin**

3.1 Parallel reduction

inductive *par-beta* :: [*dB*, *dB*] => *bool* (**infixl** => *50*)
where
var [*simp*, *intro*!]: *Var n* => *Var n*
| *abs* [*simp*, *intro*!]: *s* => *t* ==> *Abs s* => *Abs t*
| *app* [*simp*, *intro*!]: [| *s* => *s'*; *t* => *t'* |] ==> *s* ° *t* => *s'* ° *t'*
| *beta* [*simp*, *intro*!]: [| *s* => *s'*; *t* => *t'* |] ==> (*Abs s*) ° *t* => *s'*[*t'*/*0*]

inductive-cases *par-beta-cases* [*elim*!]:
Var n => *t*
Abs s => *Abs t*
(*Abs s*) ° *t* => *u*
s ° *t* => *u*
Abs s => *t*

3.2 Inclusions

beta ⊆ *par-beta* ⊆ *beta*ˆ∗

lemma *par-beta-varL* [*simp*]:
(*Var n* => *t*) = (*t* = *Var n*)
by *blast*

lemma *par-beta-refl* [*simp*]: *t* => *t*
by (*induct t*) *simp-all*

lemma *beta-subset-par-beta*: *beta* <= *par-beta*
apply (*rule predicate2I*)
apply (*erule beta.induct*)
apply (*blast intro*!: *par-beta-refl*)+
done

lemma *par-beta-subset-beta*: *par-beta* <= *beta*ˆ∗∗
apply (*rule predicate2I*)
apply (*erule par-beta.induct*)
apply *blast*
apply (*blast del*: *rtranclp.rtrancl-refl intro*: *rtranclp.rtrancl-into-rtrancl*)+
— *rtrancl-refl* complicates the proof by increasing the branching factor

```
  done
```

3.3 Misc properties of par-beta

```
lemma par-beta-lift [simp]:
    t => t' ⟹ lift t n => lift t' n
  by (induct t arbitrary: t' n) fastsimp+

lemma par-beta-subst:
    s => s' ⟹ t => t' ⟹ t[s/n] => t'[s'/n]
  apply (induct t arbitrary: s s' t' n)
    apply (simp add: subst-Var)
   apply (erule par-beta-cases)
    apply simp
   apply (simp add: subst-subst [symmetric])
   apply (fastsimp intro!: par-beta-lift)
  apply fastsimp
  done
```

3.4 Confluence (directly)

```
lemma diamond-par-beta: diamond par-beta
  apply (unfold diamond-def commute-def square-def)
  apply (rule impI [THEN allI [THEN allI]])
  apply (erule par-beta.induct)
     apply (blast intro!: par-beta-subst)+
  done
```

3.5 Complete developments

```
consts
  cd :: dB => dB
recdef cd measure size
  cd (Var n) = Var n
  cd (Var n ° t) = Var n ° cd t
  cd ((s1 ° s2) ° t) = cd (s1 ° s2) ° cd t
  cd (Abs u ° t) = (cd u)[cd t/0]
  cd (Abs s) = Abs (cd s)

lemma par-beta-cd: s => t ⟹ t => cd s
  apply (induct s arbitrary: t rule: cd.induct)
      apply auto
  apply (fast intro!: par-beta-subst)
  done
```

3.6 Confluence (via complete developments)

```
lemma diamond-par-beta2: diamond par-beta
  apply (unfold diamond-def commute-def square-def)
  apply (blast intro: par-beta-cd)
```

done

theorem *beta-confluent*: *confluent beta*
apply (*rule diamond-par-beta2 diamond-to-confluence*
par-beta-subset-beta beta-subset-par-beta)+
done

end

4 Eta-reduction

theory *Eta* **imports** *ParRed* **begin**

4.1 Definition of eta-reduction and relatives

consts
free :: *dB* => *nat* => *bool*
primrec
free (*Var j*) *i* = (*j* = *i*)
free (s ° t) *i* = (*free s i* $\vee$ *free t i*)
free (*Abs s*) *i* = *free s* (*i* + *1*)

inductive *eta* :: [*dB*, *dB*] => *bool* (**infixl** $\to_\eta$ *50*)
where
eta [*simp*, *intro*]: $\neg$ *free s 0* ==> *Abs* (*s* ° *Var 0*) $\to_\eta$ *s*[*dummy*/*0*]
| *appL* [*simp*, *intro*]: *s* $\to_\eta$ *t* ==> *s* ° *u* $\to_\eta$ *t* ° *u*
| *appR* [*simp*, *intro*]: *s* $\to_\eta$ *t* ==> *u* ° *s* $\to_\eta$ *u* ° *t*
| *abs* [*simp*, *intro*]: *s* $\to_\eta$ *t* ==> *Abs s* $\to_\eta$ *Abs t*

abbreviation
eta-reds :: [*dB*, *dB*] => *bool* (**infixl** −*e*>> *50*) **where**
s −*e*>> *t* == *eta*^** *s t*

abbreviation
eta-red0 :: [*dB*, *dB*] => *bool* (**infixl** −*e*>= *50*) **where**
s −*e*>= *t* == *eta*^== *s t*

notation (*xsymbols*)
eta-reds (**infixl** ${\to_\eta}^*$ *50*) **and**
eta-red0 (**infixl** ${\to_\eta}^=$ *50*)

inductive-cases *eta-cases* [*elim*!]:
Abs s $\to_\eta$ *z*
s ° *t* $\to_\eta$ *u*
Var i $\to_\eta$ *t*

4.2 Properties of eta, subst and free

lemma *subst-not-free* [*simp*]: $\neg$ *free s i* $\Longrightarrow$ $s[t/i] = s[u/i]$
by (*induct s arbitrary*: *i t u*) (*simp-all add*: *subst-Var*)

lemma *free-lift* [*simp*]:
free (*lift t k*) $i = (i < k \wedge$ *free t i* $\vee\ k < i \wedge$ *free t* $(i - 1))$
apply (*induct t arbitrary*: *i k*)
apply (*auto cong*: *conj-cong*)
done

lemma *free-subst* [*simp*]:
free $(s[t/k])\ i =$
(*free s k* $\wedge$ *free t i* $\vee$ *free s* (*if* $i < k$ *then* i *else* $i + 1$))
apply (*induct s arbitrary*: *i k t*)
prefer *2*
apply *simp*
apply *blast*
prefer *2*
apply *simp*
apply (*simp add*: *diff-Suc subst-Var split*: *nat.split*)
done

lemma *free-eta*: $s \rightarrow_\eta t$ ==> *free t i* = *free s i*
by (*induct arbitrary*: *i set*: *eta*) (*simp-all cong*: *conj-cong*)

lemma *not-free-eta*:
[| $s \rightarrow_\eta t$; $\neg$ *free s i* |] ==> $\neg$ *free t i*
by (*simp add*: *free-eta*)

lemma *eta-subst* [*simp*]:
$s \rightarrow_\eta t$ ==> $s[u/i] \rightarrow_\eta t[u/i]$
by (*induct arbitrary*: *u i set*: *eta*) (*simp-all add*: *subst-subst* [*symmetric*])

theorem *lift-subst-dummy*: $\neg$ *free s i* $\Longrightarrow$ *lift* (*s*[*dummy*/*i*]) $i = s$
by (*induct s arbitrary*: *i dummy*) *simp-all*

4.3 Confluence of eta

lemma *square-eta*: *square eta eta* (*eta*^==) (*eta*^==)
apply (*unfold square-def id-def*)
apply (*rule impI* [*THEN allI* [*THEN allI*]])
apply *simp*
apply (*erule eta.induct*)
apply (*slowsimp intro*: *subst-not-free eta-subst free-eta* [*THEN iffD1*])
apply *safe*
prefer *5*
apply (*blast intro*!: *eta-subst intro*: *free-eta* [*THEN iffD1*])
apply *blast+*
done

theorem *eta-confluent*: *confluent eta*
apply (*rule square-eta* [*THEN square-reflcl-confluent*])
done

4.4 Congruence rules for eta*

lemma *rtrancl-eta-Abs*: $s \rightarrow_\eta{}^* s' ==> Abs\ s \rightarrow_\eta{}^* Abs\ s'$
by (*induct set*: *rtranclp*)
(*blast intro*: *rtranclp.rtrancl-into-rtrancl*)+

lemma *rtrancl-eta-AppL*: $s \rightarrow_\eta{}^* s' ==> s \circ t \rightarrow_\eta{}^* s' \circ t$
by (*induct set*: *rtranclp*)
(*blast intro*: *rtranclp.rtrancl-into-rtrancl*)+

lemma *rtrancl-eta-AppR*: $t \rightarrow_\eta{}^* t' ==> s \circ t \rightarrow_\eta{}^* s \circ t'$
by (*induct set*: *rtranclp*) (*blast intro*: *rtranclp.rtrancl-into-rtrancl*)+

lemma *rtrancl-eta-App*:
$[|\ s \rightarrow_\eta{}^* s';\ t \rightarrow_\eta{}^* t'\ |] ==> s \circ t \rightarrow_\eta{}^* s' \circ t'$
by (*blast intro*!: *rtrancl-eta-AppL rtrancl-eta-AppR intro*: *rtranclp-trans*)

4.5 Commutation of beta and eta

lemma *free-beta*:
$s \rightarrow_\beta t ==> free\ t\ i \Longrightarrow free\ s\ i$
by (*induct arbitrary*: *i set*: *beta*) *auto*

lemma *beta-subst* [*intro*]: $s \rightarrow_\beta t ==> s[u/i] \rightarrow_\beta t[u/i]$
by (*induct arbitrary*: *u i set*: *beta*) (*simp-all add*: *subst-subst* [*symmetric*])

lemma *subst-Var-Suc* [*simp*]: $t[Var\ i/i] = t[Var(i)/i + 1]$
by (*induct t arbitrary*: *i*) (*auto elim*!: *linorder-neqE simp*: *subst-Var*)

lemma *eta-lift* [*simp*]: $s \rightarrow_\eta t ==> lift\ s\ i \rightarrow_\eta lift\ t\ i$
by (*induct arbitrary*: *i set*: *eta*) *simp-all*

lemma *rtrancl-eta-subst*: $s \rightarrow_\eta t \Longrightarrow u[s/i] \rightarrow_\eta{}^* u[t/i]$
apply (*induct u arbitrary*: *s t i*)
apply (*simp-all add*: *subst-Var*)
apply *blast*
apply (*blast intro*: *rtrancl-eta-App*)
apply (*blast intro*!: *rtrancl-eta-Abs eta-lift*)
done

lemma *rtrancl-eta-subst'*:
fixes *s t* :: *dB*
assumes *eta*: $s \rightarrow_\eta{}^* t$
shows $s[u/i] \rightarrow_\eta{}^* t[u/i]$ **using** *eta*
by *induct* (*iprover intro*: *eta-subst*)+

lemma *rtrancl-eta-subst''*:
  fixes *s t* :: *dB*
  assumes *eta*: $s \rightarrow_\eta{}^* t$
  shows $u[s/i] \rightarrow_\eta{}^* u[t/i]$ **using** *eta*
  by *induct* (*iprover intro*: *rtrancl-eta-subst rtranclp-trans*)+

lemma *square-beta-eta*: *square beta eta* (*eta*^**) (*beta*^==)
  apply (*unfold square-def*)
  apply (*rule impI* [*THEN allI* [*THEN allI*]])
  apply (*erule beta.induct*)
     apply (*slowsimp intro*: *rtrancl-eta-subst eta-subst*)
    apply (*blast intro*: *rtrancl-eta-AppL*)
   apply (*blast intro*: *rtrancl-eta-AppR*)
  apply *simp*
  apply (*slowsimp intro*: *rtrancl-eta-Abs free-beta*
    iff del: *dB.distinct simp*: *dB.distinct*)
  done

lemma *confluent-beta-eta*: *confluent* (*sup beta eta*)
  apply (*assumption* |
    rule square-rtrancl-reflcl-commute confluent-Un
      beta-confluent eta-confluent square-beta-eta)+
  done

4.6 Implicit definition of eta

Abs (*lift s 0* ° *Var 0*) $\rightarrow_\eta$ *s*

lemma *not-free-iff-lifted*:
    ($\neg$ *free s i*) = ($\exists$ *t*. *s* = *lift t i*)
  apply (*induct s arbitrary*: *i*)
    apply *simp*
    apply (*rule iffI*)
     apply (*erule linorder-neqE*)
      apply (*rule-tac x* = *Var nat* **in** *exI*)
      apply *simp*
     apply (*rule-tac x* = *Var* (*nat* − *1*) **in** *exI*)
     apply *simp*
    apply *clarify*
    apply (*rule notE*)
     prefer *2*
     apply *assumption*
    apply (*erule thin-rl*)
    apply (*case-tac t*)
      apply *simp*
     apply *simp*
    apply *simp*
   apply *simp*
   apply (*erule thin-rl*)

apply (*erule thin-rl*)
apply (*rule iffI*)
apply (*elim conjE exE*)
apply (*rename-tac u1 u2*)
apply (*rule-tac x = u1 ° u2* **in** *exI*)
apply *simp*
apply (*erule exE*)
apply (*erule rev-mp*)
apply (*case-tac t*)
apply *simp*
apply *simp*
apply *blast*
apply *simp*
apply *simp*
apply (*erule thin-rl*)
apply (*rule iffI*)
apply (*erule exE*)
apply (*rule-tac x = Abs t* **in** *exI*)
apply *simp*
apply (*erule exE*)
apply (*erule rev-mp*)
apply (*case-tac t*)
apply *simp*
apply *simp*
apply *simp*
apply *blast*
done

theorem *explicit-is-implicit*:
$(\forall\, s\ u.\ (\neg\ \mathit{free}\ s\ 0) \ {-}{-}{>}\ R\ (\mathit{Abs}\ (s\ \circ\ \mathit{Var}\ 0))\ (s[u/0])) =$
$(\forall\, s.\ R\ (\mathit{Abs}\ (\mathit{lift}\ s\ 0\ \circ\ \mathit{Var}\ 0))\ s)$
by (*auto simp add*: *not-free-iff-lifted*)

4.7 Eta-postponement theorem

Based on a paper proof due to Andreas Abel. Unlike the proof by Masako Takahashi [4], it does not use parallel eta reduction, which only seems to complicate matters unnecessarily.

theorem *eta-case*:
fixes $s :: \mathit{dB}$
assumes *free*: $\neg\ \mathit{free}\ s\ 0$
and s: $s[\mathit{dummy}/0] \Rightarrow u$
shows $\exists\, t'.\ \mathit{Abs}\ (s\ \circ\ \mathit{Var}\ 0) \Rightarrow t' \wedge t' \rightarrow_\eta{}^*\ u$
proof −
from s **have** $\mathit{lift}\ (s[\mathit{dummy}/0])\ 0 \Rightarrow \mathit{lift}\ u\ 0$ **by** (*simp del*: *lift-subst*)
with *free* **have** $s \Rightarrow \mathit{lift}\ u\ 0$ **by** (*simp add*: *lift-subst-dummy del*: *lift-subst*)
hence $\mathit{Abs}\ (s\ \circ\ \mathit{Var}\ 0) \Rightarrow \mathit{Abs}\ (\mathit{lift}\ u\ 0\ \circ\ \mathit{Var}\ 0)$ **by** *simp*
moreover have $\neg\ \mathit{free}\ (\mathit{lift}\ u\ 0)\ 0$ **by** *simp*
hence $\mathit{Abs}\ (\mathit{lift}\ u\ 0\ \circ\ \mathit{Var}\ 0) \rightarrow_\eta \mathit{lift}\ u\ 0[\mathit{dummy}/0]$

```
    by (rule eta.eta)
  hence Abs (lift u 0 ° Var 0) →η* u by simp
  ultimately show ?thesis by iprover
qed

theorem eta-par-beta:
  assumes st: s →η t
  and tu: t => u
  shows ∃ t'. s => t' ∧ t' →η* u using tu st
proof (induct arbitrary: s)
  case (var n)
  thus ?case by (iprover intro: par-beta-refl)
next
  case (abs s' t)
  note abs' = this
  from ‹s →η Abs s'› show ?case
  proof cases
    case (eta s'' dummy)
    from abs have Abs s' => Abs t by simp
    with eta have s''[dummy/0] => Abs t by simp
    with ‹¬ free s'' 0› have ∃ t'. Abs (s'' ° Var 0) => t' ∧ t' →η* Abs t
      by (rule eta-case)
    with eta show ?thesis by simp
  next
    case (abs r u)
    hence r →η s' by simp
    then obtain t' where r: r => t' and t': t' →η* t by (iprover dest: abs')
    from r have Abs r => Abs t' ..
    moreover from t' have Abs t' →η* Abs t by (rule rtrancl-eta-Abs)
    ultimately show ?thesis using abs by simp iprover
  qed simp-all
next
  case (app u u' t t')
  from ‹s →η u ° t› show ?case
  proof cases
    case (eta s' dummy)
    from app have u ° t => u' ° t' by simp
    with eta have s'[dummy/0] => u' ° t' by simp
    with ‹¬ free s' 0› have ∃ r. Abs (s' ° Var 0) => r ∧ r →η* u' ° t'
      by (rule eta-case)
    with eta show ?thesis by simp
  next
    case (appL s' t'' u'')
    hence s' →η u by simp
    then obtain r where s': s' => r and r: r →η* u' by (iprover dest: app)
    from s' and app have s' ° t => r ° t' by simp
    moreover from r have r ° t' →η* u' ° t' by (simp add: rtrancl-eta-AppL)
    ultimately show ?thesis using appL by simp iprover
  next
```

