The Isabelle/HOL Algebra Library

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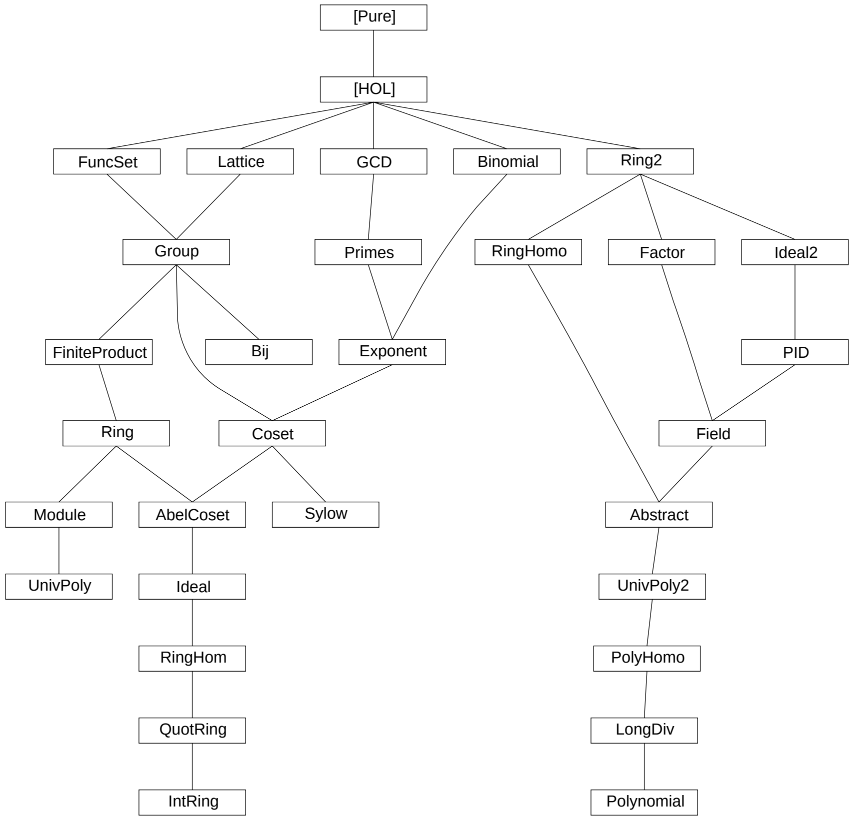
Contents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theory *Lattice* **imports** *Main* **begin**

1 Orders and Lattices

Object with a carrier set.

record *'a partial-object* =
  carrier :: *'a set*

1.1 Partial Orders

record *'a order* = *'a partial-object* +
  le :: [*'a*, *'a*] => *bool* (**infixl** $\sqsubseteq_1$ *50*)

locale *partial-order* =
  fixes *L* (**structure**)
  assumes *refl* [*intro*, *simp*]:
    $x \in carrier\ L ==> x \sqsubseteq x$
   and *anti-sym* [*intro*]:
    $[|\ x \sqsubseteq y;\ y \sqsubseteq x;\ x \in carrier\ L;\ y \in carrier\ L\ |] ==> x = y$
   and *trans* [*trans*]:
    $[|\ x \sqsubseteq y;\ y \sqsubseteq z;$
    $x \in carrier\ L;\ y \in carrier\ L;\ z \in carrier\ L\ |] ==> x \sqsubseteq z$

constdefs (**structure** *L*)
  lless :: [-, *'a*, *'a*] => *bool* (**infixl** $\sqsubset_1$ *50*)
  $x \sqsubset y == x \sqsubseteq y\ \&\ x\ {}^{\sim}\!= y$

  — Upper and lower bounds of a set.
  Upper :: [-, *'a set*] => *'a set*
  $Upper\ L\ A == \{u.\ (ALL\ x.\ x \in A \cap carrier\ L\ --> x \sqsubseteq u)\} \cap$
    $carrier\ L$

  Lower :: [-, *'a set*] => *'a set*
  $Lower\ L\ A == \{l.\ (ALL\ x.\ x \in A \cap carrier\ L\ --> l \sqsubseteq x)\} \cap$
    $carrier\ L$

  — Least and greatest, as predicate.
  least :: [-, *'a*, *'a set*] => *bool*
  $least\ L\ l\ A == A \subseteq carrier\ L\ \&\ l \in A\ \&\ (ALL\ x : A.\ l \sqsubseteq x)$

  greatest :: [-, *'a*, *'a set*] => *bool*
  $greatest\ L\ g\ A == A \subseteq carrier\ L\ \&\ g \in A\ \&\ (ALL\ x : A.\ x \sqsubseteq g)$

  — Supremum and infimum
  sup :: [-, *'a set*] => *'a* ($\bigsqcup_1$- [*90*] *90*)
  $\bigsqcup A == THE\ x.\ least\ L\ x\ (Upper\ L\ A)$

inf :: [-, *'a set*] => *'a* ($\bigsqcap$1- [*90*] *90*)
$\bigsqcap$*A* == *THE x. greatest L x* (*Lower L A*)

join :: [-, *'a*, *'a*] => *'a* (**infixl** $\sqcup$1 *65*)
x $\sqcup$ *y* == *sup L* {*x*, *y*}

meet :: [-, *'a*, *'a*] => *'a* (**infixl** $\sqcap$1 *70*)
x $\sqcap$ *y* == *inf L* {*x*, *y*}

1.1.1 Upper

lemma *Upper-closed* [*intro*, *simp*]:
Upper L A $\subseteq$ *carrier L*
⟨*proof*⟩

lemma *UpperD* [*dest*]:
fixes *L* (**structure**)
shows [| *u* $\in$ *Upper L A*; *x* $\in$ *A*; *A* $\subseteq$ *carrier L* |] ==> *x* $\sqsubseteq$ *u*
⟨*proof*⟩

lemma *Upper-memI*:
fixes *L* (**structure**)
shows [| !! *y*. *y* $\in$ *A* ==> *y* $\sqsubseteq$ *x*; *x* $\in$ *carrier L* |] ==> *x* $\in$ *Upper L A*
⟨*proof*⟩

lemma *Upper-antimono*:
A $\subseteq$ *B* ==> *Upper L B* $\subseteq$ *Upper L A*
⟨*proof*⟩

1.1.2 Lower

lemma *Lower-closed* [*intro*, *simp*]:
Lower L A $\subseteq$ *carrier L*
⟨*proof*⟩

lemma *LowerD* [*dest*]:
fixes *L* (**structure**)
shows [| *l* $\in$ *Lower L A*; *x* $\in$ *A*; *A* $\subseteq$ *carrier L* |] ==> *l* $\sqsubseteq$ *x*
⟨*proof*⟩

lemma *Lower-memI*:
fixes *L* (**structure**)
shows [| !! *y*. *y* $\in$ *A* ==> *x* $\sqsubseteq$ *y*; *x* $\in$ *carrier L* |] ==> *x* $\in$ *Lower L A*
⟨*proof*⟩

lemma *Lower-antimono*:
A $\subseteq$ *B* ==> *Lower L B* $\subseteq$ *Lower L A*
⟨*proof*⟩

1.1.3 least

lemma *least-carrier* [*intro*, *simp*]:
  shows *least L l A* ==> $l \in$ *carrier L*
  ⟨*proof*⟩

lemma *least-mem*:
  least L l A ==> $l \in A$
  ⟨*proof*⟩

lemma (**in** *partial-order*) *least-unique*:
  [| *least L x A*; *least L y A* |] ==> $x = y$
  ⟨*proof*⟩

lemma *least-le*:
  fixes *L* (**structure**)
  shows [| *least L x A*; $a \in A$ |] ==> $x \sqsubseteq a$
  ⟨*proof*⟩

lemma *least-UpperI*:
  fixes *L* (**structure**)
  assumes *above*: !! *x*. $x \in A$ ==> $x \sqsubseteq s$
    and *below*: !! *y*. $y \in$ *Upper L A* ==> $s \sqsubseteq y$
    and *L*: $A \subseteq$ *carrier L* $s \in$ *carrier L*
  shows *least L s* (*Upper L A*)
⟨*proof*⟩

1.1.4 greatest

lemma *greatest-carrier* [*intro*, *simp*]:
  shows *greatest L l A* ==> $l \in$ *carrier L*
  ⟨*proof*⟩

lemma *greatest-mem*:
  greatest L l A ==> $l \in A$
  ⟨*proof*⟩

lemma (**in** *partial-order*) *greatest-unique*:
  [| *greatest L x A*; *greatest L y A* |] ==> $x = y$
  ⟨*proof*⟩

lemma *greatest-le*:
  fixes *L* (**structure**)
  shows [| *greatest L x A*; $a \in A$ |] ==> $a \sqsubseteq x$
  ⟨*proof*⟩

lemma *greatest-LowerI*:
  fixes *L* (**structure**)
  assumes *below*: !! *x*. $x \in A$ ==> $i \sqsubseteq x$
    and *above*: !! *y*. $y \in$ *Lower L A* ==> $y \sqsubseteq i$

and L: $A \subseteq$ *carrier* L $i \in$ *carrier* L
shows *greatest* L i (*Lower* L A)
⟨*proof*⟩

1.2 Lattices

locale *lattice* = *partial-order* +
assumes *sup-of-two-exists*:
[| $x \in$ *carrier* L; $y \in$ *carrier* L |] ==> *EX* s. *least* L s (*Upper* L $\{x, y\}$)
and *inf-of-two-exists*:
[| $x \in$ *carrier* L; $y \in$ *carrier* L |] ==> *EX* s. *greatest* L s (*Lower* L $\{x, y\}$)

lemma *least-Upper-above*:
fixes L (**structure**)
shows [| *least* L s (*Upper* L A); $x \in A$; $A \subseteq$ *carrier* L |] ==> $x \sqsubseteq s$
⟨*proof*⟩

lemma *greatest-Lower-above*:
fixes L (**structure**)
shows [| *greatest* L i (*Lower* L A); $x \in A$; $A \subseteq$ *carrier* L |] ==> $i \sqsubseteq x$
⟨*proof*⟩

1.2.1 Supremum

lemma (**in** *lattice*) *joinI*:
[| !!l. *least* L l (*Upper* L $\{x, y\}$) ==> P l; $x \in$ *carrier* L; $y \in$ *carrier* L |]
==> P $(x \sqcup y)$
⟨*proof*⟩

lemma (**in** *lattice*) *join-closed* [*simp*]:
[| $x \in$ *carrier* L; $y \in$ *carrier* L |] ==> $x \sqcup y \in$ *carrier* L
⟨*proof*⟩

lemma (**in** *partial-order*) *sup-of-singletonI*:
$x \in$ *carrier* L ==> *least* L x (*Upper* L $\{x\}$)
⟨*proof*⟩

lemma (**in** *partial-order*) *sup-of-singleton* [*simp*]:
$x \in$ *carrier* L ==> $\bigsqcup\{x\} = x$
⟨*proof*⟩

Condition on A: supremum exists.

lemma (**in** *lattice*) *sup-insertI*:
[| !!s. *least* L s (*Upper* L (*insert* x A)) ==> P s;
least L a (*Upper* L A); $x \in$ *carrier* L; $A \subseteq$ *carrier* L |]
==> P ($\bigsqcup$(*insert* x A))
⟨*proof*⟩

lemma (**in** *lattice*) *finite-sup-least*:

$[|\ \mathit{finite}\ A;\ A \subseteq \mathit{carrier}\ L;\ A\ \tilde{}{=}\ \{\}\ |] ==> \mathit{least}\ L\ (\bigsqcup A)\ (\mathit{Upper}\ L\ A)$
⟨*proof*⟩

lemma (**in** *lattice*) *finite-sup-insertI*:
assumes P: $!!l.\ \mathit{least}\ L\ l\ (\mathit{Upper}\ L\ (\mathit{insert}\ x\ A)) ==> P\ l$
and xA: $\mathit{finite}\ A\ \ x \in \mathit{carrier}\ L\ \ A \subseteq \mathit{carrier}\ L$
shows $P\ (\bigsqcup\ (\mathit{insert}\ x\ A))$
⟨*proof*⟩

lemma (**in** *lattice*) *finite-sup-closed*:
$[|\ \mathit{finite}\ A;\ A \subseteq \mathit{carrier}\ L;\ A\ \tilde{}{=}\ \{\}\ |] ==> \bigsqcup A \in \mathit{carrier}\ L$
⟨*proof*⟩

lemma (**in** *lattice*) *join-left*:
$[|\ x \in \mathit{carrier}\ L;\ y \in \mathit{carrier}\ L\ |] ==> x \sqsubseteq x \sqcup y$
⟨*proof*⟩

lemma (**in** *lattice*) *join-right*:
$[|\ x \in \mathit{carrier}\ L;\ y \in \mathit{carrier}\ L\ |] ==> y \sqsubseteq x \sqcup y$
⟨*proof*⟩

lemma (**in** *lattice*) *sup-of-two-least*:
$[|\ x \in \mathit{carrier}\ L;\ y \in \mathit{carrier}\ L\ |] ==> \mathit{least}\ L\ (\bigsqcup \{x,\ y\})\ (\mathit{Upper}\ L\ \{x,\ y\})$
⟨*proof*⟩

lemma (**in** *lattice*) *join-le*:
assumes *sub*: $x \sqsubseteq z\ \ y \sqsubseteq z$
and x: $x \in \mathit{carrier}\ L$ **and** y: $y \in \mathit{carrier}\ L$ **and** z: $z \in \mathit{carrier}\ L$
shows $x \sqcup y \sqsubseteq z$
⟨*proof*⟩

lemma (**in** *lattice*) *join-assoc-lemma*:
assumes L: $x \in \mathit{carrier}\ L\ \ y \in \mathit{carrier}\ L\ \ z \in \mathit{carrier}\ L$
shows $x \sqcup (y \sqcup z) = \bigsqcup \{x,\ y,\ z\}$
⟨*proof*⟩

lemma *join-comm*:
fixes L (**structure**)
shows $x \sqcup y = y \sqcup x$
⟨*proof*⟩

lemma (**in** *lattice*) *join-assoc*:
assumes L: $x \in \mathit{carrier}\ L\ \ y \in \mathit{carrier}\ L\ \ z \in \mathit{carrier}\ L$
shows $(x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$
⟨*proof*⟩

1.2.2 Infimum

lemma (**in** *lattice*) *meetI*:

[| !!i. *greatest L i* (*Lower L* $\{x, y\}$) ==> *P i*;
$x \in$ *carrier L*; $y \in$ *carrier L* |]
==> *P* ($x \sqcap y$)
⟨*proof*⟩

lemma (**in** *lattice*) *meet-closed* [*simp*]:
[| $x \in$ *carrier L*; $y \in$ *carrier L* |] ==> $x \sqcap y \in$ *carrier L*
⟨*proof*⟩

lemma (**in** *partial-order*) *inf-of-singletonI*:
$x \in$ *carrier L* ==> *greatest L x* (*Lower L* $\{x\}$)
⟨*proof*⟩

lemma (**in** *partial-order*) *inf-of-singleton* [*simp*]:
$x \in$ *carrier L* ==> $\bigsqcap$ $\{x\} = x$
⟨*proof*⟩

Condition on A: infimum exists.

lemma (**in** *lattice*) *inf-insertI*:
[| !!i. *greatest L i* (*Lower L* (*insert x A*)) ==> *P i*;
greatest L a (*Lower L A*); $x \in$ *carrier L*; $A \subseteq$ *carrier L* |]
==> *P* ($\bigsqcap$(*insert x A*))
⟨*proof*⟩

lemma (**in** *lattice*) *finite-inf-greatest*:
[| *finite A*; $A \subseteq$ *carrier L*; A ~= {} |] ==> *greatest L* ($\bigsqcap A$) (*Lower L A*)
⟨*proof*⟩

lemma (**in** *lattice*) *finite-inf-insertI*:
assumes *P*: !!i. *greatest L i* (*Lower L* (*insert x A*)) ==> *P i*
and *xA*: *finite A* $x \in$ *carrier L* $A \subseteq$ *carrier L*
shows *P* ($\bigsqcap$ (*insert x A*))
⟨*proof*⟩

lemma (**in** *lattice*) *finite-inf-closed*:
[| *finite A*; $A \subseteq$ *carrier L*; A ~= {} |] ==> $\bigsqcap A \in$ *carrier L*
⟨*proof*⟩

lemma (**in** *lattice*) *meet-left*:
[| $x \in$ *carrier L*; $y \in$ *carrier L* |] ==> $x \sqcap y \sqsubseteq x$
⟨*proof*⟩

lemma (**in** *lattice*) *meet-right*:
[| $x \in$ *carrier L*; $y \in$ *carrier L* |] ==> $x \sqcap y \sqsubseteq y$
⟨*proof*⟩

lemma (**in** *lattice*) *inf-of-two-greatest*:
[| $x \in$ *carrier L*; $y \in$ *carrier L* |] ==>
greatest L ($\bigsqcap$ $\{x, y\}$) (*Lower L* $\{x, y\}$)

$\langle proof \rangle$

lemma (**in** *lattice*) *meet-le*:
assumes *sub*: $z \sqsubseteq x \quad z \sqsubseteq y$
and *x*: $x \in carrier\ L$ **and** *y*: $y \in carrier\ L$ **and** *z*: $z \in carrier\ L$
shows $z \sqsubseteq x \sqcap y$
$\langle proof \rangle$

lemma (**in** *lattice*) *meet-assoc-lemma*:
assumes *L*: $x \in carrier\ L \quad y \in carrier\ L \quad z \in carrier\ L$
shows $x \sqcap (y \sqcap z) = \bigsqcap\{x, y, z\}$
$\langle proof \rangle$

lemma *meet-comm*:
fixes *L* (**structure**)
shows $x \sqcap y = y \sqcap x$
$\langle proof \rangle$

lemma (**in** *lattice*) *meet-assoc*:
assumes *L*: $x \in carrier\ L \quad y \in carrier\ L \quad z \in carrier\ L$
shows $(x \sqcap y) \sqcap z = x \sqcap (y \sqcap z)$
$\langle proof \rangle$

1.3 Total Orders

locale *total-order* = *partial-order* +
assumes *total*: $[| \ x \in carrier\ L;\ y \in carrier\ L \ |] ==> x \sqsubseteq y \mid y \sqsubseteq x$

Introduction rule: the usual definition of total order

lemma (**in** *partial-order*) *total-orderI*:
assumes *total*: $!!x\ y.\ [| \ x \in carrier\ L;\ y \in carrier\ L \ |] ==> x \sqsubseteq y \mid y \sqsubseteq x$
shows *total-order L*
$\langle proof \rangle$

Total orders are lattices.

interpretation *total-order* < *lattice*
$\langle proof \rangle$

1.4 Complete lattices

locale *complete-lattice* = *lattice* +
assumes *sup-exists*:
$[| \ A \subseteq carrier\ L \ |] ==> EX\ s.\ least\ L\ s\ (Upper\ L\ A)$
and *inf-exists*:
$[| \ A \subseteq carrier\ L \ |] ==> EX\ i.\ greatest\ L\ i\ (Lower\ L\ A)$

Introduction rule: the usual definition of complete lattice

lemma (**in** *partial-order*) *complete-latticeI*:
assumes *sup-exists*:

!!*A*. [| *A* ⊆ *carrier L* |] ==> *EX s*. *least L s* (*Upper L A*)
and *inf-exists*:
!!*A*. [| *A* ⊆ *carrier L* |] ==> *EX i*. *greatest L i* (*Lower L A*)
shows *complete-lattice L*
⟨*proof*⟩

constdefs (**structure** *L*)
top :: - => *'a* (⊤1)
⊤ == *sup L* (*carrier L*)

bottom :: - => *'a* (⊥1)
⊥ == *inf L* (*carrier L*)

lemma (**in** *complete-lattice*) *supI*:
[| !!*l*. *least L l* (*Upper L A*) ==> *P l*; *A* ⊆ *carrier L* |]
==> *P* ($\bigsqcup$*A*)
⟨*proof*⟩

lemma (**in** *complete-lattice*) *sup-closed* [*simp*]:
A ⊆ *carrier L* ==> $\bigsqcup$*A* ∈ *carrier L*
⟨*proof*⟩

lemma (**in** *complete-lattice*) *top-closed* [*simp*, *intro*]:
⊤ ∈ *carrier L*
⟨*proof*⟩

lemma (**in** *complete-lattice*) *infI*:
[| !!*i*. *greatest L i* (*Lower L A*) ==> *P i*; *A* ⊆ *carrier L* |]
==> *P* ($\bigsqcap$*A*)
⟨*proof*⟩

lemma (**in** *complete-lattice*) *inf-closed* [*simp*]:
A ⊆ *carrier L* ==> $\bigsqcap$*A* ∈ *carrier L*
⟨*proof*⟩

lemma (**in** *complete-lattice*) *bottom-closed* [*simp*, *intro*]:
⊥ ∈ *carrier L*
⟨*proof*⟩

Jacobson: Theorem 8.1

lemma *Lower-empty* [*simp*]:
Lower L {} = *carrier L*
⟨*proof*⟩

lemma *Upper-empty* [*simp*]:
Upper L {} = *carrier L*
⟨*proof*⟩

theorem (**in** *partial-order*) *complete-lattice-criterion1*:
  assumes *top-exists*: *EX g. greatest L g* (*carrier L*)
    and *inf-exists*:
      $!!A$. [| $A \subseteq$ *carrier L*; A ~= {} |] ==> *EX i. greatest L i* (*Lower L A*)
  shows *complete-lattice L*
⟨*proof*⟩

1.5 Examples

1.5.1 Powerset of a Set is a Complete Lattice

theorem *powerset-is-complete-lattice*:
  complete-lattice (| *carrier* = *Pow A*, *le* = *op* $\subseteq$ |)
  (**is** *complete-lattice ?L*)
⟨*proof*⟩

An other example, that of the lattice of subgroups of a group, can be found in Group theory (Section 2.7).

end

theory *Group* **imports** *FuncSet Lattice* **begin**

2 Monoids and Groups

2.1 Definitions

Definitions follow [2].

record *'a monoid* = *'a partial-object* +
  mult :: [*'a*, *'a*] $\Rightarrow$ *'a* (**infixl** $\otimes_1$ *70*)
  one :: *'a* ($\mathbf{1}_1$)

constdefs (**structure** *G*)
  m-inv :: (*'a*, *'b*) *monoid-scheme* => *'a* => *'a* (*inv*1 - [*81*] *80*)
  inv x == (*THE y*. $y \in$ *carrier G* & $x \otimes y = \mathbf{1}$ & $y \otimes x = \mathbf{1}$)

  Units :: - => *'a set*
  — The set of invertible elements
  Units G == {y. $y \in$ *carrier G* & ($\exists x \in$ *carrier G*. $x \otimes y = \mathbf{1}$ & $y \otimes x = \mathbf{1}$)}

consts
  pow :: [(*'a*, *'m*) *monoid-scheme*, *'a*, *'b::number*] => *'a* (**infixr** '(^')1 *75*)

defs (**overloaded**)
  nat-pow-def: *pow G a n* == *nat-rec* $\mathbf{1}_G$ (%*u b*. $b \otimes_G a$) *n*
  int-pow-def: *pow G a z* ==
    let p = *nat-rec* $\mathbf{1}_G$ (%*u b*. $b \otimes_G a$)

in if neg z then inv_G $(p\ (nat\ (-z)))$ *else* $p\ (nat\ z)$

locale *monoid* =
fixes G (**structure**)
assumes *m-closed* [*intro*, *simp*]:
$[\![x \in carrier\ G;\ y \in carrier\ G]\!] \Longrightarrow x \otimes y \in carrier\ G$
and *m-assoc*:
$[\![x \in carrier\ G;\ y \in carrier\ G;\ z \in carrier\ G]\!]$
$\Longrightarrow (x \otimes y) \otimes z = x \otimes (y \otimes z)$
and *one-closed* [*intro*, *simp*]: $\mathbf{1} \in carrier\ G$
and *l-one* [*simp*]: $x \in carrier\ G \Longrightarrow \mathbf{1} \otimes x = x$
and *r-one* [*simp*]: $x \in carrier\ G \Longrightarrow x \otimes \mathbf{1} = x$

lemma *monoidI*:
fixes G (**structure**)
assumes *m-closed*:
!!$x\ y$. [| $x \in carrier\ G$; $y \in carrier\ G$ |] ==> $x \otimes y \in carrier\ G$
and *one-closed*: $\mathbf{1} \in carrier\ G$
and *m-assoc*:
!!$x\ y\ z$. [| $x \in carrier\ G$; $y \in carrier\ G$; $z \in carrier\ G$ |] ==>
$(x \otimes y) \otimes z = x \otimes (y \otimes z)$
and *l-one*: !!x. $x \in carrier\ G$ ==> $\mathbf{1} \otimes x = x$
and *r-one*: !!x. $x \in carrier\ G$ ==> $x \otimes \mathbf{1} = x$
shows *monoid* G
⟨*proof*⟩

lemma (**in** *monoid*) *Units-closed* [*dest*]:
$x \in Units\ G$ ==> $x \in carrier\ G$
⟨*proof*⟩

lemma (**in** *monoid*) *inv-unique*:
assumes *eq*: $y \otimes x = \mathbf{1}$ $\ x \otimes y' = \mathbf{1}$
and G: $x \in carrier\ G$ $\ y \in carrier\ G$ $\ y' \in carrier\ G$
shows $y = y'$
⟨*proof*⟩

lemma (**in** *monoid*) *Units-one-closed* [*intro*, *simp*]:
$\mathbf{1} \in Units\ G$
⟨*proof*⟩

lemma (**in** *monoid*) *Units-inv-closed* [*intro*, *simp*]:
$x \in Units\ G$ ==> $inv\ x \in carrier\ G$
⟨*proof*⟩

lemma (**in** *monoid*) *Units-l-inv-ex*:
$x \in Units\ G$ ==> $\exists\, y \in carrier\ G.\ y \otimes x = \mathbf{1}$
⟨*proof*⟩

lemma (**in** *monoid*) *Units-r-inv-ex*:

$x \in$ *Units G* ==> $\exists\, y \in$ *carrier G*. $x \otimes y = \mathbf{1}$
⟨*proof*⟩

lemma (**in** *monoid*) *Units-l-inv*:
$x \in$ *Units G* ==> *inv* $x \otimes x = \mathbf{1}$
⟨*proof*⟩

lemma (**in** *monoid*) *Units-r-inv*:
$x \in$ *Units G* ==> $x \otimes$ *inv* $x = \mathbf{1}$
⟨*proof*⟩

lemma (**in** *monoid*) *Units-inv-Units* [*intro*, *simp*]:
$x \in$ *Units G* ==> *inv* $x \in$ *Units G*
⟨*proof*⟩

lemma (**in** *monoid*) *Units-l-cancel* [*simp*]:
[| $x \in$ *Units G*; $y \in$ *carrier G*; $z \in$ *carrier G* |] ==>
$(x \otimes y = x \otimes z) = (y = z)$
⟨*proof*⟩

lemma (**in** *monoid*) *Units-inv-inv* [*simp*]:
$x \in$ *Units G* ==> *inv* (*inv* x) $= x$
⟨*proof*⟩

lemma (**in** *monoid*) *inv-inj-on-Units*:
inj-on (*m-inv G*) (*Units G*)
⟨*proof*⟩

lemma (**in** *monoid*) *Units-inv-comm*:
assumes *inv*: $x \otimes y = \mathbf{1}$
and *G*: $x \in$ *Units G* $y \in$ *Units G*
shows $y \otimes x = \mathbf{1}$
⟨*proof*⟩

Power

lemma (**in** *monoid*) *nat-pow-closed* [*intro*, *simp*]:
$x \in$ *carrier G* ==> x (^) (n::*nat*) $\in$ *carrier G*
⟨*proof*⟩

lemma (**in** *monoid*) *nat-pow-0* [*simp*]:
x (^) (*0*::*nat*) $= \mathbf{1}$
⟨*proof*⟩

lemma (**in** *monoid*) *nat-pow-Suc* [*simp*]:
x (^) (*Suc* n) $= x$ (^) $n \otimes x$
⟨*proof*⟩

lemma (**in** *monoid*) *nat-pow-one* [*simp*]:
$\mathbf{1}$ (^) (n::*nat*) $= \mathbf{1}$

