The Isabelle/HOL Algebra Library

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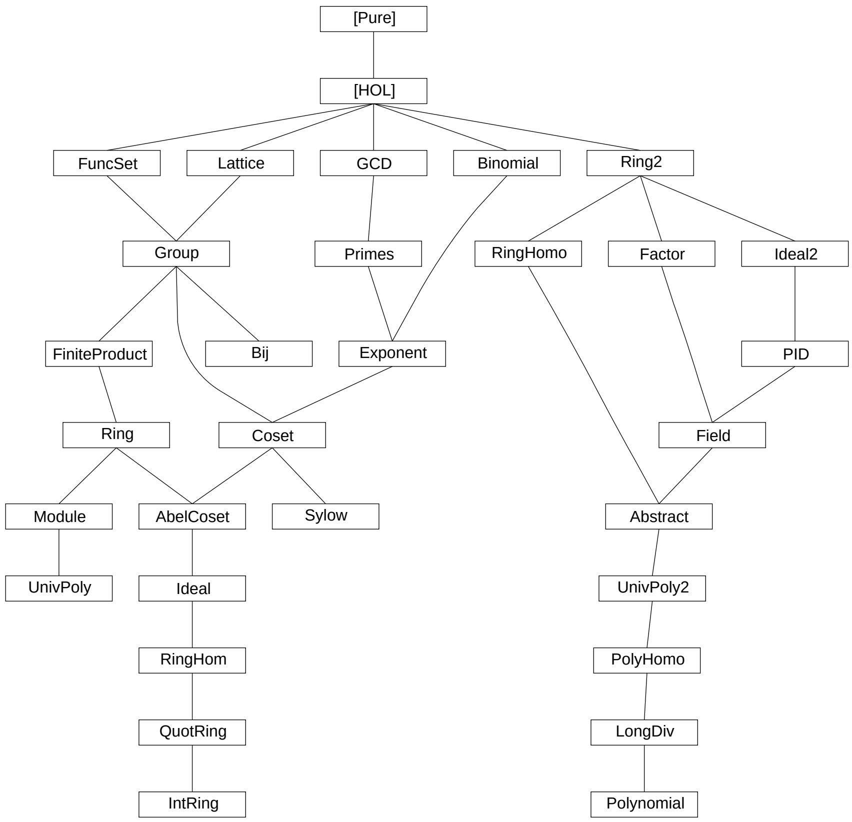
Contents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theory *Lattice* **imports** *Main* **begin**

1 Orders and Lattices

Object with a carrier set.

record *'a partial-object* =
  carrier :: *'a set*

1.1 Partial Orders

record *'a order* = *'a partial-object* +
  le :: [*'a*, *'a*] => *bool* (**infixl** $\sqsubseteq_1$ *50*)

locale *partial-order* =
  fixes *L* (**structure**)
  assumes *refl* [*intro*, *simp*]:
      $x \in carrier\ L ==> x \sqsubseteq x$
    and *anti-sym* [*intro*]:
      $[|\ x \sqsubseteq y;\ y \sqsubseteq x;\ x \in carrier\ L;\ y \in carrier\ L\ |] ==> x = y$
    and *trans* [*trans*]:
      $[|\ x \sqsubseteq y;\ y \sqsubseteq z;$
      $x \in carrier\ L;\ y \in carrier\ L;\ z \in carrier\ L\ |] ==> x \sqsubseteq z$

constdefs (**structure** *L*)
  lless :: [-, *'a*, *'a*] => *bool* (**infixl** $\sqsubset_1$ *50*)
  $x \sqsubset y == x \sqsubseteq y\ \&\ x\ \tilde{}= y$

  — Upper and lower bounds of a set.
  Upper :: [-, *'a set*] => *'a set*
  $Upper\ L\ A == \{u.\ (ALL\ x.\ x \in A \cap carrier\ L --> x \sqsubseteq u)\} \cap$
      $carrier\ L$

  Lower :: [-, *'a set*] => *'a set*
  $Lower\ L\ A == \{l.\ (ALL\ x.\ x \in A \cap carrier\ L --> l \sqsubseteq x)\} \cap$
      $carrier\ L$

  — Least and greatest, as predicate.
  least :: [-, *'a*, *'a set*] => *bool*
  $least\ L\ l\ A == A \subseteq carrier\ L\ \&\ l \in A\ \&\ (ALL\ x : A.\ l \sqsubseteq x)$

  greatest :: [-, *'a*, *'a set*] => *bool*
  $greatest\ L\ g\ A == A \subseteq carrier\ L\ \&\ g \in A\ \&\ (ALL\ x : A.\ x \sqsubseteq g)$

  — Supremum and infimum
  sup :: [-, *'a set*] => *'a* ($\bigsqcup_1$- [*90*] *90*)
  $\bigsqcup A == THE\ x.\ least\ L\ x\ (Upper\ L\ A)$

inf :: [-, *'a set*] => *'a* ($\bigsqcap$1- [*90*] *90*)
$\bigsqcap$*A* == *THE x*. *greatest L x* (*Lower L A*)

join :: [-, *'a*, *'a*] => *'a* (**infixl** $\sqcup$1 *65*)
x $\sqcup$ *y* == *sup L* {*x*, *y*}

meet :: [-, *'a*, *'a*] => *'a* (**infixl** $\sqcap$1 *70*)
x $\sqcap$ *y* == *inf L* {*x*, *y*}

1.1.1 Upper

lemma *Upper-closed* [*intro*, *simp*]:
Upper L A ⊆ *carrier L*
by (*unfold Upper-def*) *clarify*

lemma *UpperD* [*dest*]:
fixes *L* (**structure**)
shows [| *u* ∈ *Upper L A*; *x* ∈ *A*; *A* ⊆ *carrier L* |] ==> *x* ⊑ *u*
by (*unfold Upper-def*) *blast*

lemma *Upper-memI*:
fixes *L* (**structure**)
shows [| !! *y*. *y* ∈ *A* ==> *y* ⊑ *x*; *x* ∈ *carrier L* |] ==> *x* ∈ *Upper L A*
by (*unfold Upper-def*) *blast*

lemma *Upper-antimono*:
A ⊆ *B* ==> *Upper L B* ⊆ *Upper L A*
by (*unfold Upper-def*) *blast*

1.1.2 Lower

lemma *Lower-closed* [*intro*, *simp*]:
Lower L A ⊆ *carrier L*
by (*unfold Lower-def*) *clarify*

lemma *LowerD* [*dest*]:
fixes *L* (**structure**)
shows [| *l* ∈ *Lower L A*; *x* ∈ *A*; *A* ⊆ *carrier L* |] ==> *l* ⊑ *x*
by (*unfold Lower-def*) *blast*

lemma *Lower-memI*:
fixes *L* (**structure**)
shows [| !! *y*. *y* ∈ *A* ==> *x* ⊑ *y*; *x* ∈ *carrier L* |] ==> *x* ∈ *Lower L A*
by (*unfold Lower-def*) *blast*

lemma *Lower-antimono*:
A ⊆ *B* ==> *Lower L B* ⊆ *Lower L A*
by (*unfold Lower-def*) *blast*

1.1.3 least

lemma *least-carrier* [*intro*, *simp*]:
  shows *least L l A* ==> $l \in$ *carrier L*
  by (*unfold least-def*) *fast*

lemma *least-mem*:
  least L l A ==> $l \in A$
  by (*unfold least-def*) *fast*

lemma (**in** *partial-order*) *least-unique*:
  [| *least L x A*; *least L y A* |] ==> $x = y$
  by (*unfold least-def*) *blast*

lemma *least-le*:
  fixes *L* (**structure**)
  shows [| *least L x A*; $a \in A$ |] ==> $x \sqsubseteq a$
  by (*unfold least-def*) *fast*

lemma *least-UpperI*:
  fixes *L* (**structure**)
  assumes *above*: !! x. $x \in A$ ==> $x \sqsubseteq s$
    and *below*: !! y. $y \in$ *Upper L A* ==> $s \sqsubseteq y$
    and *L*: $A \subseteq$ *carrier L* $s \in$ *carrier L*
  shows *least L s* (*Upper L A*)
proof –
  have *Upper L A* $\subseteq$ *carrier L* **by** *simp*
  moreover from *above L* **have** $s \in$ *Upper L A* **by** (*simp add*: *Upper-def*)
  moreover from *below* **have** *ALL x* : *Upper L A*. $s \sqsubseteq x$ **by** *fast*
  ultimately show *?thesis* **by** (*simp add*: *least-def*)
qed

1.1.4 greatest

lemma *greatest-carrier* [*intro*, *simp*]:
  shows *greatest L l A* ==> $l \in$ *carrier L*
  by (*unfold greatest-def*) *fast*

lemma *greatest-mem*:
  greatest L l A ==> $l \in A$
  by (*unfold greatest-def*) *fast*

lemma (**in** *partial-order*) *greatest-unique*:
  [| *greatest L x A*; *greatest L y A* |] ==> $x = y$
  by (*unfold greatest-def*) *blast*

lemma *greatest-le*:
  fixes *L* (**structure**)
  shows [| *greatest L x A*; $a \in A$ |] ==> $a \sqsubseteq x$
  by (*unfold greatest-def*) *fast*

lemma *greatest-LowerI*:
  fixes L (**structure**)
  assumes *below*: !! x. $x \in A$ ==> $i \sqsubseteq x$
    and *above*: !! y. $y \in$ *Lower* L A ==> $y \sqsubseteq i$
    and L: $A \subseteq$ *carrier* L $i \in$ *carrier* L
  shows *greatest* L i (*Lower* L A)
proof −
  have *Lower* L $A \subseteq$ *carrier* L **by** *simp*
  moreover from *below* L **have** $i \in$ *Lower* L A **by** (*simp add*: *Lower-def*)
  moreover from *above* **have** *ALL* x : *Lower* L A. $x \sqsubseteq i$ **by** *fast*
  ultimately show *?thesis* **by** (*simp add*: *greatest-def*)
qed

1.2 Lattices

locale *lattice* = *partial-order* +
  assumes *sup-of-two-exists*:
    [| $x \in$ *carrier* L; $y \in$ *carrier* L |] ==> *EX* s. *least* L s (*Upper* L $\{x, y\}$)
    and *inf-of-two-exists*:
    [| $x \in$ *carrier* L; $y \in$ *carrier* L |] ==> *EX* s. *greatest* L s (*Lower* L $\{x, y\}$)

lemma *least-Upper-above*:
  fixes L (**structure**)
  shows [| *least* L s (*Upper* L A); $x \in A$; $A \subseteq$ *carrier* L |] ==> $x \sqsubseteq s$
  by (*unfold least-def*) *blast*

lemma *greatest-Lower-above*:
  fixes L (**structure**)
  shows [| *greatest* L i (*Lower* L A); $x \in A$; $A \subseteq$ *carrier* L |] ==> $i \sqsubseteq x$
  by (*unfold greatest-def*) *blast*

1.2.1 Supremum

lemma (**in** *lattice*) *joinI*:
  [| !!l. *least* L l (*Upper* L $\{x, y\}$) ==> P l; $x \in$ *carrier* L; $y \in$ *carrier* L |]
  ==> P $(x \sqcup y)$
proof (*unfold join-def sup-def*)
  assume L: $x \in$ *carrier* L $y \in$ *carrier* L
    and P: !!l. *least* L l (*Upper* L $\{x, y\}$) ==> P l
  with *sup-of-two-exists* **obtain** s **where** *least* L s (*Upper* L $\{x, y\}$) **by** *fast*
  with L **show** P (*THE* l. *least* L l (*Upper* L $\{x, y\}$))
    by (*fast intro*: *theI2 least-unique* P)
qed

lemma (**in** *lattice*) *join-closed* [*simp*]:
  [| $x \in$ *carrier* L; $y \in$ *carrier* L |] ==> $x \sqcup y \in$ *carrier* L
  by (*rule joinI*) (*rule least-carrier*)

lemma (**in** *partial-order*) *sup-of-singletonI*:

```
  x ∈ carrier L ==> least L x (Upper L {x})
  by (rule least-UpperI) fast+

lemma (in partial-order) sup-of-singleton [simp]:
  x ∈ carrier L ==> ⨆{x} = x
  by (unfold sup-def) (blast intro: least-unique least-UpperI sup-of-singletonI)
```

Condition on A: supremum exists.

```
lemma (in lattice) sup-insertI:
  [| !!s. least L s (Upper L (insert x A)) ==> P s;
  least L a (Upper L A); x ∈ carrier L; A ⊆ carrier L |]
  ==> P (⨆(insert x A))
proof (unfold sup-def)
  assume L: x ∈ carrier L  A ⊆ carrier L
    and P: !!l. least L l (Upper L (insert x A)) ==> P l
    and least-a: least L a (Upper L A)
  from L least-a have La: a ∈ carrier L by simp
  from L sup-of-two-exists least-a
  obtain s where least-s: least L s (Upper L {a, x}) by blast
  show P (THE l. least L l (Upper L (insert x A)))
  proof (rule theI2)
    show least L s (Upper L (insert x A))
    proof (rule least-UpperI)
      fix z
      assume z ∈ insert x A
      then show z ⊑ s
      proof
        assume z = x then show ?thesis
          by (simp add: least-Upper-above [OF least-s] L La)
      next
        assume z ∈ A
        with L least-s least-a show ?thesis
          by (rule-tac trans [where y = a]) (auto dest: least-Upper-above)
      qed
    next
      fix y
      assume y: y ∈ Upper L (insert x A)
      show s ⊑ y
      proof (rule least-le [OF least-s], rule Upper-memI)
        fix z
        assume z: z ∈ {a, x}
        then show z ⊑ y
        proof
          have y': y ∈ Upper L A
            apply (rule subsetD [where A = Upper L (insert x A)])
             apply (rule Upper-antimono)
             apply blast
            apply (rule y)
            done
```

```
          assume z = a
          with y' least-a show ?thesis by (fast dest: least-le)
        next
          assume z ∈ {x}
          with y L show ?thesis by blast
        qed
      qed (rule Upper-closed [THEN subsetD, OF y])
    next
      from L show insert x A ⊆ carrier L by simp
      from least-s show s ∈ carrier L by simp
    qed
  next
    fix l
    assume least-l: least L l (Upper L (insert x A))
    show l = s
    proof (rule least-unique)
      show least L s (Upper L (insert x A))
      proof (rule least-UpperI)
        fix z
        assume z ∈ insert x A
        then show z ⊑ s
        proof
          assume z = x then show ?thesis
            by (simp add: least-Upper-above [OF least-s] L La)
        next
          assume z ∈ A
          with L least-s least-a show ?thesis
            by (rule-tac trans [where y = a]) (auto dest: least-Upper-above)
        qed
      next
        fix y
        assume y: y ∈ Upper L (insert x A)
        show s ⊑ y
        proof (rule least-le [OF least-s], rule Upper-memI)
          fix z
          assume z: z ∈ {a, x}
          then show z ⊑ y
          proof
            have y': y ∈ Upper L A
              apply (rule subsetD [where A = Upper L (insert x A)])
              apply (rule Upper-antimono)
               apply blast
              apply (rule y)
              done
            assume z = a
            with y' least-a show ?thesis by (fast dest: least-le)
          next
            assume z ∈ {x}
            with y L show ?thesis by blast
```

```
        qed
      qed (rule Upper-closed [THEN subsetD, OF y])
    next
      from L show insert x A ⊆ carrier L by simp
      from least-s show s ∈ carrier L by simp
    qed
  qed (rule least-l)
 qed (rule P)
qed

lemma (in lattice) finite-sup-least:
 [| finite A; A ⊆ carrier L; A ~= {} |] ==> least L (⨆A) (Upper L A)
proof (induct set: finite)
 case empty
 then show ?case by simp
next
 case (insert x A)
 show ?case
 proof (cases A = {})
  case True
  with insert show ?thesis by (simp add: sup-of-singletonI)
 next
  case False
  with insert have least L (⨆A) (Upper L A) by simp
  with - show ?thesis
   by (rule sup-insertI) (simp-all add: insert [simplified])
 qed
qed

lemma (in lattice) finite-sup-insertI:
 assumes P: !!l. least L l (Upper L (insert x A)) ==> P l
  and xA: finite A  x ∈ carrier L  A ⊆ carrier L
 shows P (⨆ (insert x A))
proof (cases A = {})
 case True with P and xA show ?thesis
  by (simp add: sup-of-singletonI)
next
 case False with P and xA show ?thesis
  by (simp add: sup-insertI finite-sup-least)
qed

lemma (in lattice) finite-sup-closed:
 [| finite A; A ⊆ carrier L; A ~= {} |] ==> ⨆A ∈ carrier L
proof (induct set: finite)
 case empty then show ?case by simp
next
 case insert then show ?case
  by - (rule finite-sup-insertI, simp-all)
qed
```

lemma (**in** *lattice*) *join-left*:
  [| $x \in$ *carrier L*; $y \in$ *carrier L* |] ==> $x \sqsubseteq x \sqcup y$
  by (*rule joinI* [*folded join-def*]) (*blast dest*: *least-mem*)

lemma (**in** *lattice*) *join-right*:
  [| $x \in$ *carrier L*; $y \in$ *carrier L* |] ==> $y \sqsubseteq x \sqcup y$
  by (*rule joinI* [*folded join-def*]) (*blast dest*: *least-mem*)

lemma (**in** *lattice*) *sup-of-two-least*:
  [| $x \in$ *carrier L*; $y \in$ *carrier L* |] ==> *least L* ($\bigsqcup\{x, y\}$) (*Upper L* $\{x, y\}$)
proof (*unfold sup-def*)
  assume *L*: $x \in$ *carrier L* $y \in$ *carrier L*
  with *sup-of-two-exists* **obtain** *s* **where** *least L s* (*Upper L* $\{x, y\}$) **by** *fast*
  with *L* **show** *least L* (*THE xa. least L xa* (*Upper L* $\{x, y\}$)) (*Upper L* $\{x, y\}$)
  by (*fast intro*: *theI2 least-unique*)
qed

lemma (**in** *lattice*) *join-le*:
  assumes *sub*: $x \sqsubseteq z$ $y \sqsubseteq z$
    and *x*: $x \in$ *carrier L* **and** *y*: $y \in$ *carrier L* **and** *z*: $z \in$ *carrier L*
  shows $x \sqcup y \sqsubseteq z$
proof (*rule joinI* [*OF - x y*])
  fix *s*
  assume *least L s* (*Upper L* $\{x, y\}$)
  with *sub z* **show** $s \sqsubseteq z$ **by** (*fast elim*: *least-le intro*: *Upper-memI*)
qed

lemma (**in** *lattice*) *join-assoc-lemma*:
  assumes *L*: $x \in$ *carrier L* $y \in$ *carrier L* $z \in$ *carrier L*
  shows $x \sqcup (y \sqcup z) = \bigsqcup\{x, y, z\}$
proof (*rule finite-sup-insertI*)
  — The textbook argument in Jacobson I, p 457
  fix *s*
  assume *sup*: *least L s* (*Upper L* $\{x, y, z\}$)
  show $x \sqcup (y \sqcup z) = s$
  proof (*rule anti-sym*)
    from *sup L* **show** $x \sqcup (y \sqcup z) \sqsubseteq s$
      by (*fastsimp intro*!: *join-le elim*: *least-Upper-above*)
  next
    from *sup L* **show** $s \sqsubseteq x \sqcup (y \sqcup z)$
    by (*erule-tac least-le*)
      (*blast intro*!: *Upper-memI intro*: *trans join-left join-right join-closed*)
  qed (*simp-all add*: *L least-carrier* [*OF sup*])
qed (*simp-all add*: *L*)

lemma *join-comm*:
  fixes *L* (**structure**)
  shows $x \sqcup y = y \sqcup x$

by (*unfold join-def*) (*simp add*: *insert-commute*)

lemma (**in** *lattice*) *join-assoc*:
assumes *L*: $x \in$ *carrier* *L* $y \in$ *carrier* *L* $z \in$ *carrier* *L*
shows $(x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$
proof −
have $(x \sqcup y) \sqcup z = z \sqcup (x \sqcup y)$ **by** (*simp only*: *join-comm*)
also from *L* **have** ... $= \bigsqcup \{z, x, y\}$ **by** (*simp add*: *join-assoc-lemma*)
also from *L* **have** ... $= \bigsqcup \{x, y, z\}$ **by** (*simp add*: *insert-commute*)
also from *L* **have** ... $= x \sqcup (y \sqcup z)$ **by** (*simp add*: *join-assoc-lemma*)
finally show *?thesis* **.**
qed

1.2.2 Infimum

lemma (**in** *lattice*) *meetI*:
[| !!*i*. *greatest* *L* *i* (*Lower* *L* $\{x, y\}$) ==> *P* *i*;
$x \in$ *carrier* *L*; $y \in$ *carrier* *L* |]
==> *P* $(x \sqcap y)$
proof (*unfold meet-def inf-def*)
assume *L*: $x \in$ *carrier* *L* $y \in$ *carrier* *L*
and *P*: !!*g*. *greatest* *L* *g* (*Lower* *L* $\{x, y\}$) ==> *P* *g*
with *inf-of-two-exists* **obtain** *i* **where** *greatest* *L* *i* (*Lower* *L* $\{x, y\}$) **by** *fast*
with *L* **show** *P* (*THE* *g*. *greatest* *L* *g* (*Lower* *L* $\{x, y\}$))
by (*fast intro*: *theI2 greatest-unique P*)
qed

lemma (**in** *lattice*) *meet-closed* [*simp*]:
[| $x \in$ *carrier* *L*; $y \in$ *carrier* *L* |] ==> $x \sqcap y \in$ *carrier* *L*
by (*rule meetI*) (*rule greatest-carrier*)

lemma (**in** *partial-order*) *inf-of-singletonI*:
$x \in$ *carrier* *L* ==> *greatest* *L* *x* (*Lower* *L* $\{x\}$)
by (*rule greatest-LowerI*) *fast+*

lemma (**in** *partial-order*) *inf-of-singleton* [*simp*]:
$x \in$ *carrier* *L* ==> $\bigsqcap \{x\} = x$
by (*unfold inf-def*) (*blast intro*: *greatest-unique greatest-LowerI inf-of-singletonI*)

Condition on A: infimum exists.

lemma (**in** *lattice*) *inf-insertI*:
[| !!*i*. *greatest* *L* *i* (*Lower* *L* (*insert* *x* *A*)) ==> *P* *i*;
greatest *L* *a* (*Lower* *L* *A*); $x \in$ *carrier* *L*; $A \subseteq$ *carrier* *L* |]
==> *P* ($\bigsqcap$(*insert* *x* *A*))
proof (*unfold inf-def*)
assume *L*: $x \in$ *carrier* *L* $A \subseteq$ *carrier* *L*
and *P*: !!*g*. *greatest* *L* *g* (*Lower* *L* (*insert* *x* *A*)) ==> *P* *g*
and *greatest-a*: *greatest* *L* *a* (*Lower* *L* *A*)
from *L* *greatest-a* **have** *La*: $a \in$ *carrier* *L* **by** *simp*

```
from L inf-of-two-exists greatest-a
obtain i where greatest-i: greatest L i (Lower L {a, x}) by blast
show P (THE g. greatest L g (Lower L (insert x A)))
proof (rule theI2)
  show greatest L i (Lower L (insert x A))
  proof (rule greatest-LowerI)
    fix z
    assume z ∈ insert x A
    then show i ⊑ z
    proof
      assume z = x then show ?thesis
        by (simp add: greatest-Lower-above [OF greatest-i] L La)
    next
      assume z ∈ A
      with L greatest-i greatest-a show ?thesis
        by (rule-tac trans [where y = a]) (auto dest: greatest-Lower-above)
    qed
  next
    fix y
    assume y: y ∈ Lower L (insert x A)
    show y ⊑ i
    proof (rule greatest-le [OF greatest-i], rule Lower-memI)
      fix z
      assume z: z ∈ {a, x}
      then show y ⊑ z
      proof
        have y': y ∈ Lower L A
          apply (rule subsetD [where A = Lower L (insert x A)])
          apply (rule Lower-antimono)
           apply blast
          apply (rule y)
          done
        assume z = a
        with y' greatest-a show ?thesis by (fast dest: greatest-le)
      next
        assume z ∈ {x}
        with y L show ?thesis by blast
      qed
    qed (rule Lower-closed [THEN subsetD, OF y])
  next
    from L show insert x A ⊆ carrier L by simp
    from greatest-i show i ∈ carrier L by simp
  qed
next
  fix g
  assume greatest-g: greatest L g (Lower L (insert x A))
  show g = i
  proof (rule greatest-unique)
    show greatest L i (Lower L (insert x A))
```

```
      proof (rule greatest-LowerI)
        fix z
        assume z ∈ insert x A
        then show i ⊑ z
        proof
          assume z = x then show ?thesis
            by (simp add: greatest-Lower-above [OF greatest-i] L La)
        next
          assume z ∈ A
          with L greatest-i greatest-a show ?thesis
            by (rule-tac trans [where y = a]) (auto dest: greatest-Lower-above)
        qed
      next
        fix y
        assume y: y ∈ Lower L (insert x A)
        show y ⊑ i
        proof (rule greatest-le [OF greatest-i], rule Lower-memI)
          fix z
          assume z: z ∈ {a, x}
          then show y ⊑ z
          proof
            have y': y ∈ Lower L A
              apply (rule subsetD [where A = Lower L (insert x A)])
              apply (rule Lower-antimono)
               apply blast
              apply (rule y)
              done
            assume z = a
            with y' greatest-a show ?thesis by (fast dest: greatest-le)
          next
            assume z ∈ {x}
            with y L show ?thesis by blast
          qed
        qed (rule Lower-closed [THEN subsetD, OF y])
      next
        from L show insert x A ⊆ carrier L by simp
        from greatest-i show i ∈ carrier L by simp
      qed
    qed (rule greatest-g)
  qed (rule P)
qed

lemma (in lattice) finite-inf-greatest:
  [| finite A; A ⊆ carrier L; A ~= {} |] ==> greatest L (⨅ A) (Lower L A)
proof (induct set: finite)
  case empty then show ?case by simp
next
  case (insert x A)
  show ?case
```

proof (*cases* $A = \{\}$)
case *True*
with *insert* **show** *?thesis* **by** (*simp add*: *inf-of-singletonI*)
next
case *False*
from *insert* **show** *?thesis*
proof (*rule-tac inf-insertI*)
from *False insert* **show** *greatest L* ($\bigsqcap A$) (*Lower L A*) **by** *simp*
qed *simp-all*
qed
qed

lemma (**in** *lattice*) *finite-inf-insertI*:
assumes *P*: !!*i*. *greatest L i* (*Lower L* (*insert x A*)) ==> *P i*
and *xA*: *finite A* $x \in$ *carrier L* $A \subseteq$ *carrier L*
shows *P* ($\bigsqcap$ (*insert x A*))
proof (*cases* $A = \{\}$)
case *True* **with** *P* **and** *xA* **show** *?thesis*
by (*simp add*: *inf-of-singletonI*)
next
case *False* **with** *P* **and** *xA* **show** *?thesis*
by (*simp add*: *inf-insertI finite-inf-greatest*)
qed

lemma (**in** *lattice*) *finite-inf-closed*:
[| *finite A*; $A \subseteq$ *carrier L*; *A* ~= {} |] ==> $\bigsqcap A \in$ *carrier L*
proof (*induct set*: *finite*)
case *empty* **then show** *?case* **by** *simp*
next
case *insert* **then show** *?case*
by (*rule-tac finite-inf-insertI*) (*simp-all*)
qed

lemma (**in** *lattice*) *meet-left*:
[| $x \in$ *carrier L*; $y \in$ *carrier L* |] ==> $x \sqcap y \sqsubseteq x$
by (*rule meetI* [*folded meet-def*]) (*blast dest*: *greatest-mem*)

lemma (**in** *lattice*) *meet-right*:
[| $x \in$ *carrier L*; $y \in$ *carrier L* |] ==> $x \sqcap y \sqsubseteq y$
by (*rule meetI* [*folded meet-def*]) (*blast dest*: *greatest-mem*)

lemma (**in** *lattice*) *inf-of-two-greatest*:
[| $x \in$ *carrier L*; $y \in$ *carrier L* |] ==>
greatest L ($\bigsqcap$ $\{x, y\}$) (*Lower L* $\{x, y\}$)
proof (*unfold inf-def*)
assume *L*: $x \in$ *carrier L* $y \in$ *carrier L*
with *inf-of-two-exists* **obtain** *s* **where** *greatest L s* (*Lower L* $\{x, y\}$) **by** *fast*
with *L*
show *greatest L* (*THE xa*. *greatest L xa* (*Lower L* $\{x, y\}$)) (*Lower L* $\{x, y\}$)

by (*fast intro*: *theI2 greatest-unique*)
qed

lemma (**in** *lattice*) *meet-le*:
assumes *sub*: $z \sqsubseteq x \quad z \sqsubseteq y$
and *x*: $x \in$ *carrier* L **and** *y*: $y \in$ *carrier* L **and** *z*: $z \in$ *carrier* L
shows $z \sqsubseteq x \sqcap y$
proof (*rule meetI* [*OF* - *x y*])
fix *i*
assume *greatest* L i (*Lower* L $\{x, y\}$)
with *sub z* **show** $z \sqsubseteq i$ **by** (*fast elim*: *greatest-le intro*: *Lower-memI*)
qed

lemma (**in** *lattice*) *meet-assoc-lemma*:
assumes *L*: $x \in$ *carrier* $L \quad y \in$ *carrier* $L \quad z \in$ *carrier* L
shows $x \sqcap (y \sqcap z) = \bigsqcap\{x, y, z\}$
proof (*rule finite-inf-insertI*)

The textbook argument in Jacobson I, p 457

fix *i*
assume *inf*: *greatest* L i (*Lower* L $\{x, y, z\}$)
show $x \sqcap (y \sqcap z) = i$
proof (*rule anti-sym*)
from *inf L* **show** $i \sqsubseteq x \sqcap (y \sqcap z)$
by (*fastsimp intro*!: *meet-le elim*: *greatest-Lower-above*)
next
from *inf L* **show** $x \sqcap (y \sqcap z) \sqsubseteq i$
by (*erule-tac greatest-le*)
(*blast intro*!: *Lower-memI intro*: *trans meet-left meet-right meet-closed*)
qed (*simp-all add*: *L greatest-carrier* [*OF inf*])
qed (*simp-all add*: *L*)

lemma *meet-comm*:
fixes *L* (**structure**)
shows $x \sqcap y = y \sqcap x$
by (*unfold meet-def*) (*simp add*: *insert-commute*)

lemma (**in** *lattice*) *meet-assoc*:
assumes *L*: $x \in$ *carrier* $L \quad y \in$ *carrier* $L \quad z \in$ *carrier* L
shows $(x \sqcap y) \sqcap z = x \sqcap (y \sqcap z)$
proof −
have $(x \sqcap y) \sqcap z = z \sqcap (x \sqcap y)$ **by** (*simp only*: *meet-comm*)
also from *L* **have** ... $= \bigsqcap \{z, x, y\}$ **by** (*simp add*: *meet-assoc-lemma*)
also from *L* **have** ... $= \bigsqcap \{x, y, z\}$ **by** (*simp add*: *insert-commute*)
also from *L* **have** ... $= x \sqcap (y \sqcap z)$ **by** (*simp add*: *meet-assoc-lemma*)
finally show *?thesis* **.**
qed

1.3 Total Orders

locale *total-order* = *partial-order* +
  assumes *total*: [| $x \in$ *carrier L*; $y \in$ *carrier L* |] ==> $x \sqsubseteq y$ | $y \sqsubseteq x$

Introduction rule: the usual definition of total order

lemma (**in** *partial-order*) *total-orderI*:
  assumes *total*: !!x y. [| $x \in$ *carrier L*; $y \in$ *carrier L* |] ==> $x \sqsubseteq y$ | $y \sqsubseteq x$
  shows *total-order L*
  by *unfold-locales* (*rule total*)

Total orders are lattices.

interpretation *total-order* < *lattice*
proof *unfold-locales*
  fix *x y*
  assume *L*: $x \in$ *carrier L* $y \in$ *carrier L*
  show *EX s. least L s* (*Upper L* {*x*, *y*})
  proof −
    note *total L*
    moreover
    {
      assume $x \sqsubseteq y$
      with *L* **have** *least L y* (*Upper L* {*x*, *y*})
        by (*rule-tac least-UpperI*) *auto*
    }
    moreover
    {
      assume $y \sqsubseteq x$
      with *L* **have** *least L x* (*Upper L* {*x*, *y*})
        by (*rule-tac least-UpperI*) *auto*
    }
    ultimately show *?thesis* **by** *blast*
  qed
next
  fix *x y*
  assume *L*: $x \in$ *carrier L* $y \in$ *carrier L*
  show *EX i. greatest L i* (*Lower L* {*x*, *y*})
  proof −
    note *total L*
    moreover
    {
      assume $y \sqsubseteq x$
      with *L* **have** *greatest L y* (*Lower L* {*x*, *y*})
        by (*rule-tac greatest-LowerI*) *auto*
    }
    moreover
    {
      assume $x \sqsubseteq y$
      with *L* **have** *greatest L x* (*Lower L* {*x*, *y*})
        by (*rule-tac greatest-LowerI*) *auto*

    }
    ultimately show *?thesis* **by** *blast*
  qed
qed

1.4 Complete lattices

locale *complete-lattice* = *lattice* +
  assumes *sup-exists*:
    [| $A \subseteq$ *carrier L* |] ==> *EX s. least L s* (*Upper L A*)
    and *inf-exists*:
    [| $A \subseteq$ *carrier L* |] ==> *EX i. greatest L i* (*Lower L A*)