case (*appR* s' t'' u'')
hence $s' \rightarrow_\eta t$ **by** *simp*
then obtain r **where** s': $s' => r$ **and** r: $r \rightarrow_\eta{}^* t'$ **by** (*iprover dest*: *app*)
from s' **and** *app* **have** $u \circ s' => u' \circ r$ **by** *simp*
moreover from r **have** $u' \circ r \rightarrow_\eta{}^* u' \circ t'$ **by** (*simp add*: *rtrancl-eta-AppR*)
ultimately show *?thesis* **using** *appR* **by** *simp iprover*
qed *simp*
next
case (*beta* u u' t t')
from ‹$s \rightarrow_\eta Abs\ u \circ t$› **show** *?case*
proof *cases*
case (*eta* s' *dummy*)
from *beta* **have** $Abs\ u \circ t => u'[t'/0]$ **by** *simp*
with *eta* **have** $s'[dummy/0] => u'[t'/0]$ **by** *simp*
with ‹$\neg$ *free* s' *0*› **have** $\exists r.\ Abs\ (s' \circ Var\ 0) => r \wedge r \rightarrow_\eta{}^* u'[t'/0]$
by (*rule eta-case*)
with *eta* **show** *?thesis* **by** *simp*
next
case (*appL* s' t'' u'')
hence $s' \rightarrow_\eta Abs\ u$ **by** *simp*
thus *?thesis*
proof *cases*
case (*eta* s'' *dummy*)
have $Abs\ (lift\ u\ 1) = lift\ (Abs\ u)\ 0$ **by** *simp*
also from *eta* **have** $\ldots = s''$ **by** (*simp add*: *lift-subst-dummy del*: *lift-subst*)
finally have s: $s = Abs\ (Abs\ (lift\ u\ 1) \circ Var\ 0) \circ t$ **using** *appL* **and** *eta* **by** *simp*
from *beta* **have** $lift\ u\ 1 => lift\ u'\ 1$ **by** *simp*
hence $Abs\ (lift\ u\ 1) \circ Var\ 0 => lift\ u'\ 1[Var\ 0/0]$
using *par-beta.var* **..**
hence $Abs\ (Abs\ (lift\ u\ 1) \circ Var\ 0) \circ t => lift\ u'\ 1[Var\ 0/0][t'/0]$
using ‹$t => t'$› **..**
with s **have** $s => u'[t'/0]$ **by** *simp*
thus *?thesis* **by** *iprover*
next
case (*abs* r r')
hence $r \rightarrow_\eta u$ **by** *simp*
then obtain r'' **where** r: $r => r''$ **and** r'': $r'' \rightarrow_\eta{}^* u'$ **by** (*iprover dest*: *beta*)
from r **and** *beta* **have** $Abs\ r \circ t => r''[t'/0]$ **by** *simp*
moreover from r'' **have** $r''[t'/0] \rightarrow_\eta{}^* u'[t'/0]$
by (*rule rtrancl-eta-subst'*)
ultimately show *?thesis* **using** *abs* **and** *appL* **by** *simp iprover*
qed *simp-all*
next
case (*appR* s' t'' u'')
hence $s' \rightarrow_\eta t$ **by** *simp*
then obtain r **where** s': $s' => r$ **and** r: $r \rightarrow_\eta{}^* t'$ **by** (*iprover dest*: *beta*)
from s' **and** *beta* **have** $Abs\ u \circ s' => u'[r/0]$ **by** *simp*

moreover from r **have** $u'[r/0] \rightarrow_\eta{}^* u'[t'/0]$
by (*rule rtrancl-eta-subst''*)
ultimately show *?thesis* **using** *appR* **by** *simp iprover*
qed *simp*
qed

theorem *eta-postponement'*:
assumes *eta*: $s \rightarrow_\eta{}^* t$ **and** *beta*: $t => u$
shows $\exists\, t'.\ s => t' \wedge t' \rightarrow_\eta{}^* u$ **using** *eta beta*
proof (*induct arbitrary*: u)
case *1*
thus *?case* **by** *blast*
next
case (*2* s' s'' s''')
from *2* **obtain** t' **where** s': $s' => t'$ **and** t': $t' \rightarrow_\eta{}^* s'''$
by (*auto dest*: *eta-par-beta*)
from s' **obtain** t'' **where** s: $s => t''$ **and** t'': $t'' \rightarrow_\eta{}^* t'$ **using** *2*
by *blast*
from t'' **and** t' **have** $t'' \rightarrow_\eta{}^* s'''$ **by** (*rule rtranclp-trans*)
with s **show** *?case* **by** *iprover*
qed

theorem *eta-postponement*:
assumes *st*: (*sup beta eta*)** s t
shows (*eta*** *OO beta***) s t **using** *st*
proof *induct*
case *1*
show *?case* **by** *blast*
next
case (*2* s' s'')
from *2*(*3*) **obtain** t' **where** s: $s \rightarrow_\beta{}^* t'$ **and** t': $t' \rightarrow_\eta{}^* s'$ **by** *blast*
from *2*(*2*) **show** *?case*
proof
assume $s' \rightarrow_\beta s''$
with *beta-subset-par-beta* **have** $s' => s''$ **..**
with t' **obtain** t'' **where** *st*: $t' => t''$ **and** *tu*: $t'' \rightarrow_\eta{}^* s''$
by (*auto dest*: *eta-postponement'*)
from *par-beta-subset-beta st* **have** $t' \rightarrow_\beta{}^* t''$ **..**
with s **have** $s \rightarrow_\beta{}^* t''$ **by** (*rule rtranclp-trans*)
thus *?thesis* **using** *tu* **..**
next
assume $s' \rightarrow_\eta s''$
with t' **have** $t' \rightarrow_\eta{}^* s''$ **..**
with s **show** *?thesis* **..**
qed
qed

end

5 Application of a term to a list of terms

theory *ListApplication* **imports** *Lambda* **begin**

abbreviation
  list-application :: *dB* => *dB list* => *dB* (**infixl** °° *150*) **where**
  t °° *ts* == *foldl* (*op* °) *t ts*

lemma *apps-eq-tail-conv* [*iff*]: (*r* °° *ts* = *s* °° *ts*) = (*r* = *s*)
  by (*induct ts rule*: *rev-induct*) *auto*

lemma *Var-eq-apps-conv* [*iff*]: (*Var m* = *s* °° *ss*) = (*Var m* = *s* ∧ *ss* = [])
  by (*induct ss arbitrary*: *s*) *auto*

lemma *Var-apps-eq-Var-apps-conv* [*iff*]:
    (*Var m* °° *rs* = *Var n* °° *ss*) = (*m* = *n* ∧ *rs* = *ss*)
  apply (*induct rs arbitrary*: *ss rule*: *rev-induct*)
   apply *simp*
   apply *blast*
  apply (*induct-tac ss rule*: *rev-induct*)
   apply *auto*
  done

lemma *App-eq-foldl-conv*:
  (*r* ° *s* = *t* °° *ts*) =
    (*if ts* = [] *then r* ° *s* = *t*
    else (∃ *ss*. *ts* = *ss* @ [*s*] ∧ *r* = *t* °° *ss*))
  apply (*rule-tac xs* = *ts* **in** *rev-exhaust*)
   apply *auto*
  done

lemma *Abs-eq-apps-conv* [*iff*]:
    (*Abs r* = *s* °° *ss*) = (*Abs r* = *s* ∧ *ss* = [])
  by (*induct ss rule*: *rev-induct*) *auto*

lemma *apps-eq-Abs-conv* [*iff*]: (*s* °° *ss* = *Abs r*) = (*s* = *Abs r* ∧ *ss* = [])
  by (*induct ss rule*: *rev-induct*) *auto*

lemma *Abs-apps-eq-Abs-apps-conv* [*iff*]:
    (*Abs r* °° *rs* = *Abs s* °° *ss*) = (*r* = *s* ∧ *rs* = *ss*)
  apply (*induct rs arbitrary*: *ss rule*: *rev-induct*)
   apply *simp*
   apply *blast*
  apply (*induct-tac ss rule*: *rev-induct*)
   apply *auto*
  done

lemma *Abs-App-neq-Var-apps* [*iff*]:
    Abs s ° *t* ≠ *Var n* °° *ss*

by (*induct ss arbitrary*: *s t rule*: *rev-induct*) *auto*

lemma *Var-apps-neq-Abs-apps* [*iff*]:
Var n °° *ts* ≠ *Abs r* °° *ss*
apply (*induct ss arbitrary*: *ts rule*: *rev-induct*)
apply *simp*
apply (*induct-tac ts rule*: *rev-induct*)
apply *auto*
done

lemma *ex-head-tail*:
∃ *ts h*. *t* = *h* °° *ts* ∧ ((∃ *n*. *h* = *Var n*) ∨ (∃ *u*. *h* = *Abs u*))
apply (*induct t*)
apply (*rule-tac x* = [] **in** *exI*)
apply *simp*
apply *clarify*
apply (*rename-tac ts1 ts2 h1 h2*)
apply (*rule-tac x* = *ts1* @ [*h2* °° *ts2*] **in** *exI*)
apply *simp*
apply *simp*
done

lemma *size-apps* [*simp*]:
size (*r* °° *rs*) = *size r* + *foldl* (*op* +) *0* (*map size rs*) + *length rs*
by (*induct rs rule*: *rev-induct*) *auto*

lemma *lem0*: [| (*0*::*nat*) < *k*; *m* <= *n* |] ==> *m* < *n* + *k*
by *simp*

lemma *lift-map* [*simp*]:
lift (*t* °° *ts*) *i* = *lift t i* °° *map* (λ*t*. *lift t i*) *ts*
by (*induct ts arbitrary*: *t*) *simp-all*

lemma *subst-map* [*simp*]:
subst (*t* °° *ts*) *u i* = *subst t u i* °° *map* (λ*t*. *subst t u i*) *ts*
by (*induct ts arbitrary*: *t*) *simp-all*

lemma *app-last*: (*t* °° *ts*) ° *u* = *t* °° (*ts* @ [*u*])
by *simp*

A customized induction schema for °°.

lemma *lem*:
assumes !!*n ts*. ∀ *t* ∈ *set ts*. *P t* ==> *P* (*Var n* °° *ts*)
and !!*u ts*. [| *P u*; ∀ *t* ∈ *set ts*. *P t* |] ==> *P* (*Abs u* °° *ts*)
shows *size t* = *n* ⟹ *P t*
apply (*induct n arbitrary*: *t rule*: *nat-less-induct*)
apply (*cut-tac t* = *t* **in** *ex-head-tail*)
apply *clarify*
apply (*erule disjE*)

```
   apply clarify
   apply (rule assms)
   apply clarify
   apply (erule allE, erule impE)
    prefer 2
    apply (erule allE, erule mp, rule refl)
   apply simp
   apply (rule lem0)
    apply force
   apply (rule elem-le-sum)
   apply force
  apply clarify
  apply (rule assms)
   apply (erule allE, erule impE)
    prefer 2
    apply (erule allE, erule mp, rule refl)
   apply simp
  apply clarify
  apply (erule allE, erule impE)
   prefer 2
   apply (erule allE, erule mp, rule refl)
  apply simp
  apply (rule le-imp-less-Suc)
  apply (rule trans-le-add1)
  apply (rule trans-le-add2)
  apply (rule elem-le-sum)
  apply force
  done

theorem Apps-dB-induct:
  assumes !!n ts. ∀ t ∈ set ts. P t ==> P (Var n °° ts)
    and !!u ts. [| P u; ∀ t ∈ set ts. P t |] ==> P (Abs u °° ts)
  shows P t
  apply (rule-tac t = t in lem)
    prefer 3
    apply (rule refl)
    using assms apply iprover+
  done

end
```

6 Simply-typed lambda terms

theory *Type* **imports** *ListApplication* **begin**

6.1 Environments

definition

```
shift :: (nat ⇒ 'a) ⇒ nat ⇒ 'a ⇒ nat ⇒ 'a  (-<-:-> [90, 0, 0] 91) where
e<i:a> = (λj. if j < i then e j else if j = i then a else e (j − 1))

notation (xsymbols)
  shift  (-⟨-:-⟩ [90, 0, 0] 91)

notation (HTML output)
  shift  (-⟨-:-⟩ [90, 0, 0] 91)

lemma shift-eq [simp]: i = j ⟹ (e⟨i:T⟩) j = T
  by (simp add: shift-def)

lemma shift-gt [simp]: j < i ⟹ (e⟨i:T⟩) j = e j
  by (simp add: shift-def)

lemma shift-lt [simp]: i < j ⟹ (e⟨i:T⟩) j = e (j − 1)
  by (simp add: shift-def)

lemma shift-commute [simp]: e⟨i:U⟩⟨0:T⟩ = e⟨0:T⟩⟨Suc i:U⟩
  apply (rule ext)
  apply (case-tac x)
   apply simp
  apply (case-tac nat)
   apply (simp-all add: shift-def)
  done
```

6.2 Types and typing rules

```
datatype type =
    Atom nat
  | Fun type type    (infixr ⇒ 200)

inductive typing :: (nat ⇒ type) ⇒ dB ⇒ type ⇒ bool  (- ⊢ - : - [50, 50, 50] 50)
  where
    Var [intro!]: env x = T ⟹ env ⊢ Var x : T
  | Abs [intro!]: env⟨0:T⟩ ⊢ t : U ⟹ env ⊢ Abs t : (T ⇒ U)
  | App [intro!]: env ⊢ s : T ⇒ U ⟹ env ⊢ t : T ⟹ env ⊢ (s ° t) : U

inductive-cases typing-elims [elim!]:
  e ⊢ Var i : T
  e ⊢ t ° u : T
  e ⊢ Abs t : T

consts
  typings :: (nat ⇒ type) ⇒ dB list ⇒ type list ⇒ bool

abbreviation
  funs :: type list ⇒ type ⇒ type  (infixr =>> 200) where
  Ts =>> T == foldr Fun Ts T
```

abbreviation
  typings-rel :: (*nat* ⇒ *type*) ⇒ *dB list* ⇒ *type list* ⇒ *bool*
    (- ||− - : - [*50*, *50*, *50*] *50*) **where**
  env ||− *ts* : *Ts* == *typings env ts Ts*

notation (*latex*)
  funs (**infixr** ⇛ *200*) **and**
  typings-rel (- ⊩ - : - [*50*, *50*, *50*] *50*)

primrec
  (*e* ⊩ [] : *Ts*) = (*Ts* = [])
  (*e* ⊩ (*t* # *ts*) : *Ts*) =
    (*case Ts of*
      [] ⇒ *False*
    | *T* # *Ts* ⇒ *e* ⊢ *t* : *T* ∧ *e* ⊩ *ts* : *Ts*)

6.3 Some examples

lemma *e* ⊢ *Abs* (*Abs* (*Abs* (*Var 1* ° (*Var 2* ° *Var 1* ° *Var 0*)))) : *?T*
  by *force*

lemma *e* ⊢ *Abs* (*Abs* (*Abs* (*Var 2* ° *Var 0* ° (*Var 1* ° *Var 0*)))) : *?T*
  by *force*

6.4 Lists of types

lemma *lists-typings*:
    e ⊩ *ts* : *Ts* ⟹ *listsp* (λ*t*. ∃ *T*. *e* ⊢ *t* : *T*) *ts*
  apply (*induct ts arbitrary*: *Ts*)
   apply (*case-tac Ts*)
     apply *simp*
     apply (*rule listsp.Nil*)
    apply *simp*
  apply (*case-tac Ts*)
   apply *simp*
  apply *simp*
  apply (*rule listsp.Cons*)
   apply *blast*
  apply *blast*
  done

lemma *types-snoc*: *e* ⊩ *ts* : *Ts* ⟹ *e* ⊢ *t* : *T* ⟹ *e* ⊩ *ts* @ [*t*] : *Ts* @ [*T*]
  apply (*induct ts arbitrary*: *Ts*)
  apply *simp*
  apply (*case-tac Ts*)
  apply *simp+*
  done

lemma *types-snoc-eq*: *e* ⊩ *ts* @ [*t*] : *Ts* @ [*T*] =

$(e \Vdash ts : Ts \wedge e \vdash t : T)$
apply (*induct ts arbitrary*: *Ts*)
apply (*case-tac Ts*)
apply *simp+*
apply (*case-tac Ts*)
apply (*case-tac ts* @ [*t*])
apply *simp+*
done

lemma *rev-exhaust2* [*case-names Nil snoc*, *extraction-expand*]:
$(xs = [] \Longrightarrow P) \Longrightarrow (\bigwedge ys\ y.\ xs = ys\ @\ [y] \Longrightarrow P) \Longrightarrow P$
— Cannot use *rev-exhaust* from the *List* theory, since it is not constructive
apply (*subgoal-tac* $\forall ys.\ xs = rev\ ys \longrightarrow P$)
apply (*erule-tac x=rev xs* **in** *allE*)
apply *simp*
apply (*rule allI*)
apply (*rule impI*)
apply (*case-tac ys*)
apply *simp*
apply *simp*
apply *atomize*
apply (*erule allE*)+
apply (*erule mp*, *rule conjI*)
apply (*rule refl*)+
done

lemma *types-snocE*: $e \Vdash ts\ @\ [t] : Ts \Longrightarrow$
$(\bigwedge Us\ U.\ Ts = Us\ @\ [U] \Longrightarrow e \Vdash ts : Us \Longrightarrow e \vdash t : U \Longrightarrow P) \Longrightarrow P$
apply (*cases Ts rule*: *rev-exhaust2*)
apply *simp*
apply (*case-tac ts* @ [*t*])
apply (*simp add*: *types-snoc-eq*)+
apply *iprover*
done