⟨*proof*⟩

lemma (**in** *monoid*) *nat-pow-mult*:
$x \in carrier\ G \Longrightarrow x\ (\hat{}\ )\ (n{::}nat) \otimes x\ (\hat{}\ )\ m = x\ (\hat{}\ )\ (n + m)$
⟨*proof*⟩

lemma (**in** *monoid*) *nat-pow-pow*:
$x \in carrier\ G \Longrightarrow (x\ (\hat{}\ )\ n)\ (\hat{}\ )\ m = x\ (\hat{}\ )\ (n * m{::}nat)$
⟨*proof*⟩

A group is a monoid all of whose elements are invertible.

locale *group* = *monoid* +
assumes *Units*: $carrier\ G <= Units\ G$

lemma (**in** *group*) *is-group*: *group G* ⟨*proof*⟩

theorem *groupI*:
fixes *G* (**structure**)
assumes *m-closed* [*simp*]:
$!!x\ y.\ [|\ x \in carrier\ G;\ y \in carrier\ G\ |] \Longrightarrow x \otimes y \in carrier\ G$
and *one-closed* [*simp*]: $\mathbf{1} \in carrier\ G$
and *m-assoc*:
$!!x\ y\ z.\ [|\ x \in carrier\ G;\ y \in carrier\ G;\ z \in carrier\ G\ |] \Longrightarrow$
$(x \otimes y) \otimes z = x \otimes (y \otimes z)$
and *l-one* [*simp*]: $!!x.\ x \in carrier\ G \Longrightarrow \mathbf{1} \otimes x = x$
and *l-inv-ex*: $!!x.\ x \in carrier\ G \Longrightarrow \exists\, y \in carrier\ G.\ y \otimes x = \mathbf{1}$
shows *group G*
⟨*proof*⟩

lemma (**in** *monoid*) *monoid-groupI*:
assumes *l-inv-ex*:
$!!x.\ x \in carrier\ G \Longrightarrow \exists\, y \in carrier\ G.\ y \otimes x = \mathbf{1}$
shows *group G*
⟨*proof*⟩

lemma (**in** *group*) *Units-eq* [*simp*]:
$Units\ G = carrier\ G$
⟨*proof*⟩

lemma (**in** *group*) *inv-closed* [*intro*, *simp*]:
$x \in carrier\ G \Longrightarrow inv\ x \in carrier\ G$
⟨*proof*⟩

lemma (**in** *group*) *l-inv-ex* [*simp*]:
$x \in carrier\ G \Longrightarrow \exists\, y \in carrier\ G.\ y \otimes x = \mathbf{1}$
⟨*proof*⟩

lemma (**in** *group*) *r-inv-ex* [*simp*]:

$x \in carrier\ G ==> \exists\, y \in carrier\ G.\ x \otimes y = \mathbf{1}$
⟨*proof*⟩

lemma (**in** *group*) *l-inv* [*simp*]:
$x \in carrier\ G ==> inv\ x \otimes x = \mathbf{1}$
⟨*proof*⟩

2.2 Cancellation Laws and Basic Properties

lemma (**in** *group*) *l-cancel* [*simp*]:
$[|\ x \in carrier\ G;\ y \in carrier\ G;\ z \in carrier\ G\ |] ==>$
$(x \otimes y = x \otimes z) = (y = z)$
⟨*proof*⟩

lemma (**in** *group*) *r-inv* [*simp*]:
$x \in carrier\ G ==> x \otimes inv\ x = \mathbf{1}$
⟨*proof*⟩

lemma (**in** *group*) *r-cancel* [*simp*]:
$[|\ x \in carrier\ G;\ y \in carrier\ G;\ z \in carrier\ G\ |] ==>$
$(y \otimes x = z \otimes x) = (y = z)$
⟨*proof*⟩

lemma (**in** *group*) *inv-one* [*simp*]:
$inv\ \mathbf{1} = \mathbf{1}$
⟨*proof*⟩

lemma (**in** *group*) *inv-inv* [*simp*]:
$x \in carrier\ G ==> inv\ (inv\ x) = x$
⟨*proof*⟩

lemma (**in** *group*) *inv-inj*:
inj-on (*m-inv G*) (*carrier G*)
⟨*proof*⟩

lemma (**in** *group*) *inv-mult-group*:
$[|\ x \in carrier\ G;\ y \in carrier\ G\ |] ==> inv\ (x \otimes y) = inv\ y \otimes inv\ x$
⟨*proof*⟩

lemma (**in** *group*) *inv-comm*:
$[|\ x \otimes y = \mathbf{1};\ x \in carrier\ G;\ y \in carrier\ G\ |] ==> y \otimes x = \mathbf{1}$
⟨*proof*⟩

lemma (**in** *group*) *inv-equality*:
$[|y \otimes x = \mathbf{1};\ x \in carrier\ G;\ y \in carrier\ G|] ==> inv\ x = y$
⟨*proof*⟩

Power

lemma (**in** *group*) *int-pow-def2*:

a (^) $(z::int) = (if\ neg\ z\ then\ inv\ (a$ (^) $(nat\ (-z)))\ else\ a$ (^) $(nat\ z))$
⟨*proof*⟩

lemma (**in** *group*) *int-pow-0* [*simp*]:
x (^) $(0::int) = \mathbf{1}$
⟨*proof*⟩

lemma (**in** *group*) *int-pow-one* [*simp*]:
$\mathbf{1}$ (^) $(z::int) = \mathbf{1}$
⟨*proof*⟩

2.3 Subgroups

locale *subgroup* =
fixes *H* **and** *G* (**structure**)
assumes *subset*: $H \subseteq carrier\ G$
and *m-closed* [*intro*, *simp*]: $[\![x \in H; y \in H]\!] \Longrightarrow x \otimes y \in H$
and *one-closed* [*simp*]: $\mathbf{1} \in H$
and *m-inv-closed* [*intro*,*simp*]: $x \in H \Longrightarrow inv\ x \in H$

lemma (**in** *subgroup*) *is-subgroup*:
subgroup H G ⟨*proof*⟩

declare (**in** *subgroup*) *group.intro* [*intro*]

lemma (**in** *subgroup*) *mem-carrier* [*simp*]:
$x \in H \Longrightarrow x \in carrier\ G$
⟨*proof*⟩

lemma *subgroup-imp-subset*:
$subgroup\ H\ G \Longrightarrow H \subseteq carrier\ G$
⟨*proof*⟩

lemma (**in** *subgroup*) *subgroup-is-group* [*intro*]:
includes *group G*
shows $group\ (G(\!|carrier := H|\!))$
⟨*proof*⟩

Since H is nonempty, it contains some element x. Since it is closed under inverse, it contains $inv\ x$. Since it is closed under product, it contains $x \otimes inv\ x = \mathbf{1}$.

lemma (**in** *group*) *one-in-subset*:
$[|\ H \subseteq carrier\ G; H \neq \{\}; \forall a \in H.\ inv\ a \in H; \forall a{\in}H.\ \forall b{\in}H.\ a \otimes b \in H\ |]$
$==> \mathbf{1} \in H$
⟨*proof*⟩

A characterization of subgroups: closed, non-empty subset.

lemma (**in** *group*) *subgroupI*:
assumes *subset*: $H \subseteq carrier\ G$ **and** *non-empty*: $H \neq \{\}$

  and *inv*: !!a. $a \in H \Longrightarrow$ *inv* $a \in H$
  and *mult*: !!a b. $[\![a \in H; b \in H]\!] \Longrightarrow a \otimes b \in H$
 shows *subgroup H G*
⟨*proof*⟩

declare *monoid.one-closed* [*iff*] *group.inv-closed* [*simp*]
 monoid.l-one [*simp*] *monoid.r-one* [*simp*] *group.inv-inv* [*simp*]

lemma *subgroup-nonempty*:
 ~ *subgroup* {} *G*
 ⟨*proof*⟩

lemma (**in** *subgroup*) *finite-imp-card-positive*:
 finite (*carrier G*) ==> *0* < *card H*
⟨*proof*⟩

2.4 Direct Products

constdefs
 DirProd :: - ⇒ - ⇒ ($'a \times 'b$) *monoid* (**infixr** ×× *80*)
 $G \times\times H \equiv (\!|$ *carrier* = *carrier* $G \times$ *carrier* H,
  mult = $(\lambda(g, h)\ (g', h').\ (g \otimes_G g', h \otimes_H h'))$,
  one = $(\mathbf{1}_G, \mathbf{1}_H)|\!)$

lemma *DirProd-monoid*:
 includes *monoid G* + *monoid H*
 shows *monoid* ($G \times\times H$)
⟨*proof*⟩

Does not use the previous result because it's easier just to use auto.

lemma *DirProd-group*:
 includes *group G* + *group H*
 shows *group* ($G \times\times H$)
 ⟨*proof*⟩

lemma *carrier-DirProd* [*simp*]:
  carrier ($G \times\times H$) = *carrier* $G \times$ *carrier* H
 ⟨*proof*⟩

lemma *one-DirProd* [*simp*]:
  $\mathbf{1}_{G \times\times H} = (\mathbf{1}_G, \mathbf{1}_H)$
 ⟨*proof*⟩

lemma *mult-DirProd* [*simp*]:
  $(g, h) \otimes_{(G \times\times H)} (g', h') = (g \otimes_G g', h \otimes_H h')$
 ⟨*proof*⟩

lemma *inv-DirProd* [*simp*]:
 includes *group G* + *group H*

assumes g: $g \in$ *carrier* G
and h: $h \in$ *carrier* H
shows *m-inv* $(G \times\times H)$ $(g, h) = (inv_G\ g,\ inv_H\ h)$
⟨*proof*⟩

This alternative proof of the previous result demonstrates interpret. It uses *Prod.inv-equality* (available after *interpret*) instead of *group.inv-equality* [*OF DirProd-group*].

lemma
includes *group* G + *group* H
assumes g: $g \in$ *carrier* G
and h: $h \in$ *carrier* H
shows *m-inv* $(G \times\times H)$ $(g, h) = (inv_G\ g,\ inv_H\ h)$
⟨*proof*⟩

2.5 Homomorphisms and Isomorphisms

constdefs (**structure** G **and** H)
hom :: - => - => ('*a* => '*b*) *set*
hom G H ==
{h. $h \in$ *carrier* G −> *carrier* H &
($\forall x \in$ *carrier* G. $\forall y \in$ *carrier* G. $h\ (x \otimes_G y) = h\ x \otimes_H h\ y$)}

lemma *hom-mult*:
[| $h \in$ *hom* G H; $x \in$ *carrier* G; $y \in$ *carrier* G |]
==> $h\ (x \otimes_G y) = h\ x \otimes_H h\ y$
⟨*proof*⟩

lemma *hom-closed*:
[| $h \in$ *hom* G H; $x \in$ *carrier* G |] ==> $h\ x \in$ *carrier* H
⟨*proof*⟩

lemma (**in** *group*) *hom-compose*:
[|$h \in$ *hom* G H; $i \in$ *hom* H I|] ==> *compose* (*carrier* G) i $h \in$ *hom* G I
⟨*proof*⟩

constdefs
iso :: - => - => ('*a* => '*b*) *set* (**infixr** ≅ *60*)
$G \cong H$ == {h. $h \in$ *hom* G H & *bij-betw* h (*carrier* G) (*carrier* H)}

lemma *iso-refl*: (%x. x) $\in G \cong G$
⟨*proof*⟩

lemma (**in** *group*) *iso-sym*:
$h \in G \cong H \Longrightarrow$ *Inv* (*carrier* G) $h \in H \cong G$
⟨*proof*⟩

lemma (**in** *group*) *iso-trans*:
[|$h \in G \cong H$; $i \in H \cong I$|] ==> (*compose* (*carrier* G) i h) $\in G \cong I$

⟨*proof*⟩

lemma *DirProd-commute-iso*:
shows $(\lambda(x,y).\ (y,x)) \in (G \times\times H) \cong (H \times\times G)$
⟨*proof*⟩

lemma *DirProd-assoc-iso*:
shows $(\lambda(x,y,z).\ (x,(y,z))) \in (G \times\times H \times\times I) \cong (G \times\times (H \times\times I))$
⟨*proof*⟩

Basis for homomorphism proofs: we assume two groups G and H, with a homomorphism h between them

locale *group-hom* = *group G* + *group H* + *var h* +
assumes *homh*: $h \in$ *hom G H*
notes *hom-mult* [*simp*] = *hom-mult* [*OF homh*]
and *hom-closed* [*simp*] = *hom-closed* [*OF homh*]

lemma (**in** *group-hom*) *one-closed* [*simp*]:
$h\ \mathbf{1} \in$ *carrier H*
⟨*proof*⟩

lemma (**in** *group-hom*) *hom-one* [*simp*]:
$h\ \mathbf{1} = \mathbf{1}_H$
⟨*proof*⟩

lemma (**in** *group-hom*) *inv-closed* [*simp*]:
$x \in$ *carrier G* ==> *h* (*inv x*) $\in$ *carrier H*
⟨*proof*⟩

lemma (**in** *group-hom*) *hom-inv* [*simp*]:
$x \in$ *carrier G* ==> *h* (*inv x*) = inv_H (*h x*)
⟨*proof*⟩

2.6 Commutative Structures

Naming convention: multiplicative structures that are commutative are called *commutative*, additive structures are called *Abelian*.

locale *comm-monoid* = *monoid* +
assumes *m-comm*: $[\![x \in carrier\ G;\ y \in carrier\ G]\!] \Longrightarrow x \otimes y = y \otimes x$

lemma (**in** *comm-monoid*) *m-lcomm*:
$[\![x \in carrier\ G;\ y \in carrier\ G;\ z \in carrier\ G]\!] \Longrightarrow$
$x \otimes (y \otimes z) = y \otimes (x \otimes z)$
⟨*proof*⟩

lemmas (**in** *comm-monoid*) *m-ac* = *m-assoc m-comm m-lcomm*

lemma *comm-monoidI*:

fixes G (**structure**)
assumes *m-closed*:
$!!x\ y.\ [|\ x \in carrier\ G;\ y \in carrier\ G\ |] ==> x \otimes y \in carrier\ G$
and *one-closed*: $\mathbf{1} \in carrier\ G$
and *m-assoc*:
$!!x\ y\ z.\ [|\ x \in carrier\ G;\ y \in carrier\ G;\ z \in carrier\ G\ |] ==>$
$(x \otimes y) \otimes z = x \otimes (y \otimes z)$
and *l-one*: $!!x.\ x \in carrier\ G ==> \mathbf{1} \otimes x = x$
and *m-comm*:
$!!x\ y.\ [|\ x \in carrier\ G;\ y \in carrier\ G\ |] ==> x \otimes y = y \otimes x$
shows *comm-monoid G*
⟨*proof*⟩

lemma (**in** *monoid*) *monoid-comm-monoidI*:
assumes *m-comm*:
$!!x\ y.\ [|\ x \in carrier\ G;\ y \in carrier\ G\ |] ==> x \otimes y = y \otimes x$
shows *comm-monoid G*
⟨*proof*⟩

lemma (**in** *comm-monoid*) *nat-pow-distr*:
$[|\ x \in carrier\ G;\ y \in carrier\ G\ |] ==>$
$(x \otimes y)\ (\hat{})\ (n::nat) = x\ (\hat{})\ n \otimes y\ (\hat{})\ n$
⟨*proof*⟩

locale *comm-group* = *comm-monoid* + *group*

lemma (**in** *group*) *group-comm-groupI*:
assumes *m-comm*: $!!x\ y.\ [|\ x \in carrier\ G;\ y \in carrier\ G\ |] ==>$
$x \otimes y = y \otimes x$
shows *comm-group G*
⟨*proof*⟩

lemma *comm-groupI*:
fixes G (**structure**)
assumes *m-closed*:
$!!x\ y.\ [|\ x \in carrier\ G;\ y \in carrier\ G\ |] ==> x \otimes y \in carrier\ G$
and *one-closed*: $\mathbf{1} \in carrier\ G$
and *m-assoc*:
$!!x\ y\ z.\ [|\ x \in carrier\ G;\ y \in carrier\ G;\ z \in carrier\ G\ |] ==>$
$(x \otimes y) \otimes z = x \otimes (y \otimes z)$
and *m-comm*:
$!!x\ y.\ [|\ x \in carrier\ G;\ y \in carrier\ G\ |] ==> x \otimes y = y \otimes x$
and *l-one*: $!!x.\ x \in carrier\ G ==> \mathbf{1} \otimes x = x$
and *l-inv-ex*: $!!x.\ x \in carrier\ G ==> \exists\, y \in carrier\ G.\ y \otimes x = \mathbf{1}$
shows *comm-group G*
⟨*proof*⟩

lemma (**in** *comm-group*) *inv-mult*:
[| $x \in$ *carrier* G; $y \in$ *carrier* G |] ==> *inv* ($x \otimes y$) = *inv* $x \otimes$ *inv* y
⟨*proof*⟩

2.7 The Lattice of Subgroups of a Group

theorem (**in** *group*) *subgroups-partial-order*:
partial-order (| *carrier* = {H. *subgroup* H G}, *le* = *op* $\subseteq$ |)
⟨*proof*⟩

lemma (**in** *group*) *subgroup-self*:
subgroup (*carrier* G) G
⟨*proof*⟩

lemma (**in** *group*) *subgroup-imp-group*:
subgroup H G ==> *group* (G(| *carrier* := H |))
⟨*proof*⟩

lemma (**in** *group*) *is-monoid* [*intro*, *simp*]:
monoid G
⟨*proof*⟩

lemma (**in** *group*) *subgroup-inv-equality*:
[| *subgroup* H G; $x \in H$ |] ==> *m-inv* (G (| *carrier* := H |)) x = *inv* x
⟨*proof*⟩

theorem (**in** *group*) *subgroups-Inter*:
assumes *subgr*: (!!H. $H \in A$ ==> *subgroup* H G)
and *not-empty*: A ~= {}
shows *subgroup* ($\bigcap A$) G
⟨*proof*⟩

theorem (**in** *group*) *subgroups-complete-lattice*:
complete-lattice (| *carrier* = {H. *subgroup* H G}, *le* = *op* $\subseteq$ |)
(**is** *complete-lattice* *?L*)
⟨*proof*⟩

end

theory *FiniteProduct* **imports** *Group* **begin**

3 Product Operator for Commutative Monoids

3.1 Inductive Definition of a Relation for Products over Sets

Instantiation of locale *LC* of theory *Finite-Set* is not possible, because here we have explicit typing rules like $x \in$ *carrier* G. We introduce an explicit argument for the domain D.

inductive-set
foldSetD :: ['*a set*, '*b* => '*a* => '*a*, '*a*] => ('*b set* * '*a*) *set*
for D :: '*a set* **and** f :: '*b* => '*a* => '*a* **and** e :: '*a*
where
emptyI [*intro*]: $e \in D$ ==> ({}, e) $\in$ *foldSetD D f e*
| *insertI* [*intro*]: [| x ~: A; $f\ x\ y \in D$; (A, y) $\in$ *foldSetD D f e* |] ==>
(*insert x A*, $f\ x\ y$) $\in$ *foldSetD D f e*

inductive-cases *empty-foldSetDE* [*elim*!]: ({}, x) $\in$ *foldSetD D f e*

constdefs
foldD :: ['*a set*, '*b* => '*a* => '*a*, '*a*, '*b set*] => '*a*
foldD D f e A == *THE* x. (A, x) $\in$ *foldSetD D f e*

lemma *foldSetD-closed*:
[| (A, z) $\in$ *foldSetD D f e* ; $e \in D$; !!$x\ y$. [| $x \in A$; $y \in D$ |] ==> $f\ x\ y \in D$
|] ==> $z \in D$
⟨*proof*⟩

lemma *Diff1-foldSetD*:
[| (A − {x}, y) $\in$ *foldSetD D f e*; $x \in A$; $f\ x\ y \in D$ |] ==>
(A, $f\ x\ y$) $\in$ *foldSetD D f e*
⟨*proof*⟩

lemma *foldSetD-imp-finite* [*simp*]: (A, x) $\in$ *foldSetD D f e* ==> *finite A*
⟨*proof*⟩

lemma *finite-imp-foldSetD*:
[| *finite A*; $e \in D$; !!$x\ y$. [| $x \in A$; $y \in D$ |] ==> $f\ x\ y \in D$ |] ==>
EX x. (A, x) $\in$ *foldSetD D f e*
⟨*proof*⟩

3.2 Left-Commutative Operations

locale *LCD* =
fixes B :: '*b set*
and D :: '*a set*
and f :: '*b* => '*a* => '*a* (**infixl** · *70*)
assumes *left-commute*:
[| $x \in B$; $y \in B$; $z \in D$ |] ==> $x \cdot (y \cdot z) = y \cdot (x \cdot z)$
and *f-closed* [*simp*, *intro*!]: !!$x\ y$. [| $x \in B$; $y \in D$ |] ==> $f\ x\ y \in D$

lemma (**in** *LCD*) *foldSetD-closed* [*dest*]:
(A, z) $\in$ *foldSetD D f e* ==> $z \in D$
⟨*proof*⟩

lemma (**in** *LCD*) *Diff1-foldSetD*:
[| (A − {x}, y) $\in$ *foldSetD D f e*; $x \in A$; $A \subseteq B$ |] ==>

$(A, f\ x\ y) \in foldSetD\ D\ f\ e$
⟨*proof*⟩

lemma (**in** *LCD*) *foldSetD-imp-finite* [*simp*]:
$(A, x) \in foldSetD\ D\ f\ e$ ==> *finite* A
⟨*proof*⟩

lemma (**in** *LCD*) *finite-imp-foldSetD*:
[| *finite* A; $A \subseteq B$; $e \in D$ |] ==> *EX* x. $(A, x) \in foldSetD\ D\ f\ e$
⟨*proof*⟩

lemma (**in** *LCD*) *foldSetD-determ-aux*:
$e \in D$ ==> $\forall A\ x.\ A \subseteq B$ & *card* $A < n$ --> $(A, x) \in foldSetD\ D\ f\ e$ -->
($\forall y.\ (A, y) \in foldSetD\ D\ f\ e$ --> $y = x$)
⟨*proof*⟩

lemma (**in** *LCD*) *foldSetD-determ*:
[| $(A, x) \in foldSetD\ D\ f\ e$; $(A, y) \in foldSetD\ D\ f\ e$; $e \in D$; $A \subseteq B$ |]
==> $y = x$
⟨*proof*⟩

lemma (**in** *LCD*) *foldD-equality*:
[| $(A, y) \in foldSetD\ D\ f\ e$; $e \in D$; $A \subseteq B$ |] ==> *foldD* $D\ f\ e\ A = y$
⟨*proof*⟩

lemma *foldD-empty* [*simp*]:
$e \in D$ ==> *foldD* $D\ f\ e\ \{\} = e$
⟨*proof*⟩

lemma (**in** *LCD*) *foldD-insert-aux*:
[| x ~: A; $x \in B$; $e \in D$; $A \subseteq B$ |] ==>
((*insert* $x\ A$, v) $\in foldSetD\ D\ f\ e$) =
(*EX* y. $(A, y) \in foldSetD\ D\ f\ e$ & $v = f\ x\ y$)
⟨*proof*⟩

lemma (**in** *LCD*) *foldD-insert*:
[| *finite* A; x ~: A; $x \in B$; $e \in D$; $A \subseteq B$ |] ==>
foldD $D\ f\ e$ (*insert* $x\ A$) = $f\ x$ (*foldD* $D\ f\ e\ A$)
⟨*proof*⟩

lemma (**in** *LCD*) *foldD-closed* [*simp*]:
[| *finite* A; $e \in D$; $A \subseteq B$ |] ==> *foldD* $D\ f\ e\ A \in D$
⟨*proof*⟩

lemma (**in** *LCD*) *foldD-commute*:
[| *finite* A; $x \in B$; $e \in D$; $A \subseteq B$ |] ==>
$f\ x$ (*foldD* $D\ f\ e\ A$) = *foldD* $D\ f$ $(f\ x\ e)$ A
⟨*proof*⟩

lemma *Int-mono2*:
  $[\![A \subseteq C; B \subseteq C]\!] \Longrightarrow A \ Int \ B \subseteq C$
  ⟨*proof*⟩

lemma (**in** *LCD*) *foldD-nest-Un-Int*:
  $[\![$ *finite* A; *finite* C; $e \in D$; $A \subseteq B$; $C \subseteq B$ $]\!] \Longrightarrow$
   foldD D f (*foldD* D f e C) A = *foldD* D f (*foldD* D f e (A *Int* C)) (A *Un* C)
  ⟨*proof*⟩

lemma (**in** *LCD*) *foldD-nest-Un-disjoint*:
  $[\![$ *finite* A; *finite* B; A *Int* B = {}; $e \in D$; $A \subseteq B$; $C \subseteq B$ $]\!]$
    $\Longrightarrow$ *foldD* D f e (A *Un* B) = *foldD* D f (*foldD* D f e B) A
  ⟨*proof*⟩

declare *foldSetD-imp-finite* [*simp del*]
  empty-foldSetDE [*rule del*]
  foldSetD.intros [*rule del*]
declare (**in** *LCD*)
  foldSetD-closed [*rule del*]

3.3 Commutative Monoids

We enter a more restrictive context, with $f :: 'a => 'a => 'a$ instead of $'b => 'a => 'a$.

locale *ACeD* =
  fixes D :: *'a set*
    and f :: $'a => 'a => 'a$ (**infixl** $\cdot$ *70*)
    and e :: $'a$
  assumes *ident* [*simp*]: $x \in D \Longrightarrow x \cdot e = x$
    and *commute*: $[\![x \in D; y \in D]\!] \Longrightarrow x \cdot y = y \cdot x$
    and *assoc*: $[\![x \in D; y \in D; z \in D]\!] \Longrightarrow (x \cdot y) \cdot z = x \cdot (y \cdot z)$
    and *e-closed* [*simp*]: $e \in D$
    and *f-closed* [*simp*]: $[\![x \in D; y \in D]\!] \Longrightarrow x \cdot y \in D$

lemma (**in** *ACeD*) *left-commute*:
  $[\![x \in D; y \in D; z \in D]\!] \Longrightarrow x \cdot (y \cdot z) = y \cdot (x \cdot z)$
⟨*proof*⟩

lemmas (**in** *ACeD*) *AC* = *assoc commute left-commute*

lemma (**in** *ACeD*) *left-ident* [*simp*]: $x \in D \Longrightarrow e \cdot x = x$
⟨*proof*⟩

lemma (**in** *ACeD*) *foldD-Un-Int*:
  $[\![$ *finite* A; *finite* B; $A \subseteq D$; $B \subseteq D$ $]\!] \Longrightarrow$
    foldD D f e A $\cdot$ *foldD* D f e B =
    foldD D f e (A *Un* B) $\cdot$ *foldD* D f e (A *Int* B)
  ⟨*proof*⟩

lemma (**in** *ACeD*) *foldD-Un-disjoint*:
[| *finite A*; *finite B*; *A Int B* = {}; $A \subseteq D$; $B \subseteq D$ |] ==>
foldD D f e (*A Un B*) = *foldD D f e A* · *foldD D f e B*
⟨*proof*⟩

3.4 Products over Finite Sets

constdefs (**structure** *G*)
finprod :: [(*'b*, *'m*) *monoid-scheme*, *'a* => *'b*, *'a set*] => *'b*
finprod G f A == *if finite A*
then foldD (*carrier G*) (*mult G o f*) **1** *A*
else arbitrary

syntax
-finprod :: *index* => *idt* => *'a set* => *'b* => *'b*
((*3*⨂ --:-. -) [*1000*, *0*, *51*, *10*] *10*)
syntax (*xsymbols*)
-finprod :: *index* => *idt* => *'a set* => *'b* => *'b*
((*3*⨂ --∈-. -) [*1000*, *0*, *51*, *10*] *10*)
syntax (*HTML* **output**)
-finprod :: *index* => *idt* => *'a set* => *'b* => *'b*
((*3*⨂ --∈-. -) [*1000*, *0*, *51*, *10*] *10*)
translations
⨂ ı*i*:*A*. *b* == *finprod* ◇ı (%*i*. *b*) *A*
— Beware of argument permutation!