Introduction rule: the usual definition of complete lattice

lemma (**in** *partial-order*) *complete-latticeI*:
  assumes *sup-exists*:
    !!*A*. [| $A \subseteq$ *carrier L* |] ==> *EX s. least L s* (*Upper L A*)
    and *inf-exists*:
    !!*A*. [| $A \subseteq$ *carrier L* |] ==> *EX i. greatest L i* (*Lower L A*)
  shows *complete-lattice L*
proof *intro-locales*
  show *lattice-axioms L*
    by (*rule lattice-axioms.intro*) (*blast intro*: *sup-exists inf-exists*)+
qed (*rule complete-lattice-axioms.intro sup-exists inf-exists* | *assumption*)+

constdefs (**structure** *L*)
  top :: - => *'a* ($\top_1$)
  $\top$ == *sup L* (*carrier L*)

  bottom :: - => *'a* ($\bot_1$)
  $\bot$ == *inf L* (*carrier L*)

lemma (**in** *complete-lattice*) *supI*:
  [| !!*l*. *least L l* (*Upper L A*) ==> *P l*; $A \subseteq$ *carrier L* |]
  ==> *P* ($\bigsqcup$*A*)
proof (*unfold sup-def*)
  assume *L*: $A \subseteq$ *carrier L*
    and *P*: !!*l*. *least L l* (*Upper L A*) ==> *P l*
  with *sup-exists* **obtain** *s* **where** *least L s* (*Upper L A*) **by** *blast*
  with *L* **show** *P* (*THE l. least L l* (*Upper L A*))
  by (*fast intro*: *theI2 least-unique P*)
qed

lemma (**in** *complete-lattice*) *sup-closed* [*simp*]:
  $A \subseteq$ *carrier L* ==> $\bigsqcup$*A* $\in$ *carrier L*
  by (*rule supI*) *simp-all*

lemma (**in** *complete-lattice*) *top-closed* [*simp*, *intro*]:

$\top \in$ *carrier L*
by (*unfold top-def*) *simp*

lemma (**in** *complete-lattice*) *infI*:
[| !!*i*. *greatest L i* (*Lower L A*) ==> *P i*; $A \subseteq$ *carrier L* |]
==> *P* ($\bigsqcap$*A*)
proof (*unfold inf-def*)
assume *L*: $A \subseteq$ *carrier L*
and *P*: !!*l*. *greatest L l* (*Lower L A*) ==> *P l*
with *inf-exists* **obtain** *s* **where** *greatest L s* (*Lower L A*) **by** *blast*
with *L* **show** *P* (*THE l*. *greatest L l* (*Lower L A*))
by (*fast intro*: *theI2 greatest-unique P*)
qed

lemma (**in** *complete-lattice*) *inf-closed* [*simp*]:
$A \subseteq$ *carrier L* ==> $\bigsqcap$*A* $\in$ *carrier L*
by (*rule infI*) *simp-all*

lemma (**in** *complete-lattice*) *bottom-closed* [*simp*, *intro*]:
$\bot \in$ *carrier L*
by (*unfold bottom-def*) *simp*

Jacobson: Theorem 8.1

lemma *Lower-empty* [*simp*]:
Lower L {} = *carrier L*
by (*unfold Lower-def*) *simp*

lemma *Upper-empty* [*simp*]:
Upper L {} = *carrier L*
by (*unfold Upper-def*) *simp*

theorem (**in** *partial-order*) *complete-lattice-criterion1*:
assumes *top-exists*: *EX g*. *greatest L g* (*carrier L*)
and *inf-exists*:
!!*A*. [| $A \subseteq$ *carrier L*; *A* ~= {} |] ==> *EX i*. *greatest L i* (*Lower L A*)
shows *complete-lattice L*
proof (*rule complete-latticeI*)
from *top-exists* **obtain** *top* **where** *top*: *greatest L top* (*carrier L*) **..**
fix *A*
assume *L*: $A \subseteq$ *carrier L*
let *?B* = *Upper L A*
from *L top* **have** *top* $\in$ *?B* **by** (*fast intro*!: *Upper-memI intro*: *greatest-le*)
then have *B-non-empty*: *?B* ~= {} **by** *fast*
have *B-L*: *?B* $\subseteq$ *carrier L* **by** *simp*
from *inf-exists* [*OF B-L B-non-empty*]
obtain *b* **where** *b-inf-B*: *greatest L b* (*Lower L ?B*) **..**
have *least L b* (*Upper L A*)
apply (*rule least-UpperI*)
apply (*rule greatest-le* [**where** *A* = *Lower L ?B*])

apply (*rule b-inf-B*)
apply (*rule Lower-memI*)
apply (*erule UpperD*)
apply *assumption*
apply (*rule L*)
apply (*fast intro*: *L* [*THEN subsetD*])
apply (*erule greatest-Lower-above* [*OF b-inf-B*])
apply *simp*
apply (*rule L*)
apply (*rule greatest-carrier* [*OF b-inf-B*])
done
then show *EX s. least L s* (*Upper L A*) **..**
next
fix *A*
assume *L*: $A \subseteq$ *carrier L*
show *EX i. greatest L i* (*Lower L A*)
proof (*cases* $A = \{\}$)
case *True* **then show** *?thesis*
by (*simp add*: *top-exists*)
next
case *False* **with** *L* **show** *?thesis*
by (*rule inf-exists*)
qed
qed

1.5 Examples

1.5.1 Powerset of a Set is a Complete Lattice

theorem *powerset-is-complete-lattice*:
complete-lattice (| *carrier* = *Pow A*, *le* = *op* $\subseteq$ |)
(**is** *complete-lattice ?L*)
proof (*rule partial-order.complete-latticeI*)
show *partial-order ?L*
by (*rule partial-order.intro*) *auto*
next
fix *B*
assume $B \subseteq$ *carrier ?L*
then have *least ?L* ($\bigcup B$) (*Upper ?L B*)
by (*fastsimp intro*!: *least-UpperI simp*: *Upper-def*)
then show *EX s. least ?L s* (*Upper ?L B*) **..**
next
fix *B*
assume $B \subseteq$ *carrier ?L*
then have *greatest ?L* ($\bigcap B \cap A$) (*Lower ?L B*)

$\bigcap B$ is not the infimum of B: $\bigcap \{\} = UNIV$ which is in general bigger than A!

by (*fastsimp intro*!: *greatest-LowerI simp*: *Lower-def*)
then show *EX i. greatest ?L i* (*Lower ?L B*) **..**
qed

An other example, that of the lattice of subgroups of a group, can be found in Group theory (Section 2.7).

end

theory *Group* **imports** *FuncSet Lattice* **begin**

2 Monoids and Groups

2.1 Definitions

Definitions follow [2].

record *'a monoid* = *'a partial-object* +
  mult :: $['a, 'a] \Rightarrow 'a$ (**infixl** $\otimes\imath$ *70*)
  one :: *'a* ($\mathbf{1}\imath$)

constdefs (**structure** *G*)
  m-inv :: (*'a*, *'b*) *monoid-scheme* => *'a* => *'a* (*inv*ı - [*81*] *80*)
  inv x == (*THE y. y* $\in$ *carrier G* & $x \otimes y = \mathbf{1}$ & $y \otimes x = \mathbf{1}$)

  Units :: - => *'a set*
  — The set of invertible elements
  Units G == $\{y.\ y \in$ *carrier G* & ($\exists x \in$ *carrier G*. $x \otimes y = \mathbf{1}$ & $y \otimes x = \mathbf{1})\}$

consts
  pow :: [(*'a*, *'m*) *monoid-scheme*, *'a*, *'b*::*number*] => *'a* (**infixr** '(ˆ')ı *75*)

defs (**overloaded**)
  nat-pow-def: *pow G a n* == *nat-rec* $\mathbf{1}_G$ (%*u b. b* $\otimes_G$ *a*) *n*
  int-pow-def: *pow G a z* ==
    let p = *nat-rec* $\mathbf{1}_G$ (%*u b. b* $\otimes_G$ *a*)
    in if neg z then inv_G (*p* (*nat* ($-z$))) *else p* (*nat z*)

locale *monoid* =
  fixes *G* (**structure**)
  assumes *m-closed* [*intro*, *simp*]:
      $[\![x \in$ *carrier G*; $y \in$ *carrier G*$]\!] \Longrightarrow x \otimes y \in$ *carrier G*
    and *m-assoc*:
      $[\![x \in$ *carrier G*; $y \in$ *carrier G*; $z \in$ *carrier G*$]\!]$
      $\Longrightarrow (x \otimes y) \otimes z = x \otimes (y \otimes z)$
    and *one-closed* [*intro*, *simp*]: $\mathbf{1} \in$ *carrier G*
    and *l-one* [*simp*]: $x \in$ *carrier G* $\Longrightarrow \mathbf{1} \otimes x = x$
    and *r-one* [*simp*]: $x \in$ *carrier G* $\Longrightarrow x \otimes \mathbf{1} = x$

lemma *monoidI*:
  fixes *G* (**structure**)
  assumes *m-closed*:

$!!x\ y.\ [|\ x \in carrier\ G;\ y \in carrier\ G\ |] ==> x \otimes y \in carrier\ G$
and *one-closed*: $\mathbf{1} \in carrier\ G$
and *m-assoc*:
$!!x\ y\ z.\ [|\ x \in carrier\ G;\ y \in carrier\ G;\ z \in carrier\ G\ |] ==>$
$(x \otimes y) \otimes z = x \otimes (y \otimes z)$
and *l-one*: $!!x.\ x \in carrier\ G ==> \mathbf{1} \otimes x = x$
and *r-one*: $!!x.\ x \in carrier\ G ==> x \otimes \mathbf{1} = x$
shows *monoid G*
by (*fast intro*!: *monoid.intro intro*: *prems*)

lemma (**in** *monoid*) *Units-closed* [*dest*]:
$x \in Units\ G ==> x \in carrier\ G$
by (*unfold Units-def*) *fast*

lemma (**in** *monoid*) *inv-unique*:
assumes *eq*: $y \otimes x = \mathbf{1}\ \ x \otimes y' = \mathbf{1}$
and *G*: $x \in carrier\ G\ \ y \in carrier\ G\ \ y' \in carrier\ G$
shows $y = y'$
proof −
from *G eq* **have** $y = y \otimes (x \otimes y')$ **by** *simp*
also from *G* **have** ... $= (y \otimes x) \otimes y'$ **by** (*simp add*: *m-assoc*)
also from *G eq* **have** ... $= y'$ **by** *simp*
finally show *?thesis* **.**
qed

lemma (**in** *monoid*) *Units-one-closed* [*intro*, *simp*]:
$\mathbf{1} \in Units\ G$
by (*unfold Units-def*) *auto*

lemma (**in** *monoid*) *Units-inv-closed* [*intro*, *simp*]:
$x \in Units\ G ==> inv\ x \in carrier\ G$
apply (*unfold Units-def m-inv-def*, *auto*)
apply (*rule theI2*, *fast*)
apply (*fast intro*: *inv-unique*, *fast*)
done

lemma (**in** *monoid*) *Units-l-inv-ex*:
$x \in Units\ G ==> \exists\, y \in carrier\ G.\ y \otimes x = \mathbf{1}$
by (*unfold Units-def*) *auto*

lemma (**in** *monoid*) *Units-r-inv-ex*:
$x \in Units\ G ==> \exists\, y \in carrier\ G.\ x \otimes y = \mathbf{1}$
by (*unfold Units-def*) *auto*

lemma (**in** *monoid*) *Units-l-inv*:
$x \in Units\ G ==> inv\ x \otimes x = \mathbf{1}$
apply (*unfold Units-def m-inv-def*, *auto*)
apply (*rule theI2*, *fast*)
apply (*fast intro*: *inv-unique*, *fast*)

done

lemma (**in** *monoid*) *Units-r-inv*:
$x \in Units\ G ==> x \otimes inv\ x = \mathbf{1}$
apply (*unfold Units-def m-inv-def*, *auto*)
apply (*rule theI2*, *fast*)
apply (*fast intro*: *inv-unique*, *fast*)
done

lemma (**in** *monoid*) *Units-inv-Units* [*intro*, *simp*]:
$x \in Units\ G ==> inv\ x \in Units\ G$
proof −
assume *x*: $x \in Units\ G$
show $inv\ x \in Units\ G$
by (*auto simp add*: *Units-def*
intro: *Units-l-inv Units-r-inv x Units-closed* [*OF x*])
qed

lemma (**in** *monoid*) *Units-l-cancel* [*simp*]:
$[|\ x \in Units\ G;\ y \in carrier\ G;\ z \in carrier\ G\ |] ==>$
$(x \otimes y = x \otimes z) = (y = z)$
proof
assume *eq*: $x \otimes y = x \otimes z$
and *G*: $x \in Units\ G \quad y \in carrier\ G \quad z \in carrier\ G$
then have $(inv\ x \otimes x) \otimes y = (inv\ x \otimes x) \otimes z$
by (*simp add*: *m-assoc Units-closed*)
with *G* **show** $y = z$ **by** (*simp add*: *Units-l-inv*)
next
assume *eq*: $y = z$
and *G*: $x \in Units\ G \quad y \in carrier\ G \quad z \in carrier\ G$
then show $x \otimes y = x \otimes z$ **by** *simp*
qed

lemma (**in** *monoid*) *Units-inv-inv* [*simp*]:
$x \in Units\ G ==> inv\ (inv\ x) = x$
proof −
assume *x*: $x \in Units\ G$
then have $inv\ x \otimes inv\ (inv\ x) = inv\ x \otimes x$
by (*simp add*: *Units-l-inv Units-r-inv*)
with *x* **show** *?thesis* **by** (*simp add*: *Units-closed*)
qed

lemma (**in** *monoid*) *inv-inj-on-Units*:
inj-on (*m-inv G*) (*Units G*)
proof (*rule inj-onI*)
fix *x y*
assume *G*: $x \in Units\ G \quad y \in Units\ G$ **and** *eq*: $inv\ x = inv\ y$
then have $inv\ (inv\ x) = inv\ (inv\ y)$ **by** *simp*
with *G* **show** $x = y$ **by** *simp*

qed

lemma (**in** *monoid*) *Units-inv-comm*:
  assumes *inv*: $x \otimes y = \mathbf{1}$
    and *G*: $x \in$ *Units G* $y \in$ *Units G*
  shows $y \otimes x = \mathbf{1}$
proof −
  from *G* **have** $x \otimes y \otimes x = x \otimes \mathbf{1}$ **by** (*auto simp add*: *inv Units-closed*)
  with *G* **show** *?thesis* **by** (*simp del*: *r-one add*: *m-assoc Units-closed*)
qed

Power

lemma (**in** *monoid*) *nat-pow-closed* [*intro*, *simp*]:
  $x \in$ *carrier G* ==> x (ˆ) (n::*nat*) $\in$ *carrier G*
  by (*induct n*) (*simp-all add*: *nat-pow-def*)

lemma (**in** *monoid*) *nat-pow-0* [*simp*]:
  x (ˆ) (*0*::*nat*) $= \mathbf{1}$
  by (*simp add*: *nat-pow-def*)

lemma (**in** *monoid*) *nat-pow-Suc* [*simp*]:
  x (ˆ) (*Suc n*) $=$ x (ˆ) $n \otimes x$
  by (*simp add*: *nat-pow-def*)

lemma (**in** *monoid*) *nat-pow-one* [*simp*]:
  $\mathbf{1}$ (ˆ) (n::*nat*) $= \mathbf{1}$
  by (*induct n*) *simp-all*

lemma (**in** *monoid*) *nat-pow-mult*:
  $x \in$ *carrier G* ==> x (ˆ) (n::*nat*) $\otimes$ x (ˆ) m $=$ x (ˆ) ($n + m$)
  by (*induct m*) (*simp-all add*: *m-assoc* [*THEN sym*])

lemma (**in** *monoid*) *nat-pow-pow*:
  $x \in$ *carrier G* ==> (x (ˆ) n) (ˆ) m $=$ x (ˆ) ($n * m$::*nat*)
  by (*induct m*) (*simp*, *simp add*: *nat-pow-mult add-commute*)

A group is a monoid all of whose elements are invertible.

locale *group* = *monoid* +
  assumes *Units*: *carrier G* <= *Units G*

lemma (**in** *group*) *is-group*: *group G* **by** *fact*

theorem *groupI*:
  fixes *G* (**structure**)
  assumes *m-closed* [*simp*]:
      !!*x y*. [| $x \in$ *carrier G*; $y \in$ *carrier G* |] ==> $x \otimes y \in$ *carrier G*
    and *one-closed* [*simp*]: $\mathbf{1} \in$ *carrier G*
    and *m-assoc*:

```
    !!x y z. [| x ∈ carrier G; y ∈ carrier G; z ∈ carrier G |] ==>
    (x ⊗ y) ⊗ z = x ⊗ (y ⊗ z)
  and l-one [simp]: !!x. x ∈ carrier G ==> 1 ⊗ x = x
  and l-inv-ex: !!x. x ∈ carrier G ==> ∃ y ∈ carrier G. y ⊗ x = 1
 shows group G
proof −
 have l-cancel [simp]:
  !!x y z. [| x ∈ carrier G; y ∈ carrier G; z ∈ carrier G |] ==>
  (x ⊗ y = x ⊗ z) = (y = z)
 proof
  fix x y z
  assume eq: x ⊗ y = x ⊗ z
   and G: x ∈ carrier G  y ∈ carrier G  z ∈ carrier G
  with l-inv-ex obtain x-inv where xG: x-inv ∈ carrier G
   and l-inv: x-inv ⊗ x = 1 by fast
  from G eq xG have (x-inv ⊗ x) ⊗ y = (x-inv ⊗ x) ⊗ z
   by (simp add: m-assoc)
  with G show y = z by (simp add: l-inv)
 next
  fix x y z
  assume eq: y = z
   and G: x ∈ carrier G  y ∈ carrier G  z ∈ carrier G
  then show x ⊗ y = x ⊗ z by simp
 qed
 have r-one:
  !!x. x ∈ carrier G ==> x ⊗ 1 = x
 proof −
  fix x
  assume x: x ∈ carrier G
  with l-inv-ex obtain x-inv where xG: x-inv ∈ carrier G
   and l-inv: x-inv ⊗ x = 1 by fast
  from x xG have x-inv ⊗ (x ⊗ 1) = x-inv ⊗ x
   by (simp add: m-assoc [symmetric] l-inv)
  with x xG show x ⊗ 1 = x by simp
 qed
 have inv-ex:
  !!x. x ∈ carrier G ==> ∃ y ∈ carrier G. y ⊗ x = 1 & x ⊗ y = 1
 proof −
  fix x
  assume x: x ∈ carrier G
  with l-inv-ex obtain y where y: y ∈ carrier G
   and l-inv: y ⊗ x = 1 by fast
  from x y have y ⊗ (x ⊗ y) = y ⊗ 1
   by (simp add: m-assoc [symmetric] l-inv r-one)
  with x y have r-inv: x ⊗ y = 1
   by simp
  from x y show ∃ y ∈ carrier G. y ⊗ x = 1 & x ⊗ y = 1
   by (fast intro: l-inv r-inv)
 qed
```

then have *carrier-subset-Units*: *carrier G* <= *Units G*
  by (*unfold Units-def*) *fast*
show *?thesis*
  by (*fast intro*!: *group.intro monoid.intro group-axioms.intro*
    carrier-subset-Units intro: *prems r-one*)
qed

lemma (**in** *monoid*) *monoid-groupI*:
  assumes *l-inv-ex*:
    !!*x*. $x \in$ *carrier G* ==> $\exists\, y \in$ *carrier G*. $y \otimes x = \mathbf{1}$
  shows *group G*
  by (*rule groupI*) (*auto intro*: *m-assoc l-inv-ex*)

lemma (**in** *group*) *Units-eq* [*simp*]:
  Units G = *carrier G*
proof
  show *Units G* <= *carrier G* **by** *fast*
next
  show *carrier G* <= *Units G* **by** (*rule Units*)
qed

lemma (**in** *group*) *inv-closed* [*intro*, *simp*]:
  $x \in$ *carrier G* ==> *inv x* $\in$ *carrier G*
  using *Units-inv-closed* **by** *simp*

lemma (**in** *group*) *l-inv-ex* [*simp*]:
  $x \in$ *carrier G* ==> $\exists\, y \in$ *carrier G*. $y \otimes x = \mathbf{1}$
  using *Units-l-inv-ex* **by** *simp*

lemma (**in** *group*) *r-inv-ex* [*simp*]:
  $x \in$ *carrier G* ==> $\exists\, y \in$ *carrier G*. $x \otimes y = \mathbf{1}$
  using *Units-r-inv-ex* **by** *simp*

lemma (**in** *group*) *l-inv* [*simp*]:
  $x \in$ *carrier G* ==> *inv x* $\otimes\, x = \mathbf{1}$
  using *Units-l-inv* **by** *simp*

2.2 Cancellation Laws and Basic Properties

lemma (**in** *group*) *l-cancel* [*simp*]:
  [| $x \in$ *carrier G*; $y \in$ *carrier G*; $z \in$ *carrier G* |] ==>
    $(x \otimes y = x \otimes z) = (y = z)$
  using *Units-l-inv* **by** *simp*

lemma (**in** *group*) *r-inv* [*simp*]:
  $x \in$ *carrier G* ==> $x \otimes$ *inv x* $= \mathbf{1}$
proof −
  assume *x*: $x \in$ *carrier G*
  then have *inv x* $\otimes$ ($x \otimes$ *inv x*) = *inv x* $\otimes\, \mathbf{1}$

```
    by (simp add: m-assoc [symmetric] l-inv)
  with x show ?thesis by (simp del: r-one)
qed

lemma (in group) r-cancel [simp]:
  [| x ∈ carrier G; y ∈ carrier G; z ∈ carrier G |] ==>
   (y ⊗ x = z ⊗ x) = (y = z)
proof
  assume eq: y ⊗ x = z ⊗ x
    and G: x ∈ carrier G  y ∈ carrier G  z ∈ carrier G
  then have y ⊗ (x ⊗ inv x) = z ⊗ (x ⊗ inv x)
    by (simp add: m-assoc [symmetric] del: r-inv)
  with G show y = z by simp
next
  assume eq: y = z
    and G: x ∈ carrier G  y ∈ carrier G  z ∈ carrier G
  then show y ⊗ x = z ⊗ x by simp
qed

lemma (in group) inv-one [simp]:
  inv 1 = 1
proof -
  have inv 1 = 1 ⊗ (inv 1) by (simp del: r-inv)
  moreover have ... = 1 by simp
  finally show ?thesis .
qed

lemma (in group) inv-inv [simp]:
  x ∈ carrier G ==> inv (inv x) = x
  using Units-inv-inv by simp

lemma (in group) inv-inj:
  inj-on (m-inv G) (carrier G)
  using inv-inj-on-Units by simp

lemma (in group) inv-mult-group:
  [| x ∈ carrier G; y ∈ carrier G |] ==> inv (x ⊗ y) = inv y ⊗ inv x
proof -
  assume G: x ∈ carrier G  y ∈ carrier G
  then have inv (x ⊗ y) ⊗ (x ⊗ y) = (inv y ⊗ inv x) ⊗ (x ⊗ y)
    by (simp add: m-assoc l-inv) (simp add: m-assoc [symmetric])
  with G show ?thesis by (simp del: l-inv)
qed

lemma (in group) inv-comm:
  [| x ⊗ y = 1; x ∈ carrier G; y ∈ carrier G |] ==> y ⊗ x = 1
  by (rule Units-inv-comm) auto

lemma (in group) inv-equality:
```

```
  [|y ⊗ x = 1; x ∈ carrier G; y ∈ carrier G|] ==> inv x = y
apply (simp add: m-inv-def)
apply (rule the-equality)
 apply (simp add: inv-comm [of y x])
apply (rule r-cancel [THEN iffD1], auto)
done
```

Power

```
lemma (in group) int-pow-def2:
  a (^) (z::int) = (if neg z then inv (a (^) (nat (−z))) else a (^) (nat z))
  by (simp add: int-pow-def nat-pow-def Let-def)

lemma (in group) int-pow-0 [simp]:
  x (^) (0::int) = 1
  by (simp add: int-pow-def2)

lemma (in group) int-pow-one [simp]:
  1 (^) (z::int) = 1
  by (simp add: int-pow-def2)
```

2.3 Subgroups

```
locale subgroup =
  fixes H and G (structure)
  assumes subset: H ⊆ carrier G
    and m-closed [intro, simp]: [[x ∈ H; y ∈ H]] ⟹ x ⊗ y ∈ H
    and one-closed [simp]: 1 ∈ H
    and m-inv-closed [intro,simp]: x ∈ H ⟹ inv x ∈ H

lemma (in subgroup) is-subgroup:
  subgroup H G by fact

declare (in subgroup) group.intro [intro]

lemma (in subgroup) mem-carrier [simp]:
  x ∈ H ⟹ x ∈ carrier G
  using subset by blast

lemma subgroup-imp-subset:
  subgroup H G ⟹ H ⊆ carrier G
  by (rule subgroup.subset)

lemma (in subgroup) subgroup-is-group [intro]:
  includes group G
  shows group (G(|carrier := H|))
  by (rule groupI) (auto intro: m-assoc l-inv mem-carrier)
```

Since H is nonempty, it contains some element x. Since it is closed under inverse, it contains $inv\ x$. Since it is closed under product, it contains $x \otimes inv\ x = \mathbf{1}$.

lemma (**in** *group*) *one-in-subset*:
$[| \ H \subseteq carrier\ G;\ H \neq \{\};\ \forall\, a \in H.\ inv\ a \in H;\ \forall\, a{\in}H.\ \forall\, b{\in}H.\ a \otimes b \in H\ |]$
$==> \mathbf{1} \in H$
by (*force simp add*: *l-inv*)

A characterization of subgroups: closed, non-empty subset.

lemma (**in** *group*) *subgroupI*:
assumes *subset*: $H \subseteq carrier\ G$ **and** *non-empty*: $H \neq \{\}$
and *inv*: $!!a.\ a \in H \Longrightarrow inv\ a \in H$
and *mult*: $!!a\ b.\ [\![a \in H;\ b \in H]\!] \Longrightarrow a \otimes b \in H$
shows *subgroup H G*
proof (*simp add*: *subgroup-def prems*)
show $\mathbf{1} \in H$ **by** (*rule one-in-subset*) (*auto simp only*: *prems*)
qed

declare *monoid.one-closed* [*iff*] *group.inv-closed* [*simp*]
monoid.l-one [*simp*] *monoid.r-one* [*simp*] *group.inv-inv* [*simp*]

lemma *subgroup-nonempty*:
$\sim$ *subgroup* $\{\}$ *G*
by (*blast dest*: *subgroup.one-closed*)

lemma (**in** *subgroup*) *finite-imp-card-positive*:
finite (*carrier G*) $==> 0 < card\ H$
proof (*rule classical*)
assume *finite* (*carrier G*) $\sim\ 0 < card\ H$
then have *finite H* **by** (*blast intro*: *finite-subset* [*OF subset*])
with *prems* **have** *subgroup* $\{\}$ *G* **by** *simp*
with *subgroup-nonempty* **show** *?thesis* **by** *contradiction*
qed

2.4 Direct Products

constdefs
$DirProd :: \text{-} \Rightarrow \text{-} \Rightarrow ('a \times 'b)\ monoid$ (**infixr** $\times\times$ *80*)
$G \times\times H \equiv (\!|carrier = carrier\ G \times carrier\ H,$
$mult = (\lambda(g,\ h)\ (g',\ h').\ (g \otimes_G g',\ h \otimes_H h')),$
$one = (\mathbf{1}_G,\ \mathbf{1}_H)|\!)$

lemma *DirProd-monoid*:
includes *monoid G* + *monoid H*
shows *monoid* ($G \times\times H$)
proof −
from *prems*
show *?thesis* **by** (*unfold monoid-def DirProd-def*, *auto*)
qed

Does not use the previous result because it's easier just to use auto.

lemma *DirProd-group*:

includes *group G* + *group H*
shows *group* (*G* ×× *H*)
by (*rule groupI*)
(*auto intro*: *G.m-assoc H.m-assoc G.l-inv H.l-inv*
simp add: *DirProd-def*)

lemma *carrier-DirProd* [*simp*]:
carrier (*G* ×× *H*) = *carrier G* × *carrier H*
by (*simp add*: *DirProd-def*)

lemma *one-DirProd* [*simp*]:
$\mathbf{1}_{G\ \times\times\ H} = (\mathbf{1}_G, \mathbf{1}_H)$
by (*simp add*: *DirProd-def*)

lemma *mult-DirProd* [*simp*]:
$(g, h) \otimes_{(G\ \times\times\ H)} (g', h') = (g \otimes_G g', h \otimes_H h')$
by (*simp add*: *DirProd-def*)

lemma *inv-DirProd* [*simp*]:
includes *group G* + *group H*
assumes *g*: *g* ∈ *carrier G*
and *h*: *h* ∈ *carrier H*
shows *m-inv* (*G* ×× *H*) (*g*, *h*) = (inv_G *g*, inv_H *h*)
apply (*rule group.inv-equality* [*OF DirProd-group*])
apply (*simp-all add*: *prems group.l-inv*)
done

This alternative proof of the previous result demonstrates interpret. It uses *Prod.inv-equality* (available after *interpret*) instead of *group.inv-equality* [*OF DirProd-group*].

lemma
includes *group G* + *group H*
assumes *g*: *g* ∈ *carrier G*
and *h*: *h* ∈ *carrier H*
shows *m-inv* (*G* ×× *H*) (*g*, *h*) = (inv_G *g*, inv_H *h*)
proof −
interpret *Prod*: *group* [*G* ×× *H*]
by (*auto intro*: *DirProd-group group.intro group.axioms prems*)
show *?thesis* **by** (*simp add*: *Prod.inv-equality g h*)
qed

2.5 Homomorphisms and Isomorphisms

constdefs (**structure** *G* **and** *H*)
hom :: - => - => (*'a* => *'b*) *set*
hom G H ==
{*h*. *h* ∈ *carrier G* −> *carrier H* &
(∀ *x* ∈ *carrier G*. ∀ *y* ∈ *carrier G*. $h\ (x \otimes_G y) = h\ x \otimes_H h\ y$)}

lemma *hom-mult*:
  [| $h \in hom\ G\ H$; $x \in carrier\ G$; $y \in carrier\ G$ |]
  ==> $h\ (x \otimes_G y) = h\ x \otimes_H h\ y$
  by (*simp add*: *hom-def*)

lemma *hom-closed*:
  [| $h \in hom\ G\ H$; $x \in carrier\ G$ |] ==> $h\ x \in carrier\ H$
  by (*auto simp add*: *hom-def funcset-mem*)

lemma (**in** *group*) *hom-compose*:
    [|$h \in hom\ G\ H$; $i \in hom\ H\ I$|] ==> *compose* (*carrier G*) $i\ h \in hom\ G\ I$
apply (*auto simp add*: *hom-def funcset-compose*)
apply (*simp add*: *compose-def funcset-mem*)
done

constdefs
  iso :: - => - => ('*a* => '*b*) *set* (**infixr** ≅ *60*)
  $G \cong H$ == {*h*. $h \in hom\ G\ H$ & *bij-betw h* (*carrier G*) (*carrier H*)}

lemma *iso-refl*: (%*x*. *x*) $\in G \cong G$
by (*simp add*: *iso-def hom-def inj-on-def bij-betw-def Pi-def*)

lemma (**in** *group*) *iso-sym*:
    $h \in G \cong H \Longrightarrow$ *Inv* (*carrier G*) $h \in H \cong G$
apply (*simp add*: *iso-def bij-betw-Inv*)
apply (*subgoal-tac Inv* (*carrier G*) $h \in carrier\ H \to carrier\ G$)
 prefer *2* **apply** (*simp add*: *bij-betw-imp-funcset* [*OF bij-betw-Inv*])
apply (*simp add*: *hom-def bij-betw-def Inv-f-eq funcset-mem f-Inv-f*)
done

lemma (**in** *group*) *iso-trans*:
    [|$h \in G \cong H$; $i \in H \cong I$|] ==> (*compose* (*carrier G*) *i h*) $\in G \cong I$
by (*auto simp add*: *iso-def hom-compose bij-betw-compose*)

lemma *DirProd-commute-iso*:
  shows $(\lambda(x,y).\ (y,x)) \in (G \times\times H) \cong (H \times\times G)$
by (*auto simp add*: *iso-def hom-def inj-on-def bij-betw-def Pi-def*)

lemma *DirProd-assoc-iso*:
  shows $(\lambda(x,y,z).\ (x,(y,z))) \in (G \times\times H \times\times I) \cong (G \times\times (H \times\times I))$
by (*auto simp add*: *iso-def hom-def inj-on-def bij-betw-def Pi-def*)

Basis for homomorphism proofs: we assume two groups G and H, with a homomorphism h between them

locale *group-hom* = *group G* + *group H* + *var h* +
  assumes *homh*: $h \in hom\ G\ H$
  notes *hom-mult* [*simp*] = *hom-mult* [*OF homh*]
    and *hom-closed* [*simp*] = *hom-closed* [*OF homh*]

lemma (**in** *group-hom*) *one-closed* [*simp*]:
$h\ \mathbf{1} \in carrier\ H$
by *simp*

lemma (**in** *group-hom*) *hom-one* [*simp*]:
$h\ \mathbf{1} = \mathbf{1}_H$
proof −
have $h\ \mathbf{1} \otimes_H \mathbf{1}_H = h\ \mathbf{1} \otimes_H h\ \mathbf{1}$
by (*simp add*: *hom-mult* [*symmetric*] *del*: *hom-mult*)
then show *?thesis* **by** (*simp del*: *r-one*)
qed

lemma (**in** *group-hom*) *inv-closed* [*simp*]:
$x \in carrier\ G$ ==> $h\ (inv\ x) \in carrier\ H$
by *simp*

lemma (**in** *group-hom*) *hom-inv* [*simp*]:
$x \in carrier\ G$ ==> $h\ (inv\ x) = inv_H\ (h\ x)$
proof −
assume x: $x \in carrier\ G$
then have $h\ x \otimes_H h\ (inv\ x) = \mathbf{1}_H$
by (*simp add*: *hom-mult* [*symmetric*] *del*: *hom-mult*)
also from x **have** ... $= h\ x \otimes_H inv_H\ (h\ x)$
by (*simp add*: *hom-mult* [*symmetric*] *del*: *hom-mult*)
finally have $h\ x \otimes_H h\ (inv\ x) = h\ x \otimes_H inv_H\ (h\ x)$ **.**
with x **show** *?thesis* **by** (*simp del*: *H.r-inv*)
qed