6.5 n-ary function types

lemma *list-app-typeD*:
$e \vdash t \circ\circ ts : T \Longrightarrow \exists Ts.\ e \vdash t : Ts \Rrightarrow T \wedge e \Vdash ts : Ts$
apply (*induct ts arbitrary*: *t T*)
apply *simp*
apply *atomize*
apply *simp*
apply (*erule-tac* $x = t \circ a$ **in** *allE*)
apply (*erule-tac* $x = T$ **in** *allE*)
apply (*erule impE*)
apply *assumption*
apply (*elim exE conjE*)
apply (*ind-cases* $e \vdash t \circ u : T$ **for** *t u T*)

apply (*rule-tac* $x = Ta \# Ts$ **in** *exI*)
apply *simp*
done

lemma *list-app-typeE*:
$e \vdash t \circ\circ ts : T \Longrightarrow (\bigwedge Ts.\ e \vdash t : Ts \Rrightarrow T \Longrightarrow e \Vdash ts : Ts \Longrightarrow C) \Longrightarrow C$
by (*insert list-app-typeD*) *fast*

lemma *list-app-typeI*:
$e \vdash t : Ts \Rrightarrow T \Longrightarrow e \Vdash ts : Ts \Longrightarrow e \vdash t \circ\circ ts : T$
apply (*induct ts arbitrary*: $t\ T\ Ts$)
apply *simp*
apply *atomize*
apply (*case-tac Ts*)
apply *simp*
apply *simp*
apply (*erule-tac* $x = t \circ a$ **in** *allE*)
apply (*erule-tac* $x = T$ **in** *allE*)
apply (*erule-tac* $x = list$ **in** *allE*)
apply (*erule impE*)
apply (*erule conjE*)
apply (*erule typing.App*)
apply *assumption*
apply *blast*
done

For the specific case where the head of the term is a variable, the following theorems allow to infer the types of the arguments without analyzing the typing derivation. This is crucial for program extraction.

theorem *var-app-type-eq*:
$e \vdash Var\ i \circ\circ ts : T \Longrightarrow e \vdash Var\ i \circ\circ ts : U \Longrightarrow T = U$
apply (*induct ts arbitrary*: $T\ U$ *rule*: *rev-induct*)
apply *simp*
apply (*ind-cases* $e \vdash Var\ i : T$ **for** T)
apply (*ind-cases* $e \vdash Var\ i : T$ **for** T)
apply *simp*
apply *simp*
apply (*ind-cases* $e \vdash t \circ u : T$ **for** $t\ u\ T$)
apply (*ind-cases* $e \vdash t \circ u : T$ **for** $t\ u\ T$)
apply *atomize*
apply (*erule-tac* $x{=}Ta \Rightarrow T$ **in** *allE*)
apply (*erule-tac* $x{=}Tb \Rightarrow U$ **in** *allE*)
apply (*erule impE*)
apply *assumption*
apply (*erule impE*)
apply *assumption*
apply *simp*
done

```
lemma var-app-types: e ⊢ Var i °° ts °° us : T ⟹ e ⊩ ts : Ts ⟹
  e ⊢ Var i °° ts : U ⟹ ∃ Us. U = Us ⇛ T ∧ e ⊩ us : Us
  apply (induct us arbitrary: ts Ts U)
  apply simp
  apply (erule var-app-type-eq)
  apply assumption
  apply simp
  apply atomize
  apply (case-tac U)
  apply (rule FalseE)
  apply simp
  apply (erule list-app-typeE)
  apply (ind-cases e ⊢ t ° u : T for t u T)
  apply (drule-tac T=Atom nat and U=Ta ⇒ Tsa ⇛ T in var-app-type-eq)
  apply assumption
  apply simp
  apply (erule-tac x=ts @ [a] in allE)
  apply (erule-tac x=Ts @ [type1] in allE)
  apply (erule-tac x=type2 in allE)
  apply simp
  apply (erule impE)
  apply (rule types-snoc)
  apply assumption
  apply (erule list-app-typeE)
  apply (ind-cases e ⊢ t ° u : T for t u T)
  apply (drule-tac T=type1 ⇒ type2 and U=Ta ⇒ Tsa ⇛ T in var-app-type-eq)
  apply assumption
  apply simp
  apply (erule impE)
  apply (rule typing.App)
  apply assumption
  apply (erule list-app-typeE)
  apply (ind-cases e ⊢ t ° u : T for t u T)
  apply (frule-tac T=type1 ⇒ type2 and U=Ta ⇒ Tsa ⇛ T in var-app-type-eq)
  apply assumption
  apply simp
  apply (erule exE)
  apply (rule-tac x=type1 # Us in exI)
  apply simp
  apply (erule list-app-typeE)
  apply (ind-cases e ⊢ t ° u : T for t u T)
  apply (frule-tac T=type1 ⇒ Us ⇛ T and U=Ta ⇒ Tsa ⇛ T in var-app-type-eq)
  apply assumption
  apply simp
  done

lemma var-app-typesE: e ⊢ Var i °° ts : T ⟹
  (⋀Ts. e ⊢ Var i : Ts ⇛ T ⟹ e ⊩ ts : Ts ⟹ P) ⟹ P
  apply (drule var-app-types [of - - [], simplified])
```

apply (*iprover intro*: *typing.Var*)+
done

lemma *abs-typeE*: $e \vdash Abs\ t : T \Longrightarrow (\bigwedge U\ V.\ e\langle 0{:}U\rangle \vdash t : V \Longrightarrow P) \Longrightarrow P$
apply (*cases T*)
apply (*rule FalseE*)
apply (*erule typing.cases*)
apply *simp-all*
apply *atomize*
apply (*erule-tac x=type1* **in** *allE*)
apply (*erule-tac x=type2* **in** *allE*)
apply (*erule mp*)
apply (*erule typing.cases*)
apply *simp-all*
done

6.6 Lifting preserves well-typedness

lemma *lift-type* [*intro*!]: $e \vdash t : T \Longrightarrow e\langle i{:}U\rangle \vdash lift\ t\ i : T$
by (*induct arbitrary*: *i U set*: *typing*) *auto*

lemma *lift-types*:
$e \Vdash ts : Ts \Longrightarrow e\langle i{:}U\rangle \Vdash (map\ (\lambda t.\ lift\ t\ i)\ ts) : Ts$
apply (*induct ts arbitrary*: *Ts*)
apply *simp*
apply (*case-tac Ts*)
apply *auto*
done

6.7 Substitution lemmas

lemma *subst-lemma*:
$e \vdash t : T \Longrightarrow e' \vdash u : U \Longrightarrow e = e'\langle i{:}U\rangle \Longrightarrow e' \vdash t[u/i] : T$
apply (*induct arbitrary*: *e' i U u set*: *typing*)
apply (*rule-tac* $x = x$ **and** $y = i$ **in** *linorder-cases*)
apply *auto*
apply *blast*
done

lemma *substs-lemma*:
$e \vdash u : T \Longrightarrow e\langle i{:}T\rangle \Vdash ts : Ts \Longrightarrow$
$e \Vdash (map\ (\lambda t.\ t[u/i])\ ts) : Ts$
apply (*induct ts arbitrary*: *Ts*)
apply (*case-tac Ts*)
apply *simp*
apply *simp*
apply *atomize*
apply (*case-tac Ts*)
apply *simp*
apply *simp*

apply (*erule conjE*)
apply (*erule* (*1*) *subst-lemma*)
apply (*rule refl*)
done

6.8 Subject reduction

lemma *subject-reduction*: $e \vdash t : T \Longrightarrow t \rightarrow_\beta t' \Longrightarrow e \vdash t' : T$
apply (*induct arbitrary*: t' *set*: *typing*)
apply *blast*
apply *blast*
apply *atomize*
apply (*ind-cases* $s \circ t \rightarrow_\beta t'$ **for** $s\ t\ t'$)
apply *hypsubst*
apply (*ind-cases* $env \vdash Abs\ t : T \Rightarrow U$ **for** $env\ t\ T\ U$)
apply (*rule subst-lemma*)
apply *assumption*
apply *assumption*
apply (*rule ext*)
apply (*case-tac x*)
apply *auto*
done

theorem *subject-reduction'*: $t \rightarrow_\beta{}^* t' \Longrightarrow e \vdash t : T \Longrightarrow e \vdash t' : T$
by (*induct set*: *rtranclp*) (*iprover intro*: *subject-reduction*)+

6.9 Alternative induction rule for types

lemma *type-induct* [*induct type*]:
assumes
$(\bigwedge T.\ (\bigwedge T1\ T2.\ T = T1 \Rightarrow T2 \Longrightarrow P\ T1) \Longrightarrow$
$(\bigwedge T1\ T2.\ T = T1 \Rightarrow T2 \Longrightarrow P\ T2) \Longrightarrow P\ T)$
shows $P\ T$
proof (*induct T*)
case *Atom*
show *?case* **by** (*rule assms*) *simp-all*
next
case *Fun*
show *?case* **by** (*rule assms*) (*insert Fun*, *simp-all*)
qed

end

7 Lifting an order to lists of elements

theory *ListOrder* **imports** *Main* **begin**

Lifting an order to lists of elements, relating exactly one element.

definition
  step1 :: ('*a* => '*a* => *bool*) => '*a list* => '*a list* => *bool* **where**
  step1 r =
    (λ*ys xs*. $\exists$ *us z z*' *vs*. *xs* = *us* @ *z* # *vs* $\wedge$ *r z*' *z* $\wedge$ *ys* =
      us @ *z*' # *vs*)

lemma *step1-converse* [*simp*]: *step1* (*r*ˆ−−*1*) = (*step1 r*)ˆ−−*1*
  apply (*unfold step1-def*)
  apply (*blast intro*!: *order-antisym*)
  done

lemma *in-step1-converse* [*iff*]: (*step1* (*r*ˆ−−*1*) *x y*) = ((*step1 r*)ˆ−−*1* *x y*)
  apply *auto*
  done

lemma *not-Nil-step1* [*iff*]: ¬ *step1 r* [] *xs*
  apply (*unfold step1-def*)
  apply *blast*
  done

lemma *not-step1-Nil* [*iff*]: ¬ *step1 r xs* []
  apply (*unfold step1-def*)
  apply *blast*
  done

lemma *Cons-step1-Cons* [*iff*]:
    (*step1 r* (*y* # *ys*) (*x* # *xs*)) =
      (*r y x* $\wedge$ *xs* = *ys* $\vee$ *x* = *y* $\wedge$ *step1 r ys xs*)
  apply (*unfold step1-def*)
  apply (*rule iffI*)
   apply (*erule exE*)
   apply (*rename-tac ts*)
   apply (*case-tac ts*)
    apply *fastsimp*
   apply *force*
  apply (*erule disjE*)
   apply *blast*
  apply (*blast intro*: *Cons-eq-appendI*)
  done

lemma *append-step1I*:
  step1 r ys xs $\wedge$ *vs* = *us* $\vee$ *ys* = *xs* $\wedge$ *step1 r vs us*
    ==> *step1 r* (*ys* @ *vs*) (*xs* @ *us*)
  apply (*unfold step1-def*)
  apply *auto*
   apply *blast*
  apply (*blast intro*: *append-eq-appendI*)
  done

lemma *Cons-step1E* [*elim*!]:
  assumes *step1 r ys* (*x* # *xs*)
    and !!*y*. *ys* = *y* # *xs* $\Longrightarrow$ *r y x* $\Longrightarrow$ *R*
    and !!*zs*. *ys* = *x* # *zs* $\Longrightarrow$ *step1 r zs xs* $\Longrightarrow$ *R*
  shows *R*
  using *assms*
  apply (*cases ys*)
   apply (*simp add*: *step1-def*)
  apply *blast*
  done

lemma *Snoc-step1-SnocD*:
  step1 r (*ys* @ [*y*]) (*xs* @ [*x*])
    ==> (*step1 r ys xs* $\wedge$ *y* = *x* $\vee$ *ys* = *xs* $\wedge$ *r y x*)
  apply (*unfold step1-def*)
  apply (*clarify del*: *disjCI*)
  apply (*rename-tac vs*)
  apply (*rule-tac xs* = *vs* **in** *rev-exhaust*)
   apply *force*
  apply *simp*
  apply *blast*
  done

lemma *Cons-acc-step1I* [*intro*!]:
    accp r x ==> *accp* (*step1 r*) *xs* $\Longrightarrow$ *accp* (*step1 r*) (*x* # *xs*)
  apply (*induct arbitrary*: *xs set*: *accp*)
  apply (*erule thin-rl*)
  apply (*erule accp-induct*)
  apply (*rule accp.accI*)
  apply *blast*
  done

lemma *lists-accD*: *listsp* (*accp r*) *xs* ==> *accp* (*step1 r*) *xs*
  apply (*induct set*: *listsp*)
   apply (*rule accp.accI*)
   apply *simp*
  apply (*rule accp.accI*)
  apply (*fast dest*: *accp-downward*)
  done

lemma *ex-step1I*:
  [| *x* $\in$ *set xs*; *r y x* |]
    ==> $\exists$ *ys*. *step1 r ys xs* $\wedge$ *y* $\in$ *set ys*
  apply (*unfold step1-def*)
  apply (*drule in-set-conv-decomp* [*THEN iffD1*])
  apply *force*
  done

lemma *lists-accI*: *accp* (*step1 r*) *xs* ==> *listsp* (*accp r*) *xs*
  apply (*induct set*: *accp*)
  apply *clarify*
  apply (*rule accp.accI*)
  apply (*drule-tac r*=*r* **in** *ex-step1I*, *assumption*)
  apply *blast*
  done

end

8 Lifting beta-reduction to lists

theory *ListBeta* **imports** *ListApplication ListOrder* **begin**

Lifting beta-reduction to lists of terms, reducing exactly one element.

abbreviation
  list-beta :: *dB list* => *dB list* => *bool* (**infixl** => *50*) **where**
  rs => *ss* == *step1 beta rs ss*

lemma *head-Var-reduction*:
  Var n °° *rs* $\to_\beta$ *v* $\Longrightarrow$ ∃ *ss*. *rs* => *ss* ∧ *v* = *Var n* °° *ss*
  apply (*induct u* == *Var n* °° *rs v arbitrary*: *rs set*: *beta*)
      apply *simp*
    apply (*rule-tac xs* = *rs* **in** *rev-exhaust*)
      apply *simp*
    apply (*atomize*, *force intro*: *append-step1I*)
   apply (*rule-tac xs* = *rs* **in** *rev-exhaust*)
    apply *simp*
    apply (*auto 0 3 intro*: *disjI2* [*THEN append-step1I*])
  done

lemma *apps-betasE* [*elim*!]:
  assumes *major*: *r* °° *rs* $\to_\beta$ *s*
    and *cases*: !!*r'*. [| *r* $\to_\beta$ *r'*; *s* = *r'* °° *rs* |] ==> *R*
      !!*rs'*. [| *rs* => *rs'*; *s* = *r* °° *rs'* |] ==> *R*
      !!*t u us*. [| *r* = *Abs t*; *rs* = *u* # *us*; *s* = *t*[*u*/*0*] °° *us* |] ==> *R*
  shows *R*
proof −
  from *major* **have**
   (∃ *r'*. *r* $\to_\beta$ *r'* ∧ *s* = *r'* °° *rs*) ∨
    (∃ *rs'*. *rs* => *rs'* ∧ *s* = *r* °° *rs'*) ∨
    (∃ *t u us*. *r* = *Abs t* ∧ *rs* = *u* # *us* ∧ *s* = *t*[*u*/*0*] °° *us*)
    apply (*induct u* == *r* °° *rs s arbitrary*: *r rs set*: *beta*)
       apply (*case-tac r*)
         apply *simp*
        apply (*simp add*: *App-eq-foldl-conv*)
        apply (*split split-if-asm*)
         apply *simp*

```
          apply blast
         apply simp
        apply (simp add: App-eq-foldl-conv)
        apply (split split-if-asm)
         apply simp
        apply simp
       apply (drule App-eq-foldl-conv [THEN iffD1])
       apply (split split-if-asm)
        apply simp
        apply blast
       apply (force intro!: disjI1 [THEN append-step1I])
      apply (drule App-eq-foldl-conv [THEN iffD1])
      apply (split split-if-asm)
       apply simp
       apply blast
      apply (clarify, auto 0 3 intro!: exI intro: append-step1I)
     done
  with cases show ?thesis by blast
qed

lemma apps-preserves-beta [simp]:
    r →β s ==> r °° ss →β s °° ss
  by (induct ss rule: rev-induct) auto

lemma apps-preserves-beta2 [simp]:
    r ->> s ==> r °° ss ->> s °° ss
  apply (induct set: rtranclp)
   apply blast
  apply (blast intro: apps-preserves-beta rtranclp.rtrancl-into-rtrancl)
  done

lemma apps-preserves-betas [simp]:
    rs => ss ⟹ r °° rs →β r °° ss
  apply (induct rs arbitrary: ss rule: rev-induct)
   apply simp
  apply simp
  apply (rule-tac xs = ss in rev-exhaust)
   apply simp
  apply simp
  apply (drule Snoc-step1-SnocD)
  apply blast
  done

end
```