lemma (**in** *comm-monoid*) *finprod-empty* [*simp*]:
finprod G f {} = **1**
⟨*proof*⟩

declare *funcsetI* [*intro*]
funcset-mem [*dest*]

lemma (**in** *comm-monoid*) *finprod-insert* [*simp*]:
[| *finite F*; $a \notin F$; $f \in F$ −> *carrier G*; $f\ a \in$ *carrier G* |] ==>
finprod G f (*insert a F*) = $f\ a \otimes$ *finprod G f F*
⟨*proof*⟩

lemma (**in** *comm-monoid*) *finprod-one* [*simp*]:
finite A ==> (⨂ *i*:*A*. **1**) = **1**
⟨*proof*⟩

lemma (**in** *comm-monoid*) *finprod-closed* [*simp*]:
fixes *A*
assumes *fin*: *finite A* **and** *f*: $f \in A$ −> *carrier G*
shows *finprod G f A* ∈ *carrier G*
⟨*proof*⟩

lemma *funcset-Int-left* [*simp*, *intro*]:

[| $f \in A$ −> C; $f \in B$ −> C |] ==> $f \in A$ *Int* B −> C
⟨*proof*⟩

lemma *funcset-Un-left* [*iff*]:
($f \in A$ *Un* B −> C) = ($f \in A$ −> C & $f \in B$ −> C)
⟨*proof*⟩

lemma (**in** *comm-monoid*) *finprod-Un-Int*:
[| *finite* A; *finite* B; $g \in A$ −> *carrier* G; $g \in B$ −> *carrier* G |] ==>
finprod G g (A *Un* B) ⊗ *finprod* G g (A *Int* B) =
finprod G g A ⊗ *finprod* G g B
— The reversed orientation looks more natural, but LOOPS as a simprule!
⟨*proof*⟩

lemma (**in** *comm-monoid*) *finprod-Un-disjoint*:
[| *finite* A; *finite* B; A *Int* B = {};
$g \in A$ −> *carrier* G; $g \in B$ −> *carrier* G |]
==> *finprod* G g (A *Un* B) = *finprod* G g A ⊗ *finprod* G g B
⟨*proof*⟩

lemma (**in** *comm-monoid*) *finprod-multf*:
[| *finite* A; $f \in A$ −> *carrier* G; $g \in A$ −> *carrier* G |] ==>
finprod G (%x. f x ⊗ g x) A = (*finprod* G f A ⊗ *finprod* G g A)
⟨*proof*⟩

lemma (**in** *comm-monoid*) *finprod-cong'*:
[| A = B; $g \in B$ −> *carrier* G;
!!i. $i \in B$ ==> f i = g i |] ==> *finprod* G f A = *finprod* G g B
⟨*proof*⟩

lemma (**in** *comm-monoid*) *finprod-cong*:
[| A = B; $f \in B$ −> *carrier* G = *True*;
!!i. $i \in B$ ==> f i = g i |] ==> *finprod* G f A = *finprod* G g B

⟨*proof*⟩

Usually, if this rule causes a failed congruence proof error, the reason is that the premise $g \in B$ −> *carrier* G cannot be shown. Adding *Pi-def* to the simpset is often useful. For this reason, *comm-monoid.finprod-cong* is not added to the simpset by default.

declare *funcsetI* [*rule del*]
funcset-mem [*rule del*]

lemma (**in** *comm-monoid*) *finprod-0* [*simp*]:
$f \in$ {*0::nat*} −> *carrier* G ==> *finprod* G f {..*0*} = f *0*
⟨*proof*⟩

lemma (**in** *comm-monoid*) *finprod-Suc* [*simp*]:
$f \in$ {..*Suc* n} −> *carrier* G ==>

finprod G f {..Suc n} = (f (Suc n) ⊗ finprod G f {..n})
⟨*proof*⟩

lemma (**in** *comm-monoid*) *finprod-Suc2*:
f ∈ {..Suc n} −> carrier G ==>
finprod G f {..Suc n} = (finprod G (%i. f (Suc i)) {..n} ⊗ f 0)
⟨*proof*⟩

lemma (**in** *comm-monoid*) *finprod-mult* [*simp*]:
[| f ∈ {..n} −> carrier G; g ∈ {..n} −> carrier G |] ==>
finprod G (%i. f i ⊗ g i) {..n::nat} =
finprod G f {..n} ⊗ finprod G g {..n}
⟨*proof*⟩

end

theory *Exponent* **imports** *Main Primes Binomial* **begin**

4 The Combinatorial Argument Underlying the First Sylow Theorem

definition *exponent* :: *nat => nat => nat* **where**
exponent p s == if prime p then (GREATEST r. p^r dvd s) else 0

4.1 Prime Theorems

lemma *prime-imp-one-less*: *prime p ==> Suc 0 < p*
⟨*proof*⟩

lemma *prime-iff*:
(prime p) = (Suc 0 < p & (∀ a b. p dvd a∗b −−> (p dvd a) | (p dvd b)))
⟨*proof*⟩

lemma *zero-less-prime-power*: *prime p ==> 0 < p^a*
⟨*proof*⟩

lemma *zero-less-card-empty*: *[| finite S; S ≠ {} |] ==> 0 < card(S)*
⟨*proof*⟩

lemma *prime-dvd-cases*:
[| p∗k dvd m∗n; prime p |]
==> (∃ x. k dvd x∗n & m = p∗x) | (∃ y. k dvd m∗y & n = p∗y)
⟨*proof*⟩

lemma *prime-power-dvd-cases* [*rule-format* (*no-asm*)]: *prime p*
$==> \forall\, m\ n.\ p\hat{\ }c\ dvd\ m{*}n\ -->$
$(\forall\, a\ b.\ a{+}b = Suc\ c\ --> p\hat{\ }a\ dvd\ m \mid p\hat{\ }b\ dvd\ n)$
⟨*proof*⟩

lemma *div-combine*:
$[|\ prime\ p;\ \sim\ (p\ \hat{\ }\ (Suc\ r)\ dvd\ n);\ \ p\hat{\ }(a{+}r)\ dvd\ n{*}k\ |]$
$==> p\ \hat{\ }\ a\ dvd\ k$
⟨*proof*⟩

lemma *Suc-le-power*: $Suc\ 0 < p ==> Suc\ n <= p\hat{\ }n$
⟨*proof*⟩

lemma *power-dvd-bound*: $[|p\hat{\ }n\ dvd\ a;\ \ Suc\ 0 < p;\ \ a > 0|] ==> n < a$
⟨*proof*⟩

4.2 Exponent Theorems

lemma *exponent-ge* [*rule-format*]:
$[|p\hat{\ }k\ dvd\ n;\ \ prime\ p;\ \ 0{<}n|] ==> k <= exponent\ p\ n$
⟨*proof*⟩

lemma *power-exponent-dvd*: $s{>}0 ==> (p\ \hat{\ }\ exponent\ p\ s)\ dvd\ s$
⟨*proof*⟩

lemma *power-Suc-exponent-Not-dvd*:
$[|(p * p\ \hat{\ }\ exponent\ p\ s)\ dvd\ s;\ \ prime\ p\ |] ==> s{=}0$
⟨*proof*⟩

lemma *exponent-power-eq* [*simp*]: $prime\ p ==> exponent\ p\ (p\hat{\ }a) = a$
⟨*proof*⟩

lemma *exponent-equalityI*:
$!r{::}nat.\ (p\hat{\ }r\ dvd\ a) = (p\hat{\ }r\ dvd\ b) ==> exponent\ p\ a = exponent\ p\ b$
⟨*proof*⟩

lemma *exponent-eq-0* [*simp*]: $\neg\ prime\ p ==> exponent\ p\ s = 0$
⟨*proof*⟩

lemma *exponent-mult-add1*: $[|\ a > 0;\ b > 0\ |]$
$==> (exponent\ p\ a) + (exponent\ p\ b) <= exponent\ p\ (a * b)$
⟨*proof*⟩

lemma *exponent-mult-add2*: $[| \ a > 0; \ b > 0 \ |]$
$==>$ *exponent p* $(a * b) <= ($*exponent p a*$) + ($*exponent p b*$)$
⟨*proof*⟩

lemma *exponent-mult-add*: $[| \ a > 0; \ b > 0 \ |]$
$==>$ *exponent p* $(a * b) = ($*exponent p a*$) + ($*exponent p b*$)$
⟨*proof*⟩

lemma *not-divides-exponent-0*: $\sim$ $(p \ dvd \ n)$ $==>$ *exponent p n* $= 0$
⟨*proof*⟩

lemma *exponent-1-eq-0* [*simp*]: *exponent p* $(Suc \ 0) = 0$
⟨*proof*⟩

4.3 Main Combinatorial Argument

lemma *le-extend-mult*: $[| \ c > 0; \ a <= b \ |] ==> a <= b * (c::nat)$
⟨*proof*⟩

lemma *p-fac-forw-lemma*:
$[| \ (m::nat) > 0; \ k > 0; \ k < p\hat{}a; \ (p\hat{}r) \ dvd \ (p\hat{}a)* \ m - k \ |] ==> r <= a$
⟨*proof*⟩

lemma *p-fac-forw*: $[| \ (m::nat) > 0; \ k>0; \ k < p\hat{}a; \ (p\hat{}r) \ dvd \ (p\hat{}a)* \ m - k \ |]$
$==> (p\hat{}r) \ dvd \ (p\hat{}a) - k$
⟨*proof*⟩

lemma *r-le-a-forw*:
$[| \ (k::nat) > 0; \ k < p\hat{}a; \ p>0; \ (p\hat{}r) \ dvd \ (p\hat{}a) - k \ |] ==> r <= a$
⟨*proof*⟩

lemma *p-fac-backw*: $[| \ m>0; \ k>0; \ (p::nat) \neq 0; \ \ k < p\hat{}a; \ \ (p\hat{}r) \ dvd \ p\hat{}a - k \ |]$
$==> (p\hat{}r) \ dvd \ (p\hat{}a)*m - k$
⟨*proof*⟩

lemma *exponent-p-a-m-k-equation*: $[| \ m>0; \ k>0; \ (p::nat) \neq 0; \ \ k < p\hat{}a \ |]$
$==>$ *exponent p* $(p\hat{}a * m - k) =$ *exponent p* $(p\hat{}a - k)$
⟨*proof*⟩

Suc rules that we have to delete from the simpset

lemmas *bad-Sucs* = *binomial-Suc-Suc mult-Suc mult-Suc-right*

lemma *p-not-div-choose-lemma* [*rule-format*]:
$[| \ \forall i. \ Suc \ i < K \ --> $ *exponent p* $(Suc \ i) =$ *exponent p* $(Suc(j+i))|]$

==> $k<K$ --> *exponent p* $((j+k)$ *choose* $k) = 0$
⟨*proof*⟩

lemma *p-not-div-choose*:
[| $k<K$; $k<=n$;
$\forall j.\ 0<j$ & $j<K$ --> *exponent p* $(n - k + (K - j))$ = *exponent p* $(K - j)$|]
==> *exponent p* $(n$ *choose* $k) = 0$
⟨*proof*⟩

lemma *const-p-fac-right*:
$m>0$ ==> *exponent p* $((p\hat{}a * m - Suc\ 0)$ *choose* $(p\hat{}a - Suc\ 0)) = 0$
⟨*proof*⟩

lemma *const-p-fac*:
$m>0$ ==> *exponent p* $(((p\hat{}a) * m)$ *choose* $p\hat{}a)$ = *exponent p m*
⟨*proof*⟩

end

theory *Coset* **imports** *Group Exponent* **begin**

5 Cosets and Quotient Groups

constdefs (**structure** *G*)
r-coset :: [-, 'a set, 'a] ⇒ 'a set (**infixl** #>ı *60*)
H #> $a \equiv \bigcup h{\in}H.\ \{h \otimes a\}$

l-coset :: [-, 'a, 'a set] ⇒ 'a set (**infixl** <#ı *60*)
a <# $H \equiv \bigcup h{\in}H.\ \{a \otimes h\}$

RCOSETS :: [-, 'a set] ⇒ ('a set)set (*rcosets*ı - [*81*] *80*)
rcosets $H \equiv \bigcup a{\in}carrier\ G.\ \{H$ #> $a\}$

set-mult :: [-, 'a set ,'a set] ⇒ 'a set (**infixl** <#>ı *60*)
H <#> $K \equiv \bigcup h{\in}H.\ \bigcup k{\in}K.\ \{h \otimes k\}$

SET-INV :: [-,'a set] ⇒ 'a set (*set'-inv*ı - [*81*] *80*)
set-inv $H \equiv \bigcup h{\in}H.\ \{inv\ h\}$

locale *normal* = *subgroup* + *group* +
assumes *coset-eq*: $(\forall x \in carrier\ G.\ H$ #> $x = x$ <# $H)$

abbreviation
normal-rel :: [$'a$ *set*, ($'a$, $'b$) *monoid-scheme*] $\Rightarrow$ *bool* (**infixl** $\lhd$ *60*) **where**
$H \lhd G \equiv$ *normal* H G

5.1 Basic Properties of Cosets

lemma (**in** *group*) *coset-mult-assoc*:
[| $M \subseteq$ *carrier* G; $g \in$ *carrier* G; $h \in$ *carrier* G |]
==> (M #> g) #> h = M #> ($g \otimes h$)
⟨*proof*⟩

lemma (**in** *group*) *coset-mult-one* [*simp*]: $M \subseteq$ *carrier* G ==> M #> **1** = M
⟨*proof*⟩

lemma (**in** *group*) *coset-mult-inv1*:
[| M #> ($x \otimes$ (*inv* y)) = M; $x \in$ *carrier* G ; $y \in$ *carrier* G;
$M \subseteq$ *carrier* G |] ==> M #> x = M #> y
⟨*proof*⟩

lemma (**in** *group*) *coset-mult-inv2*:
[| M #> x = M #> y; $x \in$ *carrier* G; $y \in$ *carrier* G; $M \subseteq$ *carrier* G |]
==> M #> ($x \otimes$ (*inv* y)) = M
⟨*proof*⟩

lemma (**in** *group*) *coset-join1*:
[| H #> x = H; $x \in$ *carrier* G; *subgroup* H G |] ==> $x \in H$
⟨*proof*⟩

lemma (**in** *group*) *solve-equation*:
$[\![$*subgroup* H G; $x \in H$; $y \in H]\!] \Longrightarrow \exists h \in H.\ y = h \otimes x$
⟨*proof*⟩

lemma (**in** *group*) *repr-independence*:
$[\![y \in H$ #> x; $x \in$ *carrier* G; *subgroup* H $G]\!] \Longrightarrow H$ #> x = H #> y
⟨*proof*⟩

lemma (**in** *group*) *coset-join2*:
$[\![x \in$ *carrier* G; *subgroup* H G; $x \in H]\!] \Longrightarrow H$ #> x = H
— Alternative proof is to put x = **1** in *repr-independence*.
⟨*proof*⟩

lemma (**in** *monoid*) *r-coset-subset-G*:
[| $H \subseteq$ *carrier* G; $x \in$ *carrier* G |] ==> H #> $x \subseteq$ *carrier* G
⟨*proof*⟩

lemma (**in** *group*) *rcosI*:
[| $h \in H$; $H \subseteq$ *carrier* G; $x \in$ *carrier* G|] ==> $h \otimes x \in H$ #> x
⟨*proof*⟩

lemma (**in** *group*) *rcosetsI*:
$[\![H \subseteq carrier\ G;\ x \in carrier\ G]\!] \Longrightarrow H\ \#>\ x \in rcosets\ H$
$\langle proof \rangle$

Really needed?

lemma (**in** *group*) *transpose-inv*:
[| $x \otimes y = z$; $x \in carrier\ G$; $y \in carrier\ G$; $z \in carrier\ G$ |]
==> $(inv\ x) \otimes z = y$
$\langle proof \rangle$

lemma (**in** *group*) *rcos-self*: [| $x \in carrier\ G$; *subgroup* $H\ G$ |] ==> $x \in H\ \#>\ x$
$\langle proof \rangle$

Opposite of *repr-independence*

lemma (**in** *group*) *repr-independenceD*:
includes *subgroup* $H\ G$
assumes *ycarr*: $y \in carrier\ G$
and *repr*: $H\ \#>\ x = H\ \#>\ y$
shows $y \in H\ \#>\ x$
$\langle proof \rangle$

Elements of a right coset are in the carrier

lemma (**in** *subgroup*) *elemrcos-carrier*:
includes *group*
assumes *acarr*: $a \in carrier\ G$
and a': $a' \in H\ \#>\ a$
shows $a' \in carrier\ G$
$\langle proof \rangle$

lemma (**in** *subgroup*) *rcos-const*:
includes *group*
assumes *hH*: $h \in H$
shows $H\ \#>\ h = H$
$\langle proof \rangle$

Step one for lemma *rcos-module*

lemma (**in** *subgroup*) *rcos-module-imp*:
includes *group*
assumes *xcarr*: $x \in carrier\ G$
and $x'cos$: $x' \in H\ \#>\ x$
shows $(x' \otimes inv\ x) \in H$
$\langle proof \rangle$

Step two for lemma *rcos-module*

lemma (**in** *subgroup*) *rcos-module-rev*:
includes *group*
assumes *carr*: $x \in carrier\ G\ x' \in carrier\ G$
and *xixH*: $(x' \otimes inv\ x) \in H$

shows $x' \in H \ \#> \ x$
$\langle proof \rangle$

Module property of right cosets

lemma (**in** *subgroup*) *rcos-module*:
includes *group*
assumes *carr*: $x \in carrier\ G\ x' \in carrier\ G$
shows $(x' \in H \ \#> \ x) = (x' \otimes inv\ x \in H)$
$\langle proof \rangle$

Right cosets are subsets of the carrier.

lemma (**in** *subgroup*) *rcosets-carrier*:
includes *group*
assumes *XH*: $X \in rcosets\ H$
shows $X \subseteq carrier\ G$
$\langle proof \rangle$

Multiplication of general subsets

lemma (**in** *monoid*) *set-mult-closed*:
assumes *Acarr*: $A \subseteq carrier\ G$
and *Bcarr*: $B \subseteq carrier\ G$
shows $A <\#> B \subseteq carrier\ G$
$\langle proof \rangle$

lemma (**in** *comm-group*) *mult-subgroups*:
assumes *subH*: *subgroup* H G
and *subK*: *subgroup* K G
shows *subgroup* $(H <\#> K)$ G
$\langle proof \rangle$

lemma (**in** *subgroup*) *lcos-module-rev*:
includes *group*
assumes *carr*: $x \in carrier\ G\ x' \in carrier\ G$
and *xixH*: $(inv\ x \otimes x') \in H$
shows $x' \in x <\# \ H$
$\langle proof \rangle$

5.2 Normal subgroups

lemma *normal-imp-subgroup*: $H \lhd G \Longrightarrow subgroup\ H\ G$
$\langle proof \rangle$

lemma (**in** *group*) *normalI*:
$subgroup\ H\ G \Longrightarrow (\forall x \in carrier\ G.\ H \ \#> \ x = x <\# \ H) \Longrightarrow H \lhd G$
$\langle proof \rangle$

lemma (**in** *normal*) *inv-op-closed1*:
$[\![x \in carrier\ G;\ h \in H]\!] \Longrightarrow (inv\ x) \otimes h \otimes x \in H$
$\langle proof \rangle$

lemma (**in** *normal*) *inv-op-closed2*:
$[\![x \in carrier\ G;\ h \in H]\!] \Longrightarrow x \otimes h \otimes (inv\ x) \in H$
⟨*proof*⟩

Alternative characterization of normal subgroups

lemma (**in** *group*) *normal-inv-iff*:
$(N \lhd G) =$
$(subgroup\ N\ G\ \&\ (\forall x \in carrier\ G.\ \forall h \in N.\ x \otimes h \otimes (inv\ x) \in N))$
(**is** - = *?rhs*)
⟨*proof*⟩

5.3 More Properties of Cosets

lemma (**in** *group*) *lcos-m-assoc*:
$[|\ M \subseteq carrier\ G;\ g \in carrier\ G;\ h \in carrier\ G\ |]$
$==> g <\#\ (h <\#\ M) = (g \otimes h) <\#\ M$
⟨*proof*⟩

lemma (**in** *group*) *lcos-mult-one*: $M \subseteq carrier\ G ==> \mathbf{1} <\#\ M = M$
⟨*proof*⟩

lemma (**in** *group*) *l-coset-subset-G*:
$[|\ H \subseteq carrier\ G;\ x \in carrier\ G\ |] ==> x <\#\ H \subseteq carrier\ G$
⟨*proof*⟩

lemma (**in** *group*) *l-coset-swap*:
$[\![y \in x <\#\ H;\ \ x \in carrier\ G;\ \ subgroup\ H\ G]\!] \Longrightarrow x \in y <\#\ H$
⟨*proof*⟩

lemma (**in** *group*) *l-coset-carrier*:
$[|\ y \in x <\#\ H;\ \ x \in carrier\ G;\ \ subgroup\ H\ G\ |] ==> y \in carrier\ G$
⟨*proof*⟩

lemma (**in** *group*) *l-repr-imp-subset*:
assumes *y*: $y \in x <\#\ H$ **and** *x*: $x \in carrier\ G$ **and** *sb*: *subgroup H G*
shows $y <\#\ H \subseteq x <\#\ H$
⟨*proof*⟩

lemma (**in** *group*) *l-repr-independence*:
assumes *y*: $y \in x <\#\ H$ **and** *x*: $x \in carrier\ G$ **and** *sb*: *subgroup H G*
shows $x <\#\ H = y <\#\ H$
⟨*proof*⟩

lemma (**in** *group*) *setmult-subset-G*:
$[\![H \subseteq carrier\ G;\ K \subseteq carrier\ G]\!] \Longrightarrow H <\#> K \subseteq carrier\ G$
⟨*proof*⟩

lemma (**in** *group*) *subgroup-mult-id*: $subgroup\ H\ G \Longrightarrow H <\#> H = H$

$\langle proof \rangle$

5.3.1 Set of Inverses of an *r-coset*.

lemma (**in** *normal*) *rcos-inv*:
assumes x: $x \in carrier\ G$
shows $set\text{-}inv\ (H\ \#{>}\ x) = H\ \#{>}\ (inv\ x)$
$\langle proof \rangle$

5.3.2 Theorems for <#> with #> or <#.

lemma (**in** *group*) *setmult-rcos-assoc*:
$[\![H \subseteq carrier\ G;\ K \subseteq carrier\ G;\ x \in carrier\ G]\!]$
$\Longrightarrow H\ {<}\#{>}\ (K\ \#{>}\ x) = (H\ {<}\#{>}\ K)\ \#{>}\ x$
$\langle proof \rangle$

lemma (**in** *group*) *rcos-assoc-lcos*:
$[\![H \subseteq carrier\ G;\ K \subseteq carrier\ G;\ x \in carrier\ G]\!]$
$\Longrightarrow (H\ \#{>}\ x)\ {<}\#{>}\ K = H\ {<}\#{>}\ (x\ {<}\#\ K)$
$\langle proof \rangle$

lemma (**in** *normal*) *rcos-mult-step1*:
$[\![x \in carrier\ G;\ y \in carrier\ G]\!]$
$\Longrightarrow (H\ \#{>}\ x)\ {<}\#{>}\ (H\ \#{>}\ y) = (H\ {<}\#{>}\ (x\ {<}\#\ H))\ \#{>}\ y$
$\langle proof \rangle$

lemma (**in** *normal*) *rcos-mult-step2*:
$[\![x \in carrier\ G;\ y \in carrier\ G]\!]$
$\Longrightarrow (H\ {<}\#{>}\ (x\ {<}\#\ H))\ \#{>}\ y = (H\ {<}\#{>}\ (H\ \#{>}\ x))\ \#{>}\ y$
$\langle proof \rangle$

lemma (**in** *normal*) *rcos-mult-step3*:
$[\![x \in carrier\ G;\ y \in carrier\ G]\!]$
$\Longrightarrow (H\ {<}\#{>}\ (H\ \#{>}\ x))\ \#{>}\ y = H\ \#{>}\ (x \otimes y)$
$\langle proof \rangle$

lemma (**in** *normal*) *rcos-sum*:
$[\![x \in carrier\ G;\ y \in carrier\ G]\!]$
$\Longrightarrow (H\ \#{>}\ x)\ {<}\#{>}\ (H\ \#{>}\ y) = H\ \#{>}\ (x \otimes y)$
$\langle proof \rangle$

lemma (**in** *normal*) *rcosets-mult-eq*: $M \in rcosets\ H \Longrightarrow H\ {<}\#{>}\ M = M$
— generalizes *subgroup-mult-id*
$\langle proof \rangle$

5.3.3 An Equivalence Relation

constdefs (**structure** G)
r-congruent :: $[('a,'b)monoid\text{-}scheme,\ 'a\ set] \Rightarrow ('a*'a)set$
$(rcong_1\ \text{-})$

rcong H ≡ {(*x*,*y*). *x* ∈ *carrier G* & *y* ∈ *carrier G* & *inv x* ⊗ *y* ∈ *H*}

lemma (**in** *subgroup*) *equiv-rcong*:
  includes *group G*
  shows *equiv* (*carrier G*) (*rcong H*)
⟨*proof*⟩

Equivalence classes of *rcong* correspond to left cosets. Was there a mistake in the definitions? I'd have expected them to correspond to right cosets.

lemma (**in** *subgroup*) *l-coset-eq-rcong*:
  includes *group G*
  assumes *a*: *a* ∈ *carrier G*
  shows *a* <# *H* = *rcong H* ‘‘ {*a*}
⟨*proof*⟩

5.3.4 Two Distinct Right Cosets are Disjoint

lemma (**in** *group*) *rcos-equation*:
  includes *subgroup H G*
  shows
    ⟦*ha* ⊗ *a* = *h* ⊗ *b*; *a* ∈ *carrier G*; *b* ∈ *carrier G*;
      h ∈ *H*; *ha* ∈ *H*; *hb* ∈ *H*⟧
    ⟹ *hb* ⊗ *a* ∈ (⋃*h*∈*H*. {*h* ⊗ *b*})
⟨*proof*⟩

lemma (**in** *group*) *rcos-disjoint*:
  includes *subgroup H G*
  shows ⟦*a* ∈ *rcosets H*; *b* ∈ *rcosets H*; *a*≠*b*⟧ ⟹ *a* ∩ *b* = {}
⟨*proof*⟩

5.4 Further lemmas for *r-congruent*

The relation is a congruence

lemma (**in** *normal*) *congruent-rcong*:
  shows *congruent2* (*rcong H*) (*rcong H*) (λ*a b*. *a* ⊗ *b* <# *H*)
⟨*proof*⟩

5.5 Order of a Group and Lagrange's Theorem

constdefs
  order :: ('*a*, '*b*) *monoid-scheme* ⇒ *nat*
  order S ≡ *card* (*carrier S*)

lemma (**in** *group*) *rcos-self*:
  includes *subgroup*
  shows *x* ∈ *carrier G* ⟹ *x* ∈ *H* #> *x*
⟨*proof*⟩

lemma (**in** *group*) *rcosets-part-G*:
includes *subgroup*
shows $\bigcup(rcosets\ H) = carrier\ G$
⟨*proof*⟩

lemma (**in** *group*) *cosets-finite*:
$[\![c \in rcosets\ H;\ \ H \subseteq carrier\ G;\ \ finite\ (carrier\ G)]\!] \Longrightarrow finite\ c$
⟨*proof*⟩

The next two lemmas support the proof of *card-cosets-equal*.

lemma (**in** *group*) *inj-on-f*:
$[\![H \subseteq carrier\ G;\ \ a \in carrier\ G]\!] \Longrightarrow inj\text{-}on\ (\lambda y.\ y \otimes inv\ a)\ (H\ \#{>}\ a)$
⟨*proof*⟩

lemma (**in** *group*) *inj-on-g*:
$[\![H \subseteq carrier\ G;\ \ a \in carrier\ G]\!] \Longrightarrow inj\text{-}on\ (\lambda y.\ y \otimes a)\ H$
⟨*proof*⟩

lemma (**in** *group*) *card-cosets-equal*:
$[\![c \in rcosets\ H;\ \ H \subseteq carrier\ G;\ finite(carrier\ G)]\!]$
$\Longrightarrow card\ c = card\ H$
⟨*proof*⟩

lemma (**in** *group*) *rcosets-subset-PowG*:
$subgroup\ H\ G\ \Longrightarrow rcosets\ H \subseteq Pow(carrier\ G)$
⟨*proof*⟩

theorem (**in** *group*) *lagrange*:
$[\![finite(carrier\ G);\ subgroup\ H\ G]\!]$
$\Longrightarrow card(rcosets\ H) * card(H) = order(G)$
⟨*proof*⟩