2.6 Commutative Structures

Naming convention: multiplicative structures that are commutative are called *commutative*, additive structures are called *Abelian*.

locale *comm-monoid* = *monoid* +
assumes *m-comm*: $[\![x \in carrier\ G;\ y \in carrier\ G]\!] \Longrightarrow x \otimes y = y \otimes x$

lemma (**in** *comm-monoid*) *m-lcomm*:
$[\![x \in carrier\ G;\ y \in carrier\ G;\ z \in carrier\ G]\!] \Longrightarrow$
$x \otimes (y \otimes z) = y \otimes (x \otimes z)$
proof −
assume *xyz*: $x \in carrier\ G\ \ y \in carrier\ G\ \ z \in carrier\ G$
from *xyz* **have** $x \otimes (y \otimes z) = (x \otimes y) \otimes z$ **by** (*simp add*: *m-assoc*)
also from *xyz* **have** ... $= (y \otimes x) \otimes z$ **by** (*simp add*: *m-comm*)
also from *xyz* **have** ... $= y \otimes (x \otimes z)$ **by** (*simp add*: *m-assoc*)
finally show *?thesis* **.**
qed

lemmas (**in** *comm-monoid*) *m-ac* = *m-assoc m-comm m-lcomm*

lemma *comm-monoidI*:

fixes G (**structure**)
assumes *m-closed*:
$!!x\ y.\ [|\ x \in carrier\ G;\ y \in carrier\ G\ |] ==> x \otimes y \in carrier\ G$
and *one-closed*: $\mathbf{1} \in carrier\ G$
and *m-assoc*:
$!!x\ y\ z.\ [|\ x \in carrier\ G;\ y \in carrier\ G;\ z \in carrier\ G\ |] ==>$
$(x \otimes y) \otimes z = x \otimes (y \otimes z)$
and *l-one*: $!!x.\ x \in carrier\ G ==> \mathbf{1} \otimes x = x$
and *m-comm*:
$!!x\ y.\ [|\ x \in carrier\ G;\ y \in carrier\ G\ |] ==> x \otimes y = y \otimes x$
shows *comm-monoid G*
using *l-one*
by (*auto intro*!: *comm-monoid.intro comm-monoid-axioms.intro monoid.intro intro*: *prems simp*: *m-closed one-closed m-comm*)

lemma (**in** *monoid*) *monoid-comm-monoidI*:
assumes *m-comm*:
$!!x\ y.\ [|\ x \in carrier\ G;\ y \in carrier\ G\ |] ==> x \otimes y = y \otimes x$
shows *comm-monoid G*
by (*rule comm-monoidI*) (*auto intro*: *m-assoc m-comm*)

lemma (**in** *comm-monoid*) *nat-pow-distr*:
$[|\ x \in carrier\ G;\ y \in carrier\ G\ |] ==>$
$(x \otimes y)\ (\hat{})\ (n::nat) = x\ (\hat{})\ n \otimes y\ (\hat{})\ n$
by (*induct n*) (*simp*, *simp add*: *m-ac*)

locale *comm-group* = *comm-monoid* + *group*

lemma (**in** *group*) *group-comm-groupI*:
assumes *m-comm*: $!!x\ y.\ [|\ x \in carrier\ G;\ y \in carrier\ G\ |] ==>$
$x \otimes y = y \otimes x$
shows *comm-group G*
by *unfold-locales* (*simp-all add*: *m-comm*)

lemma *comm-groupI*:
fixes G (**structure**)
assumes *m-closed*:
$!!x\ y.\ [|\ x \in carrier\ G;\ y \in carrier\ G\ |] ==> x \otimes y \in carrier\ G$
and *one-closed*: $\mathbf{1} \in carrier\ G$
and *m-assoc*:
$!!x\ y\ z.\ [|\ x \in carrier\ G;\ y \in carrier\ G;\ z \in carrier\ G\ |] ==>$
$(x \otimes y) \otimes z = x \otimes (y \otimes z)$
and *m-comm*:
$!!x\ y.\ [|\ x \in carrier\ G;\ y \in carrier\ G\ |] ==> x \otimes y = y \otimes x$
and *l-one*: $!!x.\ x \in carrier\ G ==> \mathbf{1} \otimes x = x$
and *l-inv-ex*: $!!x.\ x \in carrier\ G ==> \exists\, y \in carrier\ G.\ y \otimes x = \mathbf{1}$
shows *comm-group G*

by (*fast intro*: *group.group-comm-groupI groupI prems*)

lemma (**in** *comm-group*) *inv-mult*:
[| $x \in$ *carrier G*; $y \in$ *carrier G* |] ==> *inv* $(x \otimes y)$ = *inv* $x \otimes$ *inv* y
by (*simp add*: *m-ac inv-mult-group*)

2.7 The Lattice of Subgroups of a Group

theorem (**in** *group*) *subgroups-partial-order*:
partial-order (| *carrier* = $\{H.$ *subgroup H G*$\}$, *le* = *op* $\subseteq$ |)
by (*rule partial-order.intro*) *simp-all*

lemma (**in** *group*) *subgroup-self*:
subgroup (*carrier G*) *G*
by (*rule subgroupI*) *auto*

lemma (**in** *group*) *subgroup-imp-group*:
subgroup H G ==> *group* (*G*(| *carrier* := *H* |))
by (*erule subgroup.subgroup-is-group*) (*rule* ‹*group G*›)

lemma (**in** *group*) *is-monoid* [*intro*, *simp*]:
monoid G
by (*auto intro*: *monoid.intro m-assoc*)

lemma (**in** *group*) *subgroup-inv-equality*:
[| *subgroup H G*; $x \in H$ |] ==> *m-inv* (*G* (| *carrier* := *H* |)) x = *inv* x
apply (*rule-tac inv-equality* [*THEN sym*])
apply (*rule group.l-inv* [*OF subgroup-imp-group*, *simplified*], *assumption+*)
apply (*rule subsetD* [*OF subgroup.subset*], *assumption+*)
apply (*rule subsetD* [*OF subgroup.subset*], *assumption*)
apply (*rule-tac group.inv-closed* [*OF subgroup-imp-group*, *simplified*], *assumption+*)
done

theorem (**in** *group*) *subgroups-Inter*:
assumes *subgr*: (!!H. $H \in A$ ==> *subgroup H G*)
and *not-empty*: A ~= {}
shows *subgroup* $(\bigcap A)$ *G*
proof (*rule subgroupI*)
from *subgr* [*THEN subgroup.subset*] **and** *not-empty*
show $\bigcap A \subseteq$ *carrier G* **by** *blast*
next
from *subgr* [*THEN subgroup.one-closed*]
show $\bigcap A$ ~= {} **by** *blast*
next
fix x **assume** $x \in \bigcap A$
with *subgr* [*THEN subgroup.m-inv-closed*]
show *inv* $x \in \bigcap A$ **by** *blast*
next
fix x y **assume** $x \in \bigcap A$ $y \in \bigcap A$

```
  with subgr [THEN subgroup.m-closed]
  show x ⊗ y ∈ ⋂A by blast
qed

theorem (in group) subgroups-complete-lattice:
  complete-lattice (| carrier = {H. subgroup H G}, le = op ⊆ |)
    (is complete-lattice ?L)
proof (rule partial-order.complete-lattice-criterion1)
  show partial-order ?L by (rule subgroups-partial-order)
next
  have greatest ?L (carrier G) (carrier ?L)
    by (unfold greatest-def)
      (simp add: subgroup.subset subgroup-self)
  then show ∃G. greatest ?L G (carrier ?L) ..
next
  fix A
  assume L: A ⊆ carrier ?L and non-empty: A ~= {}
  then have Int-subgroup: subgroup (⋂A) G
    by (fastsimp intro: subgroups-Inter)
  have greatest ?L (⋂A) (Lower ?L A)
    (is greatest - ?Int -)
  proof (rule greatest-LowerI)
    fix H
    assume H: H ∈ A
    with L have subgroupH: subgroup H G by auto
    from subgroupH have groupH: group (G (| carrier := H |)) (is group ?H)
      by (rule subgroup-imp-group)
    from groupH have monoidH: monoid ?H
      by (rule group.is-monoid)
    from H have Int-subset: ?Int ⊆ H by fastsimp
    then show le ?L ?Int H by simp
  next
    fix H
    assume H: H ∈ Lower ?L A
    with L Int-subgroup show le ?L H ?Int
      by (fastsimp simp: Lower-def intro: Inter-greatest)
  next
    show A ⊆ carrier ?L by (rule L)
  next
    show ?Int ∈ carrier ?L by simp (rule Int-subgroup)
  qed
  then show ∃I. greatest ?L I (Lower ?L A) ..
qed

end
```

```
theory FiniteProduct imports Group begin
```

3 Product Operator for Commutative Monoids

3.1 Inductive Definition of a Relation for Products over Sets

Instantiation of locale *LC* of theory *Finite-Set* is not possible, because here we have explicit typing rules like $x \in$ *carrier G*. We introduce an explicit argument for the domain *D*.

```
inductive-set
  foldSetD :: ['a set, 'b => 'a => 'a, 'a] => ('b set * 'a) set
  for D :: 'a set and f :: 'b => 'a => 'a and e :: 'a
  where
    emptyI [intro]: e ∈ D ==> ({}, e) ∈ foldSetD D f e
  | insertI [intro]: [| x ~: A; f x y ∈ D; (A, y) ∈ foldSetD D f e |] ==>
                      (insert x A, f x y) ∈ foldSetD D f e

inductive-cases empty-foldSetDE [elim!]: ({}, x) ∈ foldSetD D f e

constdefs
  foldD :: ['a set, 'b => 'a => 'a, 'a, 'b set] => 'a
  foldD D f e A == THE x. (A, x) ∈ foldSetD D f e

lemma foldSetD-closed:
  [| (A, z) ∈ foldSetD D f e ; e ∈ D; !!x y. [| x ∈ A; y ∈ D |] ==> f x y ∈ D
     |] ==> z ∈ D
  by (erule foldSetD.cases) auto

lemma Diff1-foldSetD:
  [| (A − {x}, y) ∈ foldSetD D f e; x ∈ A; f x y ∈ D |] ==>
   (A, f x y) ∈ foldSetD D f e
  apply (erule insert-Diff [THEN subst], rule foldSetD.intros)
    apply auto
  done

lemma foldSetD-imp-finite [simp]: (A, x) ∈ foldSetD D f e ==> finite A
  by (induct set: foldSetD) auto

lemma finite-imp-foldSetD:
  [| finite A; e ∈ D; !!x y. [| x ∈ A; y ∈ D |] ==> f x y ∈ D |] ==>
   EX x. (A, x) ∈ foldSetD D f e
proof (induct set: finite)
  case empty then show ?case by auto
next
  case (insert x F)
  then obtain y where y: (F, y) ∈ foldSetD D f e by auto
  with insert have y ∈ D by (auto dest: foldSetD-closed)
  with y and insert have (insert x F, f x y) ∈ foldSetD D f e
    by (intro foldSetD.intros) auto
  then show ?case ..
qed
```

3.2 Left-Commutative Operations

```
locale LCD =
  fixes B :: 'b set
  and D :: 'a set
  and f :: 'b => 'a => 'a    (infixl · 70)
  assumes left-commute:
    [| x ∈ B; y ∈ B; z ∈ D |] ==> x · (y · z) = y · (x · z)
  and f-closed [simp, intro!]: !!x y. [| x ∈ B; y ∈ D |] ==> f x y ∈ D

lemma (in LCD) foldSetD-closed [dest]:
  (A, z) ∈ foldSetD D f e ==> z ∈ D
  by (erule foldSetD.cases) auto

lemma (in LCD) Diff1-foldSetD:
  [| (A − {x}, y) ∈ foldSetD D f e; x ∈ A; A ⊆ B |] ==>
  (A, f x y) ∈ foldSetD D f e
  apply (subgoal-tac x ∈ B)
   prefer 2 apply fast
  apply (erule insert-Diff [THEN subst], rule foldSetD.intros)
    apply auto
  done

lemma (in LCD) foldSetD-imp-finite [simp]:
  (A, x) ∈ foldSetD D f e ==> finite A
  by (induct set: foldSetD) auto

lemma (in LCD) finite-imp-foldSetD:
  [| finite A; A ⊆ B; e ∈ D |] ==> EX x. (A, x) ∈ foldSetD D f e
proof (induct set: finite)
  case empty then show ?case by auto
next
  case (insert x F)
  then obtain y where y: (F, y) ∈ foldSetD D f e by auto
  with insert have y ∈ D by auto
  with y and insert have (insert x F, f x y) ∈ foldSetD D f e
    by (intro foldSetD.intros) auto
  then show ?case ..
qed

lemma (in LCD) foldSetD-determ-aux:
  e ∈ D ==> ∀ A x. A ⊆ B & card A < n −−> (A, x) ∈ foldSetD D f e −−>
    (∀ y. (A, y) ∈ foldSetD D f e −−> y = x)
  apply (induct n)
   apply (auto simp add: less-Suc-eq)
  apply (erule foldSetD.cases)
   apply blast
  apply (erule foldSetD.cases)
   apply blast
  apply clarify
```

force simplification of *card* $A <$ *card* (*insert* ...).

apply (*erule rev-mp*)
apply (*simp add*: *less-Suc-eq-le*)
apply (*rule impI*)
apply (*rename-tac xa Aa ya xb Ab yb*, *case-tac xa* = *xb*)
apply (*subgoal-tac Aa* = *Ab*)
prefer *2* **apply** (*blast elim*!: *equalityE*)
apply *blast*

case $xa \notin xb$.

apply (*subgoal-tac* $Aa - \{xb\} = Ab - \{xa\}$ & $xb \in Aa$ & $xa \in Ab$)
prefer *2* **apply** (*blast elim*!: *equalityE*)
apply *clarify*
apply (*subgoal-tac Aa* = *insert xb Ab* $- \{xa\}$)
prefer *2* **apply** *blast*
apply (*subgoal-tac card Aa* $\leq$ *card Ab*)
prefer *2*
apply (*rule Suc-le-mono* [*THEN subst*])
apply (*simp add*: *card-Suc-Diff1*)
apply (*rule-tac A1* = $Aa - \{xb\}$ **in** *finite-imp-foldSetD* [*THEN exE*])
apply (*blast intro*: *foldSetD-imp-finite finite-Diff*)
apply *best*
apply *assumption*
apply (*frule* (*1*) *Diff1-foldSetD*)
apply *best*
apply (*subgoal-tac ya* = *f xb x*)
prefer *2*
apply (*subgoal-tac Aa* $\subseteq$ *B*)
prefer *2* **apply** *best*
apply (*blast del*: *equalityCE*)
apply (*subgoal-tac* ($Ab - \{xa\}$, x) $\in$ *foldSetD D f e*)
prefer *2* **apply** *simp*
apply (*subgoal-tac yb* = *f xa x*)
prefer *2*
apply (*blast del*: *equalityCE dest*: *Diff1-foldSetD*)
apply (*simp* (*no-asm-simp*))
apply (*rule left-commute*)
apply *assumption*
apply *best*
apply *best*
done

lemma (**in** *LCD*) *foldSetD-determ*:
[| (A, x) $\in$ *foldSetD D f e*; (A, y) $\in$ *foldSetD D f e*; $e \in D$; $A \subseteq B$ |]
==> $y = x$
by (*blast intro*: *foldSetD-determ-aux* [*rule-format*])

lemma (**in** *LCD*) *foldD-equality*:
[| (A, y) $\in$ *foldSetD D f e*; $e \in D$; $A \subseteq B$ |] ==> *foldD D f e A* = y

```
  by (unfold foldD-def) (blast intro: foldSetD-determ)

lemma foldD-empty [simp]:
  e ∈ D ==> foldD D f e {} = e
  by (unfold foldD-def) blast

lemma (in LCD) foldD-insert-aux:
  [| x ~: A; x ∈ B; e ∈ D; A ⊆ B |] ==>
    ((insert x A, v) ∈ foldSetD D f e) =
    (EX y. (A, y) ∈ foldSetD D f e & v = f x y)
  apply auto
  apply (rule-tac A1 = A in finite-imp-foldSetD [THEN exE])
     apply (fastsimp dest: foldSetD-imp-finite)
    apply assumption
   apply assumption
  apply (blast intro: foldSetD-determ)
  done

lemma (in LCD) foldD-insert:
    [| finite A; x ~: A; x ∈ B; e ∈ D; A ⊆ B |] ==>
     foldD D f e (insert x A) = f x (foldD D f e A)
  apply (unfold foldD-def)
  apply (simp add: foldD-insert-aux)
  apply (rule the-equality)
   apply (auto intro: finite-imp-foldSetD
     cong add: conj-cong simp add: foldD-def [symmetric] foldD-equality)
  done

lemma (in LCD) foldD-closed [simp]:
  [| finite A; e ∈ D; A ⊆ B |] ==> foldD D f e A ∈ D
proof (induct set: finite)
  case empty then show ?case by (simp add: foldD-empty)
next
  case insert then show ?case by (simp add: foldD-insert)
qed

lemma (in LCD) foldD-commute:
  [| finite A; x ∈ B; e ∈ D; A ⊆ B |] ==>
   f x (foldD D f e A) = foldD D f (f x e) A
  apply (induct set: finite)
   apply simp
  apply (auto simp add: left-commute foldD-insert)
  done

lemma Int-mono2:
  [| A ⊆ C; B ⊆ C |] ==> A Int B ⊆ C
  by blast

lemma (in LCD) foldD-nest-Un-Int:
```

[| *finite* A; *finite* C; $e \in D$; $A \subseteq B$; $C \subseteq B$ |] ==>
foldD D f (*foldD* D f e C) A = *foldD* D f (*foldD* D f e (A *Int* C)) (A *Un* C)
apply (*induct set*: *finite*)
apply *simp*
apply (*simp add*: *foldD-insert foldD-commute Int-insert-left insert-absorb*
Int-mono2 Un-subset-iff)
done

lemma (**in** *LCD*) *foldD-nest-Un-disjoint*:
[| *finite* A; *finite* B; A *Int* $B = \{\}$; $e \in D$; $A \subseteq B$; $C \subseteq B$ |]
==> *foldD* D f e (A *Un* B) = *foldD* D f (*foldD* D f e B) A
by (*simp add*: *foldD-nest-Un-Int*)

— Delete rules to do with *foldSetD* relation.

declare *foldSetD-imp-finite* [*simp del*]
empty-foldSetDE [*rule del*]
foldSetD.intros [*rule del*]
declare (**in** *LCD*)
foldSetD-closed [*rule del*]

3.3 Commutative Monoids

We enter a more restrictive context, with f :: $'a$ => $'a$ => $'a$ instead of $'b$ => $'a$ => $'a$.

locale *ACeD* =
fixes D :: $'a$ *set*
and f :: $'a$ => $'a$ => $'a$ (**infixl** $\cdot$ *70*)
and e :: $'a$
assumes *ident* [*simp*]: $x \in D$ ==> $x \cdot e = x$
and *commute*: [| $x \in D$; $y \in D$ |] ==> $x \cdot y = y \cdot x$
and *assoc*: [| $x \in D$; $y \in D$; $z \in D$ |] ==> $(x \cdot y) \cdot z = x \cdot (y \cdot z)$
and *e-closed* [*simp*]: $e \in D$
and *f-closed* [*simp*]: [| $x \in D$; $y \in D$ |] ==> $x \cdot y \in D$

lemma (**in** *ACeD*) *left-commute*:
[| $x \in D$; $y \in D$; $z \in D$ |] ==> $x \cdot (y \cdot z) = y \cdot (x \cdot z)$
proof −
assume D: $x \in D$ $y \in D$ $z \in D$
then have $x \cdot (y \cdot z) = (y \cdot z) \cdot x$ **by** (*simp add*: *commute*)
also from D **have** ... $= y \cdot (z \cdot x)$ **by** (*simp add*: *assoc*)
also from D **have** $z \cdot x = x \cdot z$ **by** (*simp add*: *commute*)
finally show *?thesis* **.**
qed

lemmas (**in** *ACeD*) *AC* = *assoc commute left-commute*

lemma (**in** *ACeD*) *left-ident* [*simp*]: $x \in D$ ==> $e \cdot x = x$
proof −

assume $x \in D$
then have $x \cdot e = x$ **by** (*rule ident*)
with ‹$x \in D$› **show** *?thesis* **by** (*simp add*: *commute*)
qed

lemma (**in** *ACeD*) *foldD-Un-Int*:
[| *finite A*; *finite B*; $A \subseteq D$; $B \subseteq D$ |] ==>
foldD D f e A · *foldD D f e B* =
foldD D f e (*A Un B*) · *foldD D f e* (*A Int B*)
apply (*induct set*: *finite*)
apply (*simp add*: *left-commute LCD.foldD-closed* [*OF LCD.intro* [*of D*]])
apply (*simp add*: *AC insert-absorb Int-insert-left*
LCD.foldD-insert [*OF LCD.intro* [*of D*]]
LCD.foldD-closed [*OF LCD.intro* [*of D*]]
Int-mono2 Un-subset-iff)
done

lemma (**in** *ACeD*) *foldD-Un-disjoint*:
[| *finite A*; *finite B*; *A Int B* = {}; $A \subseteq D$; $B \subseteq D$ |] ==>
foldD D f e (*A Un B*) = *foldD D f e A* · *foldD D f e B*
by (*simp add*: *foldD-Un-Int*
left-commute LCD.foldD-closed [*OF LCD.intro* [*of D*]] *Un-subset-iff*)

3.4 Products over Finite Sets

constdefs (**structure** *G*)
finprod :: [('*b*, '*m*) *monoid-scheme*, '*a* => '*b*, '*a set*] => '*b*
finprod G f A == *if finite A*
then foldD (*carrier G*) (*mult G o f*) **1** *A*
else arbitrary

syntax
-finprod :: *index* => *idt* => '*a set* => '*b* => '*b*
((*3*⨂ --:-. -) [*1000*, *0*, *51*, *10*] *10*)
syntax (*xsymbols*)
-finprod :: *index* => *idt* => '*a set* => '*b* => '*b*
((*3*⨂ --∈-. -) [*1000*, *0*, *51*, *10*] *10*)
syntax (*HTML* **output**)
-finprod :: *index* => *idt* => '*a set* => '*b* => '*b*
((*3*⨂ --∈-. -) [*1000*, *0*, *51*, *10*] *10*)
translations
⨂ı*i*:*A*. *b* == *finprod* ◇ı (%*i*. *b*) *A*
— Beware of argument permutation!

lemma (**in** *comm-monoid*) *finprod-empty* [*simp*]:
finprod G f {} = **1**
by (*simp add*: *finprod-def*)

declare *funcsetI* [*intro*]

funcset-mem [*dest*]

lemma (**in** *comm-monoid*) *finprod-insert* [*simp*]:
[| *finite F*; $a \notin F$; $f \in F$ −> *carrier G*; $f\ a \in$ *carrier G* |] ==>
finprod G f (*insert a F*) = $f\ a \otimes$ *finprod G f F*
apply (*rule trans*)
apply (*simp add*: *finprod-def*)
apply (*rule trans*)
apply (*rule LCD.foldD-insert* [*OF LCD.intro* [*of insert a F*]])
apply *simp*
apply (*rule m-lcomm*)
apply *fast*
apply *fast*
apply *assumption*
apply (*fastsimp intro*: *m-closed*)
apply *simp+*
apply *fast*
apply (*auto simp add*: *finprod-def*)
done

lemma (**in** *comm-monoid*) *finprod-one* [*simp*]:
finite A ==> ($\bigotimes$ *i*:*A*. **1**) = **1**
proof (*induct set*: *finite*)
case *empty* **show** *?case* **by** *simp*
next
case (*insert a A*)
have (%*i*. **1**) $\in$ *A* −> *carrier G* **by** *auto*
with *insert* **show** *?case* **by** *simp*
qed

lemma (**in** *comm-monoid*) *finprod-closed* [*simp*]:
fixes *A*
assumes *fin*: *finite A* **and** *f*: $f \in A$ −> *carrier G*
shows *finprod G f A* $\in$ *carrier G*
using *fin f*
proof *induct*
case *empty* **show** *?case* **by** *simp*
next
case (*insert a A*)
then have *a*: $f\ a \in$ *carrier G* **by** *fast*
from *insert* **have** *A*: $f \in A$ −> *carrier G* **by** *fast*
from *insert A a* **show** *?case* **by** *simp*
qed

lemma *funcset-Int-left* [*simp*, *intro*]:
[| $f \in A$ −> *C*; $f \in B$ −> *C* |] ==> $f \in A$ *Int B* −> *C*
by *fast*

lemma *funcset-Un-left* [*iff*]:

$(f \in A\ Un\ B\ {-}{>}\ C) = (f \in A\ {-}{>}\ C\ \&\ f \in B\ {-}{>}\ C)$
by *fast*

lemma (**in** *comm-monoid*) *finprod-Un-Int*:
[| *finite A*; *finite B*; $g \in A\ {-}{>}$ *carrier G*; $g \in B\ {-}{>}$ *carrier G* |] ==>
finprod G g (*A Un B*) $\otimes$ *finprod G g* (*A Int B*) =
finprod G g A $\otimes$ *finprod G g B*
— The reversed orientation looks more natural, but LOOPS as a simprule!
proof (*induct set*: *finite*)
case *empty* **then show** *?case* **by** (*simp add*: *finprod-closed*)
next
case (*insert a A*)
then have *a*: $g\ a \in$ *carrier G* **by** *fast*
from *insert* **have** *A*: $g \in A\ {-}{>}$ *carrier G* **by** *fast*
from *insert A a* **show** *?case*
by (*simp add*: *m-ac Int-insert-left insert-absorb finprod-closed*
Int-mono2 Un-subset-iff)
qed

lemma (**in** *comm-monoid*) *finprod-Un-disjoint*:
[| *finite A*; *finite B*; *A Int B* = {};
$g \in A\ {-}{>}$ *carrier G*; $g \in B\ {-}{>}$ *carrier G* |]
==> *finprod G g* (*A Un B*) = *finprod G g A* $\otimes$ *finprod G g B*
apply (*subst finprod-Un-Int* [*symmetric*])
apply (*auto simp add*: *finprod-closed*)
done

lemma (**in** *comm-monoid*) *finprod-multf*:
[| *finite A*; $f \in A\ {-}{>}$ *carrier G*; $g \in A\ {-}{>}$ *carrier G* |] ==>
finprod G (%*x*. *f x* $\otimes$ *g x*) *A* = (*finprod G f A* $\otimes$ *finprod G g A*)
proof (*induct set*: *finite*)
case *empty* **show** *?case* **by** *simp*
next
case (*insert a A*) **then**
have *fA*: $f \in A\ {-}{>}$ *carrier G* **by** *fast*
from *insert* **have** *fa*: $f\ a \in$ *carrier G* **by** *fast*
from *insert* **have** *gA*: $g \in A\ {-}{>}$ *carrier G* **by** *fast*
from *insert* **have** *ga*: $g\ a \in$ *carrier G* **by** *fast*
from *insert* **have** *fgA*: (%*x*. *f x* $\otimes$ *g x*) $\in A\ {-}{>}$ *carrier G*
by (*simp add*: *Pi-def*)
show *?case*
by (*simp add*: *insert fA fa gA ga fgA m-ac*)
qed

lemma (**in** *comm-monoid*) *finprod-cong'*:
[| *A* = *B*; $g \in B\ {-}{>}$ *carrier G*;
!!*i*. $i \in B$ ==> *f i* = *g i* |] ==> *finprod G f A* = *finprod G g B*
proof −
assume *prems*: *A* = *B* $g \in B\ {-}{>}$ *carrier G*

```
    !!i. i ∈ B ==> f i = g i
  show ?thesis
  proof (cases finite B)
    case True
    then have !!A. [| A = B; g ∈ B -> carrier G;
      !!i. i ∈ B ==> f i = g i |] ==> finprod G f A = finprod G g B
    proof induct
      case empty thus ?case by simp
    next
      case (insert x B)
      then have finprod G f A = finprod G f (insert x B) by simp
      also from insert have ... = f x ⊗ finprod G f B
      proof (intro finprod-insert)
        show finite B by fact
      next
        show x ~: B by fact
      next
        assume x ~: B !!i. i ∈ insert x B ⟹ f i = g i
          g ∈ insert x B → carrier G
        thus f ∈ B -> carrier G by fastsimp
      next
        assume x ~: B !!i. i ∈ insert x B ⟹ f i = g i
          g ∈ insert x B → carrier G
        thus f x ∈ carrier G by fastsimp
      qed
      also from insert have ... = g x ⊗ finprod G g B by fastsimp
      also from insert have ... = finprod G g (insert x B)
      by (intro finprod-insert [THEN sym]) auto
      finally show ?case .
    qed
    with prems show ?thesis by simp
  next
    case False with prems show ?thesis by (simp add: finprod-def)
  qed
qed

lemma (in comm-monoid) finprod-cong:
  [| A = B; f ∈ B -> carrier G = True;
     !!i. i ∈ B ==> f i = g i |] ==> finprod G f A = finprod G g B

  by (rule finprod-cong') force+
```

Usually, if this rule causes a failed congruence proof error, the reason is that the premise *g* ∈ *B* −> *carrier G* cannot be shown. Adding *Pi-def* to the simpset is often useful. For this reason, *comm-monoid.finprod-cong* is not added to the simpset by default.

```
declare funcsetI [rule del]
  funcset-mem [rule del]
```

```
lemma (in comm-monoid) finprod-0 [simp]:
  f ∈ {0::nat} −> carrier G ==> finprod G f {..0} = f 0
by (simp add: Pi-def)

lemma (in comm-monoid) finprod-Suc [simp]:
  f ∈ {..Suc n} −> carrier G ==>
   finprod G f {..Suc n} = (f (Suc n) ⊗ finprod G f {..n})
by (simp add: Pi-def atMost-Suc)

lemma (in comm-monoid) finprod-Suc2:
  f ∈ {..Suc n} −> carrier G ==>
   finprod G f {..Suc n} = (finprod G (%i. f (Suc i)) {..n} ⊗ f 0)
proof (induct n)
  case 0 thus ?case by (simp add: Pi-def)
next
  case Suc thus ?case by (simp add: m-assoc Pi-def)
qed

lemma (in comm-monoid) finprod-mult [simp]:
  [| f ∈ {..n} −> carrier G; g ∈ {..n} −> carrier G |] ==>
     finprod G (%i. f i ⊗ g i) {..n::nat} =
     finprod G f {..n} ⊗ finprod G g {..n}
  by (induct n) (simp-all add: m-ac Pi-def)

end
```

```
theory Exponent imports Main Primes Binomial begin
```

4 The Combinatorial Argument Underlying the First Sylow Theorem

```
definition exponent :: nat => nat => nat where
exponent p s == if prime p then (GREATEST r. p^r dvd s) else 0
```

4.1 Prime Theorems

```
lemma prime-imp-one-less: prime p ==> Suc 0 < p
by (unfold prime-def, force)

lemma prime-iff:
  (prime p) = (Suc 0 < p & (∀ a b. p dvd a*b −−> (p dvd a) | (p dvd b)))
apply (auto simp add: prime-imp-one-less)
apply (blast dest!: prime-dvd-mult)
apply (auto simp add: prime-def)
apply (erule dvdE)
```

apply (*case-tac k=0*, *simp*)
apply (*drule-tac x* = *m* **in** *spec*)
apply (*drule-tac x* = *k* **in** *spec*)
apply (*simp add*: *dvd-mult-cancel1 dvd-mult-cancel2*)
done

lemma *zero-less-prime-power*: *prime p* ==> *0* < *p*ˆ*a*
by (*force simp add*: *prime-iff*)

lemma *zero-less-card-empty*: [| *finite S*; *S* ≠ {} |] ==> *0* < *card*(*S*)
by (*rule ccontr*, *simp*)

lemma *prime-dvd-cases*:
 [| *p*∗*k dvd m*∗*n*; *prime p* |]
 ==> (∃ *x*. *k dvd x*∗*n* & *m* = *p*∗*x*) | (∃ *y*. *k dvd m*∗*y* & *n* = *p*∗*y*)
apply (*simp add*: *prime-iff*)
apply (*frule dvd-mult-left*)
apply (*subgoal-tac p dvd m* | *p dvd n*)
 prefer *2* **apply** *blast*
apply (*erule disjE*)
apply (*rule disjI1*)
apply (*rule-tac* [*2*] *disjI2*)
apply (*erule-tac n* = *m* **in** *dvdE*)
apply (*erule-tac* [*2*] *n* = *n* **in** *dvdE*, *auto*)
done

lemma *prime-power-dvd-cases* [*rule-format* (*no-asm*)]: *prime p*
 ==> ∀ *m n*. *p*ˆ*c dvd m*∗*n* −−>
 (∀ *a b*. *a*+*b* = *Suc c* −−> *p*ˆ*a dvd m* | *p*ˆ*b dvd n*)
apply (*induct-tac c*)
 apply *clarify*
 apply (*case-tac a*)
 apply *simp*
 apply *simp*

apply *simp*
apply *clarify*
apply (*erule prime-dvd-cases* [*THEN disjE*], *assumption*, *auto*)

 apply (*case-tac a*)
 apply *simp*
 apply *clarify*
 apply (*drule spec*, *drule spec*, *erule* (*1*) *notE impE*)
 apply (*drule-tac x* = *nat* **in** *spec*)
 apply (*drule-tac x* = *b* **in** *spec*)
 apply *simp*

apply (*case-tac b*)
apply *simp*
apply *clarify*
apply (*drule spec*, *drule spec*, *erule* (*1*) *notE impE*)
apply (*drule-tac x* = *a* **in** *spec*)
apply (*drule-tac x* = *nat* **in** *spec*, *simp*)
done

lemma *div-combine*:
[| *prime p*; ~ (*p* ^ (*Suc r*) *dvd n*); *p*^(*a*+*r*) *dvd n*∗*k* |]
==> *p* ^ *a dvd k*
by (*drule-tac a* = *Suc r* **and** *b* = *a* **in** *prime-power-dvd-cases*, *assumption*, *auto*)

lemma *Suc-le-power*: *Suc 0* < *p* ==> *Suc n* <= *p*^*n*
apply (*induct-tac n*)
apply (*simp* (*no-asm-simp*))
apply *simp*
apply (*subgoal-tac 2* ∗ *n* + *2* <= *p* ∗ *p*^*n*, *simp*)
apply (*subgoal-tac 2* ∗ *p*^*n* <= *p* ∗ *p*^*n*)
apply *arith*
apply (*drule-tac k* = *2* **in** *mult-le-mono2*, *simp*)
done

lemma *power-dvd-bound*: [|*p*^*n dvd a*; *Suc 0* < *p*; *a* > *0*|] ==> *n* < *a*
apply (*drule dvd-imp-le*)
apply (*drule-tac* [*2*] *n* = *n* **in** *Suc-le-power*, *auto*)
done