9 Inductive characterization of terminating lambda terms

theory *InductTermi* **imports** *ListBeta* **begin**

9.1 Terminating lambda terms

```
inductive IT :: dB => bool
  where
    Var [intro]: listsp IT rs ==> IT (Var n °° rs)
  | Lambda [intro]: IT r ==> IT (Abs r)
  | Beta [intro]: IT ((r[s/0]) °° ss) ==> IT s ==> IT ((Abs r ° s) °° ss)
```

9.2 Every term in IT terminates

```
lemma double-induction-lemma [rule-format]:
  termip beta s ==> ∀ t. termip beta t -->
    (∀ r ss. t = r[s/0] °° ss --> termip beta (Abs r ° s °° ss))
  apply (erule accp-induct)
  apply (rule allI)
  apply (rule impI)
  apply (erule thin-rl)
  apply (erule accp-induct)
  apply clarify
  apply (rule accp.accI)
  apply (safe elim!: apps-betasE)
     apply assumption
    apply (blast intro: subst-preserves-beta apps-preserves-beta)
   apply (blast intro: apps-preserves-beta2 subst-preserves-beta2 rtranclp-converseI
     dest: accp-downwards)
  apply (blast dest: apps-preserves-betas)
  done
```

```
lemma IT-implies-termi: IT t ==> termip beta t
  apply (induct set: IT)
    apply (drule rev-predicate1D [OF - listsp-mono [where B=termip beta]])
    apply fast
    apply (drule lists-accD)
    apply (erule accp-induct)
    apply (rule accp.accI)
    apply (blast dest: head-Var-reduction)
   apply (erule accp-induct)
   apply (rule accp.accI)
   apply blast
  apply (blast intro: double-induction-lemma)
  done
```

9.3 Every terminating term is in IT

declare *Var-apps-neq-Abs-apps* [*symmetric*, *simp*]

lemma [*simp*, *THEN not-sym*, *simp*]: *Var n* °° *ss* ≠ *Abs r* ° *s* °° *ts*
  by (*simp add*: *foldl-Cons* [*symmetric*] *del*: *foldl-Cons*)

lemma [*simp*]:
  (*Abs r* ° *s* °° *ss* = *Abs r'* ° *s'* °° *ss'*) = (*r* = *r'* ∧ *s* = *s'* ∧ *ss* = *ss'*)
  by (*simp add*: *foldl-Cons* [*symmetric*] *del*: *foldl-Cons*)

inductive-cases [*elim*!]:
  IT (*Var n* °° *ss*)
  IT (*Abs t*)
  IT (*Abs r* ° *s* °° *ts*)

theorem *termi-implies-IT*: *termip beta r* ==> *IT r*
  apply (*erule accp-induct*)
  apply (*rename-tac r*)
  apply (*erule thin-rl*)
  apply (*erule rev-mp*)
  apply *simp*
  apply (*rule-tac t* = *r* **in** *Apps-dB-induct*)
   apply *clarify*
   apply (*rule IT.intros*)
   apply *clarify*
   apply (*drule bspec*, *assumption*)
   apply (*erule mp*)
   apply *clarify*
   apply (*drule-tac r*=*beta* **in** *conversepI*)
   apply (*drule-tac r*=*beta*ˆ−−*1* **in** *ex-step1I*, *assumption*)
   apply *clarify*
   apply (*rename-tac us*)
   apply (*erule-tac x* = *Var n* °° *us* **in** *allE*)
   apply *force*
   apply (*rename-tac u ts*)
   apply (*case-tac ts*)
    apply *simp*
    apply *blast*
   apply (*rename-tac s ss*)
   apply *simp*
   apply *clarify*
   apply (*rule IT.intros*)
    apply (*blast intro*: *apps-preserves-beta*)
   apply (*erule mp*)
   apply *clarify*
   apply (*rename-tac t*)
   apply (*erule-tac x* = *Abs u* ° *t* °° *ss* **in** *allE*)
   apply *force*
   done

end

10 Strong normalization for simply-typed lambda calculus

theory *StrongNorm* **imports** *Type InductTermi* **begin**

Formalization by Stefan Berghofer. Partly based on a paper proof by Felix Joachimski and Ralph Matthes [1].

10.1 Properties of *IT*

```
lemma lift-IT [intro!]: IT t ⟹ IT (lift t i)
  apply (induct arbitrary: i set: IT)
    apply (simp (no-asm))
    apply (rule conjI)
     apply
      (rule impI,
       rule IT.Var,
       erule listsp.induct,
       simp (no-asm),
       rule listsp.Nil,
       simp (no-asm),
       rule listsp.Cons,
       blast,
       assumption)+
     apply auto
   done

lemma lifts-IT: listsp IT ts ⟹ listsp IT (map (λt. lift t 0) ts)
  by (induct ts) auto

lemma subst-Var-IT: IT r ⟹ IT (r[Var i/j])
  apply (induct arbitrary: i j set: IT)
```

Case *Var*:

```
    apply (simp (no-asm) add: subst-Var)
    apply
    ((rule conjI impI)+,
      rule IT.Var,
      erule listsp.induct,
      simp (no-asm),
      rule listsp.Nil,
      simp (no-asm),
      rule listsp.Cons,
      fast,
      assumption)+
```

Case *Lambda*:

apply *atomize*
apply *simp*
apply (*rule IT.Lambda*)
apply *fast*

Case *Beta*:

apply *atomize*
apply (*simp* (*no-asm-use*) *add*: *subst-subst* [*symmetric*])
apply (*rule IT.Beta*)
apply *auto*
done

lemma *Var-IT*: *IT* (*Var n*)
apply (*subgoal-tac IT* (*Var n* $\circ\circ$ []))
apply *simp*
apply (*rule IT.Var*)
apply (*rule listsp.Nil*)
done

lemma *app-Var-IT*: *IT t* $\Longrightarrow$ *IT* (*t* $\circ$ *Var i*)
apply (*induct set*: *IT*)
apply (*subst app-last*)
apply (*rule IT.Var*)
apply *simp*
apply (*rule listsp.Cons*)
apply (*rule Var-IT*)
apply (*rule listsp.Nil*)
apply (*rule IT.Beta* [**where** *?ss* = [], *unfolded foldl-Nil* [*THEN eq-reflection*]])
apply (*erule subst-Var-IT*)
apply (*rule Var-IT*)
apply (*subst app-last*)
apply (*rule IT.Beta*)
apply (*subst app-last* [*symmetric*])
apply *assumption*
apply *assumption*
done

10.2 Well-typed substitution preserves termination

lemma *subst-type-IT*:
$\bigwedge t\ e\ T\ u\ i.\ IT\ t \Longrightarrow e\langle i{:}U\rangle \vdash t : T \Longrightarrow$
$IT\ u \Longrightarrow e \vdash u : U \Longrightarrow IT\ (t[u/i])$
(**is** *PROP ?P U* **is** $\bigwedge t\ e\ T\ u\ i.\ \text{-} \Longrightarrow PROP\ ?Q\ t\ e\ T\ u\ i\ U$)
proof (*induct U*)
fix *T t*
assume *MI1*: $\bigwedge T1\ T2.\ T = T1 \Rightarrow T2 \Longrightarrow PROP\ ?P\ T1$
assume *MI2*: $\bigwedge T1\ T2.\ T = T1 \Rightarrow T2 \Longrightarrow PROP\ ?P\ T2$
assume *IT t*
thus $\bigwedge e\ T'\ u\ i.\ PROP\ ?Q\ t\ e\ T'\ u\ i\ T$
proof *induct*

```
fix e T′ u i
assume uIT: IT u
assume uT: e ⊢ u : T
{
  case (Var rs n e- T′- u- i-)
  assume nT: e⟨i:T⟩ ⊢ Var n °° rs : T′
  let ?ty = λt. ∃ T′. e⟨i:T⟩ ⊢ t : T′
  let ?R = λt. ∀ e T′ u i.
    e⟨i:T⟩ ⊢ t : T′ ⟶ IT u ⟶ e ⊢ u : T ⟶ IT (t[u/i])
  show IT ((Var n °° rs)[u/i])
  proof (cases n = i)
    case True
    show ?thesis
    proof (cases rs)
      case Nil
      with uIT True show ?thesis by simp
    next
      case (Cons a as)
      with nT have e⟨i:T⟩ ⊢ Var n ° a °° as : T′ by simp
      then obtain Ts
          where headT: e⟨i:T⟩ ⊢ Var n ° a : Ts ⇛ T′
          and argsT: e⟨i:T⟩ ⊩ as : Ts
        by (rule list-app-typeE)
      from headT obtain T″
          where varT: e⟨i:T⟩ ⊢ Var n : T″ ⇒ Ts ⇛ T′
          and argT: e⟨i:T⟩ ⊢ a : T″
        by cases simp-all
      from varT True have T: T = T″ ⇒ Ts ⇛ T′
        by cases auto
      with uT have uT′: e ⊢ u : T″ ⇒ Ts ⇛ T′ by simp
      from T have IT ((Var 0 °° map (λt. lift t 0)
        (map (λt. t[u/i]) as))[(u ° a[u/i])/0])
      proof (rule MI2)
        from T have IT ((lift u 0 ° Var 0)[a[u/i]/0])
        proof (rule MI1)
          have IT (lift u 0) by (rule lift-IT [OF uIT])
          thus IT (lift u 0 ° Var 0) by (rule app-Var-IT)
          show e⟨0:T″⟩ ⊢ lift u 0 ° Var 0 : Ts ⇛ T′
          proof (rule typing.App)
            show e⟨0:T″⟩ ⊢ lift u 0 : T″ ⇒ Ts ⇛ T′
              by (rule lift-type) (rule uT′)
            show e⟨0:T″⟩ ⊢ Var 0 : T″
              by (rule typing.Var) simp
          qed
          from Var have ?R a by cases (simp-all add: Cons)
          with argT uIT uT show IT (a[u/i]) by simp
          from argT uT show e ⊢ a[u/i] : T″
            by (rule subst-lemma) simp
        qed
```

thus $IT\ (u \circ a[u/i])$ **by** *simp*
from *Var* **have** *listsp ?R as*
by *cases* (*simp-all add*: *Cons*)
moreover from *argsT* **have** *listsp ?ty as*
by (*rule lists-typings*)
ultimately have *listsp* ($\lambda t.$ *?R t* $\wedge$ *?ty t*) *as*
by *simp*
hence *listsp IT* (*map* ($\lambda t.$ *lift t 0*) (*map* ($\lambda t.$ $t[u/i]$) *as*))
(**is** *listsp IT* (*?ls as*))
proof *induct*
case *Nil*
show *?case* **by** *fastsimp*
next
case (*Cons b bs*)
hence *I*: *?R b* **by** *simp*
from *Cons* **obtain** *U* **where** $e\langle i{:}T\rangle \vdash b : U$ **by** *fast*
with *uT uIT I* **have** *IT* ($b[u/i]$) **by** *simp*
hence *IT* (*lift* ($b[u/i]$) *0*) **by** (*rule lift-IT*)
hence *listsp IT* (*lift* ($b[u/i]$) *0 # ?ls bs*)
by (*rule listsp.Cons*) (*rule Cons*)
thus *?case* **by** *simp*
qed
thus *IT* (*Var 0* $\circ\circ$ *?ls as*) **by** (*rule IT.Var*)
have $e\langle 0{:}Ts \Rrightarrow T'\rangle \vdash Var\ 0 : Ts \Rrightarrow T'$
by (*rule typing.Var*) *simp*
moreover from *uT argsT* **have** $e \Vdash map\ (\lambda t.\ t[u/i])\ as : Ts$
by (*rule substs-lemma*)
hence $e\langle 0{:}Ts \Rrightarrow T'\rangle \Vdash\ ?ls\ as : Ts$
by (*rule lift-types*)
ultimately show $e\langle 0{:}Ts \Rrightarrow T'\rangle \vdash Var\ 0 \circ\circ\ ?ls\ as : T'$
by (*rule list-app-typeI*)
from *argT uT* **have** $e \vdash a[u/i] : T''$
by (*rule subst-lemma*) (*rule refl*)
with uT' **show** $e \vdash u \circ a[u/i] : Ts \Rrightarrow T'$
by (*rule typing.App*)
qed
with *Cons True* **show** *?thesis*
by (*simp add*: *map-compose* [*symmetric*] *comp-def*)
qed
next
case *False*
from *Var* **have** *listsp ?R rs* **by** *simp*
moreover from *nT* **obtain** *Ts* **where** $e\langle i{:}T\rangle \Vdash rs : Ts$
by (*rule list-app-typeE*)
hence *listsp ?ty rs* **by** (*rule lists-typings*)
ultimately have *listsp* ($\lambda t.$ *?R t* $\wedge$ *?ty t*) *rs*
by *simp*
hence *listsp IT* (*map* ($\lambda x.$ $x[u/i]$) *rs*)
proof *induct*

```
      case Nil
      show ?case by fastsimp
    next
      case (Cons a as)
      hence I: ?R a by simp
      from Cons obtain U where e⟨i:T⟩ ⊢ a : U by fast
      with uT uIT I have IT (a[u/i]) by simp
      hence listsp IT (a[u/i] # map (λt. t[u/i]) as)
        by (rule listsp.Cons) (rule Cons)
      thus ?case by simp
    qed
    with False show ?thesis by (auto simp add: subst-Var)
  qed
next
  case (Lambda r e- T'- u- i-)
  assume e⟨i:T⟩ ⊢ Abs r : T'
    and ⋀e T' u i. PROP ?Q r e T' u i T
  with uIT uT show IT (Abs r[u/i])
    by fastsimp
next
  case (Beta r a as e- T'- u- i-)
  assume T: e⟨i:T⟩ ⊢ Abs r ° a °° as : T'
  assume SI1: ⋀e T' u i. PROP ?Q (r[a/0] °° as) e T' u i T
  assume SI2: ⋀e T' u i. PROP ?Q a e T' u i T
  have IT (Abs (r[lift u 0/Suc i]) ° a[u/i] °° map (λt. t[u/i]) as)
  proof (rule IT.Beta)
    have Abs r ° a °° as →β r[a/0] °° as
      by (rule apps-preserves-beta) (rule beta.beta)
    with T have e⟨i:T⟩ ⊢ r[a/0] °° as : T'
      by (rule subject-reduction)
    hence IT ((r[a/0] °° as)[u/i])
      using uIT uT by (rule SI1)
    thus IT (r[lift u 0/Suc i][a[u/i]/0] °° map (λt. t[u/i]) as)
      by (simp del: subst-map add: subst-subst subst-map [symmetric])
    from T obtain U where e⟨i:T⟩ ⊢ Abs r ° a : U
      by (rule list-app-typeE) fast
    then obtain T'' where e⟨i:T⟩ ⊢ a : T'' by cases simp-all
    thus IT (a[u/i]) using uIT uT by (rule SI2)
  qed
  thus IT ((Abs r ° a °° as)[u/i]) by simp
}
qed
qed
```

10.3 Well-typed terms are strongly normalizing

```
lemma type-implies-IT:
  assumes e ⊢ t : T
  shows IT t
```

```
  using assms
proof induct
  case Var
  show ?case by (rule Var-IT)
next
  case Abs
  show ?case by (rule IT.Lambda) (rule Abs)
next
  case (App e s T U t)
  have IT ((Var 0 ° lift t 0)[s/0])
  proof (rule subst-type-IT)
    have IT (lift t 0) using ‹IT t› by (rule lift-IT)
    hence listsp IT [lift t 0] by (rule listsp.Cons) (rule listsp.Nil)
    hence IT (Var 0 °° [lift t 0]) by (rule IT.Var)
    also have Var 0 °° [lift t 0] = Var 0 ° lift t 0 by simp
    finally show IT ... .
    have e⟨0:T ⇒ U⟩ ⊢ Var 0 : T ⇒ U
      by (rule typing.Var) simp
    moreover have e⟨0:T ⇒ U⟩ ⊢ lift t 0 : T
      by (rule lift-type) (rule App.hyps)
    ultimately show e⟨0:T ⇒ U⟩ ⊢ Var 0 ° lift t 0 : U
      by (rule typing.App)
    show IT s by fact
    show e ⊢ s : T ⇒ U by fact
  qed
  thus ?case by simp
qed

theorem type-implies-termi: e ⊢ t : T ⟹ termip beta t
proof −
  assume e ⊢ t : T
  hence IT t by (rule type-implies-IT)
  thus ?thesis by (rule IT-implies-termi)
qed

end
```