5.6 Quotient Groups: Factorization of a Group

constdefs
$FactGroup :: [('a,'b)\ monoid\text{-}scheme,\ 'a\ set] \Rightarrow ('a\ set)\ monoid$
(**infixl** *Mod 65*)
— Actually defined for groups rather than monoids
$FactGroup\ G\ H \equiv$
$(\!|carrier = rcosets_G\ H,\ mult = set\text{-}mult\ G,\ one = H|\!)$

lemma (**in** *normal*) *setmult-closed*:
$[\![K1 \in rcosets\ H;\ K2 \in rcosets\ H]\!] \Longrightarrow K1 <\#> K2 \in rcosets\ H$
⟨*proof*⟩

lemma (**in** *normal*) *setinv-closed*:
$K \in rcosets\ H \Longrightarrow set\text{-}inv\ K \in rcosets\ H$
⟨*proof*⟩

lemma (**in** *normal*) *rcosets-assoc*:
$[\![M1 \in \mathit{rcosets}\ H;\ M2 \in \mathit{rcosets}\ H;\ M3 \in \mathit{rcosets}\ H]\!]$
$\Longrightarrow M1 <\#> M2 <\#> M3 = M1 <\#> (M2 <\#> M3)$
⟨*proof*⟩

lemma (**in** *subgroup*) *subgroup-in-rcosets*:
includes *group G*
shows $H \in \mathit{rcosets}\ H$
⟨*proof*⟩

lemma (**in** *normal*) *rcosets-inv-mult-group-eq*:
$M \in \mathit{rcosets}\ H \Longrightarrow \mathit{set\text{-}inv}\ M <\#> M = H$
⟨*proof*⟩

theorem (**in** *normal*) *factorgroup-is-group*:
group (*G Mod H*)
⟨*proof*⟩

lemma *mult-FactGroup* [*simp*]: $X \otimes_{(G\ \mathit{Mod}\ H)} X' = X <\#>_G X'$
⟨*proof*⟩

lemma (**in** *normal*) *inv-FactGroup*:
$X \in \mathit{carrier}\ (G\ \mathit{Mod}\ H) \Longrightarrow \mathit{inv}_{G\ \mathit{Mod}\ H}\ X = \mathit{set\text{-}inv}\ X$
⟨*proof*⟩

The coset map is a homomorphism from G to the quotient group $G\ \mathit{Mod}\ H$

lemma (**in** *normal*) *r-coset-hom-Mod*:
$(\lambda a.\ H \#> a) \in \mathit{hom}\ G\ (G\ \mathit{Mod}\ H)$
⟨*proof*⟩

5.7 The First Isomorphism Theorem

The quotient by the kernel of a homomorphism is isomorphic to the range of that homomorphism.

constdefs
$\mathit{kernel} :: ('a, 'm)\ \mathit{monoid\text{-}scheme} \Rightarrow ('b, 'n)\ \mathit{monoid\text{-}scheme} \Rightarrow$
$('a \Rightarrow 'b) \Rightarrow 'a\ \mathit{set}$
— the kernel of a homomorphism
$\mathit{kernel}\ G\ H\ h \equiv \{x.\ x \in \mathit{carrier}\ G\ \&\ h\ x = \mathbf{1}_H\}$

lemma (**in** *group-hom*) *subgroup-kernel*: *subgroup* (*kernel G H h*) *G*
⟨*proof*⟩

The kernel of a homomorphism is a normal subgroup

lemma (**in** *group-hom*) *normal-kernel*: $(\mathit{kernel}\ G\ H\ h) \lhd G$
⟨*proof*⟩

lemma (**in** *group-hom*) *FactGroup-nonempty*:
  assumes X: $X \in$ *carrier* $(G\ Mod\ kernel\ G\ H\ h)$
  shows $X \neq \{\}$
⟨*proof*⟩

lemma (**in** *group-hom*) *FactGroup-contents-mem*:
  assumes X: $X \in$ *carrier* $(G\ Mod\ (kernel\ G\ H\ h))$
  shows *contents* $(h`X) \in$ *carrier* H
⟨*proof*⟩

lemma (**in** *group-hom*) *FactGroup-hom*:
  $(\lambda X.\ contents\ (h`X)) \in hom\ (G\ Mod\ (kernel\ G\ H\ h))\ H$
⟨*proof*⟩

Lemma for the following injectivity result

lemma (**in** *group-hom*) *FactGroup-subset*:
  $[\![g \in carrier\ G;\ g' \in carrier\ G;\ h\ g = h\ g']\!]$
  $\Longrightarrow\ kernel\ G\ H\ h\ \#{>}\ g \subseteq kernel\ G\ H\ h\ \#{>}\ g'$
⟨*proof*⟩

lemma (**in** *group-hom*) *FactGroup-inj-on*:
  inj-on $(\lambda X.\ contents\ (h\ `\ X))\ (carrier\ (G\ Mod\ kernel\ G\ H\ h))$
⟨*proof*⟩

If the homomorphism h is onto H, then so is the homomorphism from the quotient group

lemma (**in** *group-hom*) *FactGroup-onto*:
  assumes h: $h\ `\ carrier\ G = carrier\ H$
  shows $(\lambda X.\ contents\ (h\ `\ X))\ `\ carrier\ (G\ Mod\ kernel\ G\ H\ h) = carrier\ H$
⟨*proof*⟩

If h is a homomorphism from G onto H, then the quotient group $G\ Mod\ kernel\ G\ H\ h$ is isomorphic to H.

theorem (**in** *group-hom*) *FactGroup-iso*:
  $h\ `\ carrier\ G = carrier\ H$
  $\Longrightarrow (\lambda X.\ contents\ (h`X)) \in (G\ Mod\ (kernel\ G\ H\ h)) \cong H$
⟨*proof*⟩

end

theory *Sylow* **imports** *Coset* **begin**

6 Sylow's Theorem

See also [3].

The combinatorial argument is in theory Exponent

locale *sylow* = *group* +
  fixes *p* **and** *a* **and** *m* **and** *calM* **and** *RelM*
  assumes *prime-p*: *prime p*
    and *order-G*: *order*(*G*) = (*p* ^*a*) ∗ *m*
    and *finite-G* [*iff*]: *finite* (*carrier G*)
  defines *calM* == {*s*. *s* ⊆ *carrier*(*G*) & *card*(*s*) = *p* ^*a*}
    and *RelM* == {(*N1*,*N2*). *N1* ∈ *calM* & *N2* ∈ *calM* &
        (∃ *g* ∈ *carrier*(*G*). *N1* = (*N2* #> *g*))}

lemma (**in** *sylow*) *RelM-refl*: *refl calM RelM*
⟨*proof*⟩

lemma (**in** *sylow*) *RelM-sym*: *sym RelM*
⟨*proof*⟩

lemma (**in** *sylow*) *RelM-trans*: *trans RelM*
⟨*proof*⟩

lemma (**in** *sylow*) *RelM-equiv*: *equiv calM RelM*
⟨*proof*⟩

lemma (**in** *sylow*) *M-subset-calM-prep*: *M'* ∈ *calM* // *RelM* ==> *M'* ⊆ *calM*
⟨*proof*⟩

6.1 Main Part of the Proof

locale *sylow-central* = *sylow* +
  fixes *H* **and** *M1* **and** *M*
  assumes *M-in-quot*: *M* ∈ *calM* // *RelM*
    and *not-dvd-M*: ~(*p* ^ *Suc*(*exponent p m*) *dvd card*(*M*))
    and *M1-in-M*: *M1* ∈ *M*
  defines *H* == {*g*. *g*∈*carrier G* & *M1* #> *g* = *M1*}

lemma (**in** *sylow-central*) *M-subset-calM*: *M* ⊆ *calM*
⟨*proof*⟩

lemma (**in** *sylow-central*) *card-M1*: *card*(*M1*) = *p* ^*a*
⟨*proof*⟩

lemma *card-nonempty*: *0* < *card*(*S*) ==> *S* ≠ {}
⟨*proof*⟩

lemma (**in** *sylow-central*) *exists-x-in-M1*: ∃ *x*. *x*∈*M1*
⟨*proof*⟩

lemma (**in** *sylow-central*) *M1-subset-G* [*simp*]: $M1 \subseteq carrier\ G$
⟨*proof*⟩

lemma (**in** *sylow-central*) *M1-inj-H*: $\exists f \in H \rightarrow M1.\ inj\text{-}on\ f\ H$
⟨*proof*⟩

6.2 Discharging the Assumptions of *sylow-central*

lemma (**in** *sylow*) *EmptyNotInEquivSet*: $\{\} \notin calM\ //\ RelM$
⟨*proof*⟩

lemma (**in** *sylow*) *existsM1inM*: $M \in calM\ //\ RelM ==> \exists M1.\ M1 \in M$
⟨*proof*⟩

lemma (**in** *sylow*) *zero-less-o-G*: $0 < order(G)$
⟨*proof*⟩

lemma (**in** *sylow*) *zero-less-m*: $m > 0$
⟨*proof*⟩

lemma (**in** *sylow*) *card-calM*: $card(calM) = (p\hat{\ }a) * m\ choose\ p\hat{\ }a$
⟨*proof*⟩

lemma (**in** *sylow*) *zero-less-card-calM*: $card\ calM > 0$
⟨*proof*⟩

lemma (**in** *sylow*) *max-p-div-calM*:
$\sim (p\ \hat{}\ Suc(exponent\ p\ m)\ dvd\ card(calM))$
⟨*proof*⟩

lemma (**in** *sylow*) *finite-calM*: $finite\ calM$
⟨*proof*⟩

lemma (**in** *sylow*) *lemma-A1*:
$\exists M \in calM\ //\ RelM.\ \sim (p\ \hat{}\ Suc(exponent\ p\ m)\ dvd\ card(M))$
⟨*proof*⟩

6.2.1 Introduction and Destruct Rules for *H*

lemma (**in** *sylow-central*) *H-I*: $[|g \in carrier\ G;\ M1\ \#>\ g = M1|] ==> g \in H$
⟨*proof*⟩

lemma (**in** *sylow-central*) *H-into-carrier-G*: $x \in H ==> x \in carrier\ G$
⟨*proof*⟩

lemma (**in** *sylow-central*) *in-H-imp-eq*: $g : H ==> M1\ \#>\ g = M1$
⟨*proof*⟩

lemma (**in** *sylow-central*) *H-m-closed*: $[|\ x \in H;\ y \in H|] ==> x \otimes y \in H$

⟨*proof*⟩

lemma (**in** *sylow-central*) *H-not-empty*: $H \neq \{\}$
⟨*proof*⟩

lemma (**in** *sylow-central*) *H-is-subgroup*: *subgroup H G*
⟨*proof*⟩

lemma (**in** *sylow-central*) *rcosetGM1g-subset-G*:
  [| $g \in$ *carrier G*; $x \in$ *M1* #> g |] ==> $x \in$ *carrier G*
⟨*proof*⟩

lemma (**in** *sylow-central*) *finite-M1*: *finite M1*
⟨*proof*⟩

lemma (**in** *sylow-central*) *finite-rcosetGM1g*: $g \in$*carrier G* ==> *finite* (*M1* #> g)
⟨*proof*⟩

lemma (**in** *sylow-central*) *M1-cardeq-rcosetGM1g*:
  $g \in$ *carrier G* ==> *card*(*M1* #> g) = *card*(*M1*)
⟨*proof*⟩

lemma (**in** *sylow-central*) *M1-RelM-rcosetGM1g*:
  $g \in$ *carrier G* ==> (*M1*, *M1* #> g) $\in$ *RelM*
⟨*proof*⟩

6.3 Equal Cardinalities of M and the Set of Cosets

Injections between M and $rcosets_G\ H$ show that their cardinalities are equal.

lemma *ElemClassEquiv*:
  [| *equiv A r*; $C \in A$ // r |] ==> $\forall x \in C.\ \forall y \in C.\ (x,y) \in r$
⟨*proof*⟩

lemma (**in** *sylow-central*) *M-elem-map*:
  M2 $\in M$ ==> $\exists g.\ g \in$ *carrier G* & *M1* #> g = *M2*
⟨*proof*⟩

lemmas (**in** *sylow-central*) *M-elem-map-carrier* =
    M-elem-map [*THEN someI-ex*, *THEN conjunct1*]

lemmas (**in** *sylow-central*) *M-elem-map-eq* =
    M-elem-map [*THEN someI-ex*, *THEN conjunct2*]

lemma (**in** *sylow-central*) *M-funcset-rcosets-H*:
  (%x:M. H #> (*SOME* g. $g \in$ *carrier G* & *M1* #> g = x)) $\in M \to$ *rcosets H*
⟨*proof*⟩

lemma (**in** *sylow-central*) *inj-M-GmodH*: $\exists f \in M \to$*rcosets H*. *inj-on f M*

⟨*proof*⟩

6.3.1 The Opposite Injection

lemma (**in** *sylow-central*) *H-elem-map*:
  $H1 \in rcosets\ H \Longrightarrow \exists g.\ g \in carrier\ G\ \&\ H\ \#{>}\ g = H1$
⟨*proof*⟩

lemmas (**in** *sylow-central*) *H-elem-map-carrier* =
  H-elem-map [*THEN someI-ex*, *THEN conjunct1*]

lemmas (**in** *sylow-central*) *H-elem-map-eq* =
  H-elem-map [*THEN someI-ex*, *THEN conjunct2*]

lemma *EquivElemClass*:
  $[|equiv\ A\ r;\ M \in A//r;\ M1{\in}M;\ (M1,M2) \in r\ |] \Longrightarrow M2 \in M$
⟨*proof*⟩

lemma (**in** *sylow-central*) *rcosets-H-funcset-M*:
  $(\lambda C \in rcosets\ H.\ M1\ \#{>}\ (@g.\ g \in carrier\ G \wedge H\ \#{>}\ g = C)) \in rcosets\ H \rightarrow M$
⟨*proof*⟩

close to a duplicate of *inj-M-GmodH*

lemma (**in** *sylow-central*) *inj-GmodH-M*:
  $\exists g \in rcosets\ H{\rightarrow}M.\ inj\text{-}on\ g\ (rcosets\ H)$
⟨*proof*⟩

lemma (**in** *sylow-central*) *calM-subset-PowG*: $calM \subseteq Pow(carrier\ G)$
⟨*proof*⟩

lemma (**in** *sylow-central*) *finite-M*: *finite M*
⟨*proof*⟩

lemma (**in** *sylow-central*) *cardMeqIndexH*: $card(M) = card(rcosets\ H)$
⟨*proof*⟩

lemma (**in** *sylow-central*) *index-lem*: $card(M) * card(H) = order(G)$
⟨*proof*⟩

lemma (**in** *sylow-central*) *lemma-leq1*: $p\hat{}a \le card(H)$
⟨*proof*⟩

lemma (**in** *sylow-central*) *lemma-leq2*: $card(H) \le p\hat{}a$
⟨*proof*⟩

lemma (**in** *sylow-central*) *card-H-eq*: $card(H) = p\hat{\ }a$
⟨*proof*⟩

lemma (**in** *sylow*) *sylow-thm*: $\exists H.\ subgroup\ H\ G\ \&\ card(H) = p\hat{\ }a$
⟨*proof*⟩

Needed because the locale's automatic definition refers to *semigroup G* and *group-axioms G* rather than simply to *group G*.

lemma *sylow-eq*: $sylow\ G\ p\ a\ m = (group\ G\ \&\ sylow\text{-}axioms\ G\ p\ a\ m)$
⟨*proof*⟩

6.4 Sylow's Theorem

theorem *sylow-thm*:
$[|\ prime\ p;\ \ group(G);\ \ order(G) = (p\hat{\ }a) * m;\ finite\ (carrier\ G)|]$
$==> \exists H.\ subgroup\ H\ G\ \&\ card(H) = p\hat{\ }a$
⟨*proof*⟩

end

theory *Bij* **imports** *Group* **begin**

7 Bijections of a Set, Permutation Groups and Automorphism Groups

constdefs
$Bij :: {'a}\ set \Rightarrow ({'a} \Rightarrow {'a})\ set$
— Only extensional functions, since otherwise we get too many.
$Bij\ S \equiv extensional\ S \cap \{f.\ bij\text{-}betw\ f\ S\ S\}$

$BijGroup :: {'a}\ set \Rightarrow ({'a} \Rightarrow {'a})\ monoid$
$BijGroup\ S \equiv$
$(|carrier = Bij\ S,$
$mult = \lambda g \in Bij\ S.\ \lambda f \in Bij\ S.\ compose\ S\ g\ f,$
$one = \lambda x \in S.\ x|)$

declare *Id-compose* [*simp*] *compose-Id* [*simp*]

lemma *Bij-imp-extensional*: $f \in Bij\ S \Longrightarrow f \in extensional\ S$
⟨*proof*⟩

lemma *Bij-imp-funcset*: $f \in Bij\ S \Longrightarrow f \in S \rightarrow S$
⟨*proof*⟩

7.1 Bijections Form a Group

lemma *restrict-Inv-Bij*: $f \in Bij\ S \Longrightarrow (\lambda x \in S.\ (Inv\ S\ f)\ x) \in Bij\ S$
⟨*proof*⟩

lemma *id-Bij*: $(\lambda x{\in}S.\ x) \in Bij\ S$
⟨*proof*⟩

lemma *compose-Bij*: $[\![x \in Bij\ S;\ y \in Bij\ S]\!] \Longrightarrow compose\ S\ x\ y \in Bij\ S$
⟨*proof*⟩

lemma *Bij-compose-restrict-eq*:
$f \in Bij\ S \Longrightarrow compose\ S\ (restrict\ (Inv\ S\ f)\ S)\ f = (\lambda x{\in}S.\ x)$
⟨*proof*⟩

theorem *group-BijGroup*: $group\ (BijGroup\ S)$
⟨*proof*⟩

7.2 Automorphisms Form a Group

lemma *Bij-Inv-mem*: $[\![\ f \in Bij\ S;\ \ x \in S]\!] \Longrightarrow Inv\ S\ f\ x \in S$
⟨*proof*⟩

lemma *Bij-Inv-lemma*:
assumes *eq*: $\bigwedge x\ y.\ [\![x \in S;\ y \in S]\!] \Longrightarrow h(g\ x\ y) = g\ (h\ x)\ (h\ y)$
shows $[\![h \in Bij\ S;\ \ g \in S \to S \to S;\ \ x \in S;\ \ y \in S]\!]$
$\Longrightarrow Inv\ S\ h\ (g\ x\ y) = g\ (Inv\ S\ h\ x)\ (Inv\ S\ h\ y)$
⟨*proof*⟩

constdefs
$auto :: ('a,\ 'b)\ monoid\text{-}scheme \Rightarrow ('a \Rightarrow 'a)\ set$
$auto\ G \equiv hom\ G\ G \cap Bij\ (carrier\ G)$

$AutoGroup :: ('a,\ 'c)\ monoid\text{-}scheme \Rightarrow ('a \Rightarrow 'a)\ monoid$
$AutoGroup\ G \equiv BijGroup\ (carrier\ G)\ (\!|carrier := auto\ G|\!)$

lemma (**in** *group*) *id-in-auto*: $(\lambda x \in carrier\ G.\ x) \in auto\ G$
⟨*proof*⟩

lemma (**in** *group*) *mult-funcset*: $mult\ G \in carrier\ G \to carrier\ G \to carrier\ G$
⟨*proof*⟩

lemma (**in** *group*) *restrict-Inv-hom*:
$[\![h \in hom\ G\ G;\ h \in Bij\ (carrier\ G)]\!]$
$\Longrightarrow restrict\ (Inv\ (carrier\ G)\ h)\ (carrier\ G) \in hom\ G\ G$
⟨*proof*⟩

lemma *inv-BijGroup*:
$f \in Bij\ S \Longrightarrow m\text{-}inv\ (BijGroup\ S)\ f = (\lambda x \in S.\ (Inv\ S\ f)\ x)$

⟨*proof*⟩

lemma (**in** *group*) *subgroup-auto*:
  subgroup (*auto G*) (*BijGroup* (*carrier G*))
⟨*proof*⟩

theorem (**in** *group*) *AutoGroup*: *group* (*AutoGroup G*)
⟨*proof*⟩

end

theory *Ring* **imports** *FiniteProduct*
uses (*ringsimp.ML*) **begin**

8 Abelian Groups

record *'a ring* = *'a monoid* +
 zero :: *'a* ($\mathbf{0}_1$)
 add :: [*'a*, *'a*] => *'a* (**infixl** $\oplus_1$ *65*)

Derived operations.

constdefs (**structure** *R*)
 a-inv :: [(*'a*, *'m*) *ring-scheme*, *'a*] => *'a* ($\ominus_1$ - [*81*] *80*)
 a-inv R == *m-inv* (| *carrier* = *carrier R*, *mult* = *add R*, *one* = *zero R* |)

 a-minus :: [(*'a*, *'m*) *ring-scheme*, *'a*, *'a*] => *'a* (**infixl** $\ominus_1$ *65*)
 [| $x \in$ *carrier R*; $y \in$ *carrier R* |] ==> $x \ominus y$ == $x \oplus (\ominus\ y)$

locale *abelian-monoid* =
 fixes *G* (**structure**)
 assumes *a-comm-monoid*:
  comm-monoid (| *carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G* |)

The following definition is redundant but simple to use.

locale *abelian-group* = *abelian-monoid* +
 assumes *a-comm-group*:
  comm-group (| *carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G* |)

8.1 Basic Properties

lemma *abelian-monoidI*:
 fixes *R* (**structure**)
 assumes *a-closed*:
  !!*x y*. [| $x \in$ *carrier R*; $y \in$ *carrier R* |] ==> $x \oplus y \in$ *carrier R*
 and *zero-closed*: $\mathbf{0} \in$ *carrier R*
 and *a-assoc*:

$!!x\ y\ z.\ [|\ x \in carrier\ R;\ y \in carrier\ R;\ z \in carrier\ R\ |] ==>$
$(x \oplus y) \oplus z = x \oplus (y \oplus z)$
and *l-zero*: $!!x.\ x \in carrier\ R ==> \mathbf{0} \oplus x = x$
and *a-comm*:
$!!x\ y.\ [|\ x \in carrier\ R;\ y \in carrier\ R\ |] ==> x \oplus y = y \oplus x$
shows *abelian-monoid R*
⟨*proof*⟩

lemma *abelian-groupI*:
fixes *R* (**structure**)
assumes *a-closed*:
$!!x\ y.\ [|\ x \in carrier\ R;\ y \in carrier\ R\ |] ==> x \oplus y \in carrier\ R$
and *zero-closed*: $zero\ R \in carrier\ R$
and *a-assoc*:
$!!x\ y\ z.\ [|\ x \in carrier\ R;\ y \in carrier\ R;\ z \in carrier\ R\ |] ==>$
$(x \oplus y) \oplus z = x \oplus (y \oplus z)$
and *a-comm*:
$!!x\ y.\ [|\ x \in carrier\ R;\ y \in carrier\ R\ |] ==> x \oplus y = y \oplus x$
and *l-zero*: $!!x.\ x \in carrier\ R ==> \mathbf{0} \oplus x = x$
and *l-inv-ex*: $!!x.\ x \in carrier\ R ==> EX\ y : carrier\ R.\ y \oplus x = \mathbf{0}$
shows *abelian-group R*
⟨*proof*⟩

lemma (**in** *abelian-monoid*) *a-monoid*:
$monoid\ (|\ carrier = carrier\ G,\ mult = add\ G,\ one = zero\ G\ |)$
⟨*proof*⟩

lemma (**in** *abelian-group*) *a-group*:
$group\ (|\ carrier = carrier\ G,\ mult = add\ G,\ one = zero\ G\ |)$
⟨*proof*⟩

lemmas *monoid-record-simps* = *partial-object.simps monoid.simps*

lemma (**in** *abelian-monoid*) *a-closed* [*intro*, *simp*]:
$[\![\ x \in carrier\ G;\ y \in carrier\ G\ ]\!] \Longrightarrow x \oplus y \in carrier\ G$
⟨*proof*⟩

lemma (**in** *abelian-monoid*) *zero-closed* [*intro*, *simp*]:
$\mathbf{0} \in carrier\ G$
⟨*proof*⟩

lemma (**in** *abelian-group*) *a-inv-closed* [*intro*, *simp*]:
$x \in carrier\ G ==> \ominus\ x \in carrier\ G$
⟨*proof*⟩

lemma (**in** *abelian-group*) *minus-closed* [*intro*, *simp*]:
$[|\ x \in carrier\ G;\ y \in carrier\ G\ |] ==> x \ominus y \in carrier\ G$
⟨*proof*⟩

lemma (**in** *abelian-group*) *a-l-cancel* [*simp*]:
[| $x \in$ *carrier* G; $y \in$ *carrier* G; $z \in$ *carrier* G |] ==>
$(x \oplus y = x \oplus z) = (y = z)$
⟨*proof*⟩

lemma (**in** *abelian-group*) *a-r-cancel* [*simp*]:
[| $x \in$ *carrier* G; $y \in$ *carrier* G; $z \in$ *carrier* G |] ==>
$(y \oplus x = z \oplus x) = (y = z)$
⟨*proof*⟩

lemma (**in** *abelian-monoid*) *a-assoc*:
$[\![x \in \mathit{carrier}\ G;\ y \in \mathit{carrier}\ G;\ z \in \mathit{carrier}\ G]\!] \Longrightarrow$
$(x \oplus y) \oplus z = x \oplus (y \oplus z)$
⟨*proof*⟩

lemma (**in** *abelian-monoid*) *l-zero* [*simp*]:
$x \in$ *carrier* G ==> $\mathbf{0} \oplus x = x$
⟨*proof*⟩

lemma (**in** *abelian-group*) *l-neg*:
$x \in$ *carrier* G ==> $\ominus\ x \oplus x = \mathbf{0}$
⟨*proof*⟩

lemma (**in** *abelian-monoid*) *a-comm*:
$[\![x \in \mathit{carrier}\ G;\ y \in \mathit{carrier}\ G]\!] \Longrightarrow x \oplus y = y \oplus x$
⟨*proof*⟩

lemma (**in** *abelian-monoid*) *a-lcomm*:
$[\![x \in \mathit{carrier}\ G;\ y \in \mathit{carrier}\ G;\ z \in \mathit{carrier}\ G]\!] \Longrightarrow$
$x \oplus (y \oplus z) = y \oplus (x \oplus z)$
⟨*proof*⟩

lemma (**in** *abelian-monoid*) *r-zero* [*simp*]:
$x \in$ *carrier* G ==> $x \oplus \mathbf{0} = x$
⟨*proof*⟩

lemma (**in** *abelian-group*) *r-neg*:
$x \in$ *carrier* G ==> $x \oplus (\ominus\ x) = \mathbf{0}$
⟨*proof*⟩

lemma (**in** *abelian-group*) *minus-zero* [*simp*]:
$\ominus\ \mathbf{0} = \mathbf{0}$
⟨*proof*⟩

lemma (**in** *abelian-group*) *minus-minus* [*simp*]:
$x \in$ *carrier* G ==> $\ominus\ (\ominus\ x) = x$
⟨*proof*⟩

lemma (**in** *abelian-group*) *a-inv-inj*:

inj-on (*a-inv* G) (*carrier* G)
⟨*proof*⟩

lemma (**in** *abelian-group*) *minus-add*:
[| $x \in$ *carrier* G; $y \in$ *carrier* G |] ==> $\ominus$ ($x \oplus y$) = $\ominus x \oplus \ominus y$
⟨*proof*⟩

lemma (**in** *abelian-group*) *minus-equality*:
[| $x \in$ *carrier* G; $y \in$ *carrier* G; $y \oplus x = \mathbf{0}$ |] ==> $\ominus x = y$
⟨*proof*⟩

lemma (**in** *abelian-monoid*) *minus-unique*:
[| $x \in$ *carrier* G; $y \in$ *carrier* G; $y' \in$ *carrier* G;
$y \oplus x = \mathbf{0}$; $x \oplus y' = \mathbf{0}$ |] ==> $y = y'$
⟨*proof*⟩

lemmas (**in** *abelian-monoid*) *a-ac* = *a-assoc a-comm a-lcomm*

Derive an *abelian-group* from a *comm-group*

lemma *comm-group-abelian-groupI*:
fixes G (**structure**)
assumes *cg*: *comm-group* (|*carrier* = *carrier* G, *mult* = *add* G, *one* = *zero* G|)
shows *abelian-group* G
⟨*proof*⟩

8.2 Sums over Finite Sets

This definition makes it easy to lift lemmas from *finprod*.