4.2 Exponent Theorems

lemma *exponent-ge* [*rule-format*]:
[|*p*^*k dvd n*; *prime p*; *0*<*n*|] ==> *k* <= *exponent p n*
apply (*simp add*: *exponent-def*)
apply (*erule Greatest-le*)
apply (*blast dest*: *prime-imp-one-less power-dvd-bound*)
done

lemma *power-exponent-dvd*: *s*>*0* ==> (*p* ^ *exponent p s*) *dvd s*
apply (*simp add*: *exponent-def*)
apply *clarify*
apply (*rule-tac k* = *0* **in** *GreatestI*)
prefer *2* **apply** (*blast dest*: *prime-imp-one-less power-dvd-bound*, *simp*)
done

lemma *power-Suc-exponent-Not-dvd*:

[|(p ∗ p ˆ *exponent p s*) *dvd s*; *prime p* |] ==> *s=0*
apply (*subgoal-tac p* ˆ *Suc* (*exponent p s*) *dvd s*)
prefer *2* **apply** *simp*
apply (*rule ccontr*)
apply (*drule exponent-ge*, *auto*)
done

lemma *exponent-power-eq* [*simp*]: *prime p* ==> *exponent p* (*p*ˆ*a*) = *a*
apply (*simp* (*no-asm-simp*) *add*: *exponent-def*)
apply (*rule Greatest-equality*, *simp*)
apply (*simp* (*no-asm-simp*) *add*: *prime-imp-one-less power-dvd-imp-le*)
done

lemma *exponent-equalityI*:
!*r*::*nat*. (*p*ˆ*r dvd a*) = (*p*ˆ*r dvd b*) ==> *exponent p a* = *exponent p b*
by (*simp* (*no-asm-simp*) *add*: *exponent-def*)

lemma *exponent-eq-0* [*simp*]: ¬ *prime p* ==> *exponent p s* = *0*
by (*simp* (*no-asm-simp*) *add*: *exponent-def*)

lemma *exponent-mult-add1*: [| *a* > *0*; *b* > *0* |]
==> (*exponent p a*) + (*exponent p b*) <= *exponent p* (*a* ∗ *b*)
apply (*case-tac prime p*)
apply (*rule exponent-ge*)
apply (*auto simp add*: *power-add*)
apply (*blast intro*: *prime-imp-one-less power-exponent-dvd mult-dvd-mono*)
done

lemma *exponent-mult-add2*: [| *a* > *0*; *b* > *0* |]
==> *exponent p* (*a* ∗ *b*) <= (*exponent p a*) + (*exponent p b*)
apply (*case-tac prime p*)
apply (*rule leI*, *clarify*)
apply (*cut-tac p* = *p* **and** *s* = *a*∗*b* **in** *power-exponent-dvd*, *auto*)
apply (*subgoal-tac p* ˆ (*Suc* (*exponent p a* + *exponent p b*)) *dvd a* ∗ *b*)
apply (*rule-tac* [*2*] *le-imp-power-dvd* [*THEN dvd-trans*])
prefer *3* **apply** *assumption*
prefer *2* **apply** *simp*
apply (*frule-tac a* = *Suc* (*exponent p a*) **and** *b* = *Suc* (*exponent p b*) **in** *prime-power-dvd-cases*)
apply (*assumption*, *force*, *simp*)
apply (*blast dest*: *power-Suc-exponent-Not-dvd*)
done

lemma *exponent-mult-add*: [| *a* > *0*; *b* > *0* |]
==> *exponent p* (*a* ∗ *b*) = (*exponent p a*) + (*exponent p b*)
by (*blast intro*: *exponent-mult-add1 exponent-mult-add2 order-antisym*)

lemma *not-divides-exponent-0*: ~ (*p dvd n*) ==> *exponent p n* = *0*
apply (*case-tac exponent p n*, *simp*)
apply (*case-tac n*, *simp*)
apply (*cut-tac s* = *n* **and** *p* = *p* **in** *power-exponent-dvd*)
apply (*auto dest*: *dvd-mult-left*)
done

lemma *exponent-1-eq-0* [*simp*]: *exponent p* (*Suc 0*) = *0*
apply (*case-tac prime p*)
apply (*auto simp add*: *prime-iff not-divides-exponent-0*)
done

4.3 Main Combinatorial Argument

lemma *le-extend-mult*: [| *c* > *0*; *a* <= *b* |] ==> *a* <= *b* * (*c*::*nat*)
apply (*rule-tac P* = %*x*. *x* <= *b* * *c* **in** *subst*)
apply (*rule mult-1-right*)
apply (*rule mult-le-mono*, *auto*)
done

lemma *p-fac-forw-lemma*:
[| (*m*::*nat*) > *0*; *k* > *0*; *k* < *p*ˆ*a*; (*p*ˆ*r*) *dvd* (*p*ˆ*a*)* *m* − *k* |] ==> *r* <= *a*
apply (*rule notnotD*)
apply (*rule notI*)
apply (*drule contrapos-nn* [*OF* - *leI*, *THEN notnotD*], *assumption*)
apply (*drule less-imp-le* [*of a*])
apply (*drule le-imp-power-dvd*)
apply (*drule-tac n* = *p* ˆ *r* **in** *dvd-trans*, *assumption*)
apply(*metis dvd-diffD1 dvd-triv-right le-extend-mult linorder-linear linorder-not-less mult-commute nat-dvd-not-less neq0-conv*)
done

lemma *p-fac-forw*: [| (*m*::*nat*) > *0*; *k*>*0*; *k* < *p*ˆ*a*; (*p*ˆ*r*) *dvd* (*p*ˆ*a*)* *m* − *k* |]
==> (*p*ˆ*r*) *dvd* (*p*ˆ*a*) − *k*
apply (*frule-tac k1* = *k* **and** *i* = *p* **in** *p-fac-forw-lemma* [*THEN le-imp-power-dvd*], *auto*)
apply (*subgoal-tac p*ˆ*r dvd p*ˆ*a***m*)
prefer *2* **apply** (*blast intro*: *dvd-mult2*)
apply (*drule dvd-diffD1*)
apply *assumption*
prefer *2* **apply** (*blast intro*: *dvd-diff*)
apply (*drule gr0-implies-Suc*, *auto*)
done

lemma *r-le-a-forw*:
[| (*k*::*nat*) > *0*; *k* < *p*ˆ*a*; *p*>*0*; (*p*ˆ*r*) *dvd* (*p*ˆ*a*) − *k* |] ==> *r* <= *a*

by (*rule-tac m* = *Suc 0* **in** *p-fac-forw-lemma*, *auto*)

lemma *p-fac-backw*: [| *m*>*0*; *k*>*0*; (*p*::*nat*)≠*0*; *k* < *p* ^ *a*; (*p* ^ *r*) *dvd p* ^ *a* − *k* |]
==> (*p* ^ *r*) *dvd* (*p* ^ *a*)∗*m* − *k*
apply (*frule-tac k1* = *k* **and** *i* = *p* **in** *r-le-a-forw* [*THEN le-imp-power-dvd*], *auto*)
apply (*subgoal-tac p* ^ *r dvd p* ^ *a*∗*m*)
prefer *2* **apply** (*blast intro*: *dvd-mult2*)
apply (*drule dvd-diffD1*)
apply *assumption*
prefer *2* **apply** (*blast intro*: *dvd-diff*)
apply (*drule less-imp-Suc-add*, *auto*)
done

lemma *exponent-p-a-m-k-equation*: [| *m*>*0*; *k*>*0*; (*p*::*nat*)≠*0*; *k* < *p* ^ *a* |]
==> *exponent p* (*p* ^ *a* ∗ *m* − *k*) = *exponent p* (*p* ^ *a* − *k*)
apply (*blast intro*: *exponent-equalityI p-fac-forw p-fac-backw*)
done

Suc rules that we have to delete from the simpset

lemmas *bad-Sucs* = *binomial-Suc-Suc mult-Suc mult-Suc-right*

lemma *p-not-div-choose-lemma* [*rule-format*]:
[| ∀ *i*. *Suc i* < *K* −−> *exponent p* (*Suc i*) = *exponent p* (*Suc*(*j*+*i*))|]
==> *k*<*K* −−> *exponent p* ((*j*+*k*) *choose k*) = *0*
apply (*case-tac prime p*)
prefer *2* **apply** *simp*
apply (*induct-tac k*)
apply (*simp* (*no-asm*))

apply (*subgoal-tac* (*Suc* (*j*+*n*) *choose Suc n*) > *0*)
prefer *2* **apply** (*simp add*: *zero-less-binomial-iff*, *clarify*)
apply (*subgoal-tac exponent p* ((*Suc* (*j*+*n*) *choose Suc n*) ∗ *Suc n*) =
exponent p (*Suc n*))

First, use the assumed equation. We simplify the LHS to *exponent p* (*Suc* (*j* + *n*) *choose Suc n*) + *exponent p* (*Suc n*) the common terms cancel, proving the conclusion.

apply (*simp del*: *bad-Sucs add*: *exponent-mult-add*)

Establishing the equation requires first applying *Suc-times-binomial-eq* ...

apply (*simp del*: *bad-Sucs add*: *Suc-times-binomial-eq* [*symmetric*])

...then *exponent-mult-add* and the quantified premise.

apply (*simp del*: *bad-Sucs add*: *zero-less-binomial-iff exponent-mult-add*)
done

lemma *p-not-div-choose*:

```
  [| k<K;  k<=n;
      ∀ j. 0<j & j<K --> exponent p (n - k + (K - j)) = exponent p (K -
j)|]
   ==> exponent p (n choose k) = 0
apply (cut-tac j = n-k and k = k and p = p in p-not-div-choose-lemma)
  prefer 3 apply simp
 prefer 2 apply assumption
apply (drule-tac x = K - Suc i in spec)
apply (simp add: Suc-diff-le)
done


lemma const-p-fac-right:
  m>0 ==> exponent p ((p^a * m - Suc 0) choose (p^a - Suc 0)) = 0
apply (case-tac prime p)
 prefer 2 apply simp
apply (frule-tac a = a in zero-less-prime-power)
apply (rule-tac K = p^a in p-not-div-choose)
   apply simp
  apply simp
 apply (case-tac m)
  apply (case-tac [2] p^a)
   apply auto

apply (subgoal-tac 0<p)
 prefer 2 apply (force dest!: prime-imp-one-less)
apply (subst exponent-p-a-m-k-equation, auto)
done

lemma const-p-fac:
  m>0 ==> exponent p (((p^a) * m) choose p^a) = exponent p m
apply (case-tac prime p)
 prefer 2 apply simp
apply (subgoal-tac 0 < p^a * m & p^a <= p^a * m)
 prefer 2 apply (force simp add: prime-iff)
```

A similar trick to the one used in *p-not-div-choose-lemma*: insert an equation; use *exponent-mult-add* on the LHS; on the RHS, first transform the binomial coefficient, then use *exponent-mult-add*.

```
apply (subgoal-tac exponent p ((( (p^a) * m) choose p^a) * p^a) =
                   a + exponent p m)
 apply (simp del: bad-Sucs add: zero-less-binomial-iff exponent-mult-add prime-iff)
```

one subgoal left!

```
apply (subst times-binomial-minus1-eq, simp, simp)
apply (subst exponent-mult-add, simp)
apply (simp (no-asm-simp) add: zero-less-binomial-iff)
apply arith
apply (simp del: bad-Sucs add: exponent-mult-add const-p-fac-right)
```

done

end

theory *Coset* **imports** *Group Exponent* **begin**

5 Cosets and Quotient Groups

constdefs (**structure** G)
  r-coset :: [-, 'a set, 'a] ⇒ 'a set (**infixl** #>ı *60*)
  H #> a ≡ $\bigcup h{\in}H.\ \{h \otimes a\}$

  l-coset :: [-, 'a, 'a set] ⇒ 'a set (**infixl** <#ı *60*)
  a <# H ≡ $\bigcup h{\in}H.\ \{a \otimes h\}$

  RCOSETS :: [-, 'a set] ⇒ ('a set)set (*rcosets*ı - [*81*] *80*)
  rcosets H ≡ $\bigcup a{\in}carrier\ G.\ \{H$ #> $a\}$

  set-mult :: [-, 'a set ,'a set] ⇒ 'a set (**infixl** <#>ı *60*)
  H <#> K ≡ $\bigcup h{\in}H.\ \bigcup k{\in}K.\ \{h \otimes k\}$

  SET-INV :: [-,'a set] ⇒ 'a set (*set'-inv*ı - [*81*] *80*)
  set-inv H ≡ $\bigcup h{\in}H.\ \{inv\ h\}$

locale *normal* = *subgroup* + *group* +
  assumes *coset-eq*: (∀ x ∈ *carrier* G. H #> x = x <# H)

abbreviation
  normal-rel :: ['a set, ('a, 'b) *monoid-scheme*] ⇒ *bool* (**infixl** ◁ *60*) **where**
  $H \lhd G \equiv$ *normal* H G

5.1 Basic Properties of Cosets

lemma (**in** *group*) *coset-mult-assoc*:
  [| $M \subseteq$ *carrier* G; $g \in$ *carrier* G; $h \in$ *carrier* G |]
  ==> (M #> g) #> h = M #> ($g \otimes h$)
by (*force simp add*: *r-coset-def m-assoc*)

lemma (**in** *group*) *coset-mult-one* [*simp*]: $M \subseteq$ *carrier* G ==> M #> **1** = M
by (*force simp add*: *r-coset-def*)

lemma (**in** *group*) *coset-mult-inv1*:
  [| M #> ($x \otimes$ (*inv* y)) = M; $x \in$ *carrier* G ; $y \in$ *carrier* G;
    $M \subseteq$ *carrier* G |] ==> M #> x = M #> y
apply (*erule subst* [*of* **concl**: %z. M #> x = z #> y])

apply (*simp add*: *coset-mult-assoc m-assoc*)
done

lemma (**in** *group*) *coset-mult-inv2*:
$[| \ M \ \#> x = M \ \#> y; \ \ x \in carrier \ G; \ \ y \in carrier \ G; \ \ M \subseteq carrier \ G \ |]$
$==> M \ \#> (x \otimes (inv \ y)) = M$
apply (*simp add*: *coset-mult-assoc* [*symmetric*])
apply (*simp add*: *coset-mult-assoc*)
done

lemma (**in** *group*) *coset-join1*:
$[| \ H \ \#> x = H; \ \ x \in carrier \ G; \ \ subgroup \ H \ G \ |] ==> x \in H$
apply (*erule subst*)
apply (*simp add*: *r-coset-def*)
apply (*blast intro*: *l-one subgroup.one-closed sym*)
done

lemma (**in** *group*) *solve-equation*:
$[\![subgroup \ H \ G; x \in H; y \in H]\!] \Longrightarrow \exists \, h{\in}H. \ y = h \otimes x$
apply (*rule bexI* [*of* - $y \otimes (inv \ x)$])
apply (*auto simp add*: *subgroup.m-closed subgroup.m-inv-closed m-assoc*
subgroup.subset [*THEN subsetD*])
done

lemma (**in** *group*) *repr-independence*:
$[\![y \in H \ \#> x; \ \ x \in carrier \ G; subgroup \ H \ G]\!] \Longrightarrow H \ \#> x = H \ \#> y$
by (*auto simp add*: *r-coset-def m-assoc* [*symmetric*]
subgroup.subset [*THEN subsetD*]
subgroup.m-closed solve-equation)

lemma (**in** *group*) *coset-join2*:
$[\![x \in carrier \ G; \ \ subgroup \ H \ G; \ \ x{\in}H]\!] \Longrightarrow H \ \#> x = H$
— Alternative proof is to put $x = \mathbf{1}$ in *repr-independence*.
by (*force simp add*: *subgroup.m-closed r-coset-def solve-equation*)

lemma (**in** *monoid*) *r-coset-subset-G*:
$[| \ H \subseteq carrier \ G; x \in carrier \ G \ |] ==> H \ \#> x \subseteq carrier \ G$
by (*auto simp add*: *r-coset-def*)

lemma (**in** *group*) *rcosI*:
$[| \ h \in H; H \subseteq carrier \ G; x \in carrier \ G|] ==> h \otimes x \in H \ \#> x$
by (*auto simp add*: *r-coset-def*)

lemma (**in** *group*) *rcosetsI*:
$[\![H \subseteq carrier \ G; x \in carrier \ G]\!] \Longrightarrow H \ \#> x \in rcosets \ H$
by (*auto simp add*: *RCOSETS-def*)

Really needed?

lemma (**in** *group*) *transpose-inv*:

```
  [| x ⊗ y = z;  x ∈ carrier G;  y ∈ carrier G;  z ∈ carrier G |]
   ==> (inv x) ⊗ z = y
by (force simp add: m-assoc [symmetric])

lemma (in group) rcos-self: [| x ∈ carrier G; subgroup H G |] ==> x ∈ H #> x
apply (simp add: r-coset-def)
apply (blast intro: sym l-one subgroup.subset [THEN subsetD]
                    subgroup.one-closed)
done
```

Opposite of *repr-independence*

```
lemma (in group) repr-independenceD:
  includes subgroup H G
  assumes ycarr: y ∈ carrier G
      and repr:  H #> x = H #> y
  shows y ∈ H #> x
  apply (subst repr)
  apply (intro rcos-self)
   apply (rule ycarr)
   apply (rule is-subgroup)
  done
```

Elements of a right coset are in the carrier

```
lemma (in subgroup) elemrcos-carrier:
  includes group
  assumes acarr: a ∈ carrier G
    and a': a' ∈ H #> a
  shows a' ∈ carrier G
proof -
  from subset and acarr
  have H #> a ⊆ carrier G by (rule r-coset-subset-G)
  from this and a'
  show a' ∈ carrier G
    by fast
qed

lemma (in subgroup) rcos-const:
  includes group
  assumes hH: h ∈ H
  shows H #> h = H
  apply (unfold r-coset-def)
  apply rule
   apply rule
   apply clarsimp
   apply (intro subgroup.m-closed)
     apply (rule is-subgroup)
    apply assumption
   apply (rule hH)
  apply rule
```

```
  apply simp
proof -
  fix h'
  assume h'H: h' ∈ H
  note carr = hH[THEN mem-carrier] h'H[THEN mem-carrier]
  from carr
  have a: h' = (h' ⊗ inv h) ⊗ h by (simp add: m-assoc)
  from h'H hH
  have h' ⊗ inv h ∈ H by simp
  from this and a
  show ∃x∈H. h' = x ⊗ h by fast
qed
```

Step one for lemma *rcos-module*

```
lemma (in subgroup) rcos-module-imp:
  includes group
  assumes xcarr: x ∈ carrier G
      and x'cos: x' ∈ H #> x
  shows (x' ⊗ inv x) ∈ H
proof -
  from xcarr x'cos
      have x'carr: x' ∈ carrier G
      by (rule elemrcos-carrier[OF is-group])
  from xcarr
      have ixcarr: inv x ∈ carrier G
      by simp
  from x'cos
      have ∃h∈H. x' = h ⊗ x
      unfolding r-coset-def
      by fast
  from this
      obtain h
        where hH: h ∈ H
        and x': x' = h ⊗ x
      by auto
  from hH and subset
      have hcarr: h ∈ carrier G by fast
  note carr = xcarr x'carr hcarr
  from x' and carr
      have x' ⊗ (inv x) = (h ⊗ x) ⊗ (inv x) by fast
  also from carr
      have ... = h ⊗ (x ⊗ inv x) by (simp add: m-assoc)
  also from carr
      have ... = h ⊗ 1 by simp
  also from carr
      have ... = h by simp
  finally
      have x' ⊗ (inv x) = h by simp
  from hH this
```

show $x' \otimes (inv\ x) \in H$ **by** *simp*
qed

Step two for lemma *rcos-module*

lemma (**in** *subgroup*) *rcos-module-rev*:
includes *group*
assumes *carr*: $x \in carrier\ G\ x' \in carrier\ G$
and *xixH*: $(x' \otimes inv\ x) \in H$
shows $x' \in H\ \#{>}\ x$
proof −
from *xixH*
have $\exists h \in H.\ x' \otimes (inv\ x) = h$ **by** *fast*
from *this*
obtain h
where *hH*: $h \in H$
and *hsym*: $x' \otimes (inv\ x) = h$
by *fast*
from *hH subset* **have** *hcarr*: $h \in carrier\ G$ **by** *simp*
note *carr* = *carr hcarr*
from *hsym*[*symmetric*] **have** $h \otimes x = x' \otimes (inv\ x) \otimes x$ **by** *fast*
also from *carr*
have $\ldots = x' \otimes ((inv\ x) \otimes x)$ **by** (*simp add*: *m-assoc*)
also from *carr*
have $\ldots = x' \otimes \mathbf{1}$ **by** (*simp add*: *l-inv*)
also from *carr*
have $\ldots = x'$ **by** *simp*
finally
have $h \otimes x = x'$ **by** *simp*
from *this*[*symmetric*] **and** *hH*
show $x' \in H\ \#{>}\ x$
unfolding *r-coset-def*
by *fast*
qed

Module property of right cosets

lemma (**in** *subgroup*) *rcos-module*:
includes *group*
assumes *carr*: $x \in carrier\ G\ x' \in carrier\ G$
shows $(x' \in H\ \#{>}\ x) = (x' \otimes inv\ x \in H)$
proof
assume $x' \in H\ \#{>}\ x$
from *this* **and** *carr*
show $x' \otimes inv\ x \in H$
by (*intro rcos-module-imp*[*OF is-group*])
next
assume $x' \otimes inv\ x \in H$
from *this* **and** *carr*
show $x' \in H\ \#{>}\ x$
by (*intro rcos-module-rev*[*OF is-group*])

qed

Right cosets are subsets of the carrier.

lemma (**in** *subgroup*) *rcosets-carrier*:
  includes *group*
  assumes *XH*: $X \in$ *rcosets H*
  shows $X \subseteq$ *carrier G*
proof $-$
  from *XH* **have** $\exists x\in$ *carrier G*. $X = H$ #> x
    unfolding *RCOSETS-def*
    by *fast*
  from *this*
    obtain x
      where *xcarr*: $x\in$ *carrier G*
      and *X*: $X = H$ #> x
    by *fast*
  from *subset* **and** *xcarr*
    show $X \subseteq$ *carrier G*
    unfolding *X*
    by (*rule r-coset-subset-G*)
qed

Multiplication of general subsets

lemma (**in** *monoid*) *set-mult-closed*:
  assumes *Acarr*: $A \subseteq$ *carrier G*
    and *Bcarr*: $B \subseteq$ *carrier G*
  shows A <#> $B \subseteq$ *carrier G*
apply *rule* **apply** (*simp add*: *set-mult-def*, *clarsimp*)
proof $-$
  fix *a b*
  assume $a \in A$
  from *this* **and** *Acarr*
    have *acarr*: $a \in$ *carrier G* **by** *fast*

  assume $b \in B$
  from *this* **and** *Bcarr*
    have *bcarr*: $b \in$ *carrier G* **by** *fast*

  from *acarr bcarr*
    show $a \otimes b \in$ *carrier G* **by** (*rule m-closed*)
qed

lemma (**in** *comm-group*) *mult-subgroups*:
  assumes *subH*: *subgroup H G*
    and *subK*: *subgroup K G*
  shows *subgroup* (H <#> K) *G*
apply (*rule subgroup.intro*)
  apply (*intro set-mult-closed subgroup.subset*[*OF subH*] *subgroup.subset*[*OF subK*])
  apply (*simp add*: *set-mult-def*) **apply** *clarsimp* **defer** *1*

apply (*simp add*: *set-mult-def*) **defer** *1*
apply (*simp add*: *set-mult-def*, *clarsimp*) **defer** *1*
proof −
fix *ha hb ka kb*
assume *haH*: $ha \in H$ **and** *hbH*: $hb \in H$ **and** *kaK*: $ka \in K$ **and** *kbK*: $kb \in K$
note *carr* = *haH*[*THEN subgroup.mem-carrier*[*OF subH*]] *hbH*[*THEN subgroup.mem-carrier*[*OF subH*]]
kaK[*THEN subgroup.mem-carrier*[*OF subK*]] *kbK*[*THEN subgroup.mem-carrier*[*OF subK*]]
from *carr*
have $(ha \otimes ka) \otimes (hb \otimes kb) = ha \otimes (ka \otimes hb) \otimes kb$ **by** (*simp add*: *m-assoc*)
also from *carr*
have $\ldots = ha \otimes (hb \otimes ka) \otimes kb$ **by** (*simp add*: *m-comm*)
also from *carr*
have $\ldots = (ha \otimes hb) \otimes (ka \otimes kb)$ **by** (*simp add*: *m-assoc*)
finally
have *eq*: $(ha \otimes ka) \otimes (hb \otimes kb) = (ha \otimes hb) \otimes (ka \otimes kb)$ **.**

from *haH hbH* **have** *hH*: $ha \otimes hb \in H$ **by** (*simp add*: *subgroup.m-closed*[*OF subH*])
from *kaK kbK* **have** *kK*: $ka \otimes kb \in K$ **by** (*simp add*: *subgroup.m-closed*[*OF subK*])

from *hH* **and** *kK* **and** *eq*
show $\exists h'{\in}H.\ \exists k'{\in}K.\ (ha \otimes ka) \otimes (hb \otimes kb) = h' \otimes k'$ **by** *fast*
next
have $\mathbf{1} = \mathbf{1} \otimes \mathbf{1}$ **by** *simp*
from *subgroup.one-closed*[*OF subH*] *subgroup.one-closed*[*OF subK*] *this*
show $\exists h{\in}H.\ \exists k{\in}K.\ \mathbf{1} = h \otimes k$ **by** *fast*
next
fix *h k*
assume *hH*: $h \in H$
and *kK*: $k \in K$

from *hH*[*THEN subgroup.mem-carrier*[*OF subH*]] *kK*[*THEN subgroup.mem-carrier*[*OF subK*]]
have $inv\ (h \otimes k) = inv\ h \otimes inv\ k$ **by** (*simp add*: *inv-mult-group m-comm*)

from *subgroup.m-inv-closed*[*OF subH hH*] **and** *subgroup.m-inv-closed*[*OF subK kK*] **and** *this*
show $\exists ha{\in}H.\ \exists ka{\in}K.\ inv\ (h \otimes k) = ha \otimes ka$ **by** *fast*
qed

lemma (**in** *subgroup*) *lcos-module-rev*:
includes *group*
assumes *carr*: $x \in carrier\ G\ x' \in carrier\ G$
and *xixH*: $(inv\ x \otimes x') \in H$
shows $x' \in x <\#\ H$
proof −

from *xixH*
  have $\exists\, h{\in}H.\ (inv\ x) \otimes x' = h$ **by** *fast*
from *this*
  obtain *h*
    where *hH*: $h \in H$
    and *hsym*: $(inv\ x) \otimes x' = h$
  by *fast*

from *hH subset* **have** *hcarr*: $h \in carrier\ G$ **by** *simp*
note *carr = carr hcarr*
from *hsym*[*symmetric*] **have** $x \otimes h = x \otimes ((inv\ x) \otimes x')$ **by** *fast*
also from *carr*
  have $\ldots = (x \otimes (inv\ x)) \otimes x'$ **by** (*simp add*: *m-assoc*[*symmetric*])
also from *carr*
  have $\ldots = \mathbf{1} \otimes x'$ **by** *simp*
also from *carr*
  have $\ldots = x'$ **by** *simp*
finally
  have $x \otimes h = x'$ **by** *simp*

from *this*[*symmetric*] **and** *hH*
  show $x' \in x <\#\ H$
  unfolding *l-coset-def*
  by *fast*
qed

5.2 Normal subgroups

lemma *normal-imp-subgroup*: $H \lhd G \Longrightarrow subgroup\ H\ G$
  by (*simp add*: *normal-def subgroup-def*)

lemma (**in** *group*) *normalI*:
  $subgroup\ H\ G \Longrightarrow (\forall\, x \in carrier\ G.\ H \#{>}\ x = x <\#\ H) \Longrightarrow H \lhd G$
  by (*simp add*: *normal-def normal-axioms-def prems*)

lemma (**in** *normal*) *inv-op-closed1*:
  $[\![x \in carrier\ G;\ h \in H]\!] \Longrightarrow (inv\ x) \otimes h \otimes x \in H$
apply (*insert coset-eq*)
apply (*auto simp add*: *l-coset-def r-coset-def*)
apply (*drule bspec*, *assumption*)
apply (*drule equalityD1* [*THEN subsetD*], *blast*, *clarify*)
apply (*simp add*: *m-assoc*)
apply (*simp add*: *m-assoc* [*symmetric*])
done

lemma (**in** *normal*) *inv-op-closed2*:
  $[\![x \in carrier\ G;\ h \in H]\!] \Longrightarrow x \otimes h \otimes (inv\ x) \in H$
apply (*subgoal-tac inv* $(inv\ x) \otimes h \otimes (inv\ x) \in H$)
apply (*simp add*:)

```
apply (blast intro: inv-op-closed1)
done
```

Alternative characterization of normal subgroups

```
lemma (in group) normal-inv-iff:
   (N ◁ G) =
    (subgroup N G & (∀ x ∈ carrier G. ∀ h ∈ N. x ⊗ h ⊗ (inv x) ∈ N))
    (is - = ?rhs)
proof
  assume N: N ◁ G
  show ?rhs
    by (blast intro: N normal.inv-op-closed2 normal-imp-subgroup)
next
  assume ?rhs
  hence sg: subgroup N G
    and closed: ⋀x. x∈carrier G ⟹ ∀ h∈N. x ⊗ h ⊗ inv x ∈ N by auto
  hence sb: N ⊆ carrier G by (simp add: subgroup.subset)
  show N ◁ G
  proof (intro normalI [OF sg], simp add: l-coset-def r-coset-def, clarify)
    fix x
    assume x: x ∈ carrier G
    show (⋃h∈N. {h ⊗ x}) = (⋃h∈N. {x ⊗ h})
    proof
      show (⋃h∈N. {h ⊗ x}) ⊆ (⋃h∈N. {x ⊗ h})
      proof clarify
        fix n
        assume n: n ∈ N
        show n ⊗ x ∈ (⋃h∈N. {x ⊗ h})
        proof
          from closed [of inv x]
          show inv x ⊗ n ⊗ x ∈ N by (simp add: x n)
          show n ⊗ x ∈ {x ⊗ (inv x ⊗ n ⊗ x)}
            by (simp add: x n m-assoc [symmetric] sb [THEN subsetD])
        qed
      qed
    next
      show (⋃h∈N. {x ⊗ h}) ⊆ (⋃h∈N. {h ⊗ x})
      proof clarify
        fix n
        assume n: n ∈ N
        show x ⊗ n ∈ (⋃h∈N. {h ⊗ x})
        proof
          show x ⊗ n ⊗ inv x ∈ N by (simp add: x n closed)
          show x ⊗ n ∈ {x ⊗ n ⊗ inv x ⊗ x}
            by (simp add: x n m-assoc sb [THEN subsetD])
        qed
      qed
    qed
  qed
```

qed

5.3 More Properties of Cosets

lemma (**in** *group*) *lcos-m-assoc*:
  [| $M \subseteq$ *carrier* G; $g \in$ *carrier* G; $h \in$ *carrier* G |]
  ==> g <# (h <# M) = ($g \otimes h$) <# M
by (*force simp add*: *l-coset-def m-assoc*)

lemma (**in** *group*) *lcos-mult-one*: $M \subseteq$ *carrier* G ==> **1** <# M = M
by (*force simp add*: *l-coset-def*)

lemma (**in** *group*) *l-coset-subset-G*:
  [| $H \subseteq$ *carrier* G; $x \in$ *carrier* G |] ==> x <# $H \subseteq$ *carrier* G
by (*auto simp add*: *l-coset-def subsetD*)

lemma (**in** *group*) *l-coset-swap*:
  $[\![y \in x$ <# H; $x \in$ *carrier* G; *subgroup* H $G]\!] \Longrightarrow x \in y$ <# H
proof (*simp add*: *l-coset-def*)
 assume $\exists h \in H.\ y = x \otimes h$
  and x: $x \in$ *carrier* G
  and *sb*: *subgroup* H G
 then obtain h' **where** h': $h' \in H$ & $x \otimes h' = y$ **by** *blast*
 show $\exists h \in H.\ x = y \otimes h$
 proof
  show $x = y \otimes$ *inv* h' **using** h' x *sb*
   by (*auto simp add*: *m-assoc subgroup.subset* [*THEN subsetD*])
  show *inv* $h' \in H$ **using** h' *sb*
   by (*auto simp add*: *subgroup.subset* [*THEN subsetD*] *subgroup.m-inv-closed*)
 qed
qed

lemma (**in** *group*) *l-coset-carrier*:
  [| $y \in x$ <# H; $x \in$ *carrier* G; *subgroup* H G |] ==> $y \in$ *carrier* G
by (*auto simp add*: *l-coset-def m-assoc*
    subgroup.subset [*THEN subsetD*] *subgroup.m-closed*)

lemma (**in** *group*) *l-repr-imp-subset*:
 assumes y: $y \in x$ <# H **and** x: $x \in$ *carrier* G **and** *sb*: *subgroup* H G
 shows y <# $H \subseteq x$ <# H
proof −
 from y
 obtain h' **where** $h' \in H$ $x \otimes h' = y$ **by** (*auto simp add*: *l-coset-def*)
 thus *?thesis* **using** x *sb*
  by (*auto simp add*: *l-coset-def m-assoc*
     subgroup.subset [*THEN subsetD*] *subgroup.m-closed*)
qed

lemma (**in** *group*) *l-repr-independence*:

assumes *y*: $y \in x$ <# H **and** *x*: $x \in$ *carrier* G **and** *sb*: *subgroup* H G
shows x <# H = y <# H
proof
show x <# $H \subseteq y$ <# H
by (*rule l-repr-imp-subset*,
(*blast intro*: *l-coset-swap l-coset-carrier y x sb*)+)
show y <# $H \subseteq x$ <# H **by** (*rule l-repr-imp-subset* [*OF y x sb*])
qed

lemma (**in** *group*) *setmult-subset-G*:
⟦$H \subseteq$ *carrier* G; $K \subseteq$ *carrier* G⟧ $\Longrightarrow$ H <#> $K \subseteq$ *carrier* G
by (*auto simp add*: *set-mult-def subsetD*)

lemma (**in** *group*) *subgroup-mult-id*: *subgroup* H G $\Longrightarrow$ H <#> $H = H$
apply (*auto simp add*: *subgroup.m-closed set-mult-def Sigma-def image-def*)
apply (*rule-tac* $x = x$ **in** *bexI*)
apply (*rule bexI* [*of* - **1**])
apply (*auto simp add*: *subgroup.m-closed subgroup.one-closed*
r-one subgroup.subset [*THEN subsetD*])
done