11 Inductive characterization of lambda terms in normal form

```
theory NormalForm
imports ListBeta
begin
```

11.1 Terms in normal form

```
definition
```

listall :: $('a \Rightarrow bool) \Rightarrow 'a\ list \Rightarrow bool$ **where**
listall P xs $\equiv (\forall\, i.\ i < length\ xs \longrightarrow P\ (xs\ !\ i))$

declare *listall-def* [*extraction-expand*]

theorem *listall-nil*: *listall P* []
by (*simp add*: *listall-def*)

theorem *listall-nil-eq* [*simp*]: *listall P* [] = *True*
by (*iprover intro*: *listall-nil*)

theorem *listall-cons*: $P\ x \Longrightarrow listall\ P\ xs \Longrightarrow listall\ P\ (x\ \#\ xs)$
apply (*simp add*: *listall-def*)
apply (*rule allI impI*)+
apply (*case-tac i*)
apply *simp*+
done

theorem *listall-cons-eq* [*simp*]: $listall\ P\ (x\ \#\ xs) = (P\ x \wedge listall\ P\ xs)$
apply (*rule iffI*)
prefer *2*
apply (*erule conjE*)
apply (*erule listall-cons*)
apply *assumption*
apply (*unfold listall-def*)
apply (*rule conjI*)
apply (*erule-tac x=0* **in** *allE*)
apply *simp*
apply *simp*
apply (*rule allI*)
apply (*erule-tac x=Suc i* **in** *allE*)
apply *simp*
done

lemma *listall-conj1*: $listall\ (\lambda x.\ P\ x \wedge Q\ x)\ xs \Longrightarrow listall\ P\ xs$
by (*induct xs*) *simp-all*

lemma *listall-conj2*: $listall\ (\lambda x.\ P\ x \wedge Q\ x)\ xs \Longrightarrow listall\ Q\ xs$
by (*induct xs*) *simp-all*

lemma *listall-app*: $listall\ P\ (xs\ @\ ys) = (listall\ P\ xs \wedge listall\ P\ ys)$
apply (*induct xs*)
apply (*rule iffI*, *simp*, *simp*)
apply (*rule iffI*, *simp*, *simp*)
done

lemma *listall-snoc* [*simp*]: $listall\ P\ (xs\ @\ [x]) = (listall\ P\ xs \wedge P\ x)$
apply (*rule iffI*)
apply (*simp add*: *listall-app*)+

done

lemma *listall-cong* [*cong*, *extraction-expand*]:
xs = *ys* $\Longrightarrow$ *listall P xs* = *listall P ys*
— Currently needed for strange technical reasons
by (*unfold listall-def*) *simp*

listsp is equivalent to *listall*, but cannot be used for program extraction.

lemma *listall-listsp-eq*: *listall P xs* = *listsp P xs*
by (*induct xs*) (*auto intro*: *listsp.intros*)

inductive *NF* :: *dB* $\Rightarrow$ *bool*
where
App: *listall NF ts* $\Longrightarrow$ *NF* (*Var x* °° *ts*)
| *Abs*: *NF t* $\Longrightarrow$ *NF* (*Abs t*)
monos *listall-def*

lemma *nat-eq-dec*: $\bigwedge$*n*::*nat*. *m* = *n* $\vee$ *m* $\neq$ *n*
apply (*induct m*)
apply (*case-tac n*)
apply (*case-tac* [*3*] *n*)
apply (*simp only*: *nat.simps*, *iprover?*)+
done

lemma *nat-le-dec*: $\bigwedge$*n*::*nat*. *m* < *n* $\vee$ $\neg$ (*m* < *n*)
apply (*induct m*)
apply (*case-tac n*)
apply (*case-tac* [*3*] *n*)
apply (*simp del*: *simp-thms*, *iprover?*)+
done

lemma *App-NF-D*: **assumes** *NF*: *NF* (*Var n* °° *ts*)
shows *listall NF ts* **using** *NF*
by *cases simp-all*

11.2 Properties of *NF*

lemma *Var-NF*: *NF* (*Var n*)
apply (*subgoal-tac NF* (*Var n* °° []))
apply *simp*
apply (*rule NF.App*)
apply *simp*
done

lemma *Abs-NF*:
assumes *NF*: *NF* (*Abs t* °° *ts*)
shows *ts* = [] **using** *NF*
proof *cases*
case (*App us i*)

thus *?thesis* **by** (*simp add*: *Var-apps-neq-Abs-apps* [*THEN not-sym*])
next
case (*Abs u*)
thus *?thesis* **by** *simp*
qed

lemma *subst-terms-NF*: *listall NF ts* $\Longrightarrow$
listall ($\lambda t.\ \forall\, i\ j.\ NF\ (t[Var\ i/j])$) *ts* $\Longrightarrow$
listall NF (*map* ($\lambda t.\ t[Var\ i/j]$) *ts*)
by (*induct ts*) *simp-all*

lemma *subst-Var-NF*: *NF t* $\Longrightarrow$ *NF* ($t[Var\ i/j]$)
apply (*induct arbitrary*: *i j set*: *NF*)
apply *simp*
apply (*frule listall-conj1*)
apply (*drule listall-conj2*)
apply (*drule-tac* *i*=*i* **and** *j*=*j* **in** *subst-terms-NF*)
apply *assumption*
apply (*rule-tac* *m*=*x* **and** *n*=*j* **in** *nat-eq-dec* [*THEN disjE*, *standard*])
apply *simp*
apply (*erule NF.App*)
apply (*rule-tac* *m*=*j* **and** *n*=*x* **in** *nat-le-dec* [*THEN disjE*, *standard*])
apply *simp*
apply (*iprover intro*: *NF.App*)
apply *simp*
apply (*iprover intro*: *NF.App*)
apply *simp*
apply (*iprover intro*: *NF.Abs*)
done

lemma *app-Var-NF*: *NF t* $\Longrightarrow \exists\, t'.\ t \circ Var\ i \rightarrow_{\beta}{}^{*}\ t' \wedge NF\ t'$
apply (*induct set*: *NF*)
apply (*simplesubst app-last*) — Using *subst* makes extraction fail
apply (*rule exI*)
apply (*rule conjI*)
apply (*rule rtranclp.rtrancl-refl*)
apply (*rule NF.App*)
apply (*drule listall-conj1*)
apply (*simp add*: *listall-app*)
apply (*rule Var-NF*)
apply (*rule exI*)
apply (*rule conjI*)
apply (*rule rtranclp.rtrancl-into-rtrancl*)
apply (*rule rtranclp.rtrancl-refl*)
apply (*rule beta*)
apply (*erule subst-Var-NF*)
done

lemma *lift-terms-NF*: *listall NF ts* $\Longrightarrow$

```
    listall (λt. ∀ i. NF (lift t i)) ts ⟹
    listall NF (map (λt. lift t i) ts)
  by (induct ts) simp-all

lemma lift-NF: NF t ⟹ NF (lift t i)
  apply (induct arbitrary: i set: NF)
  apply (frule listall-conj1)
  apply (drule listall-conj2)
  apply (drule-tac i=i in lift-terms-NF)
  apply assumption
  apply (rule-tac m=x and n=i in nat-le-dec [THEN disjE, standard])
  apply simp
  apply (rule NF.App)
  apply assumption
  apply simp
  apply (rule NF.App)
  apply assumption
  apply simp
  apply (rule NF.Abs)
  apply simp
  done
```

NF characterizes exactly the terms that are in normal form.

```
lemma NF-eq: NF t = (∀ t'. ¬ t →β t')
proof
  assume NF t
  then have ⋀t'. ¬ t →β t'
  proof induct
    case (App ts t)
    show ?case
    proof
      assume Var t °° ts →β t'
      then obtain rs where ts => rs
        by (iprover dest: head-Var-reduction)
      with App show False
        by (induct rs arbitrary: ts) auto
    qed
  next
    case (Abs t)
    show ?case
    proof
      assume Abs t →β t'
      then show False using Abs by cases simp-all
    qed
  qed
  then show ∀ t'. ¬ t →β t' ..
next
  assume H: ∀ t'. ¬ t →β t'
  then show NF t
```

```
  proof (induct t rule: Apps-dB-induct)
    case (1 n ts)
    then have ∀ ts'. ¬ ts => ts'
      by (iprover intro: apps-preserves-betas)
    with 1(1) have listall NF ts
      by (induct ts) auto
    then show ?case by (rule NF.App)
  next
    case (2 u ts)
    show ?case
    proof (cases ts)
      case Nil
      from 2 have ∀ u'. ¬ u →β u'
        by (auto intro: apps-preserves-beta)
      then have NF u by (rule 2)
      then have NF (Abs u) by (rule NF.Abs)
      with Nil show ?thesis by simp
    next
      case (Cons r rs)
      have Abs u ° r →β u[r/0] ..
      then have Abs u ° r °° rs →β u[r/0] °° rs
        by (rule apps-preserves-beta)
      with Cons have Abs u °° ts →β u[r/0] °° rs
        by simp
      with 2 show ?thesis by iprover
    qed
  qed
qed

end
```

12 Standardization

```
theory Standardization
imports NormalForm
begin
```

Based on lecture notes by Ralph Matthes [3], original proof idea due to Ralph Loader [2].

12.1 Standard reduction relation

```
declare listrel-mono [mono-set]

inductive
  sred :: dB ⇒ dB ⇒ bool  (infixl →s 50)
  and sredlist :: dB list ⇒ dB list ⇒ bool  (infixl [→s] 50)
where
```

$s\ [\rightarrow_s]\ t \equiv \mathit{listrelp}\ \mathit{op} \rightarrow_s\ s\ t$
| Var: $\mathit{rs}\ [\rightarrow_s]\ \mathit{rs}' \Longrightarrow \mathit{Var}\ x\ {}^{\circ\circ}\ \mathit{rs} \rightarrow_s \mathit{Var}\ x\ {}^{\circ\circ}\ \mathit{rs}'$
| Abs: $r \rightarrow_s r' \Longrightarrow \mathit{ss}\ [\rightarrow_s]\ \mathit{ss}' \Longrightarrow \mathit{Abs}\ r\ {}^{\circ\circ}\ \mathit{ss} \rightarrow_s \mathit{Abs}\ r'\ {}^{\circ\circ}\ \mathit{ss}'$
| Beta: $r[s/0]\ {}^{\circ\circ}\ \mathit{ss} \rightarrow_s t \Longrightarrow \mathit{Abs}\ r\ {}^{\circ}\ s\ {}^{\circ\circ}\ \mathit{ss} \rightarrow_s t$

lemma *refl-listrelp*: $\forall x \in \mathit{set}\ \mathit{xs}.\ R\ x\ x \Longrightarrow \mathit{listrelp}\ R\ \mathit{xs}\ \mathit{xs}$
by (*induct xs*) (*auto intro*: *listrelp.intros*)

lemma *refl-sred*: $t \rightarrow_s t$
by (*induct t rule*: *Apps-dB-induct*) (*auto intro*: *refl-listrelp sred.intros*)

lemma *refl-sreds*: $\mathit{ts}\ [\rightarrow_s]\ \mathit{ts}$
by (*simp add*: *refl-sred refl-listrelp*)

lemma *listrelp-conj1*: $\mathit{listrelp}\ (\lambda x\ y.\ R\ x\ y \wedge S\ x\ y)\ x\ y \Longrightarrow \mathit{listrelp}\ R\ x\ y$
by (*erule listrelp.induct*) (*auto intro*: *listrelp.intros*)

lemma *listrelp-conj2*: $\mathit{listrelp}\ (\lambda x\ y.\ R\ x\ y \wedge S\ x\ y)\ x\ y \Longrightarrow \mathit{listrelp}\ S\ x\ y$
by (*erule listrelp.induct*) (*auto intro*: *listrelp.intros*)

lemma *listrelp-app*:
assumes *xsys*: *listrelp R xs ys*
shows $\mathit{listrelp}\ R\ \mathit{xs}'\ \mathit{ys}' \Longrightarrow \mathit{listrelp}\ R\ (\mathit{xs}\ @\ \mathit{xs}')\ (\mathit{ys}\ @\ \mathit{ys}')$ **using** *xsys*
by (*induct arbitrary*: $\mathit{xs}'\ \mathit{ys}'$) (*auto intro*: *listrelp.intros*)

lemma *lemma1*:
assumes r: $r \rightarrow_s r'$ **and** s: $s \rightarrow_s s'$
shows $r\ {}^{\circ}\ s \rightarrow_s r'\ {}^{\circ}\ s'$ **using** r
proof *induct*
case ($\mathit{Var}\ \mathit{rs}\ \mathit{rs}'\ x$)
then have $\mathit{rs}\ [\rightarrow_s]\ \mathit{rs}'$ **by** (*rule listrelp-conj1*)
moreover have $[s]\ [\rightarrow_s]\ [s']$ **by** (*iprover intro*: *s listrelp.intros*)
ultimately have $\mathit{rs}\ @\ [s]\ [\rightarrow_s]\ \mathit{rs}'\ @\ [s']$ **by** (*rule listrelp-app*)
hence $\mathit{Var}\ x\ {}^{\circ\circ}\ (\mathit{rs}\ @\ [s]) \rightarrow_s \mathit{Var}\ x\ {}^{\circ\circ}\ (\mathit{rs}'\ @\ [s'])$ **by** (*rule sred.Var*)
thus *?case* **by** (*simp only*: *app-last*)
next
case ($\mathit{Abs}\ r\ r'\ \mathit{ss}\ \mathit{ss}'$)
from *Abs*(*3*) **have** $\mathit{ss}\ [\rightarrow_s]\ \mathit{ss}'$ **by** (*rule listrelp-conj1*)
moreover have $[s]\ [\rightarrow_s]\ [s']$ **by** (*iprover intro*: *s listrelp.intros*)
ultimately have $\mathit{ss}\ @\ [s]\ [\rightarrow_s]\ \mathit{ss}'\ @\ [s']$ **by** (*rule listrelp-app*)
with $\langle r \rightarrow_s r' \rangle$ **have** $\mathit{Abs}\ r\ {}^{\circ\circ}\ (\mathit{ss}\ @\ [s]) \rightarrow_s \mathit{Abs}\ r'\ {}^{\circ\circ}\ (\mathit{ss}'\ @\ [s'])$
by (*rule sred.Abs*)
thus *?case* **by** (*simp only*: *app-last*)
next
case ($\mathit{Beta}\ r\ u\ \mathit{ss}\ t$)
hence $r[u/0]\ {}^{\circ\circ}\ (\mathit{ss}\ @\ [s]) \rightarrow_s t\ {}^{\circ}\ s'$ **by** (*simp only*: *app-last*)
hence $\mathit{Abs}\ r\ {}^{\circ}\ u\ {}^{\circ\circ}\ (\mathit{ss}\ @\ [s]) \rightarrow_s t\ {}^{\circ}\ s'$ **by** (*rule sred.Beta*)
thus *?case* **by** (*simp only*: *app-last*)
qed

lemma *lemma1'*:
  assumes *ts*: $ts\ [\rightarrow_s]\ ts'$
  shows $r \rightarrow_s r' \Longrightarrow r \ {}^{\circ\circ}\ ts \rightarrow_s r' \ {}^{\circ\circ}\ ts'$ **using** *ts*
  by (*induct arbitrary*: *r r'*) (*auto intro*: *lemma1*)

lemma *lemma2-1*:
  assumes *beta*: $t \rightarrow_\beta u$
  shows $t \rightarrow_s u$ **using** *beta*
proof *induct*
  case (*beta s t*)
  have $Abs\ s \ {}^\circ\ t \ {}^{\circ\circ}\ [] \rightarrow_s s[t/0] \ {}^{\circ\circ}\ []$ **by** (*iprover intro*: *sred.Beta refl-sred*)
  thus *?case* **by** *simp*
next
  case (*appL s t u*)
  thus *?case* **by** (*iprover intro*: *lemma1 refl-sred*)
next
  case (*appR s t u*)
  thus *?case* **by** (*iprover intro*: *lemma1 refl-sred*)
next
  case (*abs s t*)
  hence $Abs\ s \ {}^{\circ\circ}\ [] \rightarrow_s Abs\ t \ {}^{\circ\circ}\ []$ **by** (*iprover intro*: *sred.Abs listrelp.Nil*)
  thus *?case* **by** *simp*
qed

lemma *listrelp-betas*:
  assumes *ts*: $listrelp\ op \rightarrow_\beta{}^*\ ts\ ts'$
  shows $\bigwedge t\ t'.\ t \rightarrow_\beta{}^*\ t' \Longrightarrow t \ {}^{\circ\circ}\ ts \rightarrow_\beta{}^*\ t' \ {}^{\circ\circ}\ ts'$ **using** *ts*
  by *induct auto*

lemma *lemma2-2*:
  assumes *t*: $t \rightarrow_s u$
  shows $t \rightarrow_\beta{}^*\ u$ **using** *t*
  by *induct* (*auto dest*: *listrelp-conj2*
    intro: *listrelp-betas apps-preserves-beta converse-rtranclp-into-rtranclp*)