constdefs
finsum :: [('b, 'm) *ring-scheme*, 'a => 'b, 'a *set*] => 'b
finsum G f A == *finprod* (| *carrier* = *carrier* G,
mult = *add* G, *one* = *zero* G |) f A

syntax
-finsum :: *index* => *idt* => 'a *set* => 'b => 'b
((*3*$\bigoplus$--:-. -) [*1000*, *0*, *51*, *10*] *10*)
syntax (*xsymbols*)
-finsum :: *index* => *idt* => 'a *set* => 'b => 'b
((*3*$\bigoplus$--$\in$-. -) [*1000*, *0*, *51*, *10*] *10*)
syntax (*HTML* **output**)
-finsum :: *index* => *idt* => 'a *set* => 'b => 'b
((*3*$\bigoplus$--$\in$-. -) [*1000*, *0*, *51*, *10*] *10*)
translations
$\bigoplus_{\imath} i$:A. b == *finsum* $\diamond_{\imath}$ (%i. b) A
— Beware of argument permutation!

lemma (**in** *abelian-monoid*) *finsum-empty* [*simp*]:

finsum G f $\{\} = \mathbf{0}$
⟨*proof*⟩

lemma (**in** *abelian-monoid*) *finsum-insert* [*simp*]:
$[|$ *finite F*; $a \notin F$; $f \in F$ $->$ *carrier G*; $f\ a \in$ *carrier G* $|]$
$==>$ *finsum G f* (*insert a F*) $= f\ a \oplus$ *finsum G f F*
⟨*proof*⟩

lemma (**in** *abelian-monoid*) *finsum-zero* [*simp*]:
finite A $==>$ $(\bigoplus i{\in}A.\ \mathbf{0}) = \mathbf{0}$
⟨*proof*⟩

lemma (**in** *abelian-monoid*) *finsum-closed* [*simp*]:
fixes *A*
assumes *fin*: *finite A* **and** *f*: $f \in A$ $->$ *carrier G*
shows *finsum G f A* $\in$ *carrier G*
⟨*proof*⟩

lemma (**in** *abelian-monoid*) *finsum-Un-Int*:
$[|$ *finite A*; *finite B*; $g \in A$ $->$ *carrier G*; $g \in B$ $->$ *carrier G* $|]$ $==>$
finsum G g (*A Un B*) $\oplus$ *finsum G g* (*A Int B*) $=$
finsum G g A $\oplus$ *finsum G g B*
⟨*proof*⟩

lemma (**in** *abelian-monoid*) *finsum-Un-disjoint*:
$[|$ *finite A*; *finite B*; *A Int B* $= \{\}$;
$g \in A$ $->$ *carrier G*; $g \in B$ $->$ *carrier G* $|]$
$==>$ *finsum G g* (*A Un B*) $=$ *finsum G g A* $\oplus$ *finsum G g B*
⟨*proof*⟩

lemma (**in** *abelian-monoid*) *finsum-addf*:
$[|$ *finite A*; $f \in A$ $->$ *carrier G*; $g \in A$ $->$ *carrier G* $|]$ $==>$
finsum G (%*x*. $f\ x \oplus g\ x$) $A =$ (*finsum G f A* $\oplus$ *finsum G g A*)
⟨*proof*⟩

lemma (**in** *abelian-monoid*) *finsum-cong'*:
$[|$ $A = B$; $g : B$ $->$ *carrier G*;
!!*i*. $i : B$ $==>$ $f\ i = g\ i$ $|]$ $==>$ *finsum G f A* $=$ *finsum G g B*
⟨*proof*⟩

lemma (**in** *abelian-monoid*) *finsum-0* [*simp*]:
$f : \{0{::}nat\}$ $->$ *carrier G* $==>$ *finsum G f* $\{..0\} = f\ 0$
⟨*proof*⟩

lemma (**in** *abelian-monoid*) *finsum-Suc* [*simp*]:
$f : \{..Suc\ n\}$ $->$ *carrier G* $==>$
finsum G f $\{..Suc\ n\} = (f\ (Suc\ n) \oplus$ *finsum G f* $\{..n\})$
⟨*proof*⟩

lemma (**in** *abelian-monoid*) *finsum-Suc2*:
  $f : \{..Suc\ n\}$ $->$ *carrier* G $==>$
   finsum $G\ f\ \{..Suc\ n\} = ($*finsum* $G\ (\%i.\ f\ (Suc\ i))\ \{..n\} \oplus f\ 0)$
  ⟨*proof*⟩

lemma (**in** *abelian-monoid*) *finsum-add* [*simp*]:
  [| $f : \{..n\}$ $->$ *carrier* G; $g : \{..n\}$ $->$ *carrier* G |] $==>$
    finsum $G\ (\%i.\ f\ i \oplus g\ i)\ \{..n::nat\} =$
    finsum $G\ f\ \{..n\} \oplus$ *finsum* $G\ g\ \{..n\}$
  ⟨*proof*⟩

lemma (**in** *abelian-monoid*) *finsum-cong*:
  [| $A = B$; $f : B$ $->$ *carrier* G;
    $!!i.\ i : B$ $=simp=>$ $f\ i = g\ i$ |] $==>$ *finsum* $G\ f\ A =$ *finsum* $G\ g\ B$
  ⟨*proof*⟩

Usually, if this rule causes a failed congruence proof error, the reason is that the premise $g \in B$ $->$ *carrier* G cannot be shown. Adding *Pi-def* to the simpset is often useful.

9 The Algebraic Hierarchy of Rings

9.1 Basic Definitions

locale *ring* = *abelian-group* R + *monoid* R +
  assumes *l-distr*: [| $x \in$ *carrier* R; $y \in$ *carrier* R; $z \in$ *carrier* R |]
    $==>$ $(x \oplus y) \otimes z = x \otimes z \oplus y \otimes z$
   and *r-distr*: [| $x \in$ *carrier* R; $y \in$ *carrier* R; $z \in$ *carrier* R |]
    $==>$ $z \otimes (x \oplus y) = z \otimes x \oplus z \otimes y$

locale *cring* = *ring* + *comm-monoid* R

locale *domain* = *cring* +
  assumes *one-not-zero* [*simp*]: $\mathbf{1}$ ~= $\mathbf{0}$
   and *integral*: [| $a \otimes b = \mathbf{0}$; $a \in$ *carrier* R; $b \in$ *carrier* R |] $==>$
        $a = \mathbf{0}$ | $b = \mathbf{0}$

locale *field* = *domain* +
  assumes *field-Units*: *Units* R = *carrier* $R - \{\mathbf{0}\}$

9.2 Rings

lemma *ringI*:
  fixes R (**structure**)
  assumes *abelian-group*: *abelian-group* R
   and *monoid*: *monoid* R
   and *l-distr*: $!!x\ y\ z.$ [| $x \in$ *carrier* R; $y \in$ *carrier* R; $z \in$ *carrier* R |]
    $==>$ $(x \oplus y) \otimes z = x \otimes z \oplus y \otimes z$
   and *r-distr*: $!!x\ y\ z.$ [| $x \in$ *carrier* R; $y \in$ *carrier* R; $z \in$ *carrier* R |]

    ==> $z \otimes (x \oplus y) = z \otimes x \oplus z \otimes y$
  shows *ring R*
  ⟨*proof*⟩

lemma (**in** *ring*) *is-abelian-group*:
  abelian-group R
  ⟨*proof*⟩

lemma (**in** *ring*) *is-monoid*:
  monoid R
  ⟨*proof*⟩

lemma (**in** *ring*) *is-ring*:
  ring R
  ⟨*proof*⟩

lemmas *ring-record-simps* = *monoid-record-simps ring.simps*

lemma *cringI*:
  fixes *R* (**structure**)
  assumes *abelian-group*: *abelian-group R*
    and *comm-monoid*: *comm-monoid R*
    and *l-distr*: !!*x y z*. [| $x \in$ *carrier R*; $y \in$ *carrier R*; $z \in$ *carrier R* |]
      ==> $(x \oplus y) \otimes z = x \otimes z \oplus y \otimes z$
  shows *cring R*
⟨*proof*⟩

lemma (**in** *cring*) *is-comm-monoid*:
  comm-monoid R
  ⟨*proof*⟩

lemma (**in** *cring*) *is-cring*:
  cring R ⟨*proof*⟩

9.2.1 Normaliser for Rings

lemma (**in** *abelian-group*) *r-neg2*:
  [| $x \in$ *carrier G*; $y \in$ *carrier G* |] ==> $x \oplus (\ominus x \oplus y) = y$
⟨*proof*⟩

lemma (**in** *abelian-group*) *r-neg1*:
  [| $x \in$ *carrier G*; $y \in$ *carrier G* |] ==> $\ominus x \oplus (x \oplus y) = y$
⟨*proof*⟩

The following proofs are from Jacobson, Basic Algebra I, pp. 88–89

lemma (**in** *ring*) *l-null* [*simp*]:
  $x \in$ *carrier R* ==> $\mathbf{0} \otimes x = \mathbf{0}$
⟨*proof*⟩

lemma (**in** *ring*) *r-null* [*simp*]:
$x \in$ *carrier R* ==> $x \otimes \mathbf{0} = \mathbf{0}$
⟨*proof*⟩

lemma (**in** *ring*) *l-minus*:
[| $x \in$ *carrier R*; $y \in$ *carrier R* |] ==> $\ominus\, x \otimes y = \ominus\, (x \otimes y)$
⟨*proof*⟩

lemma (**in** *ring*) *r-minus*:
[| $x \in$ *carrier R*; $y \in$ *carrier R* |] ==> $x \otimes \ominus\, y = \ominus\, (x \otimes y)$
⟨*proof*⟩

lemma (**in** *abelian-group*) *minus-eq*:
[| $x \in$ *carrier G*; $y \in$ *carrier G* |] ==> $x \ominus y = x \oplus \ominus\, y$
⟨*proof*⟩

Setup algebra method: compute distributive normal form in locale contexts

⟨*ML*⟩

lemmas (**in** *ring*) *ring-simprules*
[*algebra ring zero R add R a-inv R a-minus R one R mult R*] =
a-closed zero-closed a-inv-closed minus-closed m-closed one-closed
a-assoc l-zero l-neg a-comm m-assoc l-one l-distr minus-eq
r-zero r-neg r-neg2 r-neg1 minus-add minus-minus minus-zero
a-lcomm r-distr l-null r-null l-minus r-minus

lemmas (**in** *cring*)
[*algebra del*: *ring zero R add R a-inv R a-minus R one R mult R*] =
\-

lemmas (**in** *cring*) *cring-simprules*
[*algebra add*: *cring zero R add R a-inv R a-minus R one R mult R*] =
a-closed zero-closed a-inv-closed minus-closed m-closed one-closed
a-assoc l-zero l-neg a-comm m-assoc l-one l-distr m-comm minus-eq
r-zero r-neg r-neg2 r-neg1 minus-add minus-minus minus-zero
a-lcomm m-lcomm r-distr l-null r-null l-minus r-minus

lemma (**in** *cring*) *nat-pow-zero*:
(*n*::*nat*) ~= *0* ==> $\mathbf{0}$ (^) $n = \mathbf{0}$
⟨*proof*⟩

lemma (**in** *ring*) *one-zeroD*:
assumes *onezero*: $\mathbf{1} = \mathbf{0}$
shows *carrier R* = $\{\mathbf{0}\}$
⟨*proof*⟩

lemma (**in** *ring*) *one-zeroI*:
assumes *carrzero*: *carrier R* = $\{\mathbf{0}\}$

shows $\mathbf{1} = \mathbf{0}$
⟨*proof*⟩

lemma (**in** *ring*) *one-zero*:
shows $(carrier\ R = \{\mathbf{0}\}) = (\mathbf{1} = \mathbf{0})$
⟨*proof*⟩

lemma (**in** *ring*) *one-not-zero*:
shows $(carrier\ R \neq \{\mathbf{0}\}) = (\mathbf{1} \neq \mathbf{0})$
⟨*proof*⟩

Two examples for use of method algebra

lemma
includes *ring* R + *cring* S
shows [| $a \in carrier\ R$; $b \in carrier\ R$; $c \in carrier\ S$; $d \in carrier\ S$ |] ==>
$a \oplus \ominus (a \oplus \ominus b) = b$ & $c \otimes_S d = d \otimes_S c$
⟨*proof*⟩

lemma
includes *cring*
shows [| $a \in carrier\ R$; $b \in carrier\ R$ |] ==> $a \ominus (a \ominus b) = b$
⟨*proof*⟩

9.2.2 Sums over Finite Sets

lemma (**in** *cring*) *finsum-ldistr*:
[| *finite* A; $a \in carrier\ R$; $f \in A$ −> *carrier* R |] ==>
finsum $R\ f\ A \otimes a$ = *finsum* R (%i. $f\ i \otimes a$) A
⟨*proof*⟩

lemma (**in** *cring*) *finsum-rdistr*:
[| *finite* A; $a \in carrier\ R$; $f \in A$ −> *carrier* R |] ==>
$a \otimes$ *finsum* $R\ f\ A$ = *finsum* R (%i. $a \otimes f\ i$) A
⟨*proof*⟩

9.3 Integral Domains

lemma (**in** *domain*) *zero-not-one* [*simp*]:
$\mathbf{0}$ ~= $\mathbf{1}$
⟨*proof*⟩

lemma (**in** *domain*) *integral-iff*:
[| $a \in carrier\ R$; $b \in carrier\ R$ |] ==> $(a \otimes b = \mathbf{0}) = (a = \mathbf{0} \mid b = \mathbf{0})$
⟨*proof*⟩

lemma (**in** *domain*) *m-lcancel*:
assumes *prem*: a ~= $\mathbf{0}$
and R: $a \in carrier\ R$ $b \in carrier\ R$ $c \in carrier\ R$
shows $(a \otimes b = a \otimes c) = (b = c)$

⟨*proof*⟩

lemma (**in** *domain*) *m-rcancel*:
  assumes *prem*: $a \sim= \mathbf{0}$
    and *R*: $a \in carrier\ R\ b \in carrier\ R\ c \in carrier\ R$
  shows *conc*: $(b \otimes a = c \otimes a) = (b = c)$
⟨*proof*⟩

9.4 Fields

Field would not need to be derived from domain, the properties for domain follow from the assumptions of field

lemma (**in** *cring*) *cring-fieldI*:
  assumes *field-Units*: $Units\ R = carrier\ R - \{\mathbf{0}\}$
  shows *field R*
⟨*proof*⟩

Another variant to show that something is a field

lemma (**in** *cring*) *cring-fieldI2*:
  assumes *notzero*: $\mathbf{0} \neq \mathbf{1}$
  and *invex*: $\bigwedge a.\ [\![a \in carrier\ R;\ a \neq \mathbf{0}]\!] \Longrightarrow \exists\, b \in carrier\ R.\ a \otimes b = \mathbf{1}$
  shows *field R*
  ⟨*proof*⟩

9.5 Morphisms

constdefs (**structure** *R S*)
  $ring\text{-}hom :: [('a, 'm)\ ring\text{-}scheme, ('b, 'n)\ ring\text{-}scheme] => ('a => 'b)\ set$
  $ring\text{-}hom\ R\ S == \{h.\ h \in carrier\ R -> carrier\ S\ \&$
    $(ALL\ x\ y.\ x \in carrier\ R\ \&\ y \in carrier\ R -->$
      $h\ (x \otimes y) = h\ x \otimes_S h\ y\ \&\ h\ (x \oplus y) = h\ x \oplus_S h\ y)\ \&$
    $h\ \mathbf{1} = \mathbf{1}_S\}$

lemma *ring-hom-memI*:
  fixes *R* (**structure**) **and** *S* (**structure**)
  assumes *hom-closed*: $!!x.\ x \in carrier\ R ==> h\ x \in carrier\ S$
    and *hom-mult*: $!!x\ y.\ [|\ x \in carrier\ R;\ y \in carrier\ R\ |] ==>$
      $h\ (x \otimes y) = h\ x \otimes_S h\ y$
    and *hom-add*: $!!x\ y.\ [|\ x \in carrier\ R;\ y \in carrier\ R\ |] ==>$
      $h\ (x \oplus y) = h\ x \oplus_S h\ y$
    and *hom-one*: $h\ \mathbf{1} = \mathbf{1}_S$
  shows $h \in ring\text{-}hom\ R\ S$
  ⟨*proof*⟩

lemma *ring-hom-closed*:
  $[|\ h \in ring\text{-}hom\ R\ S;\ x \in carrier\ R\ |] ==> h\ x \in carrier\ S$
  ⟨*proof*⟩

lemma *ring-hom-mult*:

fixes R (**structure**) **and** S (**structure**)
shows
$[| \ h \in \mathit{ring\text{-}hom}\ R\ S;\ x \in \mathit{carrier}\ R;\ y \in \mathit{carrier}\ R\ |] ==>$
$h\ (x \otimes y) = h\ x \otimes_S h\ y$
⟨*proof*⟩

lemma *ring-hom-add*:
fixes R (**structure**) **and** S (**structure**)
shows
$[| \ h \in \mathit{ring\text{-}hom}\ R\ S;\ x \in \mathit{carrier}\ R;\ y \in \mathit{carrier}\ R\ |] ==>$
$h\ (x \oplus y) = h\ x \oplus_S h\ y$
⟨*proof*⟩

lemma *ring-hom-one*:
fixes R (**structure**) **and** S (**structure**)
shows $h \in \mathit{ring\text{-}hom}\ R\ S ==> h\ \mathbf{1} = \mathbf{1}_S$
⟨*proof*⟩

locale *ring-hom-cring* = *cring* R + *cring* S +
fixes h
assumes *homh* [*simp*, *intro*]: $h \in \mathit{ring\text{-}hom}\ R\ S$
notes *hom-closed* [*simp*, *intro*] = *ring-hom-closed* [*OF homh*]
and *hom-mult* [*simp*] = *ring-hom-mult* [*OF homh*]
and *hom-add* [*simp*] = *ring-hom-add* [*OF homh*]
and *hom-one* [*simp*] = *ring-hom-one* [*OF homh*]

lemma (**in** *ring-hom-cring*) *hom-zero* [*simp*]:
$h\ \mathbf{0} = \mathbf{0}_S$
⟨*proof*⟩

lemma (**in** *ring-hom-cring*) *hom-a-inv* [*simp*]:
$x \in \mathit{carrier}\ R ==> h\ (\ominus\ x) = \ominus_S\ h\ x$
⟨*proof*⟩

lemma (**in** *ring-hom-cring*) *hom-finsum* [*simp*]:
$[| \ \mathit{finite}\ A;\ f \in A\ -> \mathit{carrier}\ R\ |] ==>$
$h\ (\mathit{finsum}\ R\ f\ A) = \mathit{finsum}\ S\ (h\ o\ f)\ A$
⟨*proof*⟩

lemma (**in** *ring-hom-cring*) *hom-finprod*:
$[| \ \mathit{finite}\ A;\ f \in A\ -> \mathit{carrier}\ R\ |] ==>$
$h\ (\mathit{finprod}\ R\ f\ A) = \mathit{finprod}\ S\ (h\ o\ f)\ A$
⟨*proof*⟩

declare *ring-hom-cring.hom-finprod* [*simp*]

lemma *id-ring-hom* [*simp*]:
$\mathit{id} \in \mathit{ring\text{-}hom}\ R\ R$
⟨*proof*⟩

end

theory *Module* **imports** *Ring* **begin**

10 Modules over an Abelian Group

10.1 Definitions

record ('*a*, '*b*) *module* = '*b ring* +
smult :: ['*a*, '*b*] => '*b* (**infixl** ⊙ı *70*)

locale *module* = *cring R* + *abelian-group M* +
assumes *smult-closed* [*simp*, *intro*]:
$[| a \in carrier\ R; x \in carrier\ M |] ==> a \odot_M x \in carrier\ M$
and *smult-l-distr*:
$[| a \in carrier\ R; b \in carrier\ R; x \in carrier\ M |] ==>$
$(a \oplus b) \odot_M x = a \odot_M x \oplus_M b \odot_M x$
and *smult-r-distr*:
$[| a \in carrier\ R; x \in carrier\ M; y \in carrier\ M |] ==>$
$a \odot_M (x \oplus_M y) = a \odot_M x \oplus_M a \odot_M y$
and *smult-assoc1*:
$[| a \in carrier\ R; b \in carrier\ R; x \in carrier\ M |] ==>$
$(a \otimes b) \odot_M x = a \odot_M (b \odot_M x)$
and *smult-one* [*simp*]:
$x \in carrier\ M ==> \mathbf{1} \odot_M x = x$

locale *algebra* = *module R M* + *cring M* +
assumes *smult-assoc2*:
$[| a \in carrier\ R; x \in carrier\ M; y \in carrier\ M |] ==>$
$(a \odot_M x) \otimes_M y = a \odot_M (x \otimes_M y)$

lemma *moduleI*:
fixes *R* (**structure**) **and** *M* (**structure**)
assumes *cring*: *cring R*
and *abelian-group*: *abelian-group M*
and *smult-closed*:
$!!a\ x.\ [| a \in carrier\ R; x \in carrier\ M |] ==> a \odot_M x \in carrier\ M$
and *smult-l-distr*:
$!!a\ b\ x.\ [| a \in carrier\ R; b \in carrier\ R; x \in carrier\ M |] ==>$
$(a \oplus b) \odot_M x = (a \odot_M x) \oplus_M (b \odot_M x)$
and *smult-r-distr*:
$!!a\ x\ y.\ [| a \in carrier\ R; x \in carrier\ M; y \in carrier\ M |] ==>$
$a \odot_M (x \oplus_M y) = (a \odot_M x) \oplus_M (a \odot_M y)$
and *smult-assoc1*:
$!!a\ b\ x.\ [| a \in carrier\ R; b \in carrier\ R; x \in carrier\ M |] ==>$
$(a \otimes b) \odot_M x = a \odot_M (b \odot_M x)$

and *smult-one*:
!!x. $x \in$ *carrier* M ==> $\mathbf{1} \odot_M x = x$
shows *module R M*
⟨*proof*⟩

lemma *algebraI*:
fixes R (**structure**) **and** M (**structure**)
assumes *R-cring*: *cring R*
and *M-cring*: *cring M*
and *smult-closed*:
!!a x. [| $a \in$ *carrier* R; $x \in$ *carrier* M |] ==> $a \odot_M x \in$ *carrier* M
and *smult-l-distr*:
!!a b x. [| $a \in$ *carrier* R; $b \in$ *carrier* R; $x \in$ *carrier* M |] ==>
$(a \oplus b) \odot_M x = (a \odot_M x) \oplus_M (b \odot_M x)$
and *smult-r-distr*:
!!a x y. [| $a \in$ *carrier* R; $x \in$ *carrier* M; $y \in$ *carrier* M |] ==>
$a \odot_M (x \oplus_M y) = (a \odot_M x) \oplus_M (a \odot_M y)$
and *smult-assoc1*:
!!a b x. [| $a \in$ *carrier* R; $b \in$ *carrier* R; $x \in$ *carrier* M |] ==>
$(a \otimes b) \odot_M x = a \odot_M (b \odot_M x)$
and *smult-one*:
!!x. $x \in$ *carrier* M ==> (*one* R) $\odot_M x = x$
and *smult-assoc2*:
!!a x y. [| $a \in$ *carrier* R; $x \in$ *carrier* M; $y \in$ *carrier* M |] ==>
$(a \odot_M x) \otimes_M y = a \odot_M (x \otimes_M y)$
shows *algebra R M*
⟨*proof*⟩

lemma (**in** *algebra*) *R-cring*:
cring R
⟨*proof*⟩

lemma (**in** *algebra*) *M-cring*:
cring M
⟨*proof*⟩

lemma (**in** *algebra*) *module*:
module R M
⟨*proof*⟩

10.2 Basic Properties of Algebras

lemma (**in** *algebra*) *smult-l-null* [*simp*]:
$x \in$ *carrier* M ==> $\mathbf{0} \odot_M x = \mathbf{0}_M$
⟨*proof*⟩

lemma (**in** *algebra*) *smult-r-null* [*simp*]:
$a \in$ *carrier* R ==> $a \odot_M \mathbf{0}_M = \mathbf{0}_M$
⟨*proof*⟩

lemma (**in** *algebra*) *smult-l-minus*:
  $[| \; a \in carrier \; R; \; x \in carrier \; M \; |] \; ==> (\ominus a) \odot_M x = \ominus_M (a \odot_M x)$
⟨*proof*⟩

lemma (**in** *algebra*) *smult-r-minus*:
  $[| \; a \in carrier \; R; \; x \in carrier \; M \; |] \; ==> a \odot_M (\ominus_M x) = \ominus_M (a \odot_M x)$
⟨*proof*⟩

end

theory *UnivPoly* **imports** *Module* **begin**

11 Univariate Polynomials

Polynomials are formalised as modules with additional operations for extracting coefficients from polynomials and for obtaining monomials from coefficients and exponents (record *up-ring*). The carrier set is a set of bounded functions from Nat to the coefficient domain. Bounded means that these functions return zero above a certain bound (the degree). There is a chapter on the formalisation of polynomials in the PhD thesis [1], which was implemented with axiomatic type classes. This was later ported to Locales.