5.3.1 Set of Inverses of an *r-coset*.

lemma (**in** *normal*) *rcos-inv*:
assumes *x*: $x \in$ *carrier* G
shows *set-inv* (H #> x) = H #> (*inv* x)
proof (*simp add*: *r-coset-def SET-INV-def x inv-mult-group*, *safe*)
fix h
assume $h \in H$
show *inv* $x \otimes$ *inv* $h \in (\bigcup j{\in}H.\ \{j \otimes \mathit{inv}\ x\})$
proof
show *inv* $x \otimes$ *inv* $h \otimes x \in H$
by (*simp add*: *inv-op-closed1 prems*)
show *inv* $x \otimes$ *inv* $h \in \{\mathit{inv}\ x \otimes \mathit{inv}\ h \otimes x \otimes \mathit{inv}\ x\}$
by (*simp add*: *prems m-assoc*)
qed
next
fix h
assume $h \in H$
show $h \otimes$ *inv* $x \in (\bigcup j{\in}H.\ \{\mathit{inv}\ x \otimes \mathit{inv}\ j\})$
proof
show $x \otimes$ *inv* $h \otimes$ *inv* $x \in H$
by (*simp add*: *inv-op-closed2 prems*)
show $h \otimes$ *inv* $x \in \{\mathit{inv}\ x \otimes \mathit{inv}\ (x \otimes \mathit{inv}\ h \otimes \mathit{inv}\ x)\}$
by (*simp add*: *prems m-assoc* [*symmetric*] *inv-mult-group*)
qed
qed

5.3.2 Theorems for <#> with #> or <#.

lemma (**in** *group*) *setmult-rcos-assoc*:
$[\![H \subseteq carrier\ G;\ K \subseteq carrier\ G;\ x \in carrier\ G]\!]$
$\Longrightarrow H <\#> (K \ \#> x) = (H <\#> K) \ \#> x$
by (*force simp add*: *r-coset-def set-mult-def m-assoc*)

lemma (**in** *group*) *rcos-assoc-lcos*:
$[\![H \subseteq carrier\ G;\ K \subseteq carrier\ G;\ x \in carrier\ G]\!]$
$\Longrightarrow (H \ \#> x) <\#> K = H <\#> (x <\# \ K)$
by (*force simp add*: *r-coset-def l-coset-def set-mult-def m-assoc*)

lemma (**in** *normal*) *rcos-mult-step1*:
$[\![x \in carrier\ G;\ y \in carrier\ G]\!]$
$\Longrightarrow (H \ \#> x) <\#> (H \ \#> y) = (H <\#> (x <\# \ H)) \ \#> y$
by (*simp add*: *setmult-rcos-assoc subset*
r-coset-subset-G l-coset-subset-G rcos-assoc-lcos)

lemma (**in** *normal*) *rcos-mult-step2*:
$[\![x \in carrier\ G;\ y \in carrier\ G]\!]$
$\Longrightarrow (H <\#> (x <\# \ H)) \ \#> y = (H <\#> (H \ \#> x)) \ \#> y$
by (*insert coset-eq*, *simp add*: *normal-def*)

lemma (**in** *normal*) *rcos-mult-step3*:
$[\![x \in carrier\ G;\ y \in carrier\ G]\!]$
$\Longrightarrow (H <\#> (H \ \#> x)) \ \#> y = H \ \#> (x \otimes y)$
by (*simp add*: *setmult-rcos-assoc coset-mult-assoc*
subgroup-mult-id normal.axioms subset prems)

lemma (**in** *normal*) *rcos-sum*:
$[\![x \in carrier\ G;\ y \in carrier\ G]\!]$
$\Longrightarrow (H \ \#> x) <\#> (H \ \#> y) = H \ \#> (x \otimes y)$
by (*simp add*: *rcos-mult-step1 rcos-mult-step2 rcos-mult-step3*)

lemma (**in** *normal*) *rcosets-mult-eq*: $M \in rcosets\ H \Longrightarrow H <\#> M = M$
— generalizes *subgroup-mult-id*
by (*auto simp add*: *RCOSETS-def subset*
setmult-rcos-assoc subgroup-mult-id normal.axioms prems)

5.3.3 An Equivalence Relation

constdefs (**structure** G)
r-congruent :: $[('a,'b)monoid\text{-}scheme,\ 'a\ set] \Rightarrow ('a*'a)set$
$(rcong\imath\ \text{-})$
$rcong\ H \equiv \{(x,y).\ x \in carrier\ G\ \&\ y \in carrier\ G\ \&\ inv\ x \otimes y \in H\}$

lemma (**in** *subgroup*) *equiv-rcong*:
includes *group G*
shows *equiv* (*carrier G*) (*rcong H*)

```
proof (intro equiv.intro)
  show refl (carrier G) (rcong H)
    by (auto simp add: r-congruent-def refl-def)
next
  show sym (rcong H)
  proof (simp add: r-congruent-def sym-def, clarify)
    fix x y
    assume [simp]: x ∈ carrier G y ∈ carrier G
       and inv x ⊗ y ∈ H
    hence inv (inv x ⊗ y) ∈ H by (simp add: m-inv-closed)
    thus inv y ⊗ x ∈ H by (simp add: inv-mult-group)
  qed
next
  show trans (rcong H)
  proof (simp add: r-congruent-def trans-def, clarify)
    fix x y z
    assume [simp]: x ∈ carrier G y ∈ carrier G z ∈ carrier G
       and inv x ⊗ y ∈ H and inv y ⊗ z ∈ H
    hence (inv x ⊗ y) ⊗ (inv y ⊗ z) ∈ H by simp
    hence inv x ⊗ (y ⊗ inv y) ⊗ z ∈ H by (simp add: m-assoc del: r-inv)
    thus inv x ⊗ z ∈ H by simp
  qed
qed
```

Equivalence classes of *rcong* correspond to left cosets. Was there a mistake in the definitions? I'd have expected them to correspond to right cosets.

```
lemma (in subgroup) l-coset-eq-rcong:
  includes group G
  assumes a: a ∈ carrier G
  shows a <# H = rcong H `` {a}
by (force simp add: r-congruent-def l-coset-def m-assoc [symmetric] a )
```

5.3.4 Two Distinct Right Cosets are Disjoint

```
lemma (in group) rcos-equation:
  includes subgroup H G
  shows
    ⟦ha ⊗ a = h ⊗ b; a ∈ carrier G;  b ∈ carrier G;
      h ∈ H;  ha ∈ H;  hb ∈ H⟧
    ⟹ hb ⊗ a ∈ (⋃h∈H. {h ⊗ b})
apply (rule UN-I [of hb ⊗ ((inv ha) ⊗ h)])
apply (simp add: )
apply (simp add: m-assoc transpose-inv)
done

lemma (in group) rcos-disjoint:
  includes subgroup H G
  shows ⟦a ∈ rcosets H; b ∈ rcosets H; a≠b⟧ ⟹ a ∩ b = {}
apply (simp add: RCOSETS-def r-coset-def)
```

```
apply (blast intro: rcos-equation prems sym)
done
```

5.4 Further lemmas for *r-congruent*

The relation is a congruence

```
lemma (in normal) congruent-rcong:
  shows congruent2 (rcong H) (rcong H) (λa b. a ⊗ b <# H)
proof (intro congruent2I[of carrier G - carrier G -] equiv-rcong is-group)
  fix a b c
  assume abrcong: (a, b) ∈ rcong H
    and ccarr: c ∈ carrier G

  from abrcong
      have acarr: a ∈ carrier G
        and bcarr: b ∈ carrier G
        and abH: inv a ⊗ b ∈ H
      unfolding r-congruent-def
      by fast+

  note carr = acarr bcarr ccarr

  from ccarr and abH
      have inv c ⊗ (inv a ⊗ b) ⊗ c ∈ H by (rule inv-op-closed1)
  moreover
      from carr and inv-closed
      have inv c ⊗ (inv a ⊗ b) ⊗ c = (inv c ⊗ inv a) ⊗ (b ⊗ c)
      by (force cong: m-assoc)
  moreover
      from carr and inv-closed
      have ... = (inv (a ⊗ c)) ⊗ (b ⊗ c)
      by (simp add: inv-mult-group)
  ultimately
      have (inv (a ⊗ c)) ⊗ (b ⊗ c) ∈ H by simp
  from carr and this
     have (b ⊗ c) ∈ (a ⊗ c) <# H
     by (simp add: lcos-module-rev[OF is-group])
  from carr and this and is-subgroup
     show (a ⊗ c) <# H = (b ⊗ c) <# H by (intro l-repr-independence, simp+)
next
  fix a b c
  assume abrcong: (a, b) ∈ rcong H
    and ccarr: c ∈ carrier G

  from ccarr have c ∈ Units G by (simp add: Units-eq)
  hence cinvc-one: inv c ⊗ c = 1 by (rule Units-l-inv)

  from abrcong
      have acarr: a ∈ carrier G
```

and *bcarr*: $b \in$ *carrier G*
and *abH*: *inv* $a \otimes b \in H$
by (*unfold r-congruent-def*, *fast+*)

note *carr* = *acarr bcarr ccarr*

from *carr* **and** *inv-closed*
have *inv* $a \otimes b =$ *inv* $a \otimes (\mathbf{1} \otimes b)$ **by** *simp*
also from *carr* **and** *inv-closed*
have $\ldots =$ *inv* $a \otimes$ (*inv* $c \otimes c) \otimes b$ **by** *simp*
also from *carr* **and** *inv-closed*
have $\ldots =$ (*inv* $a \otimes$ *inv* $c) \otimes (c \otimes b)$ **by** (*force cong*: *m-assoc*)
also from *carr* **and** *inv-closed*
have $\ldots =$ *inv* $(c \otimes a) \otimes (c \otimes b)$ **by** (*simp add*: *inv-mult-group*)
finally
have *inv* $a \otimes b =$ *inv* $(c \otimes a) \otimes (c \otimes b)$ **.**
from *abH* **and** *this*
have *inv* $(c \otimes a) \otimes (c \otimes b) \in H$ **by** *simp*

from *carr* **and** *this*
have $(c \otimes b) \in (c \otimes a)$ <# H
by (*simp add*: *lcos-module-rev*[*OF is-group*])
from *carr* **and** *this* **and** *is-subgroup*
show $(c \otimes a)$ <# $H = (c \otimes b)$ <# H **by** (*intro l-repr-independence*, *simp+*)
qed

5.5 Order of a Group and Lagrange's Theorem

constdefs
order :: ('*a*, '*b*) *monoid-scheme* $\Rightarrow$ *nat*
order S $\equiv$ *card* (*carrier S*)

lemma (**in** *group*) *rcos-self*:
includes *subgroup*
shows $x \in$ *carrier G* $\Longrightarrow x \in H$ #> x
apply (*simp add*: *r-coset-def*)
apply (*rule-tac* x=**1** **in** *bexI*)
apply (*auto simp add*:)
done

lemma (**in** *group*) *rcosets-part-G*:
includes *subgroup*
shows $\bigcup$(*rcosets H*) = *carrier G*
apply (*rule equalityI*)
apply (*force simp add*: *RCOSETS-def r-coset-def*)
apply (*auto simp add*: *RCOSETS-def intro*: *rcos-self prems*)
done

lemma (**in** *group*) *cosets-finite*:

$[\![c \in \mathit{rcosets}\ H;\ \ H \subseteq \mathit{carrier}\ G;\ \ \mathit{finite}\ (\mathit{carrier}\ G)]\!] \Longrightarrow \mathit{finite}\ c$
apply (*auto simp add*: *RCOSETS-def*)
apply (*simp add*: *r-coset-subset-G* [*THEN finite-subset*])
done

The next two lemmas support the proof of *card-cosets-equal*.

lemma (**in** *group*) *inj-on-f*:
$[\![H \subseteq \mathit{carrier}\ G;\ \ a \in \mathit{carrier}\ G]\!] \Longrightarrow \mathit{inj\text{-}on}\ (\lambda y.\ y \otimes \mathit{inv}\ a)\ (H\ \#{>}\ a)$
apply (*rule inj-onI*)
apply (*subgoal-tac* $x \in \mathit{carrier}\ G\ \&\ y \in \mathit{carrier}\ G$)
prefer *2* **apply** (*blast intro*: *r-coset-subset-G* [*THEN subsetD*])
apply (*simp add*: *subsetD*)
done

lemma (**in** *group*) *inj-on-g*:
$[\![H \subseteq \mathit{carrier}\ G;\ \ a \in \mathit{carrier}\ G]\!] \Longrightarrow \mathit{inj\text{-}on}\ (\lambda y.\ y \otimes a)\ H$
by (*force simp add*: *inj-on-def subsetD*)

lemma (**in** *group*) *card-cosets-equal*:
$[\![c \in \mathit{rcosets}\ H;\ \ H \subseteq \mathit{carrier}\ G;\ \mathit{finite}(\mathit{carrier}\ G)]\!]$
$\Longrightarrow \mathit{card}\ c = \mathit{card}\ H$
apply (*auto simp add*: *RCOSETS-def*)
apply (*rule card-bij-eq*)
apply (*rule inj-on-f*, *assumption+*)
apply (*force simp add*: *m-assoc subsetD r-coset-def*)
apply (*rule inj-on-g*, *assumption+*)
apply (*force simp add*: *m-assoc subsetD r-coset-def*)

The sets $H\ \#{>}\ a$ and H are finite.

apply (*simp add*: *r-coset-subset-G* [*THEN finite-subset*])
apply (*blast intro*: *finite-subset*)
done

lemma (**in** *group*) *rcosets-subset-PowG*:
$\mathit{subgroup}\ H\ G \ \Longrightarrow \mathit{rcosets}\ H \subseteq \mathit{Pow}(\mathit{carrier}\ G)$
apply (*simp add*: *RCOSETS-def*)
apply (*blast dest*: *r-coset-subset-G subgroup.subset*)
done

theorem (**in** *group*) *lagrange*:
$[\![\mathit{finite}(\mathit{carrier}\ G);\ \mathit{subgroup}\ H\ G]\!]$
$\Longrightarrow \mathit{card}(\mathit{rcosets}\ H) * \mathit{card}(H) = \mathit{order}(G)$
apply (*simp* (*no-asm-simp*) *add*: *order-def rcosets-part-G* [*symmetric*])
apply (*subst mult-commute*)
apply (*rule card-partition*)
apply (*simp add*: *rcosets-subset-PowG* [*THEN finite-subset*])
apply (*simp add*: *rcosets-part-G*)
apply (*simp add*: *card-cosets-equal subgroup.subset*)

apply (*simp add*: *rcos-disjoint*)
done

5.6 Quotient Groups: Factorization of a Group

constdefs
FactGroup :: [('a,'b) *monoid-scheme*, 'a *set*] ⇒ ('a *set*) *monoid*
(**infixl** *Mod 65*)
— Actually defined for groups rather than monoids
FactGroup G H ≡
(| *carrier* = *rcosets*$_G$ *H*, *mult* = *set-mult G*, *one* = *H* |)

lemma (**in** *normal*) *setmult-closed*:
⟦*K1* ∈ *rcosets H*; *K2* ∈ *rcosets H*⟧ ⟹ *K1* <#> *K2* ∈ *rcosets H*
by (*auto simp add*: *rcos-sum RCOSETS-def*)

lemma (**in** *normal*) *setinv-closed*:
K ∈ *rcosets H* ⟹ *set-inv K* ∈ *rcosets H*
by (*auto simp add*: *rcos-inv RCOSETS-def*)

lemma (**in** *normal*) *rcosets-assoc*:
⟦*M1* ∈ *rcosets H*; *M2* ∈ *rcosets H*; *M3* ∈ *rcosets H*⟧
⟹ *M1* <#> *M2* <#> *M3* = *M1* <#> (*M2* <#> *M3*)
by (*auto simp add*: *RCOSETS-def rcos-sum m-assoc*)

lemma (**in** *subgroup*) *subgroup-in-rcosets*:
includes *group G*
shows *H* ∈ *rcosets H*
proof −
from - ⟨*subgroup H G*⟩ **have** *H* #> **1** = *H*
by (*rule coset-join2*) *auto*
then show *?thesis*
by (*auto simp add*: *RCOSETS-def*)
qed

lemma (**in** *normal*) *rcosets-inv-mult-group-eq*:
M ∈ *rcosets H* ⟹ *set-inv M* <#> *M* = *H*
by (*auto simp add*: *RCOSETS-def rcos-inv rcos-sum subgroup.subset normal.axioms prems*)

theorem (**in** *normal*) *factorgroup-is-group*:
group (*G Mod H*)
apply (*simp add*: *FactGroup-def*)
apply (*rule groupI*)
apply (*simp add*: *setmult-closed*)
apply (*simp add*: *normal-imp-subgroup subgroup-in-rcosets* [*OF is-group*])
apply (*simp add*: *restrictI setmult-closed rcosets-assoc*)
apply (*simp add*: *normal-imp-subgroup*
subgroup-in-rcosets rcosets-mult-eq)

apply (*auto dest*: *rcosets-inv-mult-group-eq simp add*: *setinv-closed*)
done

lemma *mult-FactGroup* [*simp*]: $X \otimes_{(G\ Mod\ H)} X' = X <\#>_G X'$
by (*simp add*: *FactGroup-def*)

lemma (**in** *normal*) *inv-FactGroup*:
$X \in carrier\ (G\ Mod\ H) \Longrightarrow inv_{G\ Mod\ H}\ X = set\text{-}inv\ X$
apply (*rule group.inv-equality* [*OF factorgroup-is-group*])
apply (*simp-all add*: *FactGroup-def setinv-closed rcosets-inv-mult-group-eq*)
done

The coset map is a homomorphism from G to the quotient group $G\ Mod\ H$

lemma (**in** *normal*) *r-coset-hom-Mod*:
$(\lambda a.\ H\ \#>\ a) \in hom\ G\ (G\ Mod\ H)$
by (*auto simp add*: *FactGroup-def RCOSETS-def Pi-def hom-def rcos-sum*)

5.7 The First Isomorphism Theorem

The quotient by the kernel of a homomorphism is isomorphic to the range of that homomorphism.

constdefs
kernel :: $('a, 'm)$ *monoid-scheme* $\Rightarrow$ $('b, 'n)$ *monoid-scheme* $\Rightarrow$
$('a \Rightarrow 'b) \Rightarrow 'a\ set$
— the kernel of a homomorphism
$kernel\ G\ H\ h \equiv \{x.\ x \in carrier\ G\ \&\ h\ x = \mathbf{1}_H\}$

lemma (**in** *group-hom*) *subgroup-kernel*: *subgroup* (*kernel G H h*) *G*
apply (*rule subgroup.intro*)
apply (*auto simp add*: *kernel-def group.intro prems*)
done

The kernel of a homomorphism is a normal subgroup

lemma (**in** *group-hom*) *normal-kernel*: $(kernel\ G\ H\ h) \lhd G$
apply (*simp add*: *G.normal-inv-iff subgroup-kernel*)
apply (*simp add*: *kernel-def*)
done

lemma (**in** *group-hom*) *FactGroup-nonempty*:
assumes X: $X \in carrier\ (G\ Mod\ kernel\ G\ H\ h)$
shows $X \neq \{\}$
proof −
from *X*
obtain *g* **where** $g \in carrier\ G$
and $X = kernel\ G\ H\ h\ \#>\ g$
by (*auto simp add*: *FactGroup-def RCOSETS-def*)
thus *?thesis*
by (*auto simp add*: *kernel-def r-coset-def image-def intro*: *hom-one*)

qed

lemma (**in** *group-hom*) *FactGroup-contents-mem*:
  assumes X: $X \in$ *carrier* (*G Mod* (*kernel G H h*))
  shows *contents* (h‘X) $\in$ *carrier H*
proof −
  from X
  obtain g **where** g: $g \in$ *carrier G*
      and X = *kernel G H h* #> g
    by (*auto simp add*: *FactGroup-def RCOSETS-def*)
  hence h ‘ $X = \{h\ g\}$ **by** (*auto simp add*: *kernel-def r-coset-def image-def g*)
  thus *?thesis* **by** (*auto simp add*: *g*)
qed

lemma (**in** *group-hom*) *FactGroup-hom*:
  (λX. *contents* (h‘X)) $\in$ *hom* (*G Mod* (*kernel G H h*)) *H*
apply (*simp add*: *hom-def FactGroup-contents-mem normal.factorgroup-is-group* [*OF normal-kernel*] *group.axioms monoid.m-closed*)
proof (*simp add*: *hom-def funcsetI FactGroup-contents-mem*, *intro ballI*)
  fix X **and** X'
  assume X: $X \in$ *carrier* (*G Mod kernel G H h*)
    and X': $X' \in$ *carrier* (*G Mod kernel G H h*)
  then
  obtain g **and** g'
      where $g \in$ *carrier G* **and** $g' \in$ *carrier G*
      and X = *kernel G H h* #> g **and** X' = *kernel G H h* #> g'
    by (*auto simp add*: *FactGroup-def RCOSETS-def*)
  hence *all*: $\forall x \in X.\ h\ x = h\ g\ \forall x \in X'.\ h\ x = h\ g'$
    and *Xsub*: $X \subseteq$ *carrier G* **and** X'*sub*: $X' \subseteq$ *carrier G*
    by (*force simp add*: *kernel-def r-coset-def image-def*)+
  hence h ‘ (X <#> X') = $\{h\ g \otimes_H h\ g'\}$ **using** $X\ X'$
    by (*auto dest*!: *FactGroup-nonempty*
        simp add: *set-mult-def image-eq-UN*
            subsetD [*OF Xsub*] *subsetD* [*OF* X'*sub*])
  thus *contents* (h ‘ (X <#> X')) = *contents* (h ‘ X) $\otimes_H$ *contents* (h ‘ X')
    by (*simp add*: *all image-eq-UN FactGroup-nonempty* $X\ X'$)
qed

Lemma for the following injectivity result

lemma (**in** *group-hom*) *FactGroup-subset*:
  $[\![g \in$ *carrier G*; $g' \in$ *carrier G*; $h\ g = h\ g']\!]$
  $\Longrightarrow$ *kernel G H h* #> $g \subseteq$ *kernel G H h* #> g'
apply (*clarsimp simp add*: *kernel-def r-coset-def image-def*)
apply (*rename-tac y*)
apply (*rule-tac* $x = y \otimes g \otimes$ *inv* g' **in** *exI*)
apply (*simp add*: *G.m-assoc*)
done

lemma (**in** *group-hom*) *FactGroup-inj-on*:
  inj-on (λX. *contents* (h ‘ X)) (*carrier* (*G Mod kernel G H h*))
proof (*simp add*: *inj-on-def*, *clarify*)
  fix X **and** X'
  assume X: $X \in$ *carrier* (*G Mod kernel G H h*)
   and X': $X' \in$ *carrier* (*G Mod kernel G H h*)
  then
  obtain g **and** g'
     where gX: $g \in$ *carrier G* $g' \in$ *carrier G*
      X = *kernel G H h* #> g X' = *kernel G H h* #> g'
   by (*auto simp add*: *FactGroup-def RCOSETS-def*)
  hence *all*: $\forall x \in X.\ h\ x = h\ g$ $\forall x \in X'.\ h\ x = h\ g'$
   by (*force simp add*: *kernel-def r-coset-def image-def*)+
  assume *contents* (h ‘ X) = *contents* (h ‘ X')
  hence h: $h\ g = h\ g'$
   by (*simp add*: *image-eq-UN all FactGroup-nonempty X X'*)
  show $X=X'$ **by** (*rule equalityI*) (*simp-all add*: *FactGroup-subset h gX*)
qed

If the homomorphism h is onto H, then so is the homomorphism from the quotient group

lemma (**in** *group-hom*) *FactGroup-onto*:
  assumes h: h ‘ *carrier G* = *carrier H*
  shows (λX. *contents* (h ‘ X)) ‘ *carrier* (*G Mod kernel G H h*) = *carrier H*
proof
  show (λX. *contents* (h ‘ X)) ‘ *carrier* (*G Mod kernel G H h*) $\subseteq$ *carrier H*
   by (*auto simp add*: *FactGroup-contents-mem*)
  show *carrier H* $\subseteq$ (λX. *contents* (h ‘ X)) ‘ *carrier* (*G Mod kernel G H h*)
  proof
   fix y
   assume y: $y \in$ *carrier H*
   with h **obtain** g **where** g: $g \in$ *carrier G* $h\ g = y$
    by (*blast elim*: *equalityE*)
   hence ($\bigcup x\in$*kernel G H h* #> g. $\{h\ x\}$) = $\{y\}$
    by (*auto simp add*: *y kernel-def r-coset-def*)
   with g **show** $y \in$ (λX. *contents* (h ‘ X)) ‘ *carrier* (*G Mod kernel G H h*)
    by (*auto intro*!: *bexI simp add*: *FactGroup-def RCOSETS-def image-eq-UN*)
  qed
qed

If h is a homomorphism from G onto H, then the quotient group *G Mod kernel G H h* is isomorphic to H.

theorem (**in** *group-hom*) *FactGroup-iso*:
  h ‘ *carrier G* = *carrier H*
  $\Longrightarrow$ (λX. *contents* (h‘X)) $\in$ (*G Mod* (*kernel G H h*)) $\cong$ H
by (*simp add*: *iso-def FactGroup-hom FactGroup-inj-on bij-betw-def*
      FactGroup-onto)

end

theory *Sylow* **imports** *Coset* **begin**

6 Sylow's Theorem

See also [3].

The combinatorial argument is in theory Exponent

```
locale sylow = group +
  fixes p and a and m and calM and RelM
  assumes prime-p:   prime p
      and order-G:   order(G) = (p ^a) * m
      and finite-G [iff]:  finite (carrier G)
  defines calM == {s. s ⊆ carrier(G) & card(s) = p ^a}
      and RelM == {(N1,N2). N1 ∈ calM & N2 ∈ calM &
                            (∃ g ∈ carrier(G). N1 = (N2 #> g) )}

lemma (in sylow) RelM-refl: refl calM RelM
apply (auto simp add: refl-def RelM-def calM-def)
apply (blast intro!: coset-mult-one [symmetric])
done

lemma (in sylow) RelM-sym: sym RelM
proof (unfold sym-def RelM-def, clarify)
  fix y g
  assume   y ∈ calM
    and g: g ∈ carrier G
  hence y = y #> g #> (inv g) by (simp add: coset-mult-assoc calM-def)
  thus ∃ g'∈carrier G. y = y #> g #> g'
   by (blast intro: g inv-closed)
qed

lemma (in sylow) RelM-trans: trans RelM
by (auto simp add: trans-def RelM-def calM-def coset-mult-assoc)

lemma (in sylow) RelM-equiv: equiv calM RelM
apply (unfold equiv-def)
apply (blast intro: RelM-refl RelM-sym RelM-trans)
done

lemma (in sylow) M-subset-calM-prep: M' ∈ calM // RelM  ==> M' ⊆ calM
apply (unfold RelM-def)
apply (blast elim!: quotientE)
done
```

6.1 Main Part of the Proof

```
locale sylow-central = sylow +
  fixes H and M1 and M
  assumes M-in-quot:  M ∈ calM // RelM
      and not-dvd-M:  ~(p ^ Suc(exponent p m) dvd card(M))
      and M1-in-M:    M1 ∈ M
  defines H == {g. g∈carrier G & M1 #> g = M1}

lemma (in sylow-central) M-subset-calM: M ⊆ calM
by (rule M-in-quot [THEN M-subset-calM-prep])

lemma (in sylow-central) card-M1: card(M1) = p^a
apply (cut-tac M-subset-calM M1-in-M)
apply (simp add: calM-def, blast)
done

lemma card-nonempty: 0 < card(S) ==> S ≠ {}
by force

lemma (in sylow-central) exists-x-in-M1: ∃x. x∈M1
apply (subgoal-tac 0 < card M1)
 apply (blast dest: card-nonempty)
apply (cut-tac prime-p [THEN prime-imp-one-less])
apply (simp (no-asm-simp) add: card-M1)
done

lemma (in sylow-central) M1-subset-G [simp]: M1 ⊆ carrier G
apply (rule subsetD [THEN PowD])
apply (rule-tac [2] M1-in-M)
apply (rule M-subset-calM [THEN subset-trans])
apply (auto simp add: calM-def)
done

lemma (in sylow-central) M1-inj-H: ∃f ∈ H→M1. inj-on f H
  proof -
    from exists-x-in-M1 obtain m1 where m1M: m1 ∈ M1..
   have m1G: m1 ∈ carrier G by (simp add: m1M M1-subset-G [THEN subsetD])
    show ?thesis
    proof
      show inj-on (λz∈H. m1 ⊗ z) H
        by (simp add: inj-on-def l-cancel [of m1 x y, THEN iffD1] H-def m1G)
      show restrict (op ⊗ m1) H ∈ H → M1
      proof (rule restrictI)
        fix z assume zH: z ∈ H
        show m1 ⊗ z ∈ M1
        proof -
          from zH
          have zG: z ∈ carrier G and M1zeq: M1 #> z = M1
            by (auto simp add: H-def)
```

show *?thesis*
by (*rule subst* [*OF M1zeq*], *simp add*: *m1M zG rcosI*)
qed
qed
qed
qed

6.2 Discharging the Assumptions of *sylow-central*

lemma (**in** *sylow*) *EmptyNotInEquivSet*: $\{\} \notin$ *calM // RelM*
by (*blast elim*!: *quotientE dest*: *RelM-equiv* [*THEN equiv-class-self*])

lemma (**in** *sylow*) *existsM1inM*: $M \in$ *calM // RelM* ==> $\exists$ *M1*. *M1* $\in$ *M*
apply (*subgoal-tac M* $\neq$ $\{\}$)
apply *blast*
apply (*cut-tac EmptyNotInEquivSet*, *blast*)
done

lemma (**in** *sylow*) *zero-less-o-G*: $0 <$ *order*(*G*)
apply (*unfold order-def*)
apply (*blast intro*: *one-closed zero-less-card-empty*)
done

lemma (**in** *sylow*) *zero-less-m*: $m > 0$
apply (*cut-tac zero-less-o-G*)
apply (*simp add*: *order-G*)
done

lemma (**in** *sylow*) *card-calM*: *card*(*calM*) = (*p* ^*a*) * *m choose p* ^*a*
by (*simp add*: *calM-def n-subsets order-G* [*symmetric*] *order-def*)

lemma (**in** *sylow*) *zero-less-card-calM*: *card calM* > 0
by (*simp add*: *card-calM zero-less-binomial le-extend-mult zero-less-m*)

lemma (**in** *sylow*) *max-p-div-calM*:
~ (*p* ^ *Suc*(*exponent p m*) *dvd card*(*calM*))
apply (*subgoal-tac exponent p m* = *exponent p* (*card calM*))
apply (*cut-tac zero-less-card-calM prime-p*)
apply (*force dest*: *power-Suc-exponent-Not-dvd*)
apply (*simp add*: *card-calM zero-less-m* [*THEN const-p-fac*])
done

lemma (**in** *sylow*) *finite-calM*: *finite calM*
apply (*unfold calM-def*)
apply (*rule-tac B* = *Pow* (*carrier G*) **in** *finite-subset*)
apply *auto*
done

lemma (**in** *sylow*) *lemma-A1*:

$\exists\, M \in calM$ // $RelM$. ~ (p ^ Suc($exponent\ p\ m$) dvd $card$(M))
apply (*rule max-p-div-calM* [*THEN contrapos-np*])
apply (*simp add*: *finite-calM equiv-imp-dvd-card* [*OF* - *RelM-equiv*])
done