lemma *sred-lift*:
  assumes *s*: $s \rightarrow_s t$
  shows $lift\ s\ i \rightarrow_s lift\ t\ i$ **using** *s*
proof (*induct arbitrary*: *i*)
  case (*Var rs rs' x*)
  hence $map\ (\lambda t.\ lift\ t\ i)\ rs\ [\rightarrow_s]\ map\ (\lambda t.\ lift\ t\ i)\ rs'$
    by *induct* (*auto intro*: *listrelp.intros*)
  thus *?case* **by** (*cases* $x < i$) (*auto intro*: *sred.Var*)
next
  case (*Abs r r' ss ss'*)
  from *Abs(3)* **have** $map\ (\lambda t.\ lift\ t\ i)\ ss\ [\rightarrow_s]\ map\ (\lambda t.\ lift\ t\ i)\ ss'$
    by *induct* (*auto intro*: *listrelp.intros*)
  thus *?case* **by** (*auto intro*: *sred.Abs Abs*)

```
next
  case (Beta r s ss t)
  thus ?case by (auto intro: sred.Beta)
qed

lemma lemma3:
  assumes r: r →s r'
  shows s →s s' ⟹ r[s/x] →s r'[s'/x] using r
proof (induct arbitrary: s s' x)
  case (Var rs rs' y)
  hence map (λt. t[s/x]) rs [→s] map (λt. t[s'/x]) rs'
    by induct (auto intro: listrelp.intros Var)
  moreover have Var y[s/x] →s Var y[s'/x]
  proof (cases y < x)
    case True thus ?thesis by simp (rule refl-sred)
  next
    case False
    thus ?thesis
      by (cases y = x) (auto simp add: Var intro: refl-sred)
  qed
  ultimately show ?case by simp (rule lemma1')
next
  case (Abs r r' ss ss')
  from Abs(4) have lift s 0 →s lift s' 0 by (rule sred-lift)
  hence r[lift s 0/Suc x] →s r'[lift s' 0/Suc x] by (fast intro: Abs.hyps)
  moreover from Abs(3) have map (λt. t[s/x]) ss [→s] map (λt. t[s'/x]) ss'
    by induct (auto intro: listrelp.intros Abs)
  ultimately show ?case by simp (rule sred.Abs)
next
  case (Beta r u ss t)
  thus ?case by (auto simp add: subst-subst intro: sred.Beta)
qed

lemma lemma4-aux:
  assumes rs: listrelp (λt u. t →s u ∧ (∀r. u →β r ⟶ t →s r)) rs rs'
  shows rs' => ss ⟹ rs [→s] ss using rs
proof (induct arbitrary: ss)
  case Nil
  thus ?case by cases (auto intro: listrelp.Nil)
next
  case (Cons x y xs ys)
  note Cons' = Cons
  show ?case
  proof (cases ss)
    case Nil with Cons show ?thesis by simp
  next
    case (Cons y' ys')
    hence ss: ss = y' # ys' by simp
    from Cons Cons' have y →β y' ∧ ys' = ys ∨ y' = y ∧ ys => ys' by simp
```

hence $x \# xs\ [\rightarrow_s]\ y' \# ys'$
proof
assume H: $y \rightarrow_\beta y' \wedge ys' = ys$
with $Cons'$ **have** $x \rightarrow_s y'$ **by** *blast*
moreover from $Cons'$ **have** $xs\ [\rightarrow_s]\ ys$ **by** (*iprover dest*: *listrelp-conj1*)
ultimately have $x \# xs\ [\rightarrow_s]\ y' \# ys$ **by** (*rule listrelp.Cons*)
with H **show** *?thesis* **by** *simp*
next
assume H: $y' = y \wedge ys => ys'$
with $Cons'$ **have** $x \rightarrow_s y'$ **by** *blast*
moreover from H **have** $xs\ [\rightarrow_s]\ ys'$ **by** (*blast intro*: $Cons'$)
ultimately show *?thesis* **by** (*rule listrelp.Cons*)
qed
with ss **show** *?thesis* **by** *simp*
qed
qed

lemma *lemma4*:
assumes r: $r \rightarrow_s r'$
shows $r' \rightarrow_\beta r'' \Longrightarrow r \rightarrow_s r''$ **using** r
proof (*induct arbitrary*: r'')
case (*Var rs rs' x*)
then obtain ss **where** rs: $rs' => ss$ **and** r'': $r'' = Var\ x\ {}^{\circ\circ}\ ss$
by (*blast dest*: *head-Var-reduction*)
from *Var(1)* rs **have** $rs\ [\rightarrow_s]\ ss$ **by** (*rule lemma4-aux*)
hence $Var\ x\ {}^{\circ\circ}\ rs \rightarrow_s Var\ x\ {}^{\circ\circ}\ ss$ **by** (*rule sred.Var*)
with r'' **show** *?case* **by** *simp*
next
case (*Abs r r' ss ss'*)
from $\langle Abs\ r'\ {}^{\circ\circ}\ ss' \rightarrow_\beta r''\rangle$ **show** *?case*
proof
fix s
assume r'': $r'' = s\ {}^{\circ\circ}\ ss'$
assume $Abs\ r' \rightarrow_\beta s$
then obtain r''' **where** s: $s = Abs\ r'''$ **and** r''': $r' \rightarrow_\beta r'''$ **by** *cases auto*
from r''' **have** $r \rightarrow_s r'''$ **by** (*blast intro*: *Abs*)
moreover from *Abs* **have** $ss\ [\rightarrow_s]\ ss'$ **by** (*iprover dest*: *listrelp-conj1*)
ultimately have $Abs\ r\ {}^{\circ\circ}\ ss \rightarrow_s Abs\ r'''\ {}^{\circ\circ}\ ss'$ **by** (*rule sred.Abs*)
with $r''\ s$ **show** $Abs\ r\ {}^{\circ\circ}\ ss \rightarrow_s r''$ **by** *simp*
next
fix rs'
assume $ss' => rs'$
with *Abs(3)* **have** $ss\ [\rightarrow_s]\ rs'$ **by** (*rule lemma4-aux*)
with $\langle r \rightarrow_s r'\rangle$ **have** $Abs\ r\ {}^{\circ\circ}\ ss \rightarrow_s Abs\ r'\ {}^{\circ\circ}\ rs'$ **by** (*rule sred.Abs*)
moreover assume $r'' = Abs\ r'\ {}^{\circ\circ}\ rs'$
ultimately show $Abs\ r\ {}^{\circ\circ}\ ss \rightarrow_s r''$ **by** *simp*
next
fix $t\ u'\ us'$
assume $ss' = u' \# us'$

with $Abs(3)$ **obtain** u us **where**
ss: $ss = u \# us$ **and** u: $u \rightarrow_s u'$ **and** us: us $[\rightarrow_s]$ us'
by $cases$ $(auto\ dest!: listrelp\text{-}conj1)$
have $r[u/0] \rightarrow_s r'[u'/0]$ **using** $Abs(1)$ **and** u **by** $(rule\ lemma3)$
with us **have** $r[u/0] \circ\circ us \rightarrow_s r'[u'/0] \circ\circ us'$ **by** $(rule\ lemma1')$
hence $Abs\ r \circ u \circ\circ us \rightarrow_s r'[u'/0] \circ\circ us'$ **by** $(rule\ sred.Beta)$
moreover assume $Abs\ r' = Abs\ t$ **and** $r'' = t[u'/0] \circ\circ us'$
ultimately show $Abs\ r \circ\circ ss \rightarrow_s r''$ **using** ss **by** $simp$
qed
next
case $(Beta\ r\ s\ ss\ t)$
show $?case$
by $(rule\ sred.Beta)$ $(rule\ Beta)+$
qed

lemma $rtrancl\text{-}beta\text{-}sred$:
assumes r: $r \rightarrow_\beta{}^* r'$
shows $r \rightarrow_s r'$ **using** r
by $induct$ $(iprover\ intro: refl\text{-}sred\ lemma4)+$

12.2 Leftmost reduction and weakly normalizing terms

inductive
$lred :: dB \Rightarrow dB \Rightarrow bool$ (**infixl** $\rightarrow_l$ 50)
and $lredlist :: dB\ list \Rightarrow dB\ list \Rightarrow bool$ (**infixl** $[\rightarrow_l]$ 50)
where
$s\ [\rightarrow_l]\ t \equiv listrelp\ op \rightarrow_l s\ t$
| Var: $rs\ [\rightarrow_l]\ rs' \Longrightarrow Var\ x \circ\circ rs \rightarrow_l Var\ x \circ\circ rs'$
| Abs: $r \rightarrow_l r' \Longrightarrow Abs\ r \rightarrow_l Abs\ r'$
| $Beta$: $r[s/0] \circ\circ ss \rightarrow_l t \Longrightarrow Abs\ r \circ s \circ\circ ss \rightarrow_l t$

lemma $lred\text{-}imp\text{-}sred$:
assumes $lred$: $s \rightarrow_l t$
shows $s \rightarrow_s t$ **using** $lred$
proof $induct$
case $(Var\ rs\ rs'\ x)$
then have $rs\ [\rightarrow_s]\ rs'$
by $induct$ $(iprover\ intro: listrelp.intros)+$
then show $?case$ **by** $(rule\ sred.Var)$
next
case $(Abs\ r\ r')$
from $\langle r \rightarrow_s r' \rangle$
have $Abs\ r \circ\circ [] \rightarrow_s Abs\ r' \circ\circ []$ **using** $listrelp.Nil$
by $(rule\ sred.Abs)$
then show $?case$ **by** $simp$
next
case $(Beta\ r\ s\ ss\ t)$
from $\langle r[s/0] \circ\circ ss \rightarrow_s t \rangle$
show $?case$ **by** $(rule\ sred.Beta)$

qed

inductive *WN* :: *dB* => *bool*
  where
    Var: *listsp WN rs* $\Longrightarrow$ *WN* (*Var n* °° *rs*)
  | *Lambda*: *WN r* $\Longrightarrow$ *WN* (*Abs r*)
  | *Beta*: *WN* ((*r*[*s*/*0*]) °° *ss*) $\Longrightarrow$ *WN* ((*Abs r* ° *s*) °° *ss*)

lemma *listrelp-imp-listsp1*:
  assumes *H*: *listrelp* (λ*x y*. *P x*) *xs ys*
  shows *listsp P xs* **using** *H*
  by *induct auto*

lemma *listrelp-imp-listsp2*:
  assumes *H*: *listrelp* (λ*x y*. *P y*) *xs ys*
  shows *listsp P ys* **using** *H*
  by *induct auto*

lemma *lemma5*:
  assumes *lred*: $r \rightarrow_l r'$
  shows *WN r* **and** *NF r'* **using** *lred*
  by *induct*
    (*iprover dest*: *listrelp-conj1 listrelp-conj2*
     listrelp-imp-listsp1 listrelp-imp-listsp2 intro: *WN.intros*
     NF.intros [*simplified listall-listsp-eq*])+

lemma *lemma6*:
  assumes *wn*: *WN r*
  shows $\exists r'.\ r \rightarrow_l r'$ **using** *wn*
proof *induct*
  case (*Var rs n*)
  then have $\exists rs'.\ rs\ [\rightarrow_l]\ rs'$
    by *induct* (*iprover intro*: *listrelp.intros*)+
  then show *?case* **by** (*iprover intro*: *lred.Var*)
qed (*iprover intro*: *lred.intros*)+

lemma *lemma7*:
  assumes *r*: $r \rightarrow_s r'$
  shows *NF r'* $\Longrightarrow r \rightarrow_l r'$ **using** *r*
proof *induct*
  case (*Var rs rs' x*)
  from ‹*NF* (*Var x* °° *rs'*)› **have** *listall NF rs'*
    by *cases simp-all*
  with *Var*(*1*) **have** $rs\ [\rightarrow_l]\ rs'$
  proof *induct*
    case *Nil*
    show *?case* **by** (*rule listrelp.Nil*)
  next
    case (*Cons x y xs ys*)

hence $x \rightarrow_l y$ **and** $xs\ [\rightarrow_l]\ ys$ **by** *simp-all*
thus *?case* **by** (*rule listrelp.Cons*)
qed
thus *?case* **by** (*rule lred.Var*)
next
case ($Abs\ r\ r'\ ss\ ss'$)
from ‹$NF\ (Abs\ r'\ {}^{\circ\circ}\ ss')$›
have ss': $ss' = []$ **by** (*rule Abs-NF*)
from *Abs*(*3*) **have** *ss*: $ss = []$ **using** ss'
by *cases simp-all*
from ss' *Abs* **have** $NF\ (Abs\ r')$ **by** *simp*
hence $NF\ r'$ **by** *cases simp-all*
with *Abs* **have** $r \rightarrow_l r'$ **by** *simp*
hence $Abs\ r \rightarrow_l Abs\ r'$ **by** (*rule lred.Abs*)
with $ss\ ss'$ **show** *?case* **by** *simp*
next
case ($Beta\ r\ s\ ss\ t$)
hence $r[s/0]\ {}^{\circ\circ}\ ss \rightarrow_l t$ **by** *simp*
thus *?case* **by** (*rule lred.Beta*)
qed

lemma *WN-eq*: $WN\ t = (\exists\, t'.\ t \rightarrow_\beta{}^*\ t' \wedge NF\ t')$
proof
assume $WN\ t$
then have $\exists\, t'.\ t \rightarrow_l t'$ **by** (*rule lemma6*)
then obtain t' **where** t': $t \rightarrow_l t'$ **..**
then have *NF*: $NF\ t'$ **by** (*rule lemma5*)
from t' **have** $t \rightarrow_s t'$ **by** (*rule lred-imp-sred*)
then have $t \rightarrow_\beta{}^*\ t'$ **by** (*rule lemma2-2*)
with *NF* **show** $\exists\, t'.\ t \rightarrow_\beta{}^*\ t' \wedge NF\ t'$ **by** *iprover*
next
assume $\exists\, t'.\ t \rightarrow_\beta{}^*\ t' \wedge NF\ t'$
then obtain t' **where** t': $t \rightarrow_\beta{}^*\ t'$ **and** *NF*: $NF\ t'$
by *iprover*
from t' **have** $t \rightarrow_s t'$ **by** (*rule rtrancl-beta-sred*)
then have $t \rightarrow_l t'$ **using** *NF* **by** (*rule lemma7*)
then show $WN\ t$ **by** (*rule lemma5*)
qed

end

13 Weak normalization for simply-typed lambda calculus

theory *WeakNorm*
imports *Type NormalForm Code-Integer*
begin

Formalization by Stefan Berghofer. Partly based on a paper proof by Felix Joachimski and Ralph Matthes [1].

13.1 Main theorems

lemma *norm-list*:
assumes *f-compat*: $\bigwedge t\ t'.\ t \rightarrow_\beta{}^* t' \Longrightarrow f\ t \rightarrow_\beta{}^* f\ t'$
and *f-NF*: $\bigwedge t.\ NF\ t \Longrightarrow NF\ (f\ t)$
and *uNF*: $NF\ u$ **and** *uT*: $e \vdash u : T$
shows $\bigwedge Us.\ e\langle i{:}T\rangle \Vdash as : Us \Longrightarrow$
$listall\ (\lambda t.\ \forall\, e\ T'\ u\ i.\ e\langle i{:}T\rangle \vdash t : T' \longrightarrow$
$NF\ u \longrightarrow e \vdash u : T \longrightarrow (\exists\, t'.\ t[u/i] \rightarrow_\beta{}^* t' \wedge NF\ t'))\ as \Longrightarrow$
$\exists\, as'.\ \forall\, j.\ Var\ j\ {}^{\circ\circ}\ map\ (\lambda t.\ f\ (t[u/i]))\ as \rightarrow_\beta{}^*$
$Var\ j\ {}^{\circ\circ}\ map\ f\ as' \wedge NF\ (Var\ j\ {}^{\circ\circ}\ map\ f\ as')$
(**is** $\bigwedge Us.$ - $\Longrightarrow listall\ ?R\ as \Longrightarrow \exists\, as'.\ ?ex\ Us\ as\ as'$)
proof (*induct as rule*: *rev-induct*)
case (*Nil Us*)
with *Var-NF* **have** *?ex Us* [] [] **by** *simp*
thus *?case* ..
next
case (*snoc b bs Us*)
have $e\langle i{:}T\rangle \Vdash bs\ @\ [b] : Us$ **by** *fact*
then obtain *Vs W* **where** *Us*: $Us = Vs\ @\ [W]$
and *bs*: $e\langle i{:}T\rangle \Vdash bs : Vs$ **and** *bT*: $e\langle i{:}T\rangle \vdash b : W$
by (*rule types-snocE*)
from *snoc* **have** *listall ?R bs* **by** *simp*
with *bs* **have** $\exists\, bs'.\ ?ex\ Vs\ bs\ bs'$ **by** (*rule snoc*)
then obtain bs' **where**
bsred: $\bigwedge j.\ Var\ j\ {}^{\circ\circ}\ map\ (\lambda t.\ f\ (t[u/i]))\ bs \rightarrow_\beta{}^* Var\ j\ {}^{\circ\circ}\ map\ f\ bs'$
and *bsNF*: $\bigwedge j.\ NF\ (Var\ j\ {}^{\circ\circ}\ map\ f\ bs')$ **by** *iprover*
from *snoc* **have** *?R b* **by** *simp*
with *bT* **and** *uNF* **and** *uT* **have** $\exists\, b'.\ b[u/i] \rightarrow_\beta{}^* b' \wedge NF\ b'$
by *iprover*
then obtain b' **where** *bred*: $b[u/i] \rightarrow_\beta{}^* b'$ **and** *bNF*: $NF\ b'$
by *iprover*
from *bsNF* [*of 0*] **have** $listall\ NF\ (map\ f\ bs')$
by (*rule App-NF-D*)
moreover have $NF\ (f\ b')$ **using** *bNF* **by** (*rule f-NF*)
ultimately have $listall\ NF\ (map\ f\ (bs'\ @\ [b']))$
by *simp*
hence $\bigwedge j.\ NF\ (Var\ j\ {}^{\circ\circ}\ map\ f\ (bs'\ @\ [b']))$ **by** (*rule NF.App*)
moreover from *bred* **have** $f\ (b[u/i]) \rightarrow_\beta{}^* f\ b'$
by (*rule f-compat*)
with *bsred* **have**
$\bigwedge j.\ (Var\ j\ {}^{\circ\circ}\ map\ (\lambda t.\ f\ (t[u/i]))\ bs)\ {}^\circ\ f\ (b[u/i]) \rightarrow_\beta{}^*$
$(Var\ j\ {}^{\circ\circ}\ map\ f\ bs')\ {}^\circ\ f\ b'$ **by** (*rule rtrancl-beta-App*)
ultimately have $?ex\ Us\ (bs\ @\ [b])\ (bs'\ @\ [b'])$ **by** *simp*
thus *?case* ..
qed