11.1 The Constructor for Univariate Polynomials

Functions with finite support.

locale *bound* =
  fixes $z :: 'a$
    and $n :: nat$
    and $f :: nat => 'a$
  assumes *bound*: $!!m. \; n < m \Longrightarrow f \; m = z$

declare *bound.intro* [*intro*!]
  and *bound.bound* [*dest*]

lemma *bound-below*:
  assumes *bound*: *bound z m f* **and** *nonzero*: $f \; n \neq z$ **shows** $n \leq m$
⟨*proof*⟩

record ('a, 'p) *up-ring* = ('a, 'p) *module* +
  monom :: ['a, *nat*] => 'p
  coeff :: ['p, *nat*] => 'a

constdefs (**structure** *R*)
  up :: ('a, 'm) *ring-scheme* => (*nat* => 'a) *set*

$up\ R == \{f.\ f \in UNIV\ {-}{>}\ carrier\ R\ \&\ (EX\ n.\ bound\ \mathbf{0}\ n\ f)\}$
$UP :: ('a,\ 'm)\ ring\text{-}scheme\ {=}{>}\ ('a,\ nat\ {=}{>}\ 'a)\ up\text{-}ring$
$UP\ R == (|$
$carrier = up\ R,$
$mult = (\%p{:}up\ R.\ \%q{:}up\ R.\ \%n.\ \bigoplus i \in \{..n\}.\ p\ i \otimes q\ (n-i)),$
$one = (\%i.\ if\ i{=}0\ then\ \mathbf{1}\ else\ \mathbf{0}),$
$zero = (\%i.\ \mathbf{0}),$
$add = (\%p{:}up\ R.\ \%q{:}up\ R.\ \%i.\ p\ i \oplus q\ i),$
$smult = (\%a{:}carrier\ R.\ \%p{:}up\ R.\ \%i.\ a \otimes p\ i),$
$monom = (\%a{:}carrier\ R.\ \%n\ i.\ if\ i{=}n\ then\ a\ else\ \mathbf{0}),$
$coeff = (\%p{:}up\ R.\ \%n.\ p\ n)\ |)$

Properties of the set of polynomials *up*.

lemma *mem-upI* [*intro*]:
$[|\ !!n.\ f\ n \in carrier\ R;\ EX\ n.\ bound\ (zero\ R)\ n\ f\ |] ==> f \in up\ R$
⟨*proof*⟩

lemma *mem-upD* [*dest*]:
$f \in up\ R ==> f\ n \in carrier\ R$
⟨*proof*⟩

lemma (**in** *cring*) *bound-upD* [*dest*]:
$f \in up\ R ==> EX\ n.\ bound\ \mathbf{0}\ n\ f$
⟨*proof*⟩

lemma (**in** *cring*) *up-one-closed*:
$(\%n.\ if\ n = 0\ then\ \mathbf{1}\ else\ \mathbf{0}) \in up\ R$
⟨*proof*⟩

lemma (**in** *cring*) *up-smult-closed*:
$[|\ a \in carrier\ R;\ p \in up\ R\ |] ==> (\%i.\ a \otimes p\ i) \in up\ R$
⟨*proof*⟩

lemma (**in** *cring*) *up-add-closed*:
$[|\ p \in up\ R;\ q \in up\ R\ |] ==> (\%i.\ p\ i \oplus q\ i) \in up\ R$
⟨*proof*⟩

lemma (**in** *cring*) *up-a-inv-closed*:
$p \in up\ R ==> (\%i.\ \ominus\ (p\ i)) \in up\ R$
⟨*proof*⟩

lemma (**in** *cring*) *up-mult-closed*:
$[|\ p \in up\ R;\ q \in up\ R\ |] ==>$
$(\%n.\ \bigoplus i \in \{..n\}.\ p\ i \otimes q\ (n-i)) \in up\ R$
⟨*proof*⟩

11.2 Effect of Operations on Coefficients

locale *UP* =

fixes R (**structure**) **and** P (**structure**)
defines *P-def*: $P == UP\ R$

locale *UP-cring* = *UP* + *cring R*

locale *UP-domain* = *UP-cring* + *domain R*

Temporarily declare $P \equiv UP\ R$ as simp rule.

declare (**in** *UP*) *P-def* [*simp*]

lemma (**in** *UP-cring*) *coeff-monom* [*simp*]:
$a \in carrier\ R ==>$
$coeff\ P\ (monom\ P\ a\ m)\ n = (if\ m{=}n\ then\ a\ else\ \mathbf{0})$
⟨*proof*⟩

lemma (**in** *UP-cring*) *coeff-zero* [*simp*]:
$coeff\ P\ \mathbf{0}_P\ n = \mathbf{0}$
⟨*proof*⟩

lemma (**in** *UP-cring*) *coeff-one* [*simp*]:
$coeff\ P\ \mathbf{1}_P\ n = (if\ n{=}0\ then\ \mathbf{1}\ else\ \mathbf{0})$
⟨*proof*⟩

lemma (**in** *UP-cring*) *coeff-smult* [*simp*]:
$[|\ a \in carrier\ R;\ p \in carrier\ P\ |] ==>$
$coeff\ P\ (a \odot_P p)\ n = a \otimes coeff\ P\ p\ n$
⟨*proof*⟩

lemma (**in** *UP-cring*) *coeff-add* [*simp*]:
$[|\ p \in carrier\ P;\ q \in carrier\ P\ |] ==>$
$coeff\ P\ (p \oplus_P q)\ n = coeff\ P\ p\ n \oplus coeff\ P\ q\ n$
⟨*proof*⟩

lemma (**in** *UP-cring*) *coeff-mult* [*simp*]:
$[|\ p \in carrier\ P;\ q \in carrier\ P\ |] ==>$
$coeff\ P\ (p \otimes_P q)\ n = (\bigoplus i \in \{..n\}.\ coeff\ P\ p\ i \otimes coeff\ P\ q\ (n-i))$
⟨*proof*⟩

lemma (**in** *UP*) *up-eqI*:
assumes *prem*: $!!n.\ coeff\ P\ p\ n = coeff\ P\ q\ n$
and R: $p \in carrier\ P\ q \in carrier\ P$
shows $p = q$
⟨*proof*⟩

11.3 Polynomials Form a Commutative Ring.

Operations are closed over P.

lemma (**in** *UP-cring*) *UP-mult-closed* [*simp*]:
$[|\ p \in carrier\ P;\ q \in carrier\ P\ |] ==> p \otimes_P q \in carrier\ P$

⟨*proof*⟩

lemma (**in** *UP-cring*) *UP-one-closed* [*simp*]:
$\mathbf{1}_P \in$ *carrier P*
⟨*proof*⟩

lemma (**in** *UP-cring*) *UP-zero-closed* [*intro*, *simp*]:
$\mathbf{0}_P \in$ *carrier P*
⟨*proof*⟩

lemma (**in** *UP-cring*) *UP-a-closed* [*intro*, *simp*]:
[| $p \in$ *carrier P*; $q \in$ *carrier P* |] ==> $p \oplus_P q \in$ *carrier P*
⟨*proof*⟩

lemma (**in** *UP-cring*) *monom-closed* [*simp*]:
$a \in$ *carrier R* ==> *monom P a n* $\in$ *carrier P*
⟨*proof*⟩

lemma (**in** *UP-cring*) *UP-smult-closed* [*simp*]:
[| $a \in$ *carrier R*; $p \in$ *carrier P* |] ==> $a \odot_P p \in$ *carrier P*
⟨*proof*⟩

lemma (**in** *UP*) *coeff-closed* [*simp*]:
$p \in$ *carrier P* ==> *coeff P p n* $\in$ *carrier R*
⟨*proof*⟩

declare (**in** *UP*) *P-def* [*simp del*]

Algebraic ring properties

lemma (**in** *UP-cring*) *UP-a-assoc*:
assumes *R*: $p \in$ *carrier P* $q \in$ *carrier P* $r \in$ *carrier P*
shows $(p \oplus_P q) \oplus_P r = p \oplus_P (q \oplus_P r)$
⟨*proof*⟩

lemma (**in** *UP-cring*) *UP-l-zero* [*simp*]:
assumes *R*: $p \in$ *carrier P*
shows $\mathbf{0}_P \oplus_P p = p$
⟨*proof*⟩

lemma (**in** *UP-cring*) *UP-l-neg-ex*:
assumes *R*: $p \in$ *carrier P*
shows *EX q* : *carrier P*. $q \oplus_P p = \mathbf{0}_P$
⟨*proof*⟩

lemma (**in** *UP-cring*) *UP-a-comm*:
assumes *R*: $p \in$ *carrier P* $q \in$ *carrier P*
shows $p \oplus_P q = q \oplus_P p$
⟨*proof*⟩

lemma (**in** *UP-cring*) *UP-m-assoc*:
assumes *R*: $p \in carrier\ P\ q \in carrier\ P\ r \in carrier\ P$
shows $(p \otimes_P q) \otimes_P r = p \otimes_P (q \otimes_P r)$
⟨*proof*⟩

lemma (**in** *UP-cring*) *UP-l-one* [*simp*]:
assumes *R*: $p \in carrier\ P$
shows $\mathbf{1}_P \otimes_P p = p$
⟨*proof*⟩

lemma (**in** *UP-cring*) *UP-l-distr*:
assumes *R*: $p \in carrier\ P\ q \in carrier\ P\ r \in carrier\ P$
shows $(p \oplus_P q) \otimes_P r = (p \otimes_P r) \oplus_P (q \otimes_P r)$
⟨*proof*⟩

lemma (**in** *UP-cring*) *UP-m-comm*:
assumes *R*: $p \in carrier\ P\ q \in carrier\ P$
shows $p \otimes_P q = q \otimes_P p$
⟨*proof*⟩

theorem (**in** *UP-cring*) *UP-cring*:
cring P
⟨*proof*⟩

lemma (**in** *UP-cring*) *UP-ring*:

ring P
⟨*proof*⟩

lemma (**in** *UP-cring*) *UP-a-inv-closed* [*intro*, *simp*]:
$p \in carrier\ P$ ==> $\ominus_P\ p \in carrier\ P$
⟨*proof*⟩

lemma (**in** *UP-cring*) *coeff-a-inv* [*simp*]:
assumes *R*: $p \in carrier\ P$
shows $coeff\ P\ (\ominus_P\ p)\ n = \ominus\ (coeff\ P\ p\ n)$
⟨*proof*⟩

Interpretation of lemmas from *cring*. Saves lifting 43 lemmas manually.

interpretation *UP-cring* < *cring P*
⟨*proof*⟩

11.4 Polynomials Form an Algebra

lemma (**in** *UP-cring*) *UP-smult-l-distr*:
[| $a \in carrier\ R$; $b \in carrier\ R$; $p \in carrier\ P$ |] ==>
$(a \oplus b) \odot_P p = a \odot_P p \oplus_P b \odot_P p$
⟨*proof*⟩

lemma (**in** *UP-cring*) *UP-smult-r-distr*:
  [| $a \in$ *carrier* R; $p \in$ *carrier* P; $q \in$ *carrier* P |] ==>
  $a \odot_P (p \oplus_P q) = a \odot_P p \oplus_P a \odot_P q$
  ⟨*proof*⟩

lemma (**in** *UP-cring*) *UP-smult-assoc1*:
    [| $a \in$ *carrier* R; $b \in$ *carrier* R; $p \in$ *carrier* P |] ==>
    $(a \otimes b) \odot_P p = a \odot_P (b \odot_P p)$
  ⟨*proof*⟩

lemma (**in** *UP-cring*) *UP-smult-one* [*simp*]:
    $p \in$ *carrier* P ==> $\mathbf{1} \odot_P p = p$
  ⟨*proof*⟩

lemma (**in** *UP-cring*) *UP-smult-assoc2*:
  [| $a \in$ *carrier* R; $p \in$ *carrier* P; $q \in$ *carrier* P |] ==>
  $(a \odot_P p) \otimes_P q = a \odot_P (p \otimes_P q)$
  ⟨*proof*⟩

Interpretation of lemmas from *algebra*.

lemma (**in** *cring*) *cring*:
  cring R
  ⟨*proof*⟩

lemma (**in** *UP-cring*) *UP-algebra*:
  algebra R P
  ⟨*proof*⟩

interpretation *UP-cring* < *algebra R P*
  ⟨*proof*⟩

11.5 Further Lemmas Involving Monomials

lemma (**in** *UP-cring*) *monom-zero* [*simp*]:
  monom P $\mathbf{0}$ $n = \mathbf{0}_P$
  ⟨*proof*⟩

lemma (**in** *UP-cring*) *monom-mult-is-smult*:
  assumes R: $a \in$ *carrier* R $p \in$ *carrier* P
  shows *monom* P a $0 \otimes_P p = a \odot_P p$
⟨*proof*⟩

lemma (**in** *UP-cring*) *monom-add* [*simp*]:
  [| $a \in$ *carrier* R; $b \in$ *carrier* R |] ==>
  monom P $(a \oplus b)$ n = *monom* P a $n \oplus_P$ *monom* P b n
  ⟨*proof*⟩

lemma (**in** *UP-cring*) *monom-one-Suc*:
  monom P $\mathbf{1}$ (*Suc* n) = *monom* P $\mathbf{1}$ $n \otimes_P$ *monom* P $\mathbf{1}$ 1

⟨*proof*⟩

lemma (**in** *UP-cring*) *monom-mult-smult*:
[| $a \in carrier\ R$; $b \in carrier\ R$ |] ==> $monom\ P\ (a \otimes b)\ n = a \odot_P monom\ P\ b\ n$
⟨*proof*⟩

lemma (**in** *UP-cring*) *monom-one* [*simp*]:
$monom\ P\ \mathbf{1}\ 0 = \mathbf{1}_P$
⟨*proof*⟩

lemma (**in** *UP-cring*) *monom-one-mult*:
$monom\ P\ \mathbf{1}\ (n + m) = monom\ P\ \mathbf{1}\ n \otimes_P monom\ P\ \mathbf{1}\ m$
⟨*proof*⟩

lemma (**in** *UP-cring*) *monom-mult* [*simp*]:
assumes *R*: $a \in carrier\ R$ $b \in carrier\ R$
shows $monom\ P\ (a \otimes b)\ (n + m) = monom\ P\ a\ n \otimes_P monom\ P\ b\ m$
⟨*proof*⟩

lemma (**in** *UP-cring*) *monom-a-inv* [*simp*]:
$a \in carrier\ R$ ==> $monom\ P\ (\ominus\ a)\ n = \ominus_P\ monom\ P\ a\ n$
⟨*proof*⟩

lemma (**in** *UP-cring*) *monom-inj*:
inj-on (%*a*. *monom P a n*) (*carrier R*)
⟨*proof*⟩

11.6 The Degree Function

constdefs (**structure** *R*)
deg :: [('*a*, '*m*) *ring-scheme*, *nat* => '*a*] => *nat*
deg R p == *LEAST n*. *bound* **0** *n* (*coeff* (*UP R*) *p*)

lemma (**in** *UP-cring*) *deg-aboveI*:
[| (!!*m*. $n < m$ ==> *coeff P p m* = **0**); $p \in carrier\ P$ |] ==> *deg R p* <= *n*
⟨*proof*⟩

lemma (**in** *UP-cring*) *deg-aboveD*:
assumes *deg R p* < *m* **and** $p \in carrier\ P$
shows *coeff P p m* = **0**
⟨*proof*⟩

lemma (**in** *UP-cring*) *deg-belowI*:
assumes *non-zero*: *n* ~= *0* ==> *coeff P p n* ~= **0**
and *R*: $p \in carrier\ P$
shows *n* <= *deg R p*

— Logically, this is a slightly stronger version of *deg-aboveD*
⟨*proof*⟩

lemma (**in** *UP-cring*) *lcoeff-nonzero-deg*:
assumes *deg*: *deg R p* ~= *0* **and** *R*: $p \in$ *carrier P*
shows *coeff P p* (*deg R p*) ~= **0**
⟨*proof*⟩

lemma (**in** *UP-cring*) *lcoeff-nonzero-nonzero*:
assumes *deg*: *deg R p* = *0* **and** *nonzero*: *p* ~= $\mathbf{0}_P$ **and** *R*: $p \in$ *carrier P*
shows *coeff P p 0* ~= **0**
⟨*proof*⟩

lemma (**in** *UP-cring*) *lcoeff-nonzero*:
assumes *neq*: *p* ~= $\mathbf{0}_P$ **and** *R*: $p \in$ *carrier P*
shows *coeff P p* (*deg R p*) ~= **0**
⟨*proof*⟩

lemma (**in** *UP-cring*) *deg-eqI*:
[| !!*m*. *n* < *m* ==> *coeff P p m* = **0**;
!!*n*. *n* ~= *0* ==> *coeff P p n* ~= **0**; $p \in$ *carrier P* |] ==> *deg R p* = *n*
⟨*proof*⟩

Degree and polynomial operations

lemma (**in** *UP-cring*) *deg-add* [*simp*]:
assumes *R*: $p \in$ *carrier P* $q \in$ *carrier P*
shows *deg R* ($p \oplus_P q$) <= *max* (*deg R p*) (*deg R q*)
⟨*proof*⟩

lemma (**in** *UP-cring*) *deg-monom-le*:
$a \in$ *carrier R* ==> *deg R* (*monom P a n*) <= *n*
⟨*proof*⟩

lemma (**in** *UP-cring*) *deg-monom* [*simp*]:
[| *a* ~= **0**; $a \in$ *carrier R* |] ==> *deg R* (*monom P a n*) = *n*
⟨*proof*⟩

lemma (**in** *UP-cring*) *deg-const* [*simp*]:
assumes *R*: $a \in$ *carrier R* **shows** *deg R* (*monom P a 0*) = *0*
⟨*proof*⟩

lemma (**in** *UP-cring*) *deg-zero* [*simp*]:
deg R $\mathbf{0}_P$ = *0*
⟨*proof*⟩

lemma (**in** *UP-cring*) *deg-one* [*simp*]:
deg R $\mathbf{1}_P$ = *0*
⟨*proof*⟩

lemma (**in** *UP-cring*) *deg-uminus* [*simp*]:
assumes *R*: $p \in carrier\ P$ **shows** $deg\ R\ (\ominus_P\ p) = deg\ R\ p$
⟨*proof*⟩

lemma (**in** *UP-domain*) *deg-smult-ring*:
$[|\ a \in carrier\ R;\ p \in carrier\ P\ |] ==>$
$deg\ R\ (a \odot_P\ p) <= (if\ a = \mathbf{0}\ then\ 0\ else\ deg\ R\ p)$
⟨*proof*⟩

lemma (**in** *UP-domain*) *deg-smult* [*simp*]:
assumes *R*: $a \in carrier\ R\ p \in carrier\ P$
shows $deg\ R\ (a \odot_P\ p) = (if\ a = \mathbf{0}\ then\ 0\ else\ deg\ R\ p)$
⟨*proof*⟩

lemma (**in** *UP-cring*) *deg-mult-cring*:
assumes *R*: $p \in carrier\ P\ q \in carrier\ P$
shows $deg\ R\ (p \otimes_P\ q) <= deg\ R\ p + deg\ R\ q$
⟨*proof*⟩

lemma (**in** *UP-domain*) *deg-mult* [*simp*]:
$[|\ p\ \tilde{}= \mathbf{0}_P;\ q\ \tilde{}= \mathbf{0}_P;\ p \in carrier\ P;\ q \in carrier\ P\ |] ==>$
$deg\ R\ (p \otimes_P\ q) = deg\ R\ p + deg\ R\ q$
⟨*proof*⟩

lemma (**in** *UP-cring*) *coeff-finsum*:
assumes *fin*: *finite A*
shows $p \in A\ {-}{>}\ carrier\ P ==>$
$coeff\ P\ (finsum\ P\ p\ A)\ k = (\bigoplus i \in A.\ coeff\ P\ (p\ i)\ k)$
⟨*proof*⟩

lemma (**in** *UP-cring*) *up-repr*:
assumes *R*: $p \in carrier\ P$
shows $(\bigoplus_P\ i \in \{..deg\ R\ p\}.\ monom\ P\ (coeff\ P\ p\ i)\ i) = p$
⟨*proof*⟩

lemma (**in** *UP-cring*) *up-repr-le*:
$[|\ deg\ R\ p <= n;\ p \in carrier\ P\ |] ==>$
$(\bigoplus_P\ i \in \{..n\}.\ monom\ P\ (coeff\ P\ p\ i)\ i) = p$
⟨*proof*⟩

11.7 Polynomials over Integral Domains

lemma *domainI*:
assumes *cring*: *cring R*
and *one-not-zero*: $one\ R\ \tilde{}= zero\ R$
and *integral*: $!!a\ b.\ [|\ mult\ R\ a\ b = zero\ R;\ a \in carrier\ R;$
$b \in carrier\ R\ |] ==> a = zero\ R\ |\ b = zero\ R$
shows *domain R*
⟨*proof*⟩

lemma (**in** *UP-domain*) *UP-one-not-zero*:
$\mathbf{1}_P$ ~= $\mathbf{0}_P$
⟨*proof*⟩

lemma (**in** *UP-domain*) *UP-integral*:
[| $p \otimes_P q = \mathbf{0}_P$; $p \in$ *carrier* P; $q \in$ *carrier* P |] ==> $p = \mathbf{0}_P$ | $q = \mathbf{0}_P$
⟨*proof*⟩

theorem (**in** *UP-domain*) *UP-domain*:
domain P
⟨*proof*⟩

Interpretation of theorems from *domain*.

interpretation *UP-domain* < *domain P*
⟨*proof*⟩

11.8 The Evaluation Homomorphism and Universal Property

theorem (**in** *cring*) *diagonal-sum*:
[| $f \in$ {..n + m::*nat*} -> *carrier R*; $g \in$ {..n + m} -> *carrier R* |] ==>
($\bigoplus k \in$ {..n + m}. $\bigoplus i \in$ {..k}. $f\ i \otimes g\ (k - i)$) =
($\bigoplus k \in$ {..n + m}. $\bigoplus i \in$ {..$n + m - k$}. $f\ k \otimes g\ i$)
⟨*proof*⟩

lemma (**in** *abelian-monoid*) *boundD-carrier*:
[| *bound* $\mathbf{0}$ n f; $n < m$ |] ==> $f\ m \in$ *carrier G*
⟨*proof*⟩

theorem (**in** *cring*) *cauchy-product*:
assumes *bf*: *bound* $\mathbf{0}$ n f **and** *bg*: *bound* $\mathbf{0}$ m g
and *Rf*: $f \in$ {..n} -> *carrier R* **and** *Rg*: $g \in$ {..m} -> *carrier R*
shows ($\bigoplus k \in$ {..n + m}. $\bigoplus i \in$ {..k}. $f\ i \otimes g\ (k - i)$) =
($\bigoplus i \in$ {..n}. $f\ i$) $\otimes$ ($\bigoplus i \in$ {..m}. $g\ i$)
⟨*proof*⟩

lemma (**in** *UP-cring*) *const-ring-hom*:
(%*a*. *monom P a 0*) $\in$ *ring-hom R P*
⟨*proof*⟩

constdefs (**structure** *S*)
eval :: [('*a*, '*m*) *ring-scheme*, ('*b*, '*n*) *ring-scheme*,
'*a* => '*b*, '*b*, *nat* => '*a*] => '*b*
eval R S phi s == $\lambda p \in$ *carrier* (*UP R*).
$\bigoplus i \in$ {..*deg R p*}. *phi* (*coeff* (*UP R*) *p i*) $\otimes$ *s* (^) *i*

lemma (**in** *UP*) *eval-on-carrier*:

fixes S (**structure**)
shows $p \in carrier\ P$ ==>
$eval\ R\ S\ phi\ s\ p = (\bigoplus_S\ i \in \{..deg\ R\ p\}.\ phi\ (coeff\ P\ p\ i) \otimes_S s\ (\hat{\ })_S\ i)$
⟨*proof*⟩

lemma (**in** *UP*) *eval-extensional*:
eval R S phi p $\in$ *extensional* (*carrier P*)
⟨*proof*⟩

The universal property of the polynomial ring

locale *UP-pre-univ-prop* = *ring-hom-cring R S h* + *UP-cring R P*

locale *UP-univ-prop* = *UP-pre-univ-prop* +
fixes *s* **and** *Eval*
assumes *indet-img-carrier* [*simp*, *intro*]: *s* $\in$ *carrier S*
defines *Eval-def*: *Eval* == *eval R S h s*

theorem (**in** *UP-pre-univ-prop*) *eval-ring-hom*:
assumes *S*: *s* $\in$ *carrier S*
shows *eval R S h s* $\in$ *ring-hom P S*
⟨*proof*⟩

Interpretation of ring homomorphism lemmas.

interpretation *UP-univ-prop* < *ring-hom-cring P S Eval*
⟨*proof*⟩

Further properties of the evaluation homomorphism.

The following lemma could be proved in *UP-cring* with the additional assumption that h is closed.

lemma (**in** *UP-pre-univ-prop*) *eval-const*:
[| *s* $\in$ *carrier S*; *r* $\in$ *carrier R* |] ==> *eval R S h s* (*monom P r 0*) = *h r*
⟨*proof*⟩

The following proof is complicated by the fact that in arbitrary rings one might have $\mathbf{1}_R = \mathbf{0}_R$.

lemma (**in** *UP-pre-univ-prop*) *eval-monom1*:
assumes *S*: *s* $\in$ *carrier S*
shows *eval R S h s* (*monom P* **1** *1*) = *s*
⟨*proof*⟩

lemma (**in** *UP-cring*) *monom-pow*:
assumes *R*: *a* $\in$ *carrier R*
shows $(monom\ P\ a\ n)\ (\hat{\ })_P\ m = monom\ P\ (a\ (\hat{\ })\ m)\ (n * m)$
⟨*proof*⟩

lemma (**in** *ring-hom-cring*) *hom-pow* [*simp*]:
$x \in carrier\ R$ ==> $h\ (x\ (\hat{\ })\ n) = h\ x\ (\hat{\ })_S\ (n::nat)$

⟨*proof*⟩

lemma (**in** *UP-univ-prop*) *Eval-monom*:
$r \in carrier\ R ==> Eval\ (monom\ P\ r\ n) = h\ r \otimes_S s\ (\hat{\ })_S\ n$
⟨*proof*⟩

lemma (**in** *UP-pre-univ-prop*) *eval-monom*:
assumes *R*: $r \in carrier\ R$ **and** *S*: $s \in carrier\ S$
shows $eval\ R\ S\ h\ s\ (monom\ P\ r\ n) = h\ r \otimes_S s\ (\hat{\ })_S\ n$
⟨*proof*⟩

lemma (**in** *UP-univ-prop*) *Eval-smult*:
$[|\ r \in carrier\ R;\ p \in carrier\ P\ |] ==> Eval\ (r \odot_P p) = h\ r \otimes_S Eval\ p$
⟨*proof*⟩

lemma *ring-hom-cringI*:
assumes *cring R*
and *cring S*
and $h \in ring\text{-}hom\ R\ S$
shows *ring-hom-cring R S h*
⟨*proof*⟩

lemma (**in** *UP-pre-univ-prop*) *UP-hom-unique*:
includes *ring-hom-cring P S Phi*
assumes *Phi*: $Phi\ (monom\ P\ \mathbf{1}\ (Suc\ 0)) = s$
$!!r.\ r \in carrier\ R ==> Phi\ (monom\ P\ r\ 0) = h\ r$
includes *ring-hom-cring P S Psi*
assumes *Psi*: $Psi\ (monom\ P\ \mathbf{1}\ (Suc\ 0)) = s$
$!!r.\ r \in carrier\ R ==> Psi\ (monom\ P\ r\ 0) = h\ r$
and *P*: $p \in carrier\ P$ **and** *S*: $s \in carrier\ S$
shows $Phi\ p = Psi\ p$
⟨*proof*⟩

lemma (**in** *UP-pre-univ-prop*) *ring-homD*:
assumes *Phi*: $Phi \in ring\text{-}hom\ P\ S$
shows *ring-hom-cring P S Phi*
⟨*proof*⟩

theorem (**in** *UP-pre-univ-prop*) *UP-universal-property*:
assumes *S*: $s \in carrier\ S$
shows $EX!\ Phi.\ Phi \in ring\text{-}hom\ P\ S \cap extensional\ (carrier\ P)\ \&$
$Phi\ (monom\ P\ \mathbf{1}\ 1) = s\ \&$
$(ALL\ r : carrier\ R.\ Phi\ (monom\ P\ r\ 0) = h\ r)$
⟨*proof*⟩

11.9 Sample Application of Evaluation Homomorphism

lemma *UP-pre-univ-propI*:
assumes *cring R*

  and *cring S*
  and $h \in$ *ring-hom R S*
shows *UP-pre-univ-prop R S h*
⟨*proof*⟩

constdefs
INTEG :: *int ring*
INTEG == (| *carrier* = *UNIV*, *mult* = *op* ∗, *one* = *1*, *zero* = *0*, *add* = *op* + |)

lemma *INTEG-cring*:
cring INTEG
⟨*proof*⟩

lemma *INTEG-id-eval*:
UP-pre-univ-prop INTEG INTEG id
⟨*proof*⟩

Interpretation now enables to import all theorems and lemmas valid in the context of homomorphisms between *INTEG* and *UP INTEG* globally.

interpretation *INTEG*: *UP-pre-univ-prop* [*INTEG INTEG id*]
⟨*proof*⟩

lemma *INTEG-closed* [*intro*, *simp*]:
$z \in$ *carrier INTEG*
⟨*proof*⟩

lemma *INTEG-mult* [*simp*]:
mult INTEG z w = *z* ∗ *w*
⟨*proof*⟩

lemma *INTEG-pow* [*simp*]:
pow INTEG z n = *z* ˆ *n*
⟨*proof*⟩

lemma *eval INTEG INTEG id 10* (*monom* (*UP INTEG*) *5 2*) = *500*
⟨*proof*⟩

end

theory *AbelCoset*
imports *Coset Ring*
begin

12 More Lifting from Groups to Abelian Groups

12.1 Definitions

Hiding <+> from *Sum-Type* until I come up with better syntax here

hide *const Plus*

constdefs (**structure** *G*)
  a-r-coset :: [-, *'a set*, *'a*] ⇒ *'a set* (**infixl** +>ı *60*)
  a-r-coset G ≡ *r-coset* (|*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*|)

  a-l-coset :: [-, *'a*, *'a set*] ⇒ *'a set* (**infixl** <+ı *60*)
  a-l-coset G ≡ *l-coset* (|*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*|)

  A-RCOSETS :: [-, *'a set*] ⇒ (*'a set*)*set* (*a'-rcosets*ı - [*81*] *80*)
  A-RCOSETS G H ≡ *RCOSETS* (|*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*|) *H*

  set-add :: [-, *'a set* ,*'a set*] ⇒ *'a set* (**infixl** <+>ı *60*)
  set-add G ≡ *set-mult* (|*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*|)