6.2.1 Introduction and Destruct Rules for *H*

lemma (**in** *sylow-central*) *H-I*: [|$g \in$ *carrier G*; *M1* #> g = *M1*|] ==> $g \in H$
by (*simp add*: *H-def*)

lemma (**in** *sylow-central*) *H-into-carrier-G*: $x \in H$ ==> $x \in$ *carrier G*
by (*simp add*: *H-def*)

lemma (**in** *sylow-central*) *in-H-imp-eq*: g : H ==> *M1* #> g = *M1*
by (*simp add*: *H-def*)

lemma (**in** *sylow-central*) *H-m-closed*: [| $x{\in}H$; $y{\in}H$|] ==> $x \otimes y \in H$
apply (*unfold H-def*)
apply (*simp add*: *coset-mult-assoc* [*symmetric*] *m-closed*)
done

lemma (**in** *sylow-central*) *H-not-empty*: $H \neq \{\}$
apply (*simp add*: *H-def*)
apply (*rule exI* [*of* - **1**], *simp*)
done

lemma (**in** *sylow-central*) *H-is-subgroup*: *subgroup H G*
apply (*rule subgroupI*)
apply (*rule subsetI*)
apply (*erule H-into-carrier-G*)
apply (*rule H-not-empty*)
apply (*simp add*: *H-def*, *clarify*)
apply (*erule-tac P* = %*z*. ?*lhs*(*z*) = *M1* **in** *subst*)
apply (*simp add*: *coset-mult-assoc*)
apply (*blast intro*: *H-m-closed*)
done

lemma (**in** *sylow-central*) *rcosetGM1g-subset-G*:
[| $g \in$ *carrier G*; $x \in$ *M1* #> g |] ==> $x \in$ *carrier G*
by (*blast intro*: *M1-subset-G* [*THEN r-coset-subset-G*, *THEN subsetD*])

lemma (**in** *sylow-central*) *finite-M1*: *finite M1*
by (*rule finite-subset* [*OF M1-subset-G finite-G*])

lemma (**in** *sylow-central*) *finite-rcosetGM1g*: $g{\in}$*carrier G* ==> *finite* (*M1* #> g)
apply (*rule finite-subset*)
apply (*rule subsetI*)
apply (*erule rcosetGM1g-subset-G*, *assumption*)

apply (*rule finite-G*)
done

lemma (**in** *sylow-central*) *M1-cardeq-rcosetGM1g*:
$g \in carrier\ G ==> card(M1\ \#> g) = card(M1)$
by (*simp* (*no-asm-simp*) *add*: *M1-subset-G card-cosets-equal rcosetsI*)

lemma (**in** *sylow-central*) *M1-RelM-rcosetGM1g*:
$g \in carrier\ G ==> (M1,\ M1\ \#> g) \in RelM$
apply (*simp* (*no-asm*) *add*: *RelM-def calM-def card-M1 M1-subset-G*)
apply (*rule conjI*)
apply (*blast intro*: *rcosetGM1g-subset-G*)
apply (*simp* (*no-asm-simp*) *add*: *card-M1 M1-cardeq-rcosetGM1g*)
apply (*rule bexI* [*of - inv g*])
apply (*simp-all add*: *coset-mult-assoc M1-subset-G*)
done

6.3 Equal Cardinalities of M and the Set of Cosets

Injections between M and $rcosets_G\ H$ show that their cardinalities are equal.

lemma *ElemClassEquiv*:
$[|\ equiv\ A\ r;\ C \in A\ //\ r\ |] ==> \forall x \in C.\ \forall y \in C.\ (x,y) \in r$
by (*unfold equiv-def quotient-def sym-def trans-def*, *blast*)

lemma (**in** *sylow-central*) *M-elem-map*:
$M2 \in M ==> \exists g.\ g \in carrier\ G\ \&\ M1\ \#> g = M2$
apply (*cut-tac M1-in-M M-in-quot* [*THEN RelM-equiv* [*THEN ElemClassEquiv*]])
apply (*simp add*: *RelM-def*)
apply (*blast dest*!: *bspec*)
done

lemmas (**in** *sylow-central*) *M-elem-map-carrier* =
M-elem-map [*THEN someI-ex*, *THEN conjunct1*]

lemmas (**in** *sylow-central*) *M-elem-map-eq* =
M-elem-map [*THEN someI-ex*, *THEN conjunct2*]

lemma (**in** *sylow-central*) *M-funcset-rcosets-H*:
$(\%x{:}M.\ H\ \#> (SOME\ g.\ g \in carrier\ G\ \&\ M1\ \#> g = x)) \in M \to rcosets\ H$
apply (*rule rcosetsI* [*THEN restrictI*])
apply (*rule H-is-subgroup* [*THEN subgroup.subset*])
apply (*erule M-elem-map-carrier*)
done

lemma (**in** *sylow-central*) *inj-M-GmodH*: $\exists f \in M \to rcosets\ H.\ inj\text{-}on\ f\ M$
apply (*rule bexI*)
apply (*rule-tac* [*2*] *M-funcset-rcosets-H*)
apply (*rule inj-onI*, *simp*)
apply (*rule trans* [*OF - M-elem-map-eq*])

prefer *2* **apply** *assumption*
apply (*rule M-elem-map-eq* [*symmetric*, *THEN trans*], *assumption*)
apply (*rule coset-mult-inv1*)
apply (*erule-tac* [*2*] *M-elem-map-carrier*)+
apply (*rule-tac* [*2*] *M1-subset-G*)
apply (*rule coset-join1* [*THEN in-H-imp-eq*])
apply (*rule-tac* [*3*] *H-is-subgroup*)
prefer *2* **apply** (*blast intro*: *m-closed M-elem-map-carrier inv-closed*)
apply (*simp add*: *coset-mult-inv2 H-def M-elem-map-carrier subset-def*)
done

6.3.1 The Opposite Injection

lemma (**in** *sylow-central*) *H-elem-map*:
  H1 ∈ *rcosets H* ==> ∃ *g*. *g* ∈ *carrier G* & *H* #> *g* = *H1*
by (*auto simp add*: *RCOSETS-def*)

lemmas (**in** *sylow-central*) *H-elem-map-carrier* =
    H-elem-map [*THEN someI-ex*, *THEN conjunct1*]

lemmas (**in** *sylow-central*) *H-elem-map-eq* =
    H-elem-map [*THEN someI-ex*, *THEN conjunct2*]

lemma *EquivElemClass*:
  [|*equiv A r*; *M* ∈ *A*//*r*; *M1*∈*M*; (*M1*,*M2*) ∈ *r* |] ==> *M2* ∈ *M*
by (*unfold equiv-def quotient-def sym-def trans-def*, *blast*)

lemma (**in** *sylow-central*) *rcosets-H-funcset-M*:
 (λ*C* ∈ *rcosets H*. *M1* #> (@*g*. *g* ∈ *carrier G* ∧ *H* #> *g* = *C*)) ∈ *rcosets H* →
M
apply (*simp add*: *RCOSETS-def*)
apply (*fast intro*: *someI2*
      intro!: *restrictI M1-in-M*
       EquivElemClass [*OF RelM-equiv M-in-quot* - *M1-RelM-rcosetGM1g*])
done

close to a duplicate of *inj-M-GmodH*

lemma (**in** *sylow-central*) *inj-GmodH-M*:
  ∃ *g* ∈ *rcosets H*→*M*. *inj-on g* (*rcosets H*)
apply (*rule bexI*)
apply (*rule-tac* [*2*] *rcosets-H-funcset-M*)
apply (*rule inj-onI*)
apply (*simp*)
apply (*rule trans* [*OF* - *H-elem-map-eq*])
prefer *2* **apply** *assumption*
apply (*rule H-elem-map-eq* [*symmetric*, *THEN trans*], *assumption*)
apply (*rule coset-mult-inv1*)

apply (*erule-tac* [*2*] *H-elem-map-carrier*)+
apply (*rule-tac* [*2*] *H-is-subgroup* [*THEN subgroup.subset*])
apply (*rule coset-join2*)
apply (*blast intro*: *m-closed inv-closed H-elem-map-carrier*)
apply (*rule H-is-subgroup*)
apply (*simp add*: *H-I coset-mult-inv2 M1-subset-G H-elem-map-carrier*)
done

lemma (**in** *sylow-central*) *calM-subset-PowG*: *calM* $\subseteq$ *Pow*(*carrier G*)
by (*auto simp add*: *calM-def*)

lemma (**in** *sylow-central*) *finite-M*: *finite M*
apply (*rule finite-subset*)
apply (*rule M-subset-calM* [*THEN subset-trans*])
apply (*rule calM-subset-PowG*, *blast*)
done

lemma (**in** *sylow-central*) *cardMeqIndexH*: *card*(*M*) = *card*(*rcosets H*)
apply (*insert inj-M-GmodH inj-GmodH-M*)
apply (*blast intro*: *card-bij finite-M H-is-subgroup*
rcosets-subset-PowG [*THEN finite-subset*]
finite-Pow-iff [*THEN iffD2*])
done

lemma (**in** *sylow-central*) *index-lem*: *card*(*M*) $*$ *card*(*H*) = *order*(*G*)
by (*simp add*: *cardMeqIndexH lagrange H-is-subgroup*)

lemma (**in** *sylow-central*) *lemma-leq1*: *p*ˆ*a* $\leq$ *card*(*H*)
apply (*rule dvd-imp-le*)
apply (*rule div-combine* [*OF prime-p not-dvd-M*])
prefer *2* **apply** (*blast intro*: *subgroup.finite-imp-card-positive H-is-subgroup*)
apply (*simp add*: *index-lem order-G power-add mult-dvd-mono power-exponent-dvd*
zero-less-m)
done

lemma (**in** *sylow-central*) *lemma-leq2*: *card*(*H*) $\leq$ *p*ˆ*a*
apply (*subst card-M1* [*symmetric*])
apply (*cut-tac M1-inj-H*)
apply (*blast intro*!: *M1-subset-G intro*:
card-inj H-into-carrier-G finite-subset [*OF - finite-G*])
done

lemma (**in** *sylow-central*) *card-H-eq*: *card*(*H*) = *p*ˆ*a*
by (*blast intro*: *le-anti-sym lemma-leq1 lemma-leq2*)

lemma (**in** *sylow*) *sylow-thm*: $\exists$ *H*. *subgroup H G* & *card*(*H*) = *p*ˆ*a*
apply (*cut-tac lemma-A1*, *clarify*)
apply (*frule existsM1inM*, *clarify*)

apply (*subgoal-tac sylow-central G p a m M1 M*)
apply (*blast dest*: *sylow-central.H-is-subgroup sylow-central.card-H-eq*)
apply (*simp add*: *sylow-central-def sylow-central-axioms-def prems*)
done

Needed because the locale's automatic definition refers to *semigroup G* and *group-axioms G* rather than simply to *group G*.

lemma *sylow-eq*: *sylow G p a m* = (*group G* & *sylow-axioms G p a m*)
by (*simp add*: *sylow-def group-def*)

6.4 Sylow's Theorem

theorem *sylow-thm*:
  [| *prime p*; *group*(*G*); *order*(*G*) = (*p* ^ *a*) ∗ *m*; *finite* (*carrier G*)|]
  ==> ∃ *H*. *subgroup H G* & *card*(*H*) = *p* ^ *a*
apply (*rule sylow.sylow-thm* [*of G p a m*])
apply (*simp add*: *sylow-eq sylow-axioms-def*)
done

end

theory *Bij* **imports** *Group* **begin**

7 Bijections of a Set, Permutation Groups and Automorphism Groups

constdefs
 Bij :: *'a set* ⇒ (*'a* ⇒ *'a*) *set*
  — Only extensional functions, since otherwise we get too many.
 Bij S ≡ *extensional S* ∩ {*f*. *bij-betw f S S*}

 BijGroup :: *'a set* ⇒ (*'a* ⇒ *'a*) *monoid*
 BijGroup S ≡
  (|*carrier* = *Bij S*,
   mult = λ*g* ∈ *Bij S*. λ*f* ∈ *Bij S*. *compose S g f*,
   one = λ*x* ∈ *S*. *x*|)

declare *Id-compose* [*simp*] *compose-Id* [*simp*]

lemma *Bij-imp-extensional*: *f* ∈ *Bij S* ⟹ *f* ∈ *extensional S*
 by (*simp add*: *Bij-def*)

lemma *Bij-imp-funcset*: *f* ∈ *Bij S* ⟹ *f* ∈ *S* → *S*
 by (*auto simp add*: *Bij-def bij-betw-imp-funcset*)

7.1 Bijections Form a Group

lemma *restrict-Inv-Bij*: $f \in Bij\ S \Longrightarrow (\lambda x \in S.\ (Inv\ S\ f)\ x) \in Bij\ S$
  by (*simp add*: *Bij-def bij-betw-Inv*)

lemma *id-Bij*: $(\lambda x{\in}S.\ x) \in Bij\ S$
  by (*auto simp add*: *Bij-def bij-betw-def inj-on-def*)

lemma *compose-Bij*: $[\![x \in Bij\ S;\ y \in Bij\ S]\!] \Longrightarrow compose\ S\ x\ y \in Bij\ S$
  by (*auto simp add*: *Bij-def bij-betw-compose*)

lemma *Bij-compose-restrict-eq*:
    $f \in Bij\ S \Longrightarrow compose\ S\ (restrict\ (Inv\ S\ f)\ S)\ f = (\lambda x{\in}S.\ x)$
  by (*simp add*: *Bij-def compose-Inv-id*)

theorem *group-BijGroup*: *group* (*BijGroup S*)
apply (*simp add*: *BijGroup-def*)
apply (*rule groupI*)
    apply (*simp add*: *compose-Bij*)
   apply (*simp add*: *id-Bij*)
  apply (*simp add*: *compose-Bij*)
  apply (*blast intro*: *compose-assoc* [*symmetric*] *Bij-imp-funcset*)
 apply (*simp add*: *id-Bij Bij-imp-funcset Bij-imp-extensional*, *simp*)
apply (*blast intro*: *Bij-compose-restrict-eq restrict-Inv-Bij*)
done

7.2 Automorphisms Form a Group

lemma *Bij-Inv-mem*: $[\![\ f \in Bij\ S;\ \ x \in S]\!] \Longrightarrow Inv\ S\ f\ x \in S$
by (*simp add*: *Bij-def bij-betw-def Inv-mem*)

lemma *Bij-Inv-lemma*:
 assumes *eq*: $\bigwedge x\ y.\ [\![x \in S;\ y \in S]\!] \Longrightarrow h(g\ x\ y) = g\ (h\ x)\ (h\ y)$
 shows $[\![h \in Bij\ S;\ \ g \in S \rightarrow S \rightarrow S;\ \ x \in S;\ \ y \in S]\!]$
    $\Longrightarrow Inv\ S\ h\ (g\ x\ y) = g\ (Inv\ S\ h\ x)\ (Inv\ S\ h\ y)$
apply (*simp add*: *Bij-def bij-betw-def*)
apply (*subgoal-tac* $\exists\, x'{\in}S.\ \exists\, y'{\in}S.\ x = h\ x'\ \&\ y = h\ y'$, *clarify*)
 apply (*simp add*: *eq* [*symmetric*] *Inv-f-f funcset-mem* [*THEN funcset-mem*], *blast*)
done

constdefs
  $auto :: ('a,\ 'b)\ monoid\text{-}scheme \Rightarrow ('a \Rightarrow 'a)\ set$
  $auto\ G \equiv hom\ G\ G \cap Bij\ (carrier\ G)$

  $AutoGroup :: ('a,\ 'c)\ monoid\text{-}scheme \Rightarrow ('a \Rightarrow 'a)\ monoid$
  $AutoGroup\ G \equiv BijGroup\ (carrier\ G)\ (\!|carrier := auto\ G|\!)$

lemma (**in** *group*) *id-in-auto*: $(\lambda x \in carrier\ G.\ x) \in auto\ G$
  by (*simp add*: *auto-def hom-def restrictI group.axioms id-Bij*)

lemma (**in** *group*) *mult-funcset*: *mult* $G \in$ *carrier* $G \to$ *carrier* $G \to$ *carrier* G
  by (*simp add*: *Pi-I group.axioms*)

lemma (**in** *group*) *restrict-Inv-hom*:
  $[\![h \in hom\ G\ G; h \in Bij\ (carrier\ G)]\!]$
  $\Longrightarrow restrict\ (Inv\ (carrier\ G)\ h)\ (carrier\ G) \in hom\ G\ G$
  by (*simp add*: *hom-def Bij-Inv-mem restrictI mult-funcset*
      group.axioms Bij-Inv-lemma)

lemma *inv-BijGroup*:
  $f \in Bij\ S \Longrightarrow m\text{-}inv\ (BijGroup\ S)\ f = (\lambda x \in S.\ (Inv\ S\ f)\ x)$
apply (*rule group.inv-equality*)
apply (*rule group-BijGroup*)
apply (*simp-all add*: *BijGroup-def restrict-Inv-Bij Bij-compose-restrict-eq*)
done

lemma (**in** *group*) *subgroup-auto*:
  subgroup (*auto* G) (*BijGroup* (*carrier* G))
proof (*rule subgroup.intro*)
  show *auto* $G \subseteq$ *carrier* (*BijGroup* (*carrier* G))
    by (*force simp add*: *auto-def BijGroup-def*)
next
  fix $x\ y$
  assume $x \in auto\ G\ y \in auto\ G$
  thus $x \otimes_{BijGroup\ (carrier\ G)} y \in auto\ G$
    by (*force simp add*: *BijGroup-def is-group auto-def Bij-imp-funcset*
        group.hom-compose compose-Bij)
next
  show $\mathbf{1}_{BijGroup\ (carrier\ G)} \in auto\ G$ **by** (*simp add*: *BijGroup-def id-in-auto*)
next
  fix x
  assume $x \in auto\ G$
  thus $inv_{BijGroup\ (carrier\ G)}\ x \in auto\ G$
    by (*simp del*: *restrict-apply*
      add: *inv-BijGroup auto-def restrict-Inv-Bij restrict-Inv-hom*)
qed

theorem (**in** *group*) *AutoGroup*: *group* (*AutoGroup* G)
by (*simp add*: *AutoGroup-def subgroup.subgroup-is-group subgroup-auto*
    group-BijGroup)

end

theory *Ring* **imports** *FiniteProduct*
uses (*ringsimp.ML*) **begin**

8 Abelian Groups

record *'a ring* = *'a monoid* +
  zero :: *'a* (**0**ı)
  add :: [*'a*, *'a*] => *'a* (**infixl** ⊕ı *65*)

Derived operations.

constdefs (**structure** *R*)
  a-inv :: [(*'a*, *'m*) *ring-scheme*, *'a*] => *'a* (⊖ı - [*81*] *80*)
  a-inv R == *m-inv* (| *carrier* = *carrier R*, *mult* = *add R*, *one* = *zero R* |)

  a-minus :: [(*'a*, *'m*) *ring-scheme*, *'a*, *'a*] => *'a* (**infixl** ⊖ı *65*)
  [| *x* ∈ *carrier R*; *y* ∈ *carrier R* |] ==> *x* ⊖ *y* == *x* ⊕ (⊖ *y*)

locale *abelian-monoid* =
  fixes *G* (**structure**)
  assumes *a-comm-monoid*:
    comm-monoid (| *carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G* |)

The following definition is redundant but simple to use.

locale *abelian-group* = *abelian-monoid* +
  assumes *a-comm-group*:
    comm-group (| *carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G* |)

8.1 Basic Properties

lemma *abelian-monoidI*:
  fixes *R* (**structure**)
  assumes *a-closed*:
    !!*x y*. [| *x* ∈ *carrier R*; *y* ∈ *carrier R* |] ==> *x* ⊕ *y* ∈ *carrier R*
   and *zero-closed*: **0** ∈ *carrier R*
   and *a-assoc*:
    !!*x y z*. [| *x* ∈ *carrier R*; *y* ∈ *carrier R*; *z* ∈ *carrier R* |] ==>
    (*x* ⊕ *y*) ⊕ *z* = *x* ⊕ (*y* ⊕ *z*)
   and *l-zero*: !!*x*. *x* ∈ *carrier R* ==> **0** ⊕ *x* = *x*
   and *a-comm*:
    !!*x y*. [| *x* ∈ *carrier R*; *y* ∈ *carrier R* |] ==> *x* ⊕ *y* = *y* ⊕ *x*
  shows *abelian-monoid R*
  by (*auto intro*!: *abelian-monoid.intro comm-monoidI intro*: *prems*)

lemma *abelian-groupI*:
  fixes *R* (**structure**)
  assumes *a-closed*:
    !!*x y*. [| *x* ∈ *carrier R*; *y* ∈ *carrier R* |] ==> *x* ⊕ *y* ∈ *carrier R*
   and *zero-closed*: *zero R* ∈ *carrier R*
   and *a-assoc*:
    !!*x y z*. [| *x* ∈ *carrier R*; *y* ∈ *carrier R*; *z* ∈ *carrier R* |] ==>
    (*x* ⊕ *y*) ⊕ *z* = *x* ⊕ (*y* ⊕ *z*)
   and *a-comm*:

$!!x\ y.\ [|\ x \in carrier\ R;\ y \in carrier\ R\ |] ==> x \oplus y = y \oplus x$
and *l-zero*: $!!x.\ x \in carrier\ R ==> \mathbf{0} \oplus x = x$
and *l-inv-ex*: $!!x.\ x \in carrier\ R ==> EX\ y : carrier\ R.\ y \oplus x = \mathbf{0}$
shows *abelian-group R*
by (*auto intro*!: *abelian-group.intro abelian-monoidI*
abelian-group-axioms.intro comm-monoidI comm-groupI
intro: *prems*)

lemma (**in** *abelian-monoid*) *a-monoid*:
monoid (| *carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G* |)
by (*rule comm-monoid.axioms*, *rule a-comm-monoid*)

lemma (**in** *abelian-group*) *a-group*:
group (| *carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G* |)
by (*simp add*: *group-def a-monoid*)
(*simp add*: *comm-group.axioms group.axioms a-comm-group*)

lemmas *monoid-record-simps* = *partial-object.simps monoid.simps*

lemma (**in** *abelian-monoid*) *a-closed* [*intro*, *simp*]:
$[\![\ x \in carrier\ G;\ y \in carrier\ G\ ]\!] \Longrightarrow x \oplus y \in carrier\ G$
by (*rule monoid.m-closed* [*OF a-monoid*, *simplified monoid-record-simps*])

lemma (**in** *abelian-monoid*) *zero-closed* [*intro*, *simp*]:
$\mathbf{0} \in carrier\ G$
by (*rule monoid.one-closed* [*OF a-monoid*, *simplified monoid-record-simps*])

lemma (**in** *abelian-group*) *a-inv-closed* [*intro*, *simp*]:
$x \in carrier\ G ==> \ominus\ x \in carrier\ G$
by (*simp add*: *a-inv-def*
group.inv-closed [*OF a-group*, *simplified monoid-record-simps*])

lemma (**in** *abelian-group*) *minus-closed* [*intro*, *simp*]:
$[|\ x \in carrier\ G;\ y \in carrier\ G\ |] ==> x \ominus y \in carrier\ G$
by (*simp add*: *a-minus-def*)

lemma (**in** *abelian-group*) *a-l-cancel* [*simp*]:
$[|\ x \in carrier\ G;\ y \in carrier\ G;\ z \in carrier\ G\ |] ==>$
$(x \oplus y = x \oplus z) = (y = z)$
by (*rule group.l-cancel* [*OF a-group*, *simplified monoid-record-simps*])

lemma (**in** *abelian-group*) *a-r-cancel* [*simp*]:
$[|\ x \in carrier\ G;\ y \in carrier\ G;\ z \in carrier\ G\ |] ==>$
$(y \oplus x = z \oplus x) = (y = z)$
by (*rule group.r-cancel* [*OF a-group*, *simplified monoid-record-simps*])

lemma (**in** *abelian-monoid*) *a-assoc*:
$[\![x \in carrier\ G;\ y \in carrier\ G;\ z \in carrier\ G]\!] \Longrightarrow$
$(x \oplus y) \oplus z = x \oplus (y \oplus z)$

by (*rule monoid.m-assoc* [*OF a-monoid*, *simplified monoid-record-simps*])

lemma (**in** *abelian-monoid*) *l-zero* [*simp*]:
$x \in carrier\ G ==> \mathbf{0} \oplus x = x$
by (*rule monoid.l-one* [*OF a-monoid*, *simplified monoid-record-simps*])

lemma (**in** *abelian-group*) *l-neg*:
$x \in carrier\ G ==> \ominus\ x \oplus x = \mathbf{0}$
by (*simp add*: *a-inv-def*
group.l-inv [*OF a-group*, *simplified monoid-record-simps*])

lemma (**in** *abelian-monoid*) *a-comm*:
$[\![x \in carrier\ G;\ y \in carrier\ G]\!] \Longrightarrow x \oplus y = y \oplus x$
by (*rule comm-monoid.m-comm* [*OF a-comm-monoid*,
simplified monoid-record-simps])

lemma (**in** *abelian-monoid*) *a-lcomm*:
$[\![x \in carrier\ G;\ y \in carrier\ G;\ z \in carrier\ G]\!] \Longrightarrow$
$x \oplus (y \oplus z) = y \oplus (x \oplus z)$
by (*rule comm-monoid.m-lcomm* [*OF a-comm-monoid*,
simplified monoid-record-simps])

lemma (**in** *abelian-monoid*) *r-zero* [*simp*]:
$x \in carrier\ G ==> x \oplus \mathbf{0} = x$
using *monoid.r-one* [*OF a-monoid*]
by *simp*

lemma (**in** *abelian-group*) *r-neg*:
$x \in carrier\ G ==> x \oplus (\ominus\ x) = \mathbf{0}$
using *group.r-inv* [*OF a-group*]
by (*simp add*: *a-inv-def*)

lemma (**in** *abelian-group*) *minus-zero* [*simp*]:
$\ominus\ \mathbf{0} = \mathbf{0}$
by (*simp add*: *a-inv-def*
group.inv-one [*OF a-group*, *simplified monoid-record-simps*])

lemma (**in** *abelian-group*) *minus-minus* [*simp*]:
$x \in carrier\ G ==> \ominus\ (\ominus\ x) = x$
using *group.inv-inv* [*OF a-group*, *simplified monoid-record-simps*]
by (*simp add*: *a-inv-def*)

lemma (**in** *abelian-group*) *a-inv-inj*:
inj-on (*a-inv G*) (*carrier G*)
using *group.inv-inj* [*OF a-group*, *simplified monoid-record-simps*]
by (*simp add*: *a-inv-def*)

lemma (**in** *abelian-group*) *minus-add*:
$[|\ x \in carrier\ G;\ y \in carrier\ G\ |] ==> \ominus\ (x \oplus y) = \ominus\ x \oplus \ominus\ y$

```
  using comm-group.inv-mult [OF a-comm-group]
  by (simp add: a-inv-def)

lemma (in abelian-group) minus-equality:
  [| x ∈ carrier G; y ∈ carrier G; y ⊕ x = 0 |] ==> ⊖ x = y
  using group.inv-equality [OF a-group]
  by (auto simp add: a-inv-def)

lemma (in abelian-monoid) minus-unique:
  [| x ∈ carrier G; y ∈ carrier G; y' ∈ carrier G;
     y ⊕ x = 0; x ⊕ y' = 0 |] ==> y = y'
  using monoid.inv-unique [OF a-monoid]
  by (simp add: a-inv-def)

lemmas (in abelian-monoid) a-ac = a-assoc a-comm a-lcomm
```

Derive an *abelian-group* from a *comm-group*

```
lemma comm-group-abelian-groupI:
  fixes G (structure)
  assumes cg: comm-group (|carrier = carrier G, mult = add G, one = zero G|)
  shows abelian-group G
proof −
  interpret comm-group [(|carrier = carrier G, mult = add G, one = zero G|)]
    by (rule cg)
  show abelian-group G by (unfold-locales)
qed
```

8.2 Sums over Finite Sets

This definition makes it easy to lift lemmas from *finprod*.

```
constdefs
  finsum :: [('b, 'm) ring-scheme, 'a => 'b, 'a set] => 'b
  finsum G f A == finprod (| carrier = carrier G,
     mult = add G, one = zero G |) f A

syntax
  -finsum :: index => idt => 'a set => 'b => 'b
     ((3⨁--:-. -) [1000, 0, 51, 10] 10)
syntax (xsymbols)
  -finsum :: index => idt => 'a set => 'b => 'b
     ((3⨁--∈-. -) [1000, 0, 51, 10] 10)
syntax (HTML output)
  -finsum :: index => idt => 'a set => 'b => 'b
     ((3⨁--∈-. -) [1000, 0, 51, 10] 10)
translations
  ⨁ıi:A. b == finsum ◇ı (%i. b) A
  — Beware of argument permutation!
```

lemma (**in** *abelian-monoid*) *finsum-empty* [*simp*]:
finsum G f {} = **0**
by (*rule comm-monoid.finprod-empty* [*OF a-comm-monoid*,
folded finsum-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-monoid*) *finsum-insert* [*simp*]:
[| *finite* F; $a \notin F$; $f \in F$ −> *carrier* G; f $a \in$ *carrier* G |]
==> *finsum* G f (*insert* a F) = f $a \oplus$ *finsum* G f F
by (*rule comm-monoid.finprod-insert* [*OF a-comm-monoid*,
folded finsum-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-monoid*) *finsum-zero* [*simp*]:
finite A ==> ($\bigoplus i{\in}A$. **0**) = **0**
by (*rule comm-monoid.finprod-one* [*OF a-comm-monoid*, *folded finsum-def*,
simplified monoid-record-simps])

lemma (**in** *abelian-monoid*) *finsum-closed* [*simp*]:
fixes A
assumes *fin*: *finite* A **and** f: $f \in A$ −> *carrier* G
shows *finsum* G f $A \in$ *carrier* G
apply (*rule comm-monoid.finprod-closed* [*OF a-comm-monoid*,
folded finsum-def, *simplified monoid-record-simps*])
apply (*rule fin*)
apply (*rule* f)
done

lemma (**in** *abelian-monoid*) *finsum-Un-Int*:
[| *finite* A; *finite* B; $g \in A$ −> *carrier* G; $g \in B$ −> *carrier* G |] ==>
finsum G g (A *Un* B) $\oplus$ *finsum* G g (A *Int* B) =
finsum G g $A \oplus$ *finsum* G g B
by (*rule comm-monoid.finprod-Un-Int* [*OF a-comm-monoid*,
folded finsum-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-monoid*) *finsum-Un-disjoint*:
[| *finite* A; *finite* B; A *Int* B = {};
$g \in A$ −> *carrier* G; $g \in B$ −> *carrier* G |]
==> *finsum* G g (A *Un* B) = *finsum* G g $A \oplus$ *finsum* G g B
by (*rule comm-monoid.finprod-Un-disjoint* [*OF a-comm-monoid*,
folded finsum-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-monoid*) *finsum-addf*:
[| *finite* A; $f \in A$ −> *carrier* G; $g \in A$ −> *carrier* G |] ==>
finsum G (%x. f $x \oplus g$ x) A = (*finsum* G f $A \oplus$ *finsum* G g A)
by (*rule comm-monoid.finprod-multf* [*OF a-comm-monoid*,
folded finsum-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-monoid*) *finsum-cong'*:
[| $A = B$; g : B −> *carrier* G;

!!*i*. *i* : *B* ==> *f i* = *g i* |] ==> *finsum G f A* = *finsum G g B*
by (*rule comm-monoid.finprod-cong'* [*OF a-comm-monoid*,
folded finsum-def, *simplified monoid-record-simps*]) *auto*

lemma (**in** *abelian-monoid*) *finsum-0* [*simp*]:
f : {*0*::*nat*} −> *carrier G* ==> *finsum G f* {..*0*} = *f 0*
by (*rule comm-monoid.finprod-0* [*OF a-comm-monoid*, *folded finsum-def*,
simplified monoid-record-simps])

lemma (**in** *abelian-monoid*) *finsum-Suc* [*simp*]:
f : {..*Suc n*} −> *carrier G* ==>
finsum G f {..*Suc n*} = (*f* (*Suc n*) ⊕ *finsum G f* {..*n*})
by (*rule comm-monoid.finprod-Suc* [*OF a-comm-monoid*, *folded finsum-def*,
simplified monoid-record-simps])

lemma (**in** *abelian-monoid*) *finsum-Suc2*:
f : {..*Suc n*} −> *carrier G* ==>
finsum G f {..*Suc n*} = (*finsum G* (%*i*. *f* (*Suc i*)) {..*n*} ⊕ *f 0*)
by (*rule comm-monoid.finprod-Suc2* [*OF a-comm-monoid*, *folded finsum-def*,
simplified monoid-record-simps])

lemma (**in** *abelian-monoid*) *finsum-add* [*simp*]:
[| *f* : {..*n*} −> *carrier G*; *g* : {..*n*} −> *carrier G* |] ==>
finsum G (%*i*. *f i* ⊕ *g i*) {..*n*::*nat*} =
finsum G f {..*n*} ⊕ *finsum G g* {..*n*}
by (*rule comm-monoid.finprod-mult* [*OF a-comm-monoid*, *folded finsum-def*,
simplified monoid-record-simps])

lemma (**in** *abelian-monoid*) *finsum-cong*:
[| *A* = *B*; *f* : *B* −> *carrier G*;
!!*i*. *i* : *B* =*simp*=> *f i* = *g i* |] ==> *finsum G f A* = *finsum G g B*
by (*rule comm-monoid.finprod-cong* [*OF a-comm-monoid*, *folded finsum-def*,
simplified monoid-record-simps]) (*auto simp add*: *simp-implies-def*)

Usually, if this rule causes a failed congruence proof error, the reason is that the premise *g* ∈ *B* −> *carrier G* cannot be shown. Adding *Pi-def* to the simpset is often useful.