lemma *subst-type-NF*:
 $\bigwedge t\ e\ T\ u\ i.\ NF\ t \Longrightarrow e\langle i{:}U\rangle \vdash t : T \Longrightarrow NF\ u \Longrightarrow e \vdash u : U \Longrightarrow \exists\, t'.\ t[u/i]$
$\rightarrow_\beta{}^*\ t' \wedge NF\ t'$
 (**is** $PROP\ ?P\ U$ **is** $\bigwedge t\ e\ T\ u\ i.\ \text{-} \Longrightarrow PROP\ ?Q\ t\ e\ T\ u\ i\ U$)
proof (*induct U*)
 fix $T\ t$
 let $?R = \lambda t.\ \forall\, e\ T'\ u\ i.$
  $e\langle i{:}T\rangle \vdash t : T' \longrightarrow NF\ u \longrightarrow e \vdash u : T \longrightarrow (\exists\, t'.\ t[u/i] \rightarrow_\beta{}^*\ t' \wedge NF\ t')$
 assume $MI1$: $\bigwedge T1\ T2.\ T = T1 \Rightarrow T2 \Longrightarrow PROP\ ?P\ T1$
 assume $MI2$: $\bigwedge T1\ T2.\ T = T1 \Rightarrow T2 \Longrightarrow PROP\ ?P\ T2$
 assume $NF\ t$
 thus $\bigwedge e\ T'\ u\ i.\ PROP\ ?Q\ t\ e\ T'\ u\ i\ T$
 proof *induct*
  fix $e\ T'\ u\ i$ **assume** uNF: $NF\ u$ **and** uT: $e \vdash u : T$
  {
   case $(App\ ts\ x\ e\text{-}\ T'\text{-}\ u\text{-}\ i\text{-})$
   assume $e\langle i{:}T\rangle \vdash Var\ x \circ\circ ts : T'$
   then obtain Us
    where $varT$: $e\langle i{:}T\rangle \vdash Var\ x : Us \Rrightarrow T'$
    and $argsT$: $e\langle i{:}T\rangle \Vdash ts : Us$
    by (*rule var-app-typesE*)
   from *nat-eq-dec* **show** $\exists\, t'.\ (Var\ x \circ\circ ts)[u/i] \rightarrow_\beta{}^*\ t' \wedge NF\ t'$
   proof
    assume eq: $x = i$
    show *?thesis*
    proof (*cases ts*)
     case *Nil*
     with *eq* **have** $(Var\ x \circ\circ [])[u/i] \rightarrow_\beta{}^*\ u$ **by** *simp*
     with *Nil* **and** *uNF* **show** *?thesis* **by** *simp iprover*
    next
     case (*Cons a as*)
     with *argsT* **obtain** $T''\ Ts$ **where** Us: $Us = T'' \# Ts$
      by (*cases Us*) (*rule FalseE*, *simp+*, *erule that*)
     from *varT* **and** *Us* **have** $varT$: $e\langle i{:}T\rangle \vdash Var\ x : T'' \Rightarrow Ts \Rrightarrow T'$
      by *simp*
     from *varT eq* **have** T: $T = T'' \Rightarrow Ts \Rrightarrow T'$ **by** *cases auto*
     with *uT* **have** uT': $e \vdash u : T'' \Rightarrow Ts \Rrightarrow T'$ **by** *simp*
     from *argsT Us Cons* **have** $argsT'$: $e\langle i{:}T\rangle \Vdash as : Ts$ **by** *simp*
     from *argsT Us Cons* **have** $argT$: $e\langle i{:}T\rangle \vdash a : T''$ **by** *simp*
     from *argT uT refl* **have** aT: $e \vdash a[u/i] : T''$ **by** (*rule subst-lemma*)
    from *App* **and** *Cons* **have** *listall ?R as* **by** *simp* (*iprover dest*: *listall-conj2*)
     with $lift\text{-}preserves\text{-}beta'\ lift\text{-}NF\ uNF\ uT\ argsT'$
     have $\exists\, as'.\ \forall j.\ Var\ j \circ\circ map\ (\lambda t.\ lift\ (t[u/i])\ 0)\ as \rightarrow_\beta{}^*$
      $Var\ j \circ\circ map\ (\lambda t.\ lift\ t\ 0)\ as' \wedge$
      $NF\ (Var\ j \circ\circ map\ (\lambda t.\ lift\ t\ 0)\ as')$ **by** (*rule norm-list*)
     then obtain as' **where**
      $asred$: $Var\ 0 \circ\circ map\ (\lambda t.\ lift\ (t[u/i])\ 0)\ as \rightarrow_\beta{}^*$
       $Var\ 0 \circ\circ map\ (\lambda t.\ lift\ t\ 0)\ as'$

and *asNF*: $NF\ (Var\ 0\ {}^{\circ\circ}\ map\ (\lambda t.\ lift\ t\ 0)\ as')$ **by** *iprover*
from *App* **and** *Cons* **have** *?R a* **by** *simp*
with *argT* **and** *uNF* **and** *uT* **have** $\exists\, a'.\ a[u/i] \rightarrow_\beta{}^*\ a' \wedge NF\ a'$
by *iprover*
then obtain a' **where** *ared*: $a[u/i] \rightarrow_\beta{}^*\ a'$ **and** *aNF*: $NF\ a'$ **by** *iprover*
from *uNF* **have** $NF\ (lift\ u\ 0)$ **by** (*rule lift-NF*)
hence $\exists\, u'.\ lift\ u\ 0\ \circ\ Var\ 0 \rightarrow_\beta{}^*\ u' \wedge NF\ u'$ **by** (*rule app-Var-NF*)
then obtain u' **where** *ured*: $lift\ u\ 0\ \circ\ Var\ 0 \rightarrow_\beta{}^*\ u'$ **and** $u'NF$: $NF\ u'$
by *iprover*
from T **and** $u'NF$ **have** $\exists\, ua.\ u'[a'/0] \rightarrow_\beta{}^*\ ua \wedge NF\ ua$
proof (*rule MI1*)
have $e\langle 0{:}T''\rangle \vdash lift\ u\ 0\ \circ\ Var\ 0 : Ts \Rrightarrow T'$
proof (*rule typing.App*)
from uT' **show** $e\langle 0{:}T''\rangle \vdash lift\ u\ 0 : T'' \Rightarrow Ts \Rrightarrow T'$ **by** (*rule lift-type*)
show $e\langle 0{:}T''\rangle \vdash Var\ 0 : T''$ **by** (*rule typing.Var*) *simp*
qed
with *ured* **show** $e\langle 0{:}T''\rangle \vdash u' : Ts \Rrightarrow T'$ **by** (*rule subject-reduction'*)
from *ared aT* **show** $e \vdash a' : T''$ **by** (*rule subject-reduction'*)
show $NF\ a'$ **by** *fact*
qed
then obtain *ua* **where** *uared*: $u'[a'/0] \rightarrow_\beta{}^*\ ua$ **and** *uaNF*: $NF\ ua$
by *iprover*
from *ared* **have** $(lift\ u\ 0\ \circ\ Var\ 0)[a[u/i]/0] \rightarrow_\beta{}^*\ (lift\ u\ 0\ \circ\ Var\ 0)[a'/0]$
by (*rule subst-preserves-beta2'*)
also from *ured* **have** $(lift\ u\ 0\ \circ\ Var\ 0)[a'/0] \rightarrow_\beta{}^*\ u'[a'/0]$
by (*rule subst-preserves-beta'*)
also note *uared*
finally have $(lift\ u\ 0\ \circ\ Var\ 0)[a[u/i]/0] \rightarrow_\beta{}^*\ ua$ **.**
hence *uared'*: $u\ \circ\ a[u/i] \rightarrow_\beta{}^*\ ua$ **by** *simp*
from *T asNF - uaNF* **have** $\exists\, r.\ (Var\ 0\ {}^{\circ\circ}\ map\ (\lambda t.\ lift\ t\ 0)\ as')[ua/0]$
$\rightarrow_\beta{}^*\ r \wedge NF\ r$
proof (*rule MI2*)
have $e\langle 0{:}Ts \Rrightarrow T'\rangle \vdash Var\ 0\ {}^{\circ\circ}\ map\ (\lambda t.\ lift\ (t[u/i])\ 0)\ as : T'$
proof (*rule list-app-typeI*)
show $e\langle 0{:}Ts \Rrightarrow T'\rangle \vdash Var\ 0 : Ts \Rrightarrow T'$ **by** (*rule typing.Var*) *simp*
from *uT argsT'* **have** $e \Vdash map\ (\lambda t.\ t[u/i])\ as : Ts$
by (*rule substs-lemma*)
hence $e\langle 0{:}Ts \Rrightarrow T'\rangle \Vdash map\ (\lambda t.\ lift\ t\ 0)\ (map\ (\lambda t.\ t[u/i])\ as) : Ts$
by (*rule lift-types*)
thus $e\langle 0{:}Ts \Rrightarrow T'\rangle \Vdash map\ (\lambda t.\ lift\ (t[u/i])\ 0)\ as : Ts$
by (*simp-all add*: *map-compose* [*symmetric*] *o-def*)
qed
with *asred* **show** $e\langle 0{:}Ts \Rrightarrow T'\rangle \vdash Var\ 0\ {}^{\circ\circ}\ map\ (\lambda t.\ lift\ t\ 0)\ as' : T'$
by (*rule subject-reduction'*)
from *argT uT refl* **have** $e \vdash a[u/i] : T''$ **by** (*rule subst-lemma*)
with uT' **have** $e \vdash u\ \circ\ a[u/i] : Ts \Rrightarrow T'$ **by** (*rule typing.App*)
with *uared'* **show** $e \vdash ua : Ts \Rrightarrow T'$ **by** (*rule subject-reduction'*)
qed
then obtain r **where** *rred*: $(Var\ 0\ {}^{\circ\circ}\ map\ (\lambda t.\ lift\ t\ 0)\ as')[ua/0] \rightarrow_\beta{}^*\ r$

and *rnf*: *NF r* **by** *iprover*
from *asred* **have**
$(Var\ 0 \circ\circ\ map\ (\lambda t.\ lift\ (t[u/i])\ 0)\ as)[u \circ a[u/i]/0] \rightarrow_\beta{}^*$
$(Var\ 0 \circ\circ\ map\ (\lambda t.\ lift\ t\ 0)\ as')[u \circ a[u/i]/0]$
by (*rule subst-preserves-beta'*)
also from *uared'* **have** $(Var\ 0 \circ\circ\ map\ (\lambda t.\ lift\ t\ 0)\ as')[u \circ a[u/i]/0] \rightarrow_\beta{}^*$
$(Var\ 0 \circ\circ\ map\ (\lambda t.\ lift\ t\ 0)\ as')[ua/0]$ **by** (*rule subst-preserves-beta2'*)
also note *rred*
finally have $(Var\ 0 \circ\circ\ map\ (\lambda t.\ lift\ (t[u/i])\ 0)\ as)[u \circ a[u/i]/0] \rightarrow_\beta{}^*\ r$ **.**
with *rnf Cons eq* **show** *?thesis*
by (*simp add*: *map-compose* [*symmetric*] *o-def*) *iprover*
qed
next
assume *neq*: $x \neq i$
from *App* **have** *listall ?R ts* **by** (*iprover dest*: *listall-conj2*)
with *TrueI TrueI uNF uT argsT*
have $\exists ts'.\ \forall j.\ Var\ j \circ\circ\ map\ (\lambda t.\ t[u/i])\ ts \rightarrow_\beta{}^*\ Var\ j \circ\circ\ ts' \wedge$
$NF\ (Var\ j \circ\circ\ ts')$ (**is** $\exists ts'.\ ?ex\ ts'$)
by (*rule norm-list* [*of* $\lambda t.\ t$, *simplified*])
then obtain *ts'* **where** *NF*: *?ex ts'* **..**
from *nat-le-dec* **show** *?thesis*
proof
assume $i < x$
with *NF* **show** *?thesis* **by** *simp iprover*
next
assume $\neg\ (i < x)$
with *NF neq* **show** *?thesis* **by** (*simp add*: *subst-Var*) *iprover*
qed
qed
next
case (*Abs r e- T'- u- i-*)
assume *absT*: $e\langle i{:}T\rangle \vdash Abs\ r : T'$
then obtain *R S* **where** $e\langle 0{:}R\rangle\langle Suc\ i{:}T\rangle \vdash r : S$ **by** (*rule abs-typeE*) *simp*
moreover have *NF* (*lift u 0*) **using** ‹*NF u*› **by** (*rule lift-NF*)
moreover have $e\langle 0{:}R\rangle \vdash lift\ u\ 0 : T$ **using** *uT* **by** (*rule lift-type*)
ultimately have $\exists t'.\ r[lift\ u\ 0/Suc\ i] \rightarrow_\beta{}^*\ t' \wedge NF\ t'$ **by** (*rule Abs*)
thus $\exists t'.\ Abs\ r[u/i] \rightarrow_\beta{}^*\ t' \wedge NF\ t'$
by *simp* (*iprover intro*: *rtrancl-beta-Abs NF.Abs*)
}
qed
qed

— A computationally relevant copy of $e \vdash t : T$
inductive *rtyping* :: (*nat* ⇒ *type*) ⇒ *dB* ⇒ *type* ⇒ *bool* (- $\vdash_R$ - : - [*50*, *50*, *50*] *50*)
where
Var: $e\ x = T \Longrightarrow e \vdash_R Var\ x : T$
| *Abs*: $e\langle 0{:}T\rangle \vdash_R t : U \Longrightarrow e \vdash_R Abs\ t : (T \Rightarrow U)$

| App: $e \vdash_R s : T \Rightarrow U \Longrightarrow e \vdash_R t : T \Longrightarrow e \vdash_R (s \circ t) : U$

lemma *rtyping-imp-typing*: $e \vdash_R t : T \Longrightarrow e \vdash t : T$
  apply (*induct set*: *rtyping*)
  apply (*erule typing.Var*)
  apply (*erule typing.Abs*)
  apply (*erule typing.App*)
  apply *assumption*
  done

theorem *type-NF*:
  assumes $e \vdash_R t : T$
  shows $\exists t'. \ t \rightarrow_\beta{}^* \ t' \wedge NF \ t'$ **using** *assms*
proof *induct*
  case *Var*
  show *?case* **by** (*iprover intro*: *Var-NF*)
next
  case *Abs*
  thus *?case* **by** (*iprover intro*: *rtrancl-beta-Abs NF.Abs*)
next
  case (*App e s T U t*)
  from *App* **obtain** $s' \ t'$ **where**
    sred: $s \rightarrow_\beta{}^* \ s'$ **and** $NF \ s'$
    and *tred*: $t \rightarrow_\beta{}^* \ t'$ **and** *tNF*: $NF \ t'$ **by** *iprover*
  have $\exists u. \ (Var \ 0 \circ lift \ t' \ 0)[s'/0] \rightarrow_\beta{}^* \ u \wedge NF \ u$
  proof (*rule subst-type-NF*)
    have $NF \ (lift \ t' \ 0)$ **using** *tNF* **by** (*rule lift-NF*)
    hence *listall NF* $[lift \ t' \ 0]$ **by** (*rule listall-cons*) (*rule listall-nil*)
    hence $NF \ (Var \ 0 \circ\circ [lift \ t' \ 0])$ **by** (*rule NF.App*)
    thus $NF \ (Var \ 0 \circ lift \ t' \ 0)$ **by** *simp*
    show $e\langle 0{:}T \Rightarrow U\rangle \vdash Var \ 0 \circ lift \ t' \ 0 : U$
    proof (*rule typing.App*)
      show $e\langle 0{:}T \Rightarrow U\rangle \vdash Var \ 0 : T \Rightarrow U$
        by (*rule typing.Var*) *simp*
      from *tred* **have** $e \vdash t' : T$
        by (*rule subject-reduction'*) (*rule rtyping-imp-typing*, *rule App.hyps*)
      thus $e\langle 0{:}T \Rightarrow U\rangle \vdash lift \ t' \ 0 : T$
        by (*rule lift-type*)
    qed
    from *sred* **show** $e \vdash s' : T \Rightarrow U$
      by (*rule subject-reduction'*) (*rule rtyping-imp-typing*, *rule App.hyps*)
    show $NF \ s'$ **by** *fact*
  qed
  then obtain u **where** *ured*: $s' \circ t' \rightarrow_\beta{}^* \ u$ **and** *unf*: $NF \ u$ **by** *simp iprover*
  from *sred tred* **have** $s \circ t \rightarrow_\beta{}^* \ s' \circ t'$ **by** (*rule rtrancl-beta-App*)
  hence $s \circ t \rightarrow_\beta{}^* \ u$ **using** *ured* **by** (*rule rtranclp-trans*)
  with *unf* **show** *?case* **by** *iprover*
qed