  A-SET-INV :: [-,*'a set*] ⇒ *'a set* (*a'-set'-inv*ı - [*81*] *80*)
  A-SET-INV G H ≡ *SET-INV* (|*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*|) *H*

constdefs (**structure** *G*)
  a-r-congruent :: [(*'a*,*'b*)*ring-scheme*, *'a set*] ⇒ (*'a*∗*'a*)*set*
    (*racong*ı -)
  a-r-congruent G ≡ *r-congruent* (|*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*|)

constdefs
  A-FactGroup :: [(*'a*,*'b*) *ring-scheme*, *'a set*] ⇒ (*'a set*) *monoid*
    (**infixl** *A'-Mod 65*)
  — Actually defined for groups rather than monoids
  A-FactGroup G H ≡ *FactGroup* (|*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*|) *H*

constdefs
  a-kernel :: (*'a*, *'m*) *ring-scheme* ⇒ (*'b*, *'n*) *ring-scheme* ⇒
    (*'a* ⇒ *'b*) ⇒ *'a set*
  — the kernel of a homomorphism (additive)
  a-kernel G H h ≡ *kernel* (|*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*|)
    (|*carrier* = *carrier H*, *mult* = *add H*, *one* = *zero H*|) *h*

locale *abelian-group-hom* = *abelian-group G* + *abelian-group H* + *var h* +
  assumes *a-group-hom*: *group-hom* (| *carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G* |)
    (| *carrier* = *carrier H*, *mult* = *add H*, *one* = *zero H* |) *h*

lemmas *a-r-coset-defs* =
a-r-coset-def r-coset-def

lemma *a-r-coset-def'*:
includes *struct G*
shows $H +> a \equiv \bigcup h{\in}H.\ \{h \oplus a\}$
⟨*proof*⟩

lemmas *a-l-coset-defs* =
a-l-coset-def l-coset-def

lemma *a-l-coset-def'*:
includes *struct G*
shows $a <+ H \equiv \bigcup h{\in}H.\ \{a \oplus h\}$
⟨*proof*⟩

lemmas *A-RCOSETS-defs* =
A-RCOSETS-def RCOSETS-def

lemma *A-RCOSETS-def'*:
includes *struct G*
shows $a\text{-}rcosets\ H \equiv \bigcup a{\in}carrier\ G.\ \{H +> a\}$
⟨*proof*⟩

lemmas *set-add-defs* =
set-add-def set-mult-def

lemma *set-add-def'*:
includes *struct G*
shows $H <+> K \equiv \bigcup h{\in}H.\ \bigcup k{\in}K.\ \{h \oplus k\}$
⟨*proof*⟩

lemmas *A-SET-INV-defs* =
A-SET-INV-def SET-INV-def

lemma *A-SET-INV-def'*:
includes *struct G*
shows $a\text{-}set\text{-}inv\ H \equiv \bigcup h{\in}H.\ \{\ominus\ h\}$
⟨*proof*⟩

12.2 Cosets

lemma (**in** *abelian-group*) *a-coset-add-assoc*:
$[|\ M \subseteq carrier\ G;\ g \in carrier\ G;\ h \in carrier\ G\ |]$
$==> (M +> g) +> h = M +> (g \oplus h)$
⟨*proof*⟩

lemma (**in** *abelian-group*) *a-coset-add-zero* [*simp*]:

$M \subseteq carrier\ G ==> M +> \mathbf{0} = M$
⟨proof⟩

lemma (**in** *abelian-group*) *a-coset-add-inv1*:
$[| M +> (x \oplus (\ominus y)) = M;\ x \in carrier\ G\ ;\ y \in carrier\ G;$
$M \subseteq carrier\ G\ |] ==> M +> x = M +> y$
⟨proof⟩

lemma (**in** *abelian-group*) *a-coset-add-inv2*:
$[| M +> x = M +> y;\ x \in carrier\ G;\ y \in carrier\ G;\ M \subseteq carrier\ G\ |]$
$==> M +> (x \oplus (\ominus y)) = M$
⟨proof⟩

lemma (**in** *abelian-group*) *a-coset-join1*:
$[| H +> x = H;\ x \in carrier\ G;\ subgroup\ H\ (|carrier = carrier\ G,\ mult = add\ G,\ one = zero\ G|)\ |] ==> x \in H$
⟨proof⟩

lemma (**in** *abelian-group*) *a-solve-equation*:
$[\![subgroup\ H\ (|carrier = carrier\ G,\ mult = add\ G,\ one = zero\ G|);\ x \in H;\ y \in H]\!] \Longrightarrow \exists h{\in}H.\ y = h \oplus x$
⟨proof⟩

lemma (**in** *abelian-group*) *a-repr-independence*:
$[\![y \in H +> x;\ x \in carrier\ G;\ subgroup\ H\ (\!|carrier = carrier\ G,\ mult = add\ G,\ one = zero\ G|\!)\ ]\!] \Longrightarrow H +> x = H +> y$
⟨proof⟩

lemma (**in** *abelian-group*) *a-coset-join2*:
$[\![x \in carrier\ G;\ subgroup\ H\ (\!|carrier = carrier\ G,\ mult = add\ G,\ one = zero\ G|\!);\ x{\in}H]\!] \Longrightarrow H +> x = H$
⟨proof⟩

lemma (**in** *abelian-monoid*) *a-r-coset-subset-G*:
$[| H \subseteq carrier\ G;\ x \in carrier\ G\ |] ==> H +> x \subseteq carrier\ G$
⟨proof⟩

lemma (**in** *abelian-group*) *a-rcosI*:
$[| h \in H;\ H \subseteq carrier\ G;\ x \in carrier\ G|] ==> h \oplus x \in H +> x$
⟨proof⟩

lemma (**in** *abelian-group*) *a-rcosetsI*:
$[\![H \subseteq carrier\ G;\ x \in carrier\ G]\!] \Longrightarrow H +> x \in a\text{-}rcosets\ H$
⟨proof⟩

Really needed?

lemma (**in** *abelian-group*) *a-transpose-inv*:
$[| x \oplus y = z;\ x \in carrier\ G;\ y \in carrier\ G;\ z \in carrier\ G\ |]$
$==> (\ominus x) \oplus z = y$

$\langle proof \rangle$

12.3 Subgroups

locale *additive-subgroup* = *var H* + *struct G* +
assumes *a-subgroup*: *subgroup H* (|*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*|)

lemma (**in** *additive-subgroup*) *is-additive-subgroup*:
shows *additive-subgroup H G*
$\langle proof \rangle$

lemma *additive-subgroupI*:
includes *struct G*
assumes *a-subgroup*: *subgroup H* (|*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*|)
shows *additive-subgroup H G*
$\langle proof \rangle$

lemma (**in** *additive-subgroup*) *a-subset*:
$H \subseteq carrier\ G$
$\langle proof \rangle$

lemma (**in** *additive-subgroup*) *a-closed* [*intro*, *simp*]:
$[\![x \in H; y \in H]\!] \Longrightarrow x \oplus y \in H$
$\langle proof \rangle$

lemma (**in** *additive-subgroup*) *zero-closed* [*simp*]:
$\mathbf{0} \in H$
$\langle proof \rangle$

lemma (**in** *additive-subgroup*) *a-inv-closed* [*intro*,*simp*]:
$x \in H \Longrightarrow \ominus\ x \in H$
$\langle proof \rangle$

12.4 Normal additive subgroups

12.4.1 Definition of *abelian-subgroup*

Every subgroup of an *abelian-group* is normal

locale *abelian-subgroup* = *additive-subgroup H G* + *abelian-group G* +
assumes *a-normal*: *normal H* (|*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*|)

lemma (**in** *abelian-subgroup*) *is-abelian-subgroup*:
shows *abelian-subgroup H G*
$\langle proof \rangle$

lemma *abelian-subgroupI*:

assumes *a-normal*: *normal H* $(\\|$*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*$\|)$
and *a-comm*: !!*x y*. [| *x* ∈ *carrier G*; *y* ∈ *carrier G* |] ==> $x \oplus_G y = y \oplus_G x$
shows *abelian-subgroup H G*
⟨*proof*⟩

lemma *abelian-subgroupI2*:
includes *struct G*
assumes *a-comm-group*: *comm-group* $(\\|$*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*$\|)$
and *a-subgroup*: *subgroup H* $(\\|$*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*$\|)$
shows *abelian-subgroup H G*
⟨*proof*⟩

lemma *abelian-subgroupI3*:
includes *struct G*
assumes *asg*: *additive-subgroup H G*
and *ag*: *abelian-group G*
shows *abelian-subgroup H G*
⟨*proof*⟩

lemma (**in** *abelian-subgroup*) *a-coset-eq*:
(∀ *x* ∈ *carrier G*. *H* +> *x* = *x* <+ *H*)
⟨*proof*⟩

lemma (**in** *abelian-subgroup*) *a-inv-op-closed1*:
shows $[\![x \in carrier\ G;\ h \in H]\!] \Longrightarrow (\ominus\ x) \oplus h \oplus x \in H$
⟨*proof*⟩

lemma (**in** *abelian-subgroup*) *a-inv-op-closed2*:
shows $[\![x \in carrier\ G;\ h \in H]\!] \Longrightarrow x \oplus h \oplus (\ominus\ x) \in H$
⟨*proof*⟩

Alternative characterization of normal subgroups

lemma (**in** *abelian-group*) *a-normal-inv-iff*:
(*N* ◁ $(\\|$*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*$\|)$) =
(*subgroup N* $(\\|$*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*$\|)$ & (∀ *x* ∈ *carrier G*. ∀ *h* ∈ *N*. $x \oplus h \oplus (\ominus\ x) \in N$))
(**is** - = *?rhs*)
⟨*proof*⟩

lemma (**in** *abelian-group*) *a-lcos-m-assoc*:
[| *M* ⊆ *carrier G*; *g* ∈ *carrier G*; *h* ∈ *carrier G* |]
==> *g* <+ (*h* <+ *M*) = ($g \oplus h$) <+ *M*
⟨*proof*⟩

lemma (**in** *abelian-group*) *a-lcos-mult-one*:

$M \subseteq \mathit{carrier}\ G ==> \mathbf{0} <+ M = M$
$\langle \mathit{proof} \rangle$

lemma (**in** *abelian-group*) *a-l-coset-subset-G*:
$[|\ H \subseteq \mathit{carrier}\ G;\ x \in \mathit{carrier}\ G\ |] ==> x <+ H \subseteq \mathit{carrier}\ G$
$\langle \mathit{proof} \rangle$

lemma (**in** *abelian-group*) *a-l-coset-swap*:
$[\![y \in x <+ H;\ \ x \in \mathit{carrier}\ G;\ \ \mathit{subgroup}\ H\ (\!|\mathit{carrier} = \mathit{carrier}\ G,\ \mathit{mult} = \mathit{add}\ G,\ \mathit{one} = \mathit{zero}\ G|\!)]\!] \Longrightarrow x \in y <+ H$
$\langle \mathit{proof} \rangle$

lemma (**in** *abelian-group*) *a-l-coset-carrier*:
$[|\ y \in x <+ H;\ \ x \in \mathit{carrier}\ G;\ \ \mathit{subgroup}\ H\ (\!|\mathit{carrier} = \mathit{carrier}\ G,\ \mathit{mult} = \mathit{add}\ G,\ \mathit{one} = \mathit{zero}\ G|\!)\ |] ==> y \in \mathit{carrier}\ G$
$\langle \mathit{proof} \rangle$

lemma (**in** *abelian-group*) *a-l-repr-imp-subset*:
assumes y: $y \in x <+ H$ **and** x: $x \in \mathit{carrier}\ G$ **and** sb: $\mathit{subgroup}\ H\ (\!|\mathit{carrier} = \mathit{carrier}\ G,\ \mathit{mult} = \mathit{add}\ G,\ \mathit{one} = \mathit{zero}\ G|\!)$
shows $y <+ H \subseteq x <+ H$
$\langle \mathit{proof} \rangle$

lemma (**in** *abelian-group*) *a-l-repr-independence*:
assumes y: $y \in x <+ H$ **and** x: $x \in \mathit{carrier}\ G$ **and** sb: $\mathit{subgroup}\ H\ (\!|\mathit{carrier} = \mathit{carrier}\ G,\ \mathit{mult} = \mathit{add}\ G,\ \mathit{one} = \mathit{zero}\ G|\!)$
shows $x <+ H = y <+ H$
$\langle \mathit{proof} \rangle$

lemma (**in** *abelian-group*) *setadd-subset-G*:
$[\![H \subseteq \mathit{carrier}\ G;\ K \subseteq \mathit{carrier}\ G]\!] \Longrightarrow H <+> K \subseteq \mathit{carrier}\ G$
$\langle \mathit{proof} \rangle$

lemma (**in** *abelian-group*) *subgroup-add-id*: $\mathit{subgroup}\ H\ (\!|\mathit{carrier} = \mathit{carrier}\ G,\ \mathit{mult} = \mathit{add}\ G,\ \mathit{one} = \mathit{zero}\ G|\!) \Longrightarrow H <+> H = H$
$\langle \mathit{proof} \rangle$

lemma (**in** *abelian-subgroup*) *a-rcos-inv*:
assumes x: $\quad x \in \mathit{carrier}\ G$
shows $\mathit{a\text{-}set\text{-}inv}\ (H +> x) = H +> (\ominus\ x)$
$\langle \mathit{proof} \rangle$

lemma (**in** *abelian-group*) *a-setmult-rcos-assoc*:
$[\![H \subseteq \mathit{carrier}\ G;\ K \subseteq \mathit{carrier}\ G;\ x \in \mathit{carrier}\ G]\!]$
$\Longrightarrow H <+> (K +> x) = (H <+> K) +> x$
$\langle \mathit{proof} \rangle$

lemma (**in** *abelian-group*) *a-rcos-assoc-lcos*:
$\llbracket H \subseteq carrier\ G;\ K \subseteq carrier\ G;\ x \in carrier\ G \rrbracket$
$\Longrightarrow (H\ {+}{>}\ x)\ {<}{+}{>}\ K = H\ {<}{+}{>}\ (x\ {<}{+}\ K)$
⟨*proof*⟩

lemma (**in** *abelian-subgroup*) *a-rcos-sum*:
$\llbracket x \in carrier\ G;\ y \in carrier\ G \rrbracket$
$\Longrightarrow (H\ {+}{>}\ x)\ {<}{+}{>}\ (H\ {+}{>}\ y) = H\ {+}{>}\ (x \oplus y)$
⟨*proof*⟩

lemma (**in** *abelian-subgroup*) *rcosets-add-eq*:
$M \in a\text{-}rcosets\ H \Longrightarrow H\ {<}{+}{>}\ M = M$
— generalizes *subgroup-mult-id*
⟨*proof*⟩

12.5 Congruence Relation

lemma (**in** *abelian-subgroup*) *a-equiv-rcong*:
shows *equiv* (*carrier G*) (*racong H*)
⟨*proof*⟩

lemma (**in** *abelian-subgroup*) *a-l-coset-eq-rcong*:
assumes *a*: $a \in carrier\ G$
shows $a\ {<}{+}\ H = racong\ H\ ``\ \{a\}$
⟨*proof*⟩

lemma (**in** *abelian-subgroup*) *a-rcos-equation*:
shows
$\llbracket ha \oplus a = h \oplus b;\ a \in carrier\ G;\ \ b \in carrier\ G;$
$h \in H;\ \ ha \in H;\ \ hb \in H \rrbracket$
$\Longrightarrow hb \oplus a \in (\bigcup h{\in}H.\ \{h \oplus b\})$
⟨*proof*⟩

lemma (**in** *abelian-subgroup*) *a-rcos-disjoint*:
shows $\llbracket a \in a\text{-}rcosets\ H;\ b \in a\text{-}rcosets\ H;\ a{\neq}b \rrbracket \Longrightarrow a \cap b = \{\}$
⟨*proof*⟩

lemma (**in** *abelian-subgroup*) *a-rcos-self*:
shows $x \in carrier\ G \Longrightarrow x \in H\ {+}{>}\ x$
⟨*proof*⟩

lemma (**in** *abelian-subgroup*) *a-rcosets-part-G*:
shows $\bigcup(a\text{-}rcosets\ H) = carrier\ G$
⟨*proof*⟩

lemma (**in** *abelian-subgroup*) *a-cosets-finite*:
$\llbracket c \in a\text{-}rcosets\ H;\ \ H \subseteq carrier\ G;\ \ finite\ (carrier\ G) \rrbracket \Longrightarrow finite\ c$
⟨*proof*⟩

lemma (**in** *abelian-group*) *a-card-cosets-equal*:
$[\![c \in \textit{a-rcosets } H;\ \ H \subseteq \textit{carrier } G;\ \textit{finite}(\textit{carrier } G)]\!]$
$\Longrightarrow \textit{card } c = \textit{card } H$
⟨*proof*⟩

lemma (**in** *abelian-group*) *rcosets-subset-PowG*:
$\textit{additive-subgroup } H\ G \ \Longrightarrow \textit{a-rcosets } H \subseteq \textit{Pow}(\textit{carrier } G)$
⟨*proof*⟩

theorem (**in** *abelian-group*) *a-lagrange*:
$[\![\textit{finite}(\textit{carrier } G);\ \textit{additive-subgroup } H\ G]\!]$
$\Longrightarrow \textit{card}(\textit{a-rcosets } H) * \textit{card}(H) = \textit{order}(G)$
⟨*proof*⟩

12.6 Factorization

lemmas *A-FactGroup-defs* = *A-FactGroup-def FactGroup-def*

lemma *A-FactGroup-def ′*:
includes *struct G*
shows $G\ \textit{A-Mod}\ H \equiv (\!|\textit{carrier} = \textit{a-rcosets}_G\ H,\ \textit{mult} = \textit{set-add}\ G,\ \textit{one} = H|\!)$
⟨*proof*⟩

lemma (**in** *abelian-subgroup*) *a-setmult-closed*:
$[\![K1 \in \textit{a-rcosets } H;\ K2 \in \textit{a-rcosets } H]\!] \Longrightarrow K1 <\!+\!> K2 \in \textit{a-rcosets } H$
⟨*proof*⟩

lemma (**in** *abelian-subgroup*) *a-setinv-closed*:
$K \in \textit{a-rcosets } H \Longrightarrow \textit{a-set-inv } K \in \textit{a-rcosets } H$
⟨*proof*⟩

lemma (**in** *abelian-subgroup*) *a-rcosets-assoc*:
$[\![M1 \in \textit{a-rcosets } H;\ M2 \in \textit{a-rcosets } H;\ M3 \in \textit{a-rcosets } H]\!]$
$\Longrightarrow M1 <\!+\!> M2 <\!+\!> M3 = M1 <\!+\!> (M2 <\!+\!> M3)$
⟨*proof*⟩

lemma (**in** *abelian-subgroup*) *a-subgroup-in-rcosets*:
$H \in \textit{a-rcosets } H$
⟨*proof*⟩

lemma (**in** *abelian-subgroup*) *a-rcosets-inv-mult-group-eq*:
$M \in \textit{a-rcosets } H \Longrightarrow \textit{a-set-inv } M <\!+\!> M = H$
⟨*proof*⟩

theorem (**in** *abelian-subgroup*) *a-factorgroup-is-group*:
group (*G A-Mod H*)
⟨*proof*⟩

Since the Factorization is based on an *abelian* subgroup, is results in a commutative group

theorem (**in** *abelian-subgroup*) *a-factorgroup-is-comm-group*:
comm-group (*G A-Mod H*)
⟨*proof*⟩

lemma *add-A-FactGroup* [*simp*]: $X \otimes_{(G\ A\text{-}Mod\ H)} X' = X <\!+\!>_G X'$
⟨*proof*⟩

lemma (**in** *abelian-subgroup*) *a-inv-FactGroup*:
$X \in carrier\ (G\ A\text{-}Mod\ H) \Longrightarrow inv_{G\ A\text{-}Mod\ H}\ X = a\text{-}set\text{-}inv\ X$
⟨*proof*⟩

The coset map is a homomorphism from G to the quotient group $G\ Mod\ H$

lemma (**in** *abelian-subgroup*) *a-r-coset-hom-A-Mod*:
$(\lambda a.\ H +\!> a) \in hom\ (\!|carrier = carrier\ G,\ mult = add\ G,\ one = zero\ G|\!)\ (G\ A\text{-}Mod\ H)$
⟨*proof*⟩

The isomorphism theorems have been omitted from lifting, at least for now

12.7 The First Isomorphism Theorem

The quotient by the kernel of a homomorphism is isomorphic to the range of that homomorphism.

lemmas *a-kernel-defs* =
a-kernel-def kernel-def

lemma *a-kernel-def'*:
$a\text{-}kernel\ R\ S\ h \equiv \{x \in carrier\ R.\ h\ x = \mathbf{0}_S\}$
⟨*proof*⟩

12.8 Homomorphisms

lemma *abelian-group-homI*:
includes *abelian-group G*
includes *abelian-group H*
assumes *a-group-hom*: $group\text{-}hom\ (|\ carrier = carrier\ G,\ mult = add\ G,\ one = zero\ G\ |)$
$(|\ carrier = carrier\ H,\ mult = add\ H,\ one = zero\ H\ |)\ h$
shows *abelian-group-hom G H h*
⟨*proof*⟩

lemma (**in** *abelian-group-hom*) *is-abelian-group-hom*:
abelian-group-hom G H h
⟨*proof*⟩

lemma (**in** *abelian-group-hom*) *hom-add* [*simp*]:
$[|\ x : carrier\ G;\ y : carrier\ G\ |]$

$==> h\ (x \oplus_G y) = h\ x \oplus_H h\ y$
$\langle proof \rangle$

lemma (**in** *abelian-group-hom*) *hom-closed* [*simp*]:
$x \in carrier\ G \Longrightarrow h\ x \in carrier\ H$
$\langle proof \rangle$

lemma (**in** *abelian-group-hom*) *zero-closed* [*simp*]:
$h\ \mathbf{0} \in carrier\ H$
$\langle proof \rangle$

lemma (**in** *abelian-group-hom*) *hom-zero* [*simp*]:
$h\ \mathbf{0} = \mathbf{0}_H$
$\langle proof \rangle$

lemma (**in** *abelian-group-hom*) *a-inv-closed* [*simp*]:
$x \in carrier\ G ==> h\ (\ominus x) \in carrier\ H$
$\langle proof \rangle$

lemma (**in** *abelian-group-hom*) *hom-a-inv* [*simp*]:
$x \in carrier\ G ==> h\ (\ominus x) = \ominus_H\ (h\ x)$
$\langle proof \rangle$

lemma (**in** *abelian-group-hom*) *additive-subgroup-a-kernel*:
additive-subgroup (*a-kernel G H h*) *G*
$\langle proof \rangle$

The kernel of a homomorphism is an abelian subgroup

lemma (**in** *abelian-group-hom*) *abelian-subgroup-a-kernel*:
abelian-subgroup (*a-kernel G H h*) *G*
$\langle proof \rangle$

lemma (**in** *abelian-group-hom*) *A-FactGroup-nonempty*:
assumes *X*: $X \in carrier\ (G\ A\text{-}Mod\ a\text{-}kernel\ G\ H\ h)$
shows $X \neq \{\}$
$\langle proof \rangle$

lemma (**in** *abelian-group-hom*) *FactGroup-contents-mem*:
assumes *X*: $X \in carrier\ (G\ A\text{-}Mod\ (a\text{-}kernel\ G\ H\ h))$
shows $contents\ (h`X) \in carrier\ H$
$\langle proof \rangle$

lemma (**in** *abelian-group-hom*) *A-FactGroup-hom*:
$(\lambda X.\ contents\ (h`X)) \in hom\ (G\ A\text{-}Mod\ (a\text{-}kernel\ G\ H\ h))$
$(\!|carrier = carrier\ H,\ mult = add\ H,\ one = zero\ H|\!)$
$\langle proof \rangle$

lemma (**in** *abelian-group-hom*) *A-FactGroup-inj-on*:
$inj\text{-}on\ (\lambda X.\ contents\ (h\ `\ X))\ (carrier\ (G\ A\text{-}Mod\ a\text{-}kernel\ G\ H\ h))$

⟨*proof*⟩

If the homomorphism h is onto H, then so is the homomorphism from the quotient group

lemma (**in** *abelian-group-hom*) *A-FactGroup-onto*:
  assumes *h*: h ‘ *carrier* G = *carrier* H
  shows $(\lambda X.\ contents\ (h\ ‘\ X))$ ‘ *carrier* $(G$ *A-Mod a-kernel* $G\ H\ h)$ = *carrier* H
⟨*proof*⟩

If h is a homomorphism from G onto H, then the quotient group G *Mod kernel* $G\ H\ h$ is isomorphic to H.

theorem (**in** *abelian-group-hom*) *A-FactGroup-iso*:
  h ‘ *carrier* G = *carrier* H
   $\Longrightarrow (\lambda X.\ contents\ (h‘X)) \in (G\ A\text{-}Mod\ (a\text{-}kernel\ G\ H\ h)) \cong$
      (| *carrier* = *carrier* H, *mult* = *add* H, *one* = *zero* H |)
⟨*proof*⟩

13 Lemmas Lifted from CosetExt.thy

Not eveything from `CosetExt.thy` is lifted here.