9 The Algebraic Hierarchy of Rings

9.1 Basic Definitions

locale *ring* = *abelian-group R* + *monoid R* +
assumes *l-distr*: [| *x* ∈ *carrier R*; *y* ∈ *carrier R*; *z* ∈ *carrier R* |]
==> (*x* ⊕ *y*) ⊗ *z* = *x* ⊗ *z* ⊕ *y* ⊗ *z*
and *r-distr*: [| *x* ∈ *carrier R*; *y* ∈ *carrier R*; *z* ∈ *carrier R* |]
==> *z* ⊗ (*x* ⊕ *y*) = *z* ⊗ *x* ⊕ *z* ⊗ *y*

locale *cring* = *ring* + *comm-monoid R*

locale *domain* = *cring* +
  assumes *one-not-zero* [*simp*]: **1** ~= **0**
    and *integral*: [| $a \otimes b = \mathbf{0}$; $a \in$ *carrier R*; $b \in$ *carrier R* |] ==>
        $a = \mathbf{0}$ | $b = \mathbf{0}$

locale *field* = *domain* +
  assumes *field-Units*: *Units R* = *carrier R* − {**0**}

9.2 Rings

lemma *ringI*:
  fixes *R* (**structure**)
  assumes *abelian-group*: *abelian-group R*
    and *monoid*: *monoid R*
    and *l-distr*: !!*x y z*. [| $x \in$ *carrier R*; $y \in$ *carrier R*; $z \in$ *carrier R* |]
      ==> $(x \oplus y) \otimes z = x \otimes z \oplus y \otimes z$
    and *r-distr*: !!*x y z*. [| $x \in$ *carrier R*; $y \in$ *carrier R*; $z \in$ *carrier R* |]
      ==> $z \otimes (x \oplus y) = z \otimes x \oplus z \otimes y$
  shows *ring R*
  by (*auto intro*: *ring.intro*
    abelian-group.axioms ring-axioms.intro prems)

lemma (**in** *ring*) *is-abelian-group*:
  abelian-group R
  by (*auto intro*!: *abelian-groupI a-assoc a-comm l-neg*)

lemma (**in** *ring*) *is-monoid*:
  monoid R
  by (*auto intro*!: *monoidI m-assoc*)

lemma (**in** *ring*) *is-ring*:
  ring R
  by *fact*

lemmas *ring-record-simps* = *monoid-record-simps ring.simps*

lemma *cringI*:
  fixes *R* (**structure**)
  assumes *abelian-group*: *abelian-group R*
    and *comm-monoid*: *comm-monoid R*
    and *l-distr*: !!*x y z*. [| $x \in$ *carrier R*; $y \in$ *carrier R*; $z \in$ *carrier R* |]
      ==> $(x \oplus y) \otimes z = x \otimes z \oplus y \otimes z$
  shows *cring R*
proof (*intro cring.intro ring.intro*)
  show *ring-axioms R*
    — Right-distributivity follows from left-distributivity and commutativity.
  proof (*rule ring-axioms.intro*)
    fix *x y z*

assume *R*: $x \in$ *carrier R* $y \in$ *carrier R* $z \in$ *carrier R*
note [*simp*] = *comm-monoid.axioms* [*OF comm-monoid*]
abelian-group.axioms [*OF abelian-group*]
abelian-monoid.a-closed

from *R* **have** $z \otimes (x \oplus y) = (x \oplus y) \otimes z$
by (*simp add*: *comm-monoid.m-comm* [*OF comm-monoid.intro*])
also from *R* **have** ... $= x \otimes z \oplus y \otimes z$ **by** (*simp add*: *l-distr*)
also from *R* **have** ... $= z \otimes x \oplus z \otimes y$
by (*simp add*: *comm-monoid.m-comm* [*OF comm-monoid.intro*])
finally show $z \otimes (x \oplus y) = z \otimes x \oplus z \otimes y$ **.**
qed (*rule l-distr*)
qed (*auto intro*: *cring.intro*
abelian-group.axioms comm-monoid.axioms ring-axioms.intro prems)

lemma (**in** *cring*) *is-comm-monoid*:
comm-monoid R
by (*auto intro*!: *comm-monoidI m-assoc m-comm*)

lemma (**in** *cring*) *is-cring*:
cring R **by** *fact*

9.2.1 Normaliser for Rings

lemma (**in** *abelian-group*) *r-neg2*:
[| $x \in$ *carrier G*; $y \in$ *carrier G* |] ==> $x \oplus (\ominus x \oplus y) = y$
proof –
assume *G*: $x \in$ *carrier G* $y \in$ *carrier G*
then have $(x \oplus \ominus x) \oplus y = y$
by (*simp only*: *r-neg l-zero*)
with *G* **show** *?thesis*
by (*simp add*: *a-ac*)
qed

lemma (**in** *abelian-group*) *r-neg1*:
[| $x \in$ *carrier G*; $y \in$ *carrier G* |] ==> $\ominus x \oplus (x \oplus y) = y$
proof –
assume *G*: $x \in$ *carrier G* $y \in$ *carrier G*
then have $(\ominus x \oplus x) \oplus y = y$
by (*simp only*: *l-neg l-zero*)
with *G* **show** *?thesis* **by** (*simp add*: *a-ac*)
qed

The following proofs are from Jacobson, Basic Algebra I, pp. 88–89

lemma (**in** *ring*) *l-null* [*simp*]:
$x \in$ *carrier R* ==> $\mathbf{0} \otimes x = \mathbf{0}$
proof –
assume *R*: $x \in$ *carrier R*
then have $\mathbf{0} \otimes x \oplus \mathbf{0} \otimes x = (\mathbf{0} \oplus \mathbf{0}) \otimes x$

by (*simp add*: *l-distr del*: *l-zero r-zero*)
also from R **have** ... $= \mathbf{0} \otimes x \oplus \mathbf{0}$ **by** *simp*
finally have $\mathbf{0} \otimes x \oplus \mathbf{0} \otimes x = \mathbf{0} \otimes x \oplus \mathbf{0}$ **.**
with R **show** *?thesis* **by** (*simp del*: *r-zero*)
qed

lemma (**in** *ring*) *r-null* [*simp*]:
$x \in carrier\ R ==> x \otimes \mathbf{0} = \mathbf{0}$
proof −
assume R: $x \in carrier\ R$
then have $x \otimes \mathbf{0} \oplus x \otimes \mathbf{0} = x \otimes (\mathbf{0} \oplus \mathbf{0})$
by (*simp add*: *r-distr del*: *l-zero r-zero*)
also from R **have** ... $= x \otimes \mathbf{0} \oplus \mathbf{0}$ **by** *simp*
finally have $x \otimes \mathbf{0} \oplus x \otimes \mathbf{0} = x \otimes \mathbf{0} \oplus \mathbf{0}$ **.**
with R **show** *?thesis* **by** (*simp del*: *r-zero*)
qed

lemma (**in** *ring*) *l-minus*:
$[|\ x \in carrier\ R;\ y \in carrier\ R\ |] ==> \ominus\ x \otimes y = \ominus\ (x \otimes y)$
proof −
assume R: $x \in carrier\ R\ y \in carrier\ R$
then have $(\ominus\ x) \otimes y \oplus x \otimes y = (\ominus\ x \oplus x) \otimes y$ **by** (*simp add*: *l-distr*)
also from R **have** ... $= \mathbf{0}$ **by** (*simp add*: *l-neg l-null*)
finally have $(\ominus\ x) \otimes y \oplus x \otimes y = \mathbf{0}$ **.**
with R **have** $(\ominus\ x) \otimes y \oplus x \otimes y \oplus \ominus\ (x \otimes y) = \mathbf{0} \oplus \ominus\ (x \otimes y)$ **by** *simp*
with R **show** *?thesis* **by** (*simp add*: *a-assoc r-neg*)
qed

lemma (**in** *ring*) *r-minus*:
$[|\ x \in carrier\ R;\ y \in carrier\ R\ |] ==> x \otimes \ominus\ y = \ominus\ (x \otimes y)$
proof −
assume R: $x \in carrier\ R\ y \in carrier\ R$
then have $x \otimes (\ominus\ y) \oplus x \otimes y = x \otimes (\ominus\ y \oplus y)$ **by** (*simp add*: *r-distr*)
also from R **have** ... $= \mathbf{0}$ **by** (*simp add*: *l-neg r-null*)
finally have $x \otimes (\ominus\ y) \oplus x \otimes y = \mathbf{0}$ **.**
with R **have** $x \otimes (\ominus\ y) \oplus x \otimes y \oplus \ominus\ (x \otimes y) = \mathbf{0} \oplus \ominus\ (x \otimes y)$ **by** *simp*
with R **show** *?thesis* **by** (*simp add*: *a-assoc r-neg*)
qed

lemma (**in** *abelian-group*) *minus-eq*:
$[|\ x \in carrier\ G;\ y \in carrier\ G\ |] ==> x \ominus y = x \oplus \ominus\ y$
by (*simp only*: *a-minus-def*)

Setup algebra method: compute distributive normal form in locale contexts

use *ringsimp.ML*

setup *Algebra.setup*

lemmas (**in** *ring*) *ring-simprules*

[*algebra ring zero R add R a-inv R a-minus R one R mult R*] =
a-closed zero-closed a-inv-closed minus-closed m-closed one-closed
a-assoc l-zero l-neg a-comm m-assoc l-one l-distr minus-eq
r-zero r-neg r-neg2 r-neg1 minus-add minus-minus minus-zero
a-lcomm r-distr l-null r-null l-minus r-minus

lemmas (**in** *cring*)
[*algebra del*: *ring zero R add R a-inv R a-minus R one R mult R*] =
-

lemmas (**in** *cring*) *cring-simprules*
[*algebra add*: *cring zero R add R a-inv R a-minus R one R mult R*] =
a-closed zero-closed a-inv-closed minus-closed m-closed one-closed
a-assoc l-zero l-neg a-comm m-assoc l-one l-distr m-comm minus-eq
r-zero r-neg r-neg2 r-neg1 minus-add minus-minus minus-zero
a-lcomm m-lcomm r-distr l-null r-null l-minus r-minus

lemma (**in** *cring*) *nat-pow-zero*:
(*n*::*nat*) ~= *0* ==> **0** (^) *n* = **0**
by (*induct n*) *simp-all*

lemma (**in** *ring*) *one-zeroD*:
assumes *onezero*: **1** = **0**
shows *carrier R* = {**0**}
proof (*rule*, *rule*)
fix *x*
assume *xcarr*: $x \in$ *carrier R*
from *xcarr*
have $x = x \otimes$ **1** **by** *simp*
from *this* **and** *onezero*
have $x = x \otimes$ **0** **by** *simp*
from *this* **and** *xcarr*
have $x =$ **0** **by** *simp*
thus $x \in$ {**0**} **by** *fast*
qed *fast*

lemma (**in** *ring*) *one-zeroI*:
assumes *carrzero*: *carrier R* = {**0**}
shows **1** = **0**
proof −
from *one-closed* **and** *carrzero*
show **1** = **0** **by** *simp*
qed

lemma (**in** *ring*) *one-zero*:
shows (*carrier R* = {**0**}) = (**1** = **0**)
by (*rule*, *erule one-zeroI*, *erule one-zeroD*)

lemma (**in** *ring*) *one-not-zero*:
  shows $(carrier\ R \neq \{\mathbf{0}\}) = (\mathbf{1} \neq \mathbf{0})$
  by (*simp add*: *one-zero*)

Two examples for use of method algebra

lemma
  includes *ring R* + *cring S*
  shows [| $a \in carrier\ R$; $b \in carrier\ R$; $c \in carrier\ S$; $d \in carrier\ S$ |] ==>
  $a \oplus \ominus (a \oplus \ominus b) = b$ & $c \otimes_S d = d \otimes_S c$
  by *algebra*

lemma
  includes *cring*
  shows [| $a \in carrier\ R$; $b \in carrier\ R$ |] ==> $a \ominus (a \ominus b) = b$
  by *algebra*

9.2.2 Sums over Finite Sets

lemma (**in** *cring*) *finsum-ldistr*:
  [| *finite A*; $a \in carrier\ R$; $f \in A$ −> *carrier R* |] ==>
   finsum R f A $\otimes$ a = *finsum R* (%*i*. $f\ i \otimes a$) *A*
proof (*induct set*: *finite*)
  case *empty* **then show** *?case* **by** *simp*
next
  case (*insert x F*) **then show** *?case* **by** (*simp add*: *Pi-def l-distr*)
qed

lemma (**in** *cring*) *finsum-rdistr*:
  [| *finite A*; $a \in carrier\ R$; $f \in A$ −> *carrier R* |] ==>
   $a \otimes$ *finsum R f A* = *finsum R* (%*i*. $a \otimes f\ i$) *A*
proof (*induct set*: *finite*)
  case *empty* **then show** *?case* **by** *simp*
next
  case (*insert x F*) **then show** *?case* **by** (*simp add*: *Pi-def r-distr*)
qed

9.3 Integral Domains

lemma (**in** *domain*) *zero-not-one* [*simp*]:
  $\mathbf{0}$ ~= $\mathbf{1}$
  by (*rule not-sym*) *simp*

lemma (**in** *domain*) *integral-iff*:
  [| $a \in carrier\ R$; $b \in carrier\ R$ |] ==> $(a \otimes b = \mathbf{0}) = (a = \mathbf{0} \mid b = \mathbf{0})$
proof
  assume $a \in carrier\ R$ $b \in carrier\ R$ $a \otimes b = \mathbf{0}$
  then show $a = \mathbf{0} \mid b = \mathbf{0}$ **by** (*simp add*: *integral*)
next
  assume $a \in carrier\ R$ $b \in carrier\ R$ $a = \mathbf{0} \mid b = \mathbf{0}$

then show $a \otimes b = \mathbf{0}$ **by** *auto*
qed

lemma (**in** *domain*) *m-lcancel*:
assumes *prem*: $a \,\tilde{}{=}\, \mathbf{0}$
and *R*: $a \in$ *carrier R* $b \in$ *carrier R* $c \in$ *carrier R*
shows $(a \otimes b = a \otimes c) = (b = c)$
proof
assume *eq*: $a \otimes b = a \otimes c$
with *R* **have** $a \otimes (b \ominus c) = \mathbf{0}$ **by** *algebra*
with *R* **have** $a = \mathbf{0} \mid (b \ominus c) = \mathbf{0}$ **by** (*simp add*: *integral-iff*)
with *prem* **and** *R* **have** $b \ominus c = \mathbf{0}$ **by** *auto*
with *R* **have** $b = b \ominus (b \ominus c)$ **by** *algebra*
also from *R* **have** $b \ominus (b \ominus c) = c$ **by** *algebra*
finally show $b = c$ **.**
next
assume $b = c$ **then show** $a \otimes b = a \otimes c$ **by** *simp*
qed

lemma (**in** *domain*) *m-rcancel*:
assumes *prem*: $a \,\tilde{}{=}\, \mathbf{0}$
and *R*: $a \in$ *carrier R* $b \in$ *carrier R* $c \in$ *carrier R*
shows *conc*: $(b \otimes a = c \otimes a) = (b = c)$
proof −
from *prem* **and** *R* **have** $(a \otimes b = a \otimes c) = (b = c)$ **by** (*rule m-lcancel*)
with *R* **show** *?thesis* **by** *algebra*
qed

9.4 Fields

Field would not need to be derived from domain, the properties for domain follow from the assumptions of field

lemma (**in** *cring*) *cring-fieldI*:
assumes *field-Units*: *Units R* = *carrier R* − $\{\mathbf{0}\}$
shows *field R*
proof *unfold-locales*
from *field-Units*
have *a*: $\mathbf{0} \notin$ *Units R* **by** *fast*
have $\mathbf{1} \in$ *Units R* **by** *fast*
from *this* **and** *a*
show $\mathbf{1} \neq \mathbf{0}$ **by** *force*
next
fix *a b*
assume *acarr*: $a \in$ *carrier R*
and *bcarr*: $b \in$ *carrier R*
and *ab*: $a \otimes b = \mathbf{0}$
show $a = \mathbf{0} \vee b = \mathbf{0}$
proof (*cases* $a = \mathbf{0}$, *simp*)
assume $a \neq \mathbf{0}$

from *this* **and** *field-Units* **and** *acarr*
have *aUnit*: $a \in$ *Units R* **by** *fast*
from *bcarr*
have $b = \mathbf{1} \otimes b$ **by** *algebra*
also from *aUnit acarr*
have ... $=$ (*inv* $a \otimes a) \otimes b$ **by** (*simp add*: *Units-l-inv*)
also from *acarr bcarr aUnit*[*THEN Units-inv-closed*]
have ... $=$ (*inv* $a) \otimes (a \otimes b)$ **by** *algebra*
also from *ab* **and** *acarr bcarr aUnit*
have ... $=$ (*inv* $a) \otimes \mathbf{0}$ **by** *simp*
also from *aUnit*[*THEN Units-inv-closed*]
have ... $= \mathbf{0}$ **by** *algebra*
finally
have $b = \mathbf{0}$ **.**
thus $a = \mathbf{0} \vee b = \mathbf{0}$ **by** *simp*
qed
qed (*rule field-Units*)

Another variant to show that something is a field

lemma (**in** *cring*) *cring-fieldI2*:
assumes *notzero*: $\mathbf{0} \neq \mathbf{1}$
and *invex*: $\bigwedge a.\ [\![a \in \mathit{carrier}\ R;\ a \neq \mathbf{0}]\!] \Longrightarrow \exists\, b{\in}\mathit{carrier}\ R.\ a \otimes b = \mathbf{1}$
shows *field R*
apply (*rule cring-fieldI*, *simp add*: *Units-def*)
apply (*rule*, *clarsimp*)
apply (*simp add*: *notzero*)
proof (*clarsimp*)
fix x
assume *xcarr*: $x \in$ *carrier R*
and $x \neq \mathbf{0}$
from *this*
have $\exists\, y{\in}\mathit{carrier}\ R.\ x \otimes y = \mathbf{1}$ **by** (*rule invex*)
from *this*
obtain y
where *ycarr*: $y \in$ *carrier R*
and *xy*: $x \otimes y = \mathbf{1}$
by *fast*
from *xy xcarr ycarr* **have** $y \otimes x = \mathbf{1}$ **by** (*simp add*: *m-comm*)
from *ycarr* **and** *this* **and** *xy*
show $\exists\, y{\in}\mathit{carrier}\ R.\ y \otimes x = \mathbf{1} \wedge x \otimes y = \mathbf{1}$ **by** *fast*
qed

9.5 Morphisms

constdefs (**structure** *R S*)
ring-hom :: [(*'a*, *'m*) *ring-scheme*, (*'b*, *'n*) *ring-scheme*] => (*'a* => *'b*) *set*
ring-hom R S == {*h*. *h* $\in$ *carrier R* −> *carrier S* &
(*ALL x y*. $x \in$ *carrier R* & $y \in$ *carrier R* −−>
$h\ (x \otimes y) = h\ x \otimes_S h\ y$ & $h\ (x \oplus y) = h\ x \oplus_S h\ y$) &

h **1** = $\mathbf{1}_S$}

lemma *ring-hom-memI*:
fixes R (**structure**) **and** S (**structure**)
assumes *hom-closed*: !!x. $x \in$ *carrier* R ==> $h\ x \in$ *carrier* S
and *hom-mult*: !!x y. [| $x \in$ *carrier* R; $y \in$ *carrier* R |] ==>
$h\ (x \otimes y) = h\ x \otimes_S h\ y$
and *hom-add*: !!x y. [| $x \in$ *carrier* R; $y \in$ *carrier* R |] ==>
$h\ (x \oplus y) = h\ x \oplus_S h\ y$
and *hom-one*: h **1** = $\mathbf{1}_S$
shows $h \in$ *ring-hom* R S
by (*auto simp add*: *ring-hom-def prems Pi-def*)

lemma *ring-hom-closed*:
[| $h \in$ *ring-hom* R S; $x \in$ *carrier* R |] ==> $h\ x \in$ *carrier* S
by (*auto simp add*: *ring-hom-def funcset-mem*)

lemma *ring-hom-mult*:
fixes R (**structure**) **and** S (**structure**)
shows
[| $h \in$ *ring-hom* R S; $x \in$ *carrier* R; $y \in$ *carrier* R |] ==>
$h\ (x \otimes y) = h\ x \otimes_S h\ y$
by (*simp add*: *ring-hom-def*)

lemma *ring-hom-add*:
fixes R (**structure**) **and** S (**structure**)
shows
[| $h \in$ *ring-hom* R S; $x \in$ *carrier* R; $y \in$ *carrier* R |] ==>
$h\ (x \oplus y) = h\ x \oplus_S h\ y$
by (*simp add*: *ring-hom-def*)

lemma *ring-hom-one*:
fixes R (**structure**) **and** S (**structure**)
shows $h \in$ *ring-hom* R S ==> h **1** = $\mathbf{1}_S$
by (*simp add*: *ring-hom-def*)

locale *ring-hom-cring* = *cring* R + *cring* S +
fixes h
assumes *homh* [*simp*, *intro*]: $h \in$ *ring-hom* R S
notes *hom-closed* [*simp*, *intro*] = *ring-hom-closed* [*OF homh*]
and *hom-mult* [*simp*] = *ring-hom-mult* [*OF homh*]
and *hom-add* [*simp*] = *ring-hom-add* [*OF homh*]
and *hom-one* [*simp*] = *ring-hom-one* [*OF homh*]

lemma (**in** *ring-hom-cring*) *hom-zero* [*simp*]:
h **0** = $\mathbf{0}_S$
proof −
have $h\ \mathbf{0} \oplus_S h\ \mathbf{0} = h\ \mathbf{0} \oplus_S \mathbf{0}_S$
by (*simp add*: *hom-add* [*symmetric*] *del*: *hom-add*)

then show *?thesis* **by** (*simp del*: *S.r-zero*)
qed

lemma (**in** *ring-hom-cring*) *hom-a-inv* [*simp*]:
$x \in$ *carrier R* ==> h ($\ominus$ x) = $\ominus_S$ h x
proof −
assume *R*: $x \in$ *carrier R*
then have $h\ x \oplus_S h\ (\ominus\ x) = h\ x \oplus_S (\ominus_S\ h\ x)$
by (*simp add*: *hom-add* [*symmetric*] *R.r-neg S.r-neg del*: *hom-add*)
with *R* **show** *?thesis* **by** *simp*
qed

lemma (**in** *ring-hom-cring*) *hom-finsum* [*simp*]:
[| *finite A*; $f \in A$ −> *carrier R* |] ==>
h (*finsum R f A*) = *finsum S* (*h o f*) *A*
proof (*induct set*: *finite*)
case *empty* **then show** *?case* **by** *simp*
next
case *insert* **then show** *?case* **by** (*simp add*: *Pi-def*)
qed

lemma (**in** *ring-hom-cring*) *hom-finprod*:
[| *finite A*; $f \in A$ −> *carrier R* |] ==>
h (*finprod R f A*) = *finprod S* (*h o f*) *A*
proof (*induct set*: *finite*)
case *empty* **then show** *?case* **by** *simp*
next
case *insert* **then show** *?case* **by** (*simp add*: *Pi-def*)
qed

declare *ring-hom-cring.hom-finprod* [*simp*]

lemma *id-ring-hom* [*simp*]:
id $\in$ *ring-hom R R*
by (*auto intro*!: *ring-hom-memI*)

end

theory *Module* **imports** *Ring* **begin**

10 Modules over an Abelian Group

10.1 Definitions

record ('*a*, '*b*) *module* = '*b ring* +
smult :: ['*a*, '*b*] => '*b* (**infixl** $\odot_1$ *70*)

locale *module* = *cring R* + *abelian-group M* +
assumes *smult-closed* [*simp*, *intro*]:
$[| \ a \in \mathit{carrier}\ R;\ x \in \mathit{carrier}\ M\ |] ==> a \odot_M x \in \mathit{carrier}\ M$
and *smult-l-distr*:
$[| \ a \in \mathit{carrier}\ R;\ b \in \mathit{carrier}\ R;\ x \in \mathit{carrier}\ M\ |] ==>$
$(a \oplus b) \odot_M x = a \odot_M x \oplus_M b \odot_M x$
and *smult-r-distr*:
$[| \ a \in \mathit{carrier}\ R;\ x \in \mathit{carrier}\ M;\ y \in \mathit{carrier}\ M\ |] ==>$
$a \odot_M (x \oplus_M y) = a \odot_M x \oplus_M a \odot_M y$
and *smult-assoc1*:
$[| \ a \in \mathit{carrier}\ R;\ b \in \mathit{carrier}\ R;\ x \in \mathit{carrier}\ M\ |] ==>$
$(a \otimes b) \odot_M x = a \odot_M (b \odot_M x)$
and *smult-one* [*simp*]:
$x \in \mathit{carrier}\ M ==> \mathbf{1} \odot_M x = x$

locale *algebra* = *module R M* + *cring M* +
assumes *smult-assoc2*:
$[| \ a \in \mathit{carrier}\ R;\ x \in \mathit{carrier}\ M;\ y \in \mathit{carrier}\ M\ |] ==>$
$(a \odot_M x) \otimes_M y = a \odot_M (x \otimes_M y)$

lemma *moduleI*:
fixes *R* (**structure**) **and** *M* (**structure**)
assumes *cring*: *cring R*
and *abelian-group*: *abelian-group M*
and *smult-closed*:
$!!a\ x.\ [| \ a \in \mathit{carrier}\ R;\ x \in \mathit{carrier}\ M\ |] ==> a \odot_M x \in \mathit{carrier}\ M$
and *smult-l-distr*:
$!!a\ b\ x.\ [| \ a \in \mathit{carrier}\ R;\ b \in \mathit{carrier}\ R;\ x \in \mathit{carrier}\ M\ |] ==>$
$(a \oplus b) \odot_M x = (a \odot_M x) \oplus_M (b \odot_M x)$
and *smult-r-distr*:
$!!a\ x\ y.\ [| \ a \in \mathit{carrier}\ R;\ x \in \mathit{carrier}\ M;\ y \in \mathit{carrier}\ M\ |] ==>$
$a \odot_M (x \oplus_M y) = (a \odot_M x) \oplus_M (a \odot_M y)$
and *smult-assoc1*:
$!!a\ b\ x.\ [| \ a \in \mathit{carrier}\ R;\ b \in \mathit{carrier}\ R;\ x \in \mathit{carrier}\ M\ |] ==>$
$(a \otimes b) \odot_M x = a \odot_M (b \odot_M x)$
and *smult-one*:
$!!x.\ x \in \mathit{carrier}\ M ==> \mathbf{1} \odot_M x = x$
shows *module R M*
by (*auto intro*: *module.intro cring.axioms abelian-group.axioms*
module-axioms.intro prems)

lemma *algebraI*:
fixes *R* (**structure**) **and** *M* (**structure**)
assumes *R-cring*: *cring R*
and *M-cring*: *cring M*
and *smult-closed*:
$!!a\ x.\ [| \ a \in \mathit{carrier}\ R;\ x \in \mathit{carrier}\ M\ |] ==> a \odot_M x \in \mathit{carrier}\ M$
and *smult-l-distr*:
$!!a\ b\ x.\ [| \ a \in \mathit{carrier}\ R;\ b \in \mathit{carrier}\ R;\ x \in \mathit{carrier}\ M\ |] ==>$

$(a \oplus b) \odot_M x = (a \odot_M x) \oplus_M (b \odot_M x)$
and *smult-r-distr*:
!!*a x y*. [| $a \in$ *carrier R*; $x \in$ *carrier M*; $y \in$ *carrier M* |] ==>
$a \odot_M (x \oplus_M y) = (a \odot_M x) \oplus_M (a \odot_M y)$
and *smult-assoc1*:
!!*a b x*. [| $a \in$ *carrier R*; $b \in$ *carrier R*; $x \in$ *carrier M* |] ==>
$(a \otimes b) \odot_M x = a \odot_M (b \odot_M x)$
and *smult-one*:
!!*x*. $x \in$ *carrier M* ==> (*one R*) $\odot_M x = x$
and *smult-assoc2*:
!!*a x y*. [| $a \in$ *carrier R*; $x \in$ *carrier M*; $y \in$ *carrier M* |] ==>
$(a \odot_M x) \otimes_M y = a \odot_M (x \otimes_M y)$
shows *algebra R M*
apply *intro-locales*
apply (*rule cring.axioms ring.axioms abelian-group.axioms comm-monoid.axioms prems*)+
apply (*rule module-axioms.intro*)
apply (*simp add*: *smult-closed*)
apply (*simp add*: *smult-l-distr*)
apply (*simp add*: *smult-r-distr*)
apply (*simp add*: *smult-assoc1*)
apply (*simp add*: *smult-one*)
apply (*rule cring.axioms ring.axioms abelian-group.axioms comm-monoid.axioms prems*)+
apply (*rule algebra-axioms.intro*)
apply (*simp add*: *smult-assoc2*)
done

lemma (**in** *algebra*) *R-cring*:
cring R
by *unfold-locales*

lemma (**in** *algebra*) *M-cring*:
cring M
by *unfold-locales*

lemma (**in** *algebra*) *module*:
module R M
by (*auto intro*: *moduleI R-cring is-abelian-group*
smult-l-distr smult-r-distr smult-assoc1)

10.2 Basic Properties of Algebras

lemma (**in** *algebra*) *smult-l-null* [*simp*]:
$x \in$ *carrier M* ==> $\mathbf{0} \odot_M x = \mathbf{0}_M$
proof −
assume *M*: $x \in$ *carrier M*
note *facts* = *M smult-closed* [*OF R.zero-closed*]
from *facts* **have** $\mathbf{0} \odot_M x = (\mathbf{0} \odot_M x \oplus_M \mathbf{0} \odot_M x) \oplus_M \ominus_M (\mathbf{0} \odot_M x)$ **by** *algebra*

also from M **have** ... $= (\mathbf{0} \oplus \mathbf{0}) \odot_M x \oplus_M \ominus_M (\mathbf{0} \odot_M x)$
by (*simp add*: *smult-l-distr del*: *R.l-zero R.r-zero*)
also from *facts* **have** ... $= \mathbf{0}_M$ **apply** *algebra* **apply** *algebra* **done**
finally show *?thesis* **.**
qed

lemma (**in** *algebra*) *smult-r-null* [*simp*]:
$a \in$ *carrier* R ==> $a \odot_M \mathbf{0}_M = \mathbf{0}_M$
proof –
assume R: $a \in$ *carrier* R
note *facts* = R *smult-closed*
from *facts* **have** $a \odot_M \mathbf{0}_M = (a \odot_M \mathbf{0}_M \oplus_M a \odot_M \mathbf{0}_M) \oplus_M \ominus_M (a \odot_M \mathbf{0}_M)$
by *algebra*
also from R **have** ... $= a \odot_M (\mathbf{0}_M \oplus_M \mathbf{0}_M) \oplus_M \ominus_M (a \odot_M \mathbf{0}_M)$
by (*simp add*: *smult-r-distr del*: *M.l-zero M.r-zero*)
also from *facts* **have** ... $= \mathbf{0}_M$ **by** *algebra*
finally show *?thesis* **.**
qed

lemma (**in** *algebra*) *smult-l-minus*:
[| $a \in$ *carrier* R; $x \in$ *carrier* M |] ==> $(\ominus a) \odot_M x = \ominus_M (a \odot_M x)$
proof –
assume *RM*: $a \in$ *carrier* R $x \in$ *carrier* M
from *RM* **have** *a-smult*: $a \odot_M x \in$ *carrier* M **by** *simp*
from *RM* **have** *ma-smult*: $\ominus a \odot_M x \in$ *carrier* M **by** *simp*
note *facts* = *RM a-smult ma-smult*
from *facts* **have** $(\ominus a) \odot_M x = (\ominus a \odot_M x \oplus_M a \odot_M x) \oplus_M \ominus_M(a \odot_M x)$
by *algebra*
also from *RM* **have** ... $= (\ominus a \oplus a) \odot_M x \oplus_M \ominus_M(a \odot_M x)$
by (*simp add*: *smult-l-distr*)
also from *facts smult-l-null* **have** ... $= \ominus_M(a \odot_M x)$
apply *algebra* **apply** *algebra* **done**
finally show *?thesis* **.**
qed

lemma (**in** *algebra*) *smult-r-minus*:
[| $a \in$ *carrier* R; $x \in$ *carrier* M |] ==> $a \odot_M (\ominus_M x) = \ominus_M (a \odot_M x)$
proof –
assume *RM*: $a \in$ *carrier* R $x \in$ *carrier* M
note *facts* = *RM smult-closed*
from *facts* **have** $a \odot_M (\ominus_M x) = (a \odot_M \ominus_M x \oplus_M a \odot_M x) \oplus_M \ominus_M(a \odot_M x)$
by *algebra*
also from *RM* **have** ... $= a \odot_M (\ominus_M x \oplus_M x) \oplus_M \ominus_M(a \odot_M x)$
by (*simp add*: *smult-r-distr*)
also from *facts smult-r-null* **have** ... $= \ominus_M(a \odot_M x)$ **by** *algebra*
finally show *?thesis* **.**
qed

end

theory *UnivPoly* **imports** *Module* **begin**

11 Univariate Polynomials

Polynomials are formalised as modules with additional operations for extracting coefficients from polynomials and for obtaining monomials from coefficients and exponents (record *up-ring*). The carrier set is a set of bounded functions from Nat to the coefficient domain. Bounded means that these functions return zero above a certain bound (the degree). There is a chapter on the formalisation of polynomials in the PhD thesis [1], which was implemented with axiomatic type classes. This was later ported to Locales.