13.2 Extracting the program

declare *NF.induct* [*ind-realizer*]
declare *rtranclp.induct* [*ind-realizer irrelevant*]
declare *rtyping.induct* [*ind-realizer*]
lemmas [*extraction-expand*] = *conj-assoc listall-cons-eq*

extract *type-NF*

lemma *rtranclR-rtrancl-eq*: $rtranclpR\ r\ a\ b = r^{**}\ a\ b$
  apply (*rule iffI*)
  apply (*erule rtranclpR.induct*)
  apply (*rule rtranclp.rtrancl-refl*)
  apply (*erule rtranclp.rtrancl-into-rtrancl*)
  apply *assumption*
  apply (*erule rtranclp.induct*)
  apply (*rule rtranclpR.rtrancl-refl*)
  apply (*erule rtranclpR.rtrancl-into-rtrancl*)
  apply *assumption*
  done

lemma *NFR-imp-NF*: $NFR\ nf\ t \Longrightarrow NF\ t$
  apply (*erule NFR.induct*)
  apply (*rule NF.intros*)
  apply (*simp add*: *listall-def*)
  apply (*erule NF.intros*)
  done

The program corresponding to the proof of the central lemma, which performs substitution and normalization, is shown in Figure 1. The correctness theorem corresponding to the program *subst-type-NF* is

$\bigwedge x.\ NFR\ x\ t \Longrightarrow$
  $e\langle i{:}U\rangle \vdash t : T \Longrightarrow$
  $(\bigwedge xa.\ NFR\ xa\ u \Longrightarrow$
    $e \vdash u : U \Longrightarrow$
    $t[u/i] \rightarrow_\beta{}^*\ fst\ (subst\text{-}type\text{-}NF\ t\ e\ i\ U\ T\ u\ x\ xa) \wedge$
    $NFR\ (snd\ (subst\text{-}type\text{-}NF\ t\ e\ i\ U\ T\ u\ x\ xa))\ (fst\ (subst\text{-}type\text{-}NF\ t\ e\ i\ U$
$T\ u\ x\ xa)))$

where *NFR* is the realizability predicate corresponding to the datatype *NFT*, which is inductively defined by the rules

```
subst-type-NF ≡
λx xa xb xc xd xe H Ha.
   type-induct-P xc
    (λx H2 H2a xa xb xc xd xe H.
        NFT-rec arbitrary
         (λts xa xaa r xb xc xd xe H.
             var-app-typesE-P (xb⟨xe:x⟩) xa ts
              (λUs--. case nat-eq-dec xa xe of
                       Left ⇒ case ts of [] ⇒ (xd, H)
                               | a # list ⇒
                                   case Us-- of [] ⇒ arbitrary
                                   | T''-- # Ts-- ⇒
                                       let (x, y) =
                                             norm-list (λt. lift t 0) xd xb xe list Ts--
                                              (λt. lift-NF 0) H
                                              (listall-conj2-P-Q list (λi. (xaa (Suc i), r (Suc i))));
                                           (xa, ya) = snd (xaa 0, r 0) xb T''-- xd xe H;
                                           (xd, yb) = app-Var-NF 0 (lift-NF 0 H);
                                           (xa, ya) =
                                             H2 T''-- (Ts-- ⇛ xc) xd xb (Ts-- ⇛ xc) xa 0 yb ya;
                                           (x, y) =
                                             H2a T''-- (Ts-- ⇛ xc) (dB.Var 0 °° map (λt. lift t 0) x)
                                              xb xc xa 0 (y 0) ya
                                       in (x, y)
                       | Right ⇒
                           let (x, y) =
                                 let (x, y) =
                                       norm-list (λt. t) xd xb xe ts Us-- (λx H. H) H
                                        (listall-conj2-P-Q ts (λz. (xaa z, r z)))
                                 in (x, λx. y x)
                           in case nat-le-dec xe xa of
                              Left ⇒ (dB.Var (xa − Suc 0) °° x, y (xa − Suc 0))
                              | Right ⇒ (dB.Var xa °° x, y xa)))
         (λt x r xa xb xc xd H.
             abs-typeE-P xb
              (λU V. let (x, y) =
                          let (x, y) = r (λa. (xa⟨0:U⟩) a) V (lift xc 0) (Suc xd) (lift-NF 0 H)
                          in (dB.Abs x, NFT.Abs x y)
                      in (x, y)))
         H (λa. xb a) xc xd xe)
    x xa xd xe xb H Ha
```

Figure 1: Program extracted from *subst-type-NF*

```
subst-Var-NF ≡
λx xa H.
  NFT-rec arbitrary
   (λts x xa r xb xc.
      case nat-eq-dec x xc of
      Left ⇒ NFT.App (map (λt. t[dB.Var xb/xc]) ts) xb
              (subst-terms-NF ts xb xc (listall-conj1-P-Q ts (λz. (xa z, r z)))
                (listall-conj2-P-Q ts (λz. (xa z, r z))))
      | Right ⇒
          case nat-le-dec xc x of
          Left ⇒ NFT.App (map (λt. t[dB.Var xb/xc]) ts) (x − Suc 0)
                  (subst-terms-NF ts xb xc (listall-conj1-P-Q ts (λz. (xa z, r z)))
                    (listall-conj2-P-Q ts (λz. (xa z, r z))))
          | Right ⇒
              NFT.App (map (λt. t[dB.Var xb/xc]) ts) x
               (subst-terms-NF ts xb xc (listall-conj1-P-Q ts (λz. (xa z, r z)))
                 (listall-conj2-P-Q ts (λz. (xa z, r z)))))
   (λt x r xa xb. NFT.Abs (t[dB.Var (Suc xa)/Suc xb]) (r (Suc xa) (Suc xb))) H x xa

app-Var-NF ≡
λx. NFT-rec arbitrary
      (λts xa xaa r.
          (dB.Var xa °° (ts @ [dB.Var x]),
           NFT.App (ts @ [dB.Var x]) xa
            (snd (listall-app-P ts)
              (listall-conj1-P-Q ts (λz. (xaa z, r z)),
               listall-cons-P (Var-NF x) listall-nil-eq-P))))
      (λt xa r. (t[dB.Var x/0], subst-Var-NF x 0 xa))

lift-NF ≡
λx H. NFT-rec arbitrary
         (λts x xa r xb.
             case nat-le-dec x xb of
             Left ⇒ NFT.App (map (λt. lift t xb) ts) x
                     (lift-terms-NF ts xb (listall-conj1-P-Q ts (λz. (xa z, r z)))
                       (listall-conj2-P-Q ts (λz. (xa z, r z))))
             | Right ⇒
                 NFT.App (map (λt. lift t xb) ts) (Suc x)
                  (lift-terms-NF ts xb (listall-conj1-P-Q ts (λz. (xa z, r z)))
                    (listall-conj2-P-Q ts (λz. (xa z, r z)))))
         (λt x r xa. NFT.Abs (lift t (Suc xa)) (r (Suc xa))) H x

type-NF ≡
λH. rtypingT-rec (λe x T. (dB.Var x, Var-NF x))
       (λe T t U x r. let (x, y) = r in (dB.Abs x, NFT.Abs x y))
       (λe s T U t x xa r ra.
           let (x, y) = r; (xa, ya) = ra;
               (x, y) =
                 let (x, y) =
                       subst-type-NF (dB.App (dB.Var 0) (lift xa 0)) e 0 (T ⇒ U) U x
                        (NFT.App [lift xa 0] 0 (listall-cons-P (lift-NF 0 ya) listall-nil-P)) y
                 in (x, y)
           in (x, y))
       H
```

Figure 2: Program extracted from lemmas and main theorem

$\forall\, i<length\ ts.\ NFR\ (nfs\ i)\ (ts\ !\ i) \Longrightarrow NFR\ (NFT.App\ ts\ x\ nfs)\ (dB.Var\ x\ {}^{\circ\circ}\ ts)$
$NFR\ nf\ t \Longrightarrow NFR\ (NFT.Abs\ t\ nf)\ (dB.Abs\ t)$

The programs corresponding to the main theorem *type-NF*, as well as to some lemmas, are shown in Figure 2. The correctness statement for the main function *type-NF* is

$\bigwedge x.\ rtypingR\ x\ e\ t\ T \Longrightarrow t \rightarrow_{\beta}{}^{*}\ fst\ (type\text{-}NF\ x) \wedge NFR\ (snd\ (type\text{-}NF\ x))\ (fst\ (type\text{-}NF\ x))$

where the realizability predicate *rtypingR* corresponding to the computationally relevant version of the typing judgement is inductively defined by the rules

$e\ x = T \Longrightarrow rtypingR\ (rtypingT.Var\ e\ x\ T)\ e\ (dB.Var\ x)\ T$
$rtypingR\ ty\ (e\langle 0{:}T\rangle)\ t\ U \Longrightarrow rtypingR\ (rtypingT.Abs\ e\ T\ t\ U\ ty)\ e\ (dB.Abs\ t)\ (T \Rightarrow U)$
$rtypingR\ ty\ e\ s\ (T \Rightarrow U) \Longrightarrow$
$rtypingR\ ty'\ e\ t\ T \Longrightarrow rtypingR\ (rtypingT.App\ e\ s\ T\ U\ t\ ty\ ty')\ e\ (dB.App\ s\ t)\ U$

13.3 Generating executable code

consts-code
  arbitrary :: *'a* ((*error arbitrary*))
  arbitrary :: *'a* ⇒ *'b* ((*fn '- => error arbitrary*))

code-module *Norm*
contains
  test = *type-NF*

The following functions convert between Isabelle's built-in `term` datatype and the generated `dB` datatype. This allows to generate example terms using Isabelle's parser and inspect normalized terms using Isabelle's pretty printer.

ML ⟪
```
fun nat-of-int 0 = Norm.zero
  | nat-of-int n = Norm.Suc (nat-of-int (n−1));

fun int-of-nat Norm.zero = 0
  | int-of-nat (Norm.Suc n) = 1 + int-of-nat n;

fun dBtype-of-typ (Type (fun, [T, U])) =
      Norm.Fun (dBtype-of-typ T, dBtype-of-typ U)
  | dBtype-of-typ (TFree (s, -)) = (case explode s of
        [', a] => Norm.Atom (nat-of-int (ord a − 97))
      | - => error dBtype-of-typ: variable name)
  | dBtype-of-typ - = error dBtype-of-typ: bad type;
```

```
fun dB-of-term (Bound i) = Norm.dB-Var (nat-of-int i)
  | dB-of-term (t $ u) = Norm.App (dB-of-term t, dB-of-term u)
  | dB-of-term (Abs (-, -, t)) = Norm.Abs (dB-of-term t)
  | dB-of-term - = error dB-of-term: bad term;

fun term-of-dB Ts (Type (fun, [T, U])) (Norm.Abs dBt) =
      Abs (x, T, term-of-dB (T :: Ts) U dBt)
  | term-of-dB Ts - dBt = term-of-dB' Ts dBt
and term-of-dB' Ts (Norm.dB-Var n) = Bound (int-of-nat n)
  | term-of-dB' Ts (Norm.App (dBt, dBu)) =
      let val t = term-of-dB' Ts dBt
      in case fastype-of1 (Ts, t) of
          Type (fun, [T, U]) => t $ term-of-dB Ts T dBu
        | - => error term-of-dB: function type expected
      end
  | term-of-dB' - - = error term-of-dB: term not in normal form;

fun typing-of-term Ts e (Bound i) =
      Norm.Var (e, nat-of-int i, dBtype-of-typ (List.nth (Ts, i)))
  | typing-of-term Ts e (t $ u) = (case fastype-of1 (Ts, t) of
        Type (fun, [T, U]) => Norm.rtypingT-App (e, dB-of-term t,
          dBtype-of-typ T, dBtype-of-typ U, dB-of-term u,
          typing-of-term Ts e t, typing-of-term Ts e u)
      | - => error typing-of-term: function type expected)
  | typing-of-term Ts e (Abs (s, T, t)) =
      let val dBT = dBtype-of-typ T
      in Norm.rtypingT-Abs (e, dBT, dB-of-term t,
        dBtype-of-typ (fastype-of1 (T :: Ts, t)),
        typing-of-term (T :: Ts) (Norm.shift e Norm.zero dBT) t)
      end
  | typing-of-term - - - = error typing-of-term: bad term;

fun dummyf - = error dummy;
⟫
```

We now try out the extracted program *type-NF* on some example terms.

```
ML ⟪
val ct1 = @{cterm %f. ((%f x. f (f (f x))) ((%f x. f (f (f (f x)))) f))};
val (dB1, -) = Norm.type-NF (typing-of-term [] dummyf (term-of ct1));
val ct1' = cterm-of @{theory} (term-of-dB [] (#T (rep-cterm ct1)) dB1);

val ct2 = @{cterm %f x. (%x. f x x) ((%x. f x x) ((%x. f x x) ((%x. f x x) ((%x.
f x x) ((%x. f x x) x)))))};
val (dB2, -) = Norm.type-NF (typing-of-term [] dummyf (term-of ct2));
val ct2' = cterm-of @{theory} (term-of-dB [] (#T (rep-cterm ct2)) dB2);
⟫
```

The same story again for code next generation.

setup ⟪
  CodeTarget.add-undefined SML arbitrary (*raise Fail* *arbitrary*\)
⟫

definition
  int-of-nat :: *nat* ⇒ *int* **where**
  int-of-nat = *of-nat*

export-code *type-NF nat int-of-nat* **in** *SML* **module-name** *Norm*

ML ⟪

```
val nat-of-int = Norm.nat;
val int-of-nat = Norm.int-of-nat;

fun dBtype-of-typ (Type (fun, [T, U])) =
      Norm.Fun (dBtype-of-typ T, dBtype-of-typ U)
  | dBtype-of-typ (TFree (s, -)) = (case explode s of
        [', a] => Norm.Atom (nat-of-int (ord a − 97))
      | - => error dBtype-of-typ: variable name)
  | dBtype-of-typ - = error dBtype-of-typ: bad type;

fun dB-of-term (Bound i) = Norm.Var (nat-of-int i)
  | dB-of-term (t $ u) = Norm.App (dB-of-term t, dB-of-term u)
  | dB-of-term (Abs (-, -, t)) = Norm.Abs (dB-of-term t)
  | dB-of-term - = error dB-of-term: bad term;

fun term-of-dB Ts (Type (fun, [T, U])) (Norm.Abs dBt) =
      Abs (x, T, term-of-dB (T :: Ts) U dBt)
  | term-of-dB Ts - dBt = term-of-dB′ Ts dBt
and term-of-dB′ Ts (Norm.Var n) = Bound (int-of-nat n)
  | term-of-dB′ Ts (Norm.App (dBt, dBu)) =
      let val t = term-of-dB′ Ts dBt
      in case fastype-of1 (Ts, t) of
          Type (fun, [T, U]) => t $ term-of-dB Ts T dBu
        | - => error term-of-dB: function type expected
      end
  | term-of-dB′ - - = error term-of-dB: term not in normal form;

fun typing-of-term Ts e (Bound i) =
      Norm.Vara (e, nat-of-int i, dBtype-of-typ (nth Ts i))
  | typing-of-term Ts e (t $ u) = (case fastype-of1 (Ts, t) of
        Type (fun, [T, U]) => Norm.Appb (e, dB-of-term t,
          dBtype-of-typ T, dBtype-of-typ U, dB-of-term u,
          typing-of-term Ts e t, typing-of-term Ts e u)
      | - => error typing-of-term: function type expected)
  | typing-of-term Ts e (Abs (s, T, t)) =
      let val dBT = dBtype-of-typ T
      in Norm.Absb (e, dBT, dB-of-term t,
        dBtype-of-typ (fastype-of1 (T :: Ts, t)),
```

```
    typing-of-term (T :: Ts) (Norm.shift e Norm.Zero-nat dBT) t)
   end
 | typing-of-term - - - = error typing-of-term: bad term;

fun dummyf - = error dummy;
⟫

ML ⟪
val ct1 = @{cterm %f. ((%f x. f (f (f x))) ((%f x. f (f (f (f x)))) f))};
val (dB1, -) = Norm.type-NF (typing-of-term [] dummyf (term-of ct1));
val ct1' = cterm-of @{theory} (term-of-dB [] (#T (rep-cterm ct1)) dB1);

val ct2 = @{cterm %f x. (%x. f x x) ((%x. f x x) ((%x. f x x) ((%x. f x x) ((%x.
f x x) ((%x. f x x) x)))))};
val (dB2, -) = Norm.type-NF (typing-of-term [] dummyf (term-of ct2));
val ct2' = cterm-of @{theory} (term-of-dB [] (#T (rep-cterm ct2)) dB2);
⟫

end
```

References

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