13.1 General Lemmas from `AlgebraExt.thy`

lemma (**in** *additive-subgroup*) *a-Hcarr* [*simp*]:
  assumes *hH*: $h \in H$
  shows $h \in$ *carrier* G
⟨*proof*⟩

13.2 Lemmas for Right Cosets

lemma (**in** *abelian-subgroup*) *a-elemrcos-carrier*:
  assumes *acarr*: $a \in$ *carrier* G
    and a': $a' \in H$ +> a
  shows $a' \in$ *carrier* G
⟨*proof*⟩

lemma (**in** *abelian-subgroup*) *a-rcos-const*:
  assumes *hH*: $h \in H$
  shows H +> $h = H$
⟨*proof*⟩

lemma (**in** *abelian-subgroup*) *a-rcos-module-imp*:
  assumes *xcarr*: $x \in$ *carrier* G
    and *x'cos*: $x' \in H$ +> x
  shows $(x' \oplus \ominus x) \in H$
⟨*proof*⟩

lemma (**in** *abelian-subgroup*) *a-rcos-module-rev*:

assumes $x \in$ *carrier* G $x' \in$ *carrier* G
and $(x' \oplus \ominus x) \in H$
shows $x' \in H$ +> x
⟨*proof*⟩

lemma (**in** *abelian-subgroup*) *a-rcos-module*:
assumes $x \in$ *carrier* G $x' \in$ *carrier* G
shows $(x' \in H$ +> $x) = (x' \oplus \ominus x \in H)$
⟨*proof*⟩
lemma (**in** *abelian-subgroup*) *a-rcos-module-minus*:
includes *ring* G
assumes *carr*: $x \in$ *carrier* G $x' \in$ *carrier* G
shows $(x' \in H$ +> $x) = (x' \ominus x \in H)$
⟨*proof*⟩

lemma (**in** *abelian-subgroup*) *a-repr-independence'*:
assumes *y*: $y \in H$ +> x
and *xcarr*: $x \in$ *carrier* G
shows H +> $x = H$ +> y
⟨*proof*⟩

lemma (**in** *abelian-subgroup*) *a-repr-independenceD*:
assumes *ycarr*: $y \in$ *carrier* G
and *repr*: H +> $x = H$ +> y
shows $y \in H$ +> x
⟨*proof*⟩

13.3 Lemmas for the Set of Right Cosets

lemma (**in** *abelian-subgroup*) *a-rcosets-carrier*:
$X \in$ *a-rcosets* $H \Longrightarrow X \subseteq$ *carrier* G
⟨*proof*⟩

13.4 Addition of Subgroups

lemma (**in** *abelian-monoid*) *set-add-closed*:
assumes *Acarr*: $A \subseteq$ *carrier* G
and *Bcarr*: $B \subseteq$ *carrier* G
shows A <+> $B \subseteq$ *carrier* G
⟨*proof*⟩

lemma (**in** *abelian-group*) *add-additive-subgroups*:
assumes *subH*: *additive-subgroup* H G
and *subK*: *additive-subgroup* K G
shows *additive-subgroup* $(H$ <+> $K)$ G
⟨*proof*⟩

end

theory *Ideal*
imports *Ring AbelCoset*
begin

14 Ideals

14.1 General definition

locale *ideal* = *additive-subgroup I R* + *ring R* +
  assumes *I-l-closed*: $[\![a \in I; x \in carrier\ R]\!] \Longrightarrow x \otimes a \in I$
    and *I-r-closed*: $[\![a \in I; x \in carrier\ R]\!] \Longrightarrow a \otimes x \in I$

interpretation *ideal* ⊆ *abelian-subgroup I R*
⟨*proof*⟩

lemma (**in** *ideal*) *is-ideal*:
  ideal I R
⟨*proof*⟩

lemma *idealI*:
  includes *ring*
  assumes *a-subgroup*: *subgroup I* (|*carrier* = *carrier R*, *mult* = *add R*, *one* = *zero R*|)
    and *I-l-closed*: $\bigwedge a\ x.\ [\![a \in I; x \in carrier\ R]\!] \Longrightarrow x \otimes a \in I$
    and *I-r-closed*: $\bigwedge a\ x.\ [\![a \in I; x \in carrier\ R]\!] \Longrightarrow a \otimes x \in I$
  shows *ideal I R*
  ⟨*proof*⟩

14.2 Ideals Generated by a Subset of *carrier R*

constdefs (**structure** *R*)
  genideal :: (*'a*, *'b*) *ring-scheme* ⇒ *'a set* ⇒ *'a set* (*Idl*ı - [*80*] *79*)
  genideal R S ≡ *Inter* {*I*. *ideal I R* ∧ *S* ⊆ *I*}

14.3 Principal Ideals

locale *principalideal* = *ideal* +
  assumes *generate*: ∃ *i* ∈ *carrier R*. *I* = *Idl* {*i*}

lemma (**in** *principalideal*) *is-principalideal*:
  shows *principalideal I R*
⟨*proof*⟩

lemma *principalidealI*:
  includes *ideal*
  assumes *generate*: ∃ *i* ∈ *carrier R*. *I* = *Idl* {*i*}
  shows *principalideal I R*
  ⟨*proof*⟩

14.4 Maximal Ideals

locale *maximalideal* = *ideal* +
assumes *I-notcarr*: *carrier* $R \neq I$
and *I-maximal*: $[\![$*ideal* J R; $I \subseteq J$; $J \subseteq$ *carrier* $R]\!] \Longrightarrow J = I \vee J =$ *carrier* R

lemma (**in** *maximalideal*) *is-maximalideal*:
shows *maximalideal* I R
⟨*proof*⟩

lemma *maximalidealI*:
includes *ideal*
assumes *I-notcarr*: *carrier* $R \neq I$
and *I-maximal*: $\bigwedge J$. $[\![$*ideal* J R; $I \subseteq J$; $J \subseteq$ *carrier* $R]\!] \Longrightarrow J = I \vee J =$ *carrier* R
shows *maximalideal* I R
⟨*proof*⟩

14.5 Prime Ideals

locale *primeideal* = *ideal* + *cring* +
assumes *I-notcarr*: *carrier* $R \neq I$
and *I-prime*: $[\![a \in$ *carrier* R; $b \in$ *carrier* R; $a \otimes b \in I]\!] \Longrightarrow a \in I \vee b \in I$

lemma (**in** *primeideal*) *is-primeideal*:
shows *primeideal* I R
⟨*proof*⟩

lemma *primeidealI*:
includes *ideal*
includes *cring*
assumes *I-notcarr*: *carrier* $R \neq I$
and *I-prime*: $\bigwedge a\ b$. $[\![a \in$ *carrier* R; $b \in$ *carrier* R; $a \otimes b \in I]\!] \Longrightarrow a \in I \vee b \in I$
shows *primeideal* I R
⟨*proof*⟩

lemma *primeidealI2*:
includes *additive-subgroup* I R
includes *cring*
assumes *I-l-closed*: $\bigwedge a\ x$. $[\![a \in I$; $x \in$ *carrier* $R]\!] \Longrightarrow x \otimes a \in I$
and *I-r-closed*: $\bigwedge a\ x$. $[\![a \in I$; $x \in$ *carrier* $R]\!] \Longrightarrow a \otimes x \in I$
and *I-notcarr*: *carrier* $R \neq I$
and *I-prime*: $\bigwedge a\ b$. $[\![a \in$ *carrier* R; $b \in$ *carrier* R; $a \otimes b \in I]\!] \Longrightarrow a \in I \vee b \in I$
shows *primeideal* I R
⟨*proof*⟩

15 Properties of Ideals

15.1 Special Ideals

lemma (**in** *ring*) *zeroideal*:
  shows *ideal* $\{\mathbf{0}\}$ *R*
⟨*proof*⟩

lemma (**in** *ring*) *oneideal*:
  shows *ideal* (*carrier R*) *R*
⟨*proof*⟩

lemma (**in** *domain*) *zeroprimeideal*:
 shows *primeideal* $\{\mathbf{0}\}$ *R*
⟨*proof*⟩

15.2 General Ideal Properies

lemma (**in** *ideal*) *one-imp-carrier*:
  assumes *I-one-closed*: $\mathbf{1} \in I$
  shows $I =$ *carrier R*
⟨*proof*⟩

lemma (**in** *ideal*) *Icarr*:
  assumes *iI*: $i \in I$
  shows $i \in$ *carrier R*
⟨*proof*⟩

15.3 Intersection of Ideals

Intersection of two ideals The intersection of any two ideals is again an ideal in R

lemma (**in** *ring*) *i-intersect*:
  includes *ideal I R*
  includes *ideal J R*
  shows *ideal* $(I \cap J)$ *R*
⟨*proof*⟩

15.3.1 Intersection of a Set of Ideals

The intersection of any Number of Ideals is again an Ideal in R

lemma (**in** *ring*) *i-Intersect*:
  assumes *Sideals*: $\bigwedge I.\ I \in S \Longrightarrow$ *ideal I R*
    and *notempty*: $S \neq \{\}$
  shows *ideal* (*Inter S*) *R*
⟨*proof*⟩

15.4 Addition of Ideals

lemma (**in** *ring*) *add-ideals*:
assumes *idealI*: *ideal I R*
and *idealJ*: *ideal J R*
shows *ideal* (*I* <+> *J*) *R*
⟨*proof*⟩

15.5 Ideals generated by a subset of *carrier R*

15.5.1 Generation of Ideals in General Rings

genideal generates an ideal

lemma (**in** *ring*) *genideal-ideal*:
assumes *Scarr*: $S \subseteq$ *carrier R*
shows *ideal* (*Idl S*) *R*
⟨*proof*⟩

lemma (**in** *ring*) *genideal-self*:
assumes $S \subseteq$ *carrier R*
shows $S \subseteq$ *Idl S*
⟨*proof*⟩

lemma (**in** *ring*) *genideal-self'*:
assumes *carr*: $i \in$ *carrier R*
shows $i \in$ *Idl* $\{i\}$
⟨*proof*⟩

genideal generates the minimal ideal

lemma (**in** *ring*) *genideal-minimal*:
assumes *a*: *ideal I R*
and *b*: $S \subseteq I$
shows *Idl* $S \subseteq I$
⟨*proof*⟩

Generated ideals and subsets

lemma (**in** *ring*) *Idl-subset-ideal*:
assumes *Iideal*: *ideal I R*
and *Hcarr*: $H \subseteq$ *carrier R*
shows (*Idl* $H \subseteq I$) = ($H \subseteq I$)
⟨*proof*⟩

lemma (**in** *ring*) *subset-Idl-subset*:
assumes *Icarr*: $I \subseteq$ *carrier R*
and *HI*: $H \subseteq I$
shows *Idl* $H \subseteq$ *Idl* I
⟨*proof*⟩

lemma (**in** *ring*) *Idl-subset-ideal'*:

assumes *acarr*: $a \in carrier\ R$ **and** *bcarr*: $b \in carrier\ R$
shows $(Idl\ \{a\} \subseteq Idl\ \{b\}) = (a \in Idl\ \{b\})$
⟨*proof*⟩

lemma (**in** *ring*) *genideal-zero*:
$Idl\ \{\mathbf{0}\} = \{\mathbf{0}\}$
⟨*proof*⟩

lemma (**in** *ring*) *genideal-one*:
$Idl\ \{\mathbf{1}\} = carrier\ R$
⟨*proof*⟩

15.5.2 Generation of Principal Ideals in Commutative Rings

constdefs (**structure** *R*)
cgenideal :: $('a, 'b)$ *monoid-scheme* $\Rightarrow$ $'a \Rightarrow 'a$ *set* (*PIdl*ı - [*80*] *79*)
$cgenideal\ R\ a \equiv \{\ x \otimes a \mid x.\ x \in carrier\ R\ \}$

genhideal (?) really generates an ideal

lemma (**in** *cring*) *cgenideal-ideal*:
assumes *acarr*: $a \in carrier\ R$
shows *ideal* (*PIdl* *a*) *R*
⟨*proof*⟩

lemma (**in** *ring*) *cgenideal-self*:
assumes *icarr*: $i \in carrier\ R$
shows $i \in PIdl\ i$
⟨*proof*⟩

cgenideal is minimal

lemma (**in** *ring*) *cgenideal-minimal*:
includes *ideal J R*
assumes *aJ*: $a \in J$
shows $PIdl\ a \subseteq J$
⟨*proof*⟩

lemma (**in** *cring*) *cgenideal-eq-genideal*:
assumes *icarr*: $i \in carrier\ R$
shows $PIdl\ i = Idl\ \{i\}$
⟨*proof*⟩

lemma (**in** *cring*) *cgenideal-eq-rcos*:
$PIdl\ i = carrier\ R\ \#\!>\ i$
⟨*proof*⟩

lemma (**in** *cring*) *cgenideal-is-principalideal*:
assumes *icarr*: $i \in carrier\ R$
shows *principalideal* (*PIdl* *i*) *R*
⟨*proof*⟩

15.6 Union of Ideals

lemma (**in** *ring*) *union-genideal*:
 assumes *idealI*: *ideal I R*
  and *idealJ*: *ideal J R*
 shows $Idl\ (I \cup J) = I <\!+\!> J$
⟨*proof*⟩

15.7 Properties of Principal Ideals

0 generates the zero ideal

lemma (**in** *ring*) *zero-genideal*:
 shows $Idl\ \{\mathbf{0}\} = \{\mathbf{0}\}$
⟨*proof*⟩

1 generates the unit ideal

lemma (**in** *ring*) *one-genideal*:
 shows $Idl\ \{\mathbf{1}\} = carrier\ R$
⟨*proof*⟩

The zero ideal is a principal ideal

corollary (**in** *ring*) *zeropideal*:
 shows $principalideal\ \{\mathbf{0}\}\ R$
⟨*proof*⟩

The unit ideal is a principal ideal

corollary (**in** *ring*) *onepideal*:
 shows $principalideal\ (carrier\ R)\ R$
⟨*proof*⟩

Every principal ideal is a right coset of the carrier

lemma (**in** *principalideal*) *rcos-generate*:
 includes *cring*
 shows $\exists x{\in}I.\ I = carrier\ R\ \#\!>\ x$
⟨*proof*⟩

15.8 Prime Ideals

lemma (**in** *ideal*) *primeidealCD*:
 includes *cring*
 assumes *notprime*: $\neg\ primeideal\ I\ R$
 shows $carrier\ R = I \lor (\exists\, a\ b.\ a \in carrier\ R \land b \in carrier\ R \land a \otimes b \in I \land a \notin I \land b \notin I)$
⟨*proof*⟩

lemma (**in** *ideal*) *primeidealCE*:
 includes *cring*
 assumes *notprime*: $\neg\ primeideal\ I\ R$

obtains *carrier* $R = I$
  | $\exists\, a\ b.\ a \in carrier\ R \wedge b \in carrier\ R \wedge a \otimes b \in I \wedge a \notin I \wedge b \notin I$
⟨*proof*⟩

If $\{\mathbf{0}\}$ is a prime ideal of a commutative ring, the ring is a domain

lemma (**in** *cring*) *zeroprimeideal-domainI*:
  assumes *pi*: *primeideal* $\{\mathbf{0}\}$ *R*
  shows *domain R*
⟨*proof*⟩

corollary (**in** *cring*) *domain-eq-zeroprimeideal*:
  shows *domain* $R =$ *primeideal* $\{\mathbf{0}\}$ *R*
⟨*proof*⟩

15.9 Maximal Ideals

lemma (**in** *ideal*) *helper-I-closed*:
  assumes *carr*: $a \in carrier\ R\ x \in carrier\ R\ y \in carrier\ R$
    and *axI*: $a \otimes x \in I$
  shows $a \otimes (x \otimes y) \in I$
⟨*proof*⟩

lemma (**in** *ideal*) *helper-max-prime*:
  includes *cring*
  assumes *acarr*: $a \in carrier\ R$
  shows *ideal* $\{x{\in}carrier\ R.\ a \otimes x \in I\}$ *R*
⟨*proof*⟩

In a cring every maximal ideal is prime

lemma (**in** *cring*) *maximalideal-is-prime*:
  includes *maximalideal*
  shows *primeideal I R*
⟨*proof*⟩

15.10 Derived Theorems Involving Ideals

— A non-zero cring that has only the two trivial ideals is a field
lemma (**in** *cring*) *trivialideals-fieldI*:
  assumes *carrnzero*: $carrier\ R \neq \{\mathbf{0}\}$
    and *haveideals*: $\{I.\ ideal\ I\ R\} = \{\{\mathbf{0}\},\ carrier\ R\}$
  shows *field R*
⟨*proof*⟩

lemma (**in** *field*) *all-ideals*:
  shows $\{I.\ ideal\ I\ R\} = \{\{\mathbf{0}\},\ carrier\ R\}$
⟨*proof*⟩
lemma (**in** *cring*) *trivialideals-eq-field*:
  assumes *carrnzero*: $carrier\ R \neq \{\mathbf{0}\}$
  shows $(\{I.\ ideal\ I\ R\} = \{\{\mathbf{0}\},\ carrier\ R\}) = field\ R$

⟨*proof*⟩

Like zeroprimeideal for domains

lemma (**in** *field*) *zeromaximalideal*:
  maximalideal {**0**} *R*
⟨*proof*⟩

lemma (**in** *cring*) *zeromaximalideal-fieldI*:
  assumes *zeromax*: *maximalideal* {**0**} *R*
  shows *field R*
⟨*proof*⟩

lemma (**in** *cring*) *zeromaximalideal-eq-field*:
  maximalideal {**0**} *R* = *field R*
⟨*proof*⟩

end

theory *RingHom*
imports *Ideal*
begin

16 Homomorphisms of Non-Commutative Rings

Lifting existing lemmas in a *ring-hom-ring* locale

locale *ring-hom-ring* = *ring R* + *ring S* + *var h* +
  assumes *homh*: *h* ∈ *ring-hom R S*
  notes *hom-mult* [*simp*] = *ring-hom-mult* [*OF homh*]
    and *hom-one* [*simp*] = *ring-hom-one* [*OF homh*]

interpretation *ring-hom-cring* ⊆ *ring-hom-ring*
  ⟨*proof*⟩

interpretation *ring-hom-ring* ⊆ *abelian-group-hom R S*
⟨*proof*⟩

lemma (**in** *ring-hom-ring*) *is-ring-hom-ring*:
  includes *struct R* + *struct S*
  shows *ring-hom-ring R S h*
⟨*proof*⟩

lemma *ring-hom-ringI*:
  includes *ring R* + *ring S*
  assumes
    hom-closed: !!*x*. *x* ∈ *carrier R* ==> *h x* ∈ *carrier S*

and *compatible-mult*: !!x y. [| x : *carrier* R; y : *carrier* R |] ==> h ($x \otimes y$) = h $x \otimes_S h$ y
and *compatible-add*: !!x y. [| x : *carrier* R; y : *carrier* R |] ==> h ($x \oplus y$) = h $x \oplus_S h$ y
and *compatible-one*: h $\mathbf{1} = \mathbf{1}_S$
shows *ring-hom-ring* R S h
⟨*proof*⟩

lemma *ring-hom-ringI2*:
includes *ring* R + *ring* S
assumes h: $h \in$ *ring-hom* R S
shows *ring-hom-ring* R S h
⟨*proof*⟩

lemma *ring-hom-ringI3*:
includes *abelian-group-hom* R S + *ring* R + *ring* S
assumes *compatible-mult*: !!x y. [| x : *carrier* R; y : *carrier* R |] ==> h ($x \otimes y$) = h $x \otimes_S h$ y
and *compatible-one*: h $\mathbf{1} = \mathbf{1}_S$
shows *ring-hom-ring* R S h
⟨*proof*⟩

lemma *ring-hom-cringI*:
includes *ring-hom-ring* R S h + *cring* R + *cring* S
shows *ring-hom-cring* R S h
⟨*proof*⟩

16.1 The kernel of a ring homomorphism

— the kernel of a ring homomorphism is an ideal
lemma (**in** *ring-hom-ring*) *kernel-is-ideal*:
shows *ideal* (*a-kernel* R S h) R
⟨*proof*⟩

Elements of the kernel are mapped to zero

lemma (**in** *abelian-group-hom*) *kernel-zero* [*simp*]:
$i \in$ *a-kernel* R S $h \Longrightarrow h\ i = \mathbf{0}_S$
⟨*proof*⟩

16.2 Cosets

Cosets of the kernel correspond to the elements of the image of the homomorphism

lemma (**in** *ring-hom-ring*) *rcos-imp-homeq*:
assumes *acarr*: $a \in$ *carrier* R
and *xrcos*: $x \in$ *a-kernel* R S h +> a
shows $h\ x = h\ a$
⟨*proof*⟩

lemma (**in** *ring-hom-ring*) *homeq-imp-rcos*:
  assumes *acarr*: $a \in carrier\ R$
    and *xcarr*: $x \in carrier\ R$
    and *hx*: $h\ x = h\ a$
  shows $x \in a\text{-}kernel\ R\ S\ h\ {+>}\ a$
$\langle proof \rangle$

corollary (**in** *ring-hom-ring*) *rcos-eq-homeq*:
  assumes *acarr*: $a \in carrier\ R$
  shows $(a\text{-}kernel\ R\ S\ h)\ {+>}\ a = \{x \in carrier\ R.\ h\ x = h\ a\}$
$\langle proof \rangle$

end

17 QuotRing: Quotient Rings

theory *QuotRing*
imports *RingHom*
begin

17.1 Multiplication on Cosets

constdefs (**structure** *R*)
  $rcoset\text{-}mult :: [('a, \text{-})\ ring\text{-}scheme, 'a\ set, 'a\ set, 'a\ set] \Rightarrow 'a\ set$
    $([mod\ \text{-}:]\ \text{-}\ \bigotimes\iota\ \text{-}\ [81,81,81]\ 80)$
  $rcoset\text{-}mult\ R\ I\ A\ B \equiv \bigcup a{\in}A.\ \bigcup b{\in}B.\ I\ {+>}\ (a \otimes b)$

rcoset-mult fulfils the properties required by congruences

lemma (**in** *ideal*) *rcoset-mult-add*:
  $[\![x \in carrier\ R;\ y \in carrier\ R]\!] \Longrightarrow [mod\ I:]\ (I\ {+>}\ x) \bigotimes (I\ {+>}\ y) = I\ {+>}\ (x \otimes y)$
$\langle proof \rangle$

17.2 Quotient Ring Definition

constdefs (**structure** *R*)
  $FactRing :: [('a,'b)\ ring\text{-}scheme, 'a\ set] \Rightarrow ('a\ set)\ ring$
    (**infixl** *Quot 65*)
  $FactRing\ R\ I \equiv$
    $(\!|carrier = a\text{-}rcosets\ I,\ mult = rcoset\text{-}mult\ R\ I,\ one = (I\ {+>}\ \mathbf{1}),\ zero = I,\ add = set\text{-}add\ R|\!)$

17.3 Factorization over General Ideals

The quotient is a ring

lemma (**in** *ideal*) *quotient-is-ring*:
  shows $ring\ (R\ Quot\ I)$

⟨*proof*⟩

This is a ring homomorphism

lemma (**in** *ideal*) *rcos-ring-hom*:
 $(op +> I) \in ring\text{-}hom\ R\ (R\ Quot\ I)$
⟨*proof*⟩

lemma (**in** *ideal*) *rcos-ring-hom-ring*:
 $ring\text{-}hom\text{-}ring\ R\ (R\ Quot\ I)\ (op +> I)$
⟨*proof*⟩

The quotient of a cring is also commutative

lemma (**in** *ideal*) *quotient-is-cring*:
 includes *cring*
 shows *cring* $(R\ Quot\ I)$
⟨*proof*⟩

Cosets as a ring homomorphism on crings

lemma (**in** *ideal*) *rcos-ring-hom-cring*:
 includes *cring*
 shows $ring\text{-}hom\text{-}cring\ R\ (R\ Quot\ I)\ (op +> I)$
⟨*proof*⟩

17.4 Factorization over Prime Ideals

The quotient ring generated by a prime ideal is a domain

lemma (**in** *primeideal*) *quotient-is-domain*:
 shows *domain* $(R\ Quot\ I)$
⟨*proof*⟩

Generating right cosets of a prime ideal is a homomorphism on commutative rings

lemma (**in** *primeideal*) *rcos-ring-hom-cring*:
 shows $ring\text{-}hom\text{-}cring\ R\ (R\ Quot\ I)\ (op +> I)$
⟨*proof*⟩

17.5 Factorization over Maximal Ideals

In a commutative ring, the quotient ring over a maximal ideal is a field. The proof follows "W. Adkins, S. Weintraub: Algebra – An Approach via Module Theory"

lemma (**in** *maximalideal*) *quotient-is-field*:
 includes *cring*
 shows *field* $(R\ Quot\ I)$
⟨*proof*⟩

end

theory *IntRing*
imports *QuotRing IntDef*
begin

18 The Ring of Integers

18.1 Some properties of *int*

lemma *dvds-imp-abseq*:
$[\![l \ dvd \ k;\ k \ dvd \ l]\!] \Longrightarrow abs\ l = abs\ (k{::}int)$
$\langle proof \rangle$

lemma *abseq-imp-dvd*:
assumes *a-lk*: $abs\ l = abs\ (k{::}int)$
shows $l\ dvd\ k$
$\langle proof \rangle$

lemma *dvds-eq-abseq*:
$(l\ dvd\ k \wedge k\ dvd\ l) = (abs\ l = abs\ (k{::}int))$
$\langle proof \rangle$

18.2 The Set of Integers as Algebraic Structure

18.2.1 Definition of $\mathcal{Z}$

constdefs
$int\text{-}ring :: int\ ring\ (\mathcal{Z})$
$int\text{-}ring \equiv (\!|carrier = UNIV,\ mult = op\ *,\ one = 1,\ zero = 0,\ add = op\ +|\!)$

lemma *int-Zcarr* [*intro*!, *simp*]:
$k \in carrier\ \mathcal{Z}$
$\langle proof \rangle$

lemma *int-is-cring*:
$cring\ \mathcal{Z}$
$\langle proof \rangle$

18.2.2 Interpretations

Since definitions of derived operations are global, their interpretation needs to be done as early as possible — that is, with as few assumptions as possible.

interpretation *int*: *monoid* $[\mathcal{Z}]$
where $carrier\ \mathcal{Z} = UNIV$
and $mult\ \mathcal{Z}\ x\ y = x * y$
and $one\ \mathcal{Z} = 1$
and $pow\ \mathcal{Z}\ x\ n = x\hat{}n$

⟨*proof*⟩

interpretation *int*: *comm-monoid* [$\mathcal{Z}$]
where *finprod* $\mathcal{Z}$ *f A* = (*if finite A then setprod f A else arbitrary*)
⟨*proof*⟩

interpretation *int*: *abelian-monoid* [$\mathcal{Z}$]
where *zero* $\mathcal{Z}$ = *0*
and *add* $\mathcal{Z}$ x y = $x + y$
and *finsum* $\mathcal{Z}$ *f A* = (*if finite A then setsum f A else arbitrary*)
⟨*proof*⟩

interpretation *int*: *abelian-group* [$\mathcal{Z}$]
where *a-inv* $\mathcal{Z}$ $x = -\ x$
and *a-minus* $\mathcal{Z}$ $x\ y = x\ -\ y$
⟨*proof*⟩

interpretation *int*: *domain* [$\mathcal{Z}$]
⟨*proof*⟩

Removal of occurrences of *UNIV* in interpretation result — experimental.

lemma *UNIV*:
$x \in$ *UNIV* = *True*
$A \subseteq$ *UNIV* = *True*
(*ALL x* : *UNIV*. *P x*) = (*ALL x*. *P x*)
(*EX x* : *UNIV*. *P x*) = (*EX x*. *P x*)
(*True* −−> *Q*) = *Q*
(*True* ==> *PROP R*) == *PROP R*
⟨*proof*⟩

interpretation *int* [*unfolded UNIV*]:
partial-order [(| *carrier* = *UNIV*::*int set*, *le* = *op* ≤ |)]
where *carrier* (| *carrier* = *UNIV*::*int set*, *le* = *op* ≤ |) = *UNIV*
and *le* (| *carrier* = *UNIV*::*int set*, *le* = *op* ≤ |) *x y* = (*x* ≤ *y*)
and *lless* (| *carrier* = *UNIV*::*int set*, *le* = *op* ≤ |) *x y* = (*x* < *y*)
⟨*proof*⟩

interpretation *int* [*unfolded UNIV*]:
lattice [(| *carrier* = *UNIV*::*int set*, *le* = *op* ≤ |)]
where *join* (| *carrier* = *UNIV*::*int set*, *le* = *op* ≤ |) *x y* = *max x y*
and *meet* (| *carrier* = *UNIV*::*int set*, *le* = *op* ≤ |) *x y* = *min x y*
⟨*proof*⟩

interpretation *int* [*unfolded UNIV*]:
total-order [(| *carrier* = *UNIV*::*int set*, *le* = *op* ≤ |)]
⟨*proof*⟩

18.2.3 Generated Ideals of $\mathcal{Z}$

lemma *int-Idl*:
$Idl_{\mathcal{Z}}\ \{a\} = \{x * a \mid x.\ True\}$
⟨*proof*⟩

lemma *multiples-principalideal*:
principalideal $\{x * a \mid x.\ True\ \}\ \mathcal{Z}$
⟨*proof*⟩

lemma *prime-primeideal*:
assumes *prime*: *prime* $(nat\ p)$
shows *primeideal* $(Idl_{\mathcal{Z}}\ \{p\})\ \mathcal{Z}$
⟨*proof*⟩

18.2.4 Ideals and Divisibility

lemma *int-Idl-subset-ideal*:
$Idl_{\mathcal{Z}}\ \{k\} \subseteq Idl_{\mathcal{Z}}\ \{l\} = (k \in Idl_{\mathcal{Z}}\ \{l\})$
⟨*proof*⟩

lemma *Idl-subset-eq-dvd*:
$(Idl_{\mathcal{Z}}\ \{k\} \subseteq Idl_{\mathcal{Z}}\ \{l\}) = (l\ dvd\ k)$
⟨*proof*⟩

lemma *dvds-eq-Idl*:
$(l\ dvd\ k \wedge k\ dvd\ l) = (Idl_{\mathcal{Z}}\ \{k\} = Idl_{\mathcal{Z}}\ \{l\})$
⟨*proof*⟩

lemma *Idl-eq-abs*:
$(Idl_{\mathcal{Z}}\ \{k\} = Idl_{\mathcal{Z}}\ \{l\}) = (abs\ l = abs\ k)$
⟨*proof*⟩

18.2.5 Ideals and the Modulus

constdefs
ZMod :: *int* => *int* => *int set*
$ZMod\ k\ r == (Idl_{\mathcal{Z}}\ \{k\})\ +>_{\mathcal{Z}}\ r$

lemmas *ZMod-defs* =
ZMod-def genideal-def

lemma *rcos-zfact*:
assumes *kIl*: $k \in ZMod\ l\ r$
shows $EX\ x.\ k = x * l + r$
⟨*proof*⟩

lemma *ZMod-imp-zmod*:
assumes *zmods*: $ZMod\ m\ a = ZMod\ m\ b$
shows $a\ mod\ m = b\ mod\ m$

⟨*proof*⟩

lemma *ZMod-mod*:
  shows *ZMod m a* = *ZMod m* (*a mod m*)
⟨*proof*⟩

lemma *zmod-imp-ZMod*:
  assumes *modeq*: *a mod m* = *b mod m*
  shows *ZMod m a* = *ZMod m b*
⟨*proof*⟩

corollary *ZMod-eq-mod*:
  shows (*ZMod m a* = *ZMod m b*) = (*a mod m* = *b mod m*)
⟨*proof*⟩

18.2.6 Factorization

constdefs
  ZFact :: *int* ⇒ *int set ring*
  ZFact k == $\mathcal{Z}$ *Quot* (*Idl*$_{\mathcal{Z}}$ {*k*})

lemmas *ZFact-defs* = *ZFact-def FactRing-def*

lemma *ZFact-is-cring*:
  shows *cring* (*ZFact k*)
⟨*proof*⟩

lemma *ZFact-zero*:
  carrier (*ZFact 0*) = ($\bigcup$ *a*. {{*a*}})
⟨*proof*⟩

lemma *ZFact-one*:
  carrier (*ZFact 1*) = {*UNIV*}
⟨*proof*⟩

lemma *ZFact-prime-is-domain*:
  assumes *pprime*: *prime* (*nat p*)
  shows *domain* (*ZFact p*)
⟨*proof*⟩

end

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