11.1 The Constructor for Univariate Polynomials

Functions with finite support.

locale *bound* =
  fixes z :: $'a$
    and n :: *nat*
    and f :: *nat* => $'a$
  assumes *bound*: !!m. $n < m \Longrightarrow f\ m = z$

declare *bound.intro* [*intro*!]
  and *bound.bound* [*dest*]

lemma *bound-below*:
  assumes *bound*: *bound* $z\ m\ f$ **and** *nonzero*: $f\ n \neq z$ **shows** $n \leq m$
proof (*rule classical*)
  assume ~ *?thesis*
  then have $m < n$ **by** *arith*
  with *bound* **have** $f\ n = z$ **..**
  with *nonzero* **show** *?thesis* **by** *contradiction*
qed

record ($'a$, $'p$) *up-ring* = ($'a$, $'p$) *module* +
  monom :: [$'a$, *nat*] => $'p$
  coeff :: [$'p$, *nat*] => $'a$

constdefs (**structure** R)
  up :: ($'a$, $'m$) *ring-scheme* => (*nat* => $'a$) *set*
  up R == {f. $f \in$ *UNIV* −> *carrier* R & (*EX* n. *bound* **0** n f)}
  UP :: ($'a$, $'m$) *ring-scheme* => ($'a$, *nat* => $'a$) *up-ring*
  UP R == (|
    carrier = *up* R,
    mult = (%p:*up* R. %q:*up* R. %n. $\bigoplus i \in \{..n\}.\ p\ i \otimes q\ (n-i)$),

```
  one = (%i. if i=0 then 1 else 0),
  zero = (%i. 0),
  add = (%p:up R. %q:up R. %i. p i ⊕ q i),
  smult = (%a:carrier R. %p:up R. %i. a ⊗ p i),
  monom = (%a:carrier R. %n i. if i=n then a else 0),
  coeff = (%p:up R. %n. p n) |)
```

Properties of the set of polynomials *up*.

```
lemma mem-upI [intro]:
  [| !!n. f n ∈ carrier R; EX n. bound (zero R) n f |] ==> f ∈ up R
  by (simp add: up-def Pi-def)

lemma mem-upD [dest]:
  f ∈ up R ==> f n ∈ carrier R
  by (simp add: up-def Pi-def)

lemma (in cring) bound-upD [dest]:
  f ∈ up R ==> EX n. bound 0 n f
  by (simp add: up-def)

lemma (in cring) up-one-closed:
   (%n. if n = 0 then 1 else 0) ∈ up R
  using up-def by force

lemma (in cring) up-smult-closed:
  [| a ∈ carrier R; p ∈ up R |] ==> (%i. a ⊗ p i) ∈ up R
  by force

lemma (in cring) up-add-closed:
  [| p ∈ up R; q ∈ up R |] ==> (%i. p i ⊕ q i) ∈ up R
proof
  fix n
  assume p ∈ up R and q ∈ up R
  then show p n ⊕ q n ∈ carrier R
    by auto
next
  assume UP: p ∈ up R q ∈ up R
  show EX n. bound 0 n (%i. p i ⊕ q i)
  proof −
    from UP obtain n where boundn: bound 0 n p by fast
    from UP obtain m where boundm: bound 0 m q by fast
    have bound 0 (max n m) (%i. p i ⊕ q i)
    proof
      fix i
      assume max n m < i
      with boundn and boundm and UP show p i ⊕ q i = 0 by fastsimp
    qed
    then show ?thesis ..
  qed
```

qed

lemma (**in** *cring*) *up-a-inv-closed*:
  $p \in up\ R ==> (\%i.\ \ominus\ (p\ i)) \in up\ R$
proof
  assume *R*: $p \in up\ R$
  then obtain n **where** *bound* **0** $n\ p$ **by** *auto*
  then have *bound* **0** n $(\%i.\ \ominus\ p\ i)$ **by** *auto*
  then show *EX* n. *bound* **0** n $(\%i.\ \ominus\ p\ i)$ **by** *auto*
qed *auto*

lemma (**in** *cring*) *up-mult-closed*:
  $[|\ p \in up\ R;\ q \in up\ R\ |] ==>$
  $(\%n.\ \bigoplus i \in \{..n\}.\ p\ i \otimes q\ (n-i)) \in up\ R$
proof
  fix n
  assume $p \in up\ R\ q \in up\ R$
  then show $(\bigoplus i \in \{..n\}.\ p\ i \otimes q\ (n-i)) \in carrier\ R$
    by (*simp add*: *mem-upD funcsetI*)
next
  assume *UP*: $p \in up\ R\ q \in up\ R$
  show *EX* n. *bound* **0** n $(\%n.\ \bigoplus i \in \{..n\}.\ p\ i \otimes q\ (n-i))$
  proof $-$
    from *UP* **obtain** n **where** *boundn*: *bound* **0** $n\ p$ **by** *fast*
    from *UP* **obtain** m **where** *boundm*: *bound* **0** $m\ q$ **by** *fast*
    have *bound* **0** $(n + m)$ $(\%n.\ \bigoplus i \in \{..n\}.\ p\ i \otimes q\ (n - i))$
    proof
      fix k **assume** *bound*: $n + m < k$
      {
        fix i
        have $p\ i \otimes q\ (k-i) =$ **0**
        proof (*cases* $n < i$)
          case *True*
          with *boundn* **have** $p\ i =$ **0** **by** *auto*
          moreover from *UP* **have** $q\ (k-i) \in carrier\ R$ **by** *auto*
          ultimately show *?thesis* **by** *simp*
        next
          case *False*
          with *bound* **have** $m < k-i$ **by** *arith*
          with *boundm* **have** $q\ (k-i) =$ **0** **by** *auto*
          moreover from *UP* **have** $p\ i \in carrier\ R$ **by** *auto*
          ultimately show *?thesis* **by** *simp*
        qed
      }
      then show $(\bigoplus i \in \{..k\}.\ p\ i \otimes q\ (k-i)) =$ **0**
        by (*simp add*: *Pi-def*)
    qed
    then show *?thesis* **by** *fast*
  qed

qed

11.2 Effect of Operations on Coefficients

locale *UP* =
  fixes *R* (**structure**) **and** *P* (**structure**)
  defines *P-def*: *P* == *UP R*

locale *UP-cring* = *UP* + *cring R*

locale *UP-domain* = *UP-cring* + *domain R*

Temporarily declare $P \equiv UP\ R$ as simp rule.

declare (**in** *UP*) *P-def* [*simp*]

lemma (**in** *UP-cring*) *coeff-monom* [*simp*]:
  $a \in carrier\ R$ ==>
  coeff P (*monom P a m*) *n* = (*if m=n then a else* **0**)
proof −
  assume *R*: $a \in carrier\ R$
  then have (%*n*. *if n* = *m then a else* **0**) $\in$ *up R*
    using *up-def* **by** *force*
  with *R* **show** *?thesis* **by** (*simp add*: *UP-def*)
qed

lemma (**in** *UP-cring*) *coeff-zero* [*simp*]:
  coeff P $\mathbf{0}_P$ *n* = **0**
  by (*auto simp add*: *UP-def*)

lemma (**in** *UP-cring*) *coeff-one* [*simp*]:
  coeff P $\mathbf{1}_P$ *n* = (*if n=0 then* **1** *else* **0**)
  using *up-one-closed* **by** (*simp add*: *UP-def*)

lemma (**in** *UP-cring*) *coeff-smult* [*simp*]:
  [| $a \in carrier\ R$; $p \in carrier\ P$ |] ==>
  coeff P ($a \odot_P p$) *n* = $a \otimes$ *coeff P p n*
  by (*simp add*: *UP-def up-smult-closed*)

lemma (**in** *UP-cring*) *coeff-add* [*simp*]:
  [| $p \in carrier\ P$; $q \in carrier\ P$ |] ==>
  coeff P ($p \oplus_P q$) *n* = *coeff P p n* $\oplus$ *coeff P q n*
  by (*simp add*: *UP-def up-add-closed*)

lemma (**in** *UP-cring*) *coeff-mult* [*simp*]:
  [| $p \in carrier\ P$; $q \in carrier\ P$ |] ==>
  coeff P ($p \otimes_P q$) *n* = ($\bigoplus i \in \{..n\}$. *coeff P p i* $\otimes$ *coeff P q* ($n-i$))
  by (*simp add*: *UP-def up-mult-closed*)

lemma (**in** *UP*) *up-eqI*:

assumes *prem*: !!n. *coeff P p n* = *coeff P q n*
  and *R*: $p \in$ *carrier P* $q \in$ *carrier P*
shows $p = q$
proof
fix x
from *prem* **and** *R* **show** $p\ x = q\ x$ **by** (*simp add*: *UP-def*)
qed

11.3 Polynomials Form a Commutative Ring.

Operations are closed over P.

lemma (**in** *UP-cring*) *UP-mult-closed* [*simp*]:
[| $p \in$ *carrier P*; $q \in$ *carrier P* |] ==> $p \otimes_P q \in$ *carrier P*
by (*simp add*: *UP-def up-mult-closed*)

lemma (**in** *UP-cring*) *UP-one-closed* [*simp*]:
$\mathbf{1}_P \in$ *carrier P*
by (*simp add*: *UP-def up-one-closed*)

lemma (**in** *UP-cring*) *UP-zero-closed* [*intro*, *simp*]:
$\mathbf{0}_P \in$ *carrier P*
by (*auto simp add*: *UP-def*)

lemma (**in** *UP-cring*) *UP-a-closed* [*intro*, *simp*]:
[| $p \in$ *carrier P*; $q \in$ *carrier P* |] ==> $p \oplus_P q \in$ *carrier P*
by (*simp add*: *UP-def up-add-closed*)

lemma (**in** *UP-cring*) *monom-closed* [*simp*]:
$a \in$ *carrier R* ==> *monom P a n* $\in$ *carrier P*
by (*auto simp add*: *UP-def up-def Pi-def*)

lemma (**in** *UP-cring*) *UP-smult-closed* [*simp*]:
[| $a \in$ *carrier R*; $p \in$ *carrier P* |] ==> $a \odot_P p \in$ *carrier P*
by (*simp add*: *UP-def up-smult-closed*)

lemma (**in** *UP*) *coeff-closed* [*simp*]:
$p \in$ *carrier P* ==> *coeff P p n* $\in$ *carrier R*
by (*auto simp add*: *UP-def*)

declare (**in** *UP*) *P-def* [*simp del*]

Algebraic ring properties

lemma (**in** *UP-cring*) *UP-a-assoc*:
assumes *R*: $p \in$ *carrier P* $q \in$ *carrier P* $r \in$ *carrier P*
shows $(p \oplus_P q) \oplus_P r = p \oplus_P (q \oplus_P r)$
by (*rule up-eqI*, *simp add*: *a-assoc R*, *simp-all add*: *R*)

lemma (**in** *UP-cring*) *UP-l-zero* [*simp*]:
assumes *R*: $p \in$ *carrier P*

```
  shows 0_P ⊕_P p = p
  by (rule up-eqI, simp-all add: R)

lemma (in UP-cring) UP-l-neg-ex:
  assumes R: p ∈ carrier P
  shows EX q : carrier P. q ⊕_P p = 0_P
proof −
  let ?q = %i. ⊖ (p i)
  from R have closed: ?q ∈ carrier P
    by (simp add: UP-def P-def up-a-inv-closed)
  from R have coeff: !!n. coeff P ?q n = ⊖ (coeff P p n)
    by (simp add: UP-def P-def up-a-inv-closed)
  show ?thesis
  proof
    show ?q ⊕_P p = 0_P
      by (auto intro!: up-eqI simp add: R closed coeff R.l-neg)
  qed (rule closed)
qed

lemma (in UP-cring) UP-a-comm:
  assumes R: p ∈ carrier P q ∈ carrier P
  shows p ⊕_P q = q ⊕_P p
  by (rule up-eqI, simp add: a-comm R, simp-all add: R)

lemma (in UP-cring) UP-m-assoc:
  assumes R: p ∈ carrier P q ∈ carrier P r ∈ carrier P
  shows (p ⊗_P q) ⊗_P r = p ⊗_P (q ⊗_P r)
proof (rule up-eqI)
  fix n
  {
    fix k and a b c :: nat=>'a
    assume R: a ∈ UNIV −> carrier R b ∈ UNIV −> carrier R
      c ∈ UNIV −> carrier R
    then have k <= n ==>
      (⨁ j ∈ {..k}. (⨁ i ∈ {..j}. a i ⊗ b (j−i)) ⊗ c (n−j)) =
      (⨁ j ∈ {..k}. a j ⊗ (⨁ i ∈ {..k−j}. b i ⊗ c (n−j−i)))
      (is - ⟹ ?eq k)
    proof (induct k)
      case 0 then show ?case by (simp add: Pi-def m-assoc)
    next
      case (Suc k)
      then have k <= n by arith
      from this R have ?eq k by (rule Suc)
      with R show ?case
        by (simp cong: finsum-cong
             add: Suc-diff-le Pi-def l-distr r-distr m-assoc)
          (simp cong: finsum-cong add: Pi-def a-ac finsum-ldistr m-assoc)
    qed
  }
```

```
  with R show coeff P ((p ⊗P q) ⊗P r) n = coeff P (p ⊗P (q ⊗P r)) n
    by (simp add: Pi-def)
qed (simp-all add: R)

lemma (in UP-cring) UP-l-one [simp]:
  assumes R: p ∈ carrier P
  shows 1P ⊗P p = p
proof (rule up-eqI)
  fix n
  show coeff P (1P ⊗P p) n = coeff P p n
  proof (cases n)
    case 0 with R show ?thesis by simp
  next
    case Suc with R show ?thesis
      by (simp del: finsum-Suc add: finsum-Suc2 Pi-def)
  qed
qed (simp-all add: R)

lemma (in UP-cring) UP-l-distr:
  assumes R: p ∈ carrier P q ∈ carrier P r ∈ carrier P
  shows (p ⊕P q) ⊗P r = (p ⊗P r) ⊕P (q ⊗P r)
  by (rule up-eqI) (simp add: l-distr R Pi-def, simp-all add: R)

lemma (in UP-cring) UP-m-comm:
  assumes R: p ∈ carrier P q ∈ carrier P
  shows p ⊗P q = q ⊗P p
proof (rule up-eqI)
  fix n
  {
    fix k and a b :: nat=>'a
    assume R: a ∈ UNIV -> carrier R b ∈ UNIV -> carrier R
    then have k <= n ==>
      (⨁ i ∈ {..k}. a i ⊗ b (n-i)) =
      (⨁ i ∈ {..k}. a (k-i) ⊗ b (i+n-k))
      (is - ⟹ ?eq k)
    proof (induct k)
      case 0 then show ?case by (simp add: Pi-def)
    next
      case (Suc k) then show ?case
        by (subst (2) finsum-Suc2) (simp add: Pi-def a-comm)+
    qed
  }
  note l = this
  from R show coeff P (p ⊗P q) n =  coeff P (q ⊗P p) n
    apply (simp add: Pi-def)
    apply (subst l)
    apply (auto simp add: Pi-def)
    apply (simp add: m-comm)
    done
```

qed (*simp-all add*: *R*)

theorem (**in** *UP-cring*) *UP-cring*:
cring P
by (*auto intro*!: *cringI abelian-groupI comm-monoidI UP-a-assoc UP-l-zero UP-l-neg-ex UP-a-comm UP-m-assoc UP-l-one UP-m-comm UP-l-distr*)

lemma (**in** *UP-cring*) *UP-ring*:

ring P
by (*auto intro*: *ring.intro cring.axioms UP-cring*)

lemma (**in** *UP-cring*) *UP-a-inv-closed* [*intro*, *simp*]:
$p \in$ *carrier P* ==> $\ominus_P$ $p \in$ *carrier P*
by (*rule abelian-group.a-inv-closed*
[*OF ring.is-abelian-group* [*OF UP-ring*]])

lemma (**in** *UP-cring*) *coeff-a-inv* [*simp*]:
assumes *R*: $p \in$ *carrier P*
shows *coeff P* ($\ominus_P$ *p*) *n* = $\ominus$ (*coeff P p n*)
proof −
from *R coeff-closed UP-a-inv-closed* **have**
coeff P ($\ominus_P$ *p*) *n* = $\ominus$ *coeff P p n* $\oplus$ (*coeff P p n* $\oplus$ *coeff P* ($\ominus_P$ *p*) *n*)
by *algebra*
also from *R* **have** ... = $\ominus$ (*coeff P p n*)
by (*simp del*: *coeff-add add*: *coeff-add* [*THEN sym*]
abelian-group.r-neg [*OF ring.is-abelian-group* [*OF UP-ring*]])
finally show *?thesis* **.**
qed

Interpretation of lemmas from *cring*. Saves lifting 43 lemmas manually.

interpretation *UP-cring* < *cring P*
by *intro-locales*
(*rule cring.axioms ring.axioms abelian-group.axioms comm-monoid.axioms UP-cring*)+

11.4 Polynomials Form an Algebra

lemma (**in** *UP-cring*) *UP-smult-l-distr*:
[| $a \in$ *carrier R*; $b \in$ *carrier R*; $p \in$ *carrier P* |] ==>
$(a \oplus b) \odot_P p = a \odot_P p \oplus_P b \odot_P p$
by (*rule up-eqI*) (*simp-all add*: *R.l-distr*)

lemma (**in** *UP-cring*) *UP-smult-r-distr*:
[| $a \in$ *carrier R*; $p \in$ *carrier P*; $q \in$ *carrier P* |] ==>
$a \odot_P (p \oplus_P q) = a \odot_P p \oplus_P a \odot_P q$
by (*rule up-eqI*) (*simp-all add*: *R.r-distr*)

lemma (**in** *UP-cring*) *UP-smult-assoc1*:
[| $a \in$ *carrier R*; $b \in$ *carrier R*; $p \in$ *carrier P* |] ==>

$(a \otimes b) \odot_P p = a \odot_P (b \odot_P p)$
by (*rule up-eqI*) (*simp-all add*: *R.m-assoc*)

lemma (**in** *UP-cring*) *UP-smult-one* [*simp*]:
$p \in carrier\ P$ ==> $\mathbf{1} \odot_P p = p$
by (*rule up-eqI*) *simp-all*

lemma (**in** *UP-cring*) *UP-smult-assoc2*:
[| $a \in carrier\ R$; $p \in carrier\ P$; $q \in carrier\ P$ |] ==>
$(a \odot_P p) \otimes_P q = a \odot_P (p \otimes_P q)$
by (*rule up-eqI*) (*simp-all add*: *R.finsum-rdistr R.m-assoc Pi-def*)

Interpretation of lemmas from *algebra*.

lemma (**in** *cring*) *cring*:
cring R
by (*fast intro*: *cring.intro prems*)

lemma (**in** *UP-cring*) *UP-algebra*:
algebra R P
by (*auto intro*: *algebraI R.cring UP-cring UP-smult-l-distr UP-smult-r-distr UP-smult-assoc1 UP-smult-assoc2*)

interpretation *UP-cring* < *algebra R P*
by *intro-locales*
(*rule module.axioms algebra.axioms UP-algebra*)+

11.5 Further Lemmas Involving Monomials

lemma (**in** *UP-cring*) *monom-zero* [*simp*]:
$monom\ P\ \mathbf{0}\ n = \mathbf{0}_P$
by (*simp add*: *UP-def P-def*)

lemma (**in** *UP-cring*) *monom-mult-is-smult*:
assumes *R*: $a \in carrier\ R$ $p \in carrier\ P$
shows $monom\ P\ a\ 0 \otimes_P p = a \odot_P p$
proof (*rule up-eqI*)
fix *n*
have $coeff\ P\ (p \otimes_P monom\ P\ a\ 0)\ n = coeff\ P\ (a \odot_P p)\ n$
proof (*cases n*)
case *0* **with** *R* **show** *?thesis* **by** (*simp add*: *R.m-comm*)
next
case *Suc* **with** *R* **show** *?thesis*
by (*simp cong*: *R.finsum-cong add*: *R.r-null Pi-def*)
(*simp add*: *R.m-comm*)
qed
with *R* **show** $coeff\ P\ (monom\ P\ a\ 0 \otimes_P p)\ n = coeff\ P\ (a \odot_P p)\ n$
by (*simp add*: *UP-m-comm*)
qed (*simp-all add*: *R*)

lemma (**in** *UP-cring*) *monom-add* [*simp*]:
  [| $a \in$ *carrier R*; $b \in$ *carrier R* |] ==>
  monom P $(a \oplus b)$ n = *monom P a n* $\oplus_P$ *monom P b n*
  by (*rule up-eqI*) *simp-all*

lemma (**in** *UP-cring*) *monom-one-Suc*:
  monom P **1** (*Suc n*) = *monom P* **1** *n* $\otimes_P$ *monom P* **1** *1*
proof (*rule up-eqI*)
  fix *k*
  show *coeff P* (*monom P* **1** (*Suc n*)) *k* = *coeff P* (*monom P* **1** *n* $\otimes_P$ *monom P* **1** *1*) *k*
  proof (*cases k* = *Suc n*)
    case *True* **show** *?thesis*
    proof –
      from *True* **have** *less-add-diff*:
        !!*i*. [| $n < i$; $i <= n + m$ |] ==> $n + m - i < m$ **by** *arith*
      from *True* **have** *coeff P* (*monom P* **1** (*Suc n*)) *k* = **1** **by** *simp*
      also from *True*
      have ... = ($\bigoplus i \in \{..<n\} \cup \{n\}$. *coeff P* (*monom P* **1** *n*) $i \otimes$
        coeff P (*monom P* **1** *1*) $(k - i)$)
        by (*simp cong*: *R.finsum-cong add*: *Pi-def*)
      also have ... = ($\bigoplus i \in \{..n\}$. *coeff P* (*monom P* **1** *n*) $i \otimes$
        coeff P (*monom P* **1** *1*) $(k - i)$)
        by (*simp only*: *ivl-disj-un-singleton*)
      also from *True*
      have ... = ($\bigoplus i \in \{..n\} \cup \{n<..k\}$. *coeff P* (*monom P* **1** *n*) $i \otimes$
        coeff P (*monom P* **1** *1*) $(k - i)$)
        by (*simp cong*: *R.finsum-cong add*: *R.finsum-Un-disjoint ivl-disj-int-one*
          order-less-imp-not-eq Pi-def)
      also from *True* **have** ... = *coeff P* (*monom P* **1** *n* $\otimes_P$ *monom P* **1** *1*) *k*
        by (*simp add*: *ivl-disj-un-one*)
      finally show *?thesis* **.**
    qed
  next
    case *False*
    note *neq* = *False*
    let *?s* =
      λi. (*if* $n = i$ *then* **1** *else* **0**) $\otimes$ (*if Suc 0* = $k - i$ *then* **1** *else* **0**)
    from *neq* **have** *coeff P* (*monom P* **1** (*Suc n*)) *k* = **0** **by** *simp*
    also have ... = ($\bigoplus i \in \{..k\}$. *?s i*)
    proof –
      have *f1*: ($\bigoplus i \in \{..<n\}$. *?s i*) = **0**
        by (*simp cong*: *R.finsum-cong add*: *Pi-def*)
      from *neq* **have** *f2*: ($\bigoplus i \in \{n\}$. *?s i*) = **0**
        by (*simp cong*: *R.finsum-cong add*: *Pi-def*) *arith*
      have *f3*: $n < k$ ==> ($\bigoplus i \in \{n<..k\}$. *?s i*) = **0**
        by (*simp cong*: *R.finsum-cong add*: *order-less-imp-not-eq Pi-def*)
      show *?thesis*
      proof (*cases* $k < n$)

```
      case True then show ?thesis by (simp cong: R.finsum-cong add: Pi-def)
    next
      case False then have n-le-k: n <= k by arith
      show ?thesis
      proof (cases n = k)
        case True
        then have 0 = (⨁ i ∈ {..<n} ∪ {n}. ?s i)
          by (simp cong: R.finsum-cong add: ivl-disj-int-singleton Pi-def)
        also from True have ... = (⨁ i ∈ {..k}. ?s i)
          by (simp only: ivl-disj-un-singleton)
        finally show ?thesis .
      next
        case False with n-le-k have n-less-k: n < k by arith
        with neq have 0 = (⨁ i ∈ {..<n} ∪ {n}. ?s i)
          by (simp add: R.finsum-Un-disjoint f1 f2
            ivl-disj-int-singleton Pi-def del: Un-insert-right)
        also have ... = (⨁ i ∈ {..n}. ?s i)
          by (simp only: ivl-disj-un-singleton)
        also from n-less-k neq have ... = (⨁ i ∈ {..n} ∪ {n<..k}. ?s i)
          by (simp add: R.finsum-Un-disjoint f3 ivl-disj-int-one Pi-def)
        also from n-less-k have ... = (⨁ i ∈ {..k}. ?s i)
          by (simp only: ivl-disj-un-one)
        finally show ?thesis .
      qed
    qed
  qed
  also have ... = coeff P (monom P 1 n ⊗P monom P 1 1) k by simp
  finally show ?thesis .
 qed
qed (simp-all)

lemma (in UP-cring) monom-mult-smult:
 [| a ∈ carrier R; b ∈ carrier R |] ==> monom P (a ⊗ b) n = a ⊙P monom P
b n
 by (rule up-eqI) simp-all

lemma (in UP-cring) monom-one [simp]:
 monom P 1 0 = 1P
 by (rule up-eqI) simp-all

lemma (in UP-cring) monom-one-mult:
 monom P 1 (n + m) = monom P 1 n ⊗P monom P 1 m
proof (induct n)
 case 0 show ?case by simp
next
 case Suc then show ?case
  by (simp only: add-Suc monom-one-Suc) (simp add: P.m-ac)
qed
```

lemma (**in** *UP-cring*) *monom-mult* [*simp*]:
 assumes *R*: $a \in$ *carrier R* $b \in$ *carrier R*
 shows *monom P* $(a \otimes b)$ $(n + m)$ = *monom P a n* $\otimes_P$ *monom P b m*
proof −
 from *R* **have** *monom P* $(a \otimes b)$ $(n + m)$ = *monom P* $(a \otimes b \otimes \mathbf{1})$ $(n + m)$
by *simp*
 also from *R* **have** ... = $a \otimes b \odot_P$ *monom P* **1** $(n + m)$
  by (*simp add*: *monom-mult-smult del*: *R.r-one*)
 also have ... = $a \otimes b \odot_P$ (*monom P* **1** *n* $\otimes_P$ *monom P* **1** *m*)
  by (*simp only*: *monom-one-mult*)
 also from *R* **have** ... = $a \odot_P$ ($b \odot_P$ (*monom P* **1** *n* $\otimes_P$ *monom P* **1** *m*))
  by (*simp add*: *UP-smult-assoc1*)
 also from *R* **have** ... = $a \odot_P$ ($b \odot_P$ (*monom P* **1** *m* $\otimes_P$ *monom P* **1** *n*))
  by (*simp add*: *P.m-comm*)
 also from *R* **have** ... = $a \odot_P$ (($b \odot_P$ *monom P* **1** *m*) $\otimes_P$ *monom P* **1** *n*)
  by (*simp add*: *UP-smult-assoc2*)
 also from *R* **have** ... = $a \odot_P$ (*monom P* **1** *n* $\otimes_P$ ($b \odot_P$ *monom P* **1** *m*))
  by (*simp add*: *P.m-comm*)
 also from *R* **have** ... = ($a \odot_P$ *monom P* **1** *n*) $\otimes_P$ ($b \odot_P$ *monom P* **1** *m*)
  by (*simp add*: *UP-smult-assoc2*)
 also from *R* **have** ... = *monom P* $(a \otimes \mathbf{1})$ *n* $\otimes_P$ *monom P* $(b \otimes \mathbf{1})$ *m*
  by (*simp add*: *monom-mult-smult del*: *R.r-one*)
 also from *R* **have** ... = *monom P a n* $\otimes_P$ *monom P b m* **by** *simp*
 finally show *?thesis* **.**
qed

lemma (**in** *UP-cring*) *monom-a-inv* [*simp*]:
 $a \in$ *carrier R* ==> *monom P* $(\ominus a)$ *n* = $\ominus_P$ *monom P a n*
 by (*rule up-eqI*) *simp-all*

lemma (**in** *UP-cring*) *monom-inj*:
 inj-on (%*a*. *monom P a n*) (*carrier R*)
proof (*rule inj-onI*)
 fix *x y*
 assume *R*: $x \in$ *carrier R* $y \in$ *carrier R* **and** *eq*: *monom P x n* = *monom P y n*
 then have *coeff P* (*monom P x n*) *n* = *coeff P* (*monom P y n*) *n* **by** *simp*
 with *R* **show** $x = y$ **by** *simp*
qed

11.6 The Degree Function

constdefs (**structure** *R*)
 deg :: [('*a*, '*m*) *ring-scheme*, *nat* => '*a*] => *nat*
 deg R p == *LEAST n*. *bound* **0** *n* (*coeff* (*UP R*) *p*)

lemma (**in** *UP-cring*) *deg-aboveI*:
 [| (!!*m*. $n < m$ ==> *coeff P p m* = **0**); $p \in$ *carrier P* |] ==> *deg R p* <= *n*
 by (*unfold deg-def P-def*) (*fast intro*: *Least-le*)

```
lemma (in UP-cring) deg-aboveD:
  assumes deg R p < m and p ∈ carrier P
  shows coeff P p m = 0
proof –
  from ‹p ∈ carrier P› obtain n where bound 0 n (coeff P p)
    by (auto simp add: UP-def P-def)
  then have bound 0 (deg R p) (coeff P p)
    by (auto simp: deg-def P-def dest: LeastI)
  from this and ‹deg R p < m› show ?thesis ..
qed

lemma (in UP-cring) deg-belowI:
  assumes non-zero: n ~= 0 ==> coeff P p n ~= 0
    and R: p ∈ carrier P
  shows n <= deg R p
— Logically, this is a slightly stronger version of deg-aboveD
proof (cases n=0)
  case True then show ?thesis by simp
next
  case False then have coeff P p n ~= 0 by (rule non-zero)
  then have ~ deg R p < n by (fast dest: deg-aboveD intro: R)
  then show ?thesis by arith
qed

lemma (in UP-cring) lcoeff-nonzero-deg:
  assumes deg: deg R p ~= 0 and R: p ∈ carrier P
  shows coeff P p (deg R p) ~= 0
proof –
  from R obtain m where deg R p <= m and m-coeff: coeff P p m ~= 0
  proof –
    have minus: !!(n::nat) m. n ~= 0 ==> (n – Suc 0 < m) = (n <= m)
      by arith

    from deg have deg R p – 1 < (LEAST n. bound 0 n (coeff P p))
      by (unfold deg-def P-def) arith
    then have ~ bound 0 (deg R p – 1) (coeff P p) by (rule not-less-Least)
    then have EX m. deg R p – 1 < m & coeff P p m ~= 0
      by (unfold bound-def) fast
     then have EX m. deg R p <= m & coeff P p m ~= 0 by (simp add: deg
minus)
    then show ?thesis by (auto intro: that)
  qed
  with deg-belowI R have deg R p = m by fastsimp
  with m-coeff show ?thesis by simp
qed

lemma (in UP-cring) lcoeff-nonzero-nonzero:
```

assumes *deg*: *deg R p = 0* **and** *nonzero*: *p* ~= $\mathbf{0}_P$ **and** *R*: *p* ∈ *carrier P*
shows *coeff P p 0* ~= **0**
proof −
have *EX m. coeff P p m* ~= **0**
proof (*rule classical*)
assume ~ *?thesis*
with *R* **have** *p* = $\mathbf{0}_P$ **by** (*auto intro*: *up-eqI*)
with *nonzero* **show** *?thesis* **by** *contradiction*
qed
then obtain *m* **where** *coeff*: *coeff P p m* ~= **0 ..**
from *this* **and** *R* **have** *m* <= *deg R p* **by** (*rule deg-belowI*)
then have *m = 0* **by** (*simp add*: *deg*)
with *coeff* **show** *?thesis* **by** *simp*
qed

lemma (**in** *UP-cring*) *lcoeff-nonzero*:
assumes *neq*: *p* ~= $\mathbf{0}_P$ **and** *R*: *p* ∈ *carrier P*
shows *coeff P p* (*deg R p*) ~= **0**
proof (*cases deg R p = 0*)
case *True* **with** *neq R* **show** *?thesis* **by** (*simp add*: *lcoeff-nonzero-nonzero*)
next
case *False* **with** *neq R* **show** *?thesis* **by** (*simp add*: *lcoeff-nonzero-deg*)
qed

lemma (**in** *UP-cring*) *deg-eqI*:
[| !!*m. n* < *m* ==> *coeff P p m* = **0**;
!!*n. n* ~= *0* ==> *coeff P p n* ~= **0**; *p* ∈ *carrier P* |] ==> *deg R p* = *n*
by (*fast intro*: *le-anti-sym deg-aboveI deg-belowI*)

Degree and polynomial operations

lemma (**in** *UP-cring*) *deg-add* [*simp*]:
assumes *R*: *p* ∈ *carrier P q* ∈ *carrier P*
shows *deg R* ($p \oplus_P q$) <= *max* (*deg R p*) (*deg R q*)
proof (*cases deg R p* <= *deg R q*)
case *True* **show** *?thesis*
by (*rule deg-aboveI*) (*simp-all add*: *True R deg-aboveD*)
next
case *False* **show** *?thesis*
by (*rule deg-aboveI*) (*simp-all add*: *False R deg-aboveD*)
qed

lemma (**in** *UP-cring*) *deg-monom-le*:
a ∈ *carrier R* ==> *deg R* (*monom P a n*) <= *n*
by (*intro deg-aboveI*) *simp-all*

lemma (**in** *UP-cring*) *deg-monom* [*simp*]:
[| *a* ~= **0**; *a* ∈ *carrier R* |] ==> *deg R* (*monom P a n*) = *n*
by (*fastsimp intro*: *le-anti-sym deg-aboveI deg-belowI*)

lemma (**in** *UP-cring*) *deg-const* [*simp*]:
  assumes *R*: $a \in$ *carrier R* **shows** *deg R* (*monom P a 0*) = *0*
proof (*rule le-anti-sym*)
  show *deg R* (*monom P a 0*) <= *0* **by** (*rule deg-aboveI*) (*simp-all add*: *R*)
next
  show *0* <= *deg R* (*monom P a 0*) **by** (*rule deg-belowI*) (*simp-all add*: *R*)
qed

lemma (**in** *UP-cring*) *deg-zero* [*simp*]:
  deg R $\mathbf{0}_P$ = *0*
proof (*rule le-anti-sym*)
  show *deg R* $\mathbf{0}_P$ <= *0* **by** (*rule deg-aboveI*) *simp-all*
next
  show *0* <= *deg R* $\mathbf{0}_P$ **by** (*rule deg-belowI*) *simp-all*
qed

lemma (**in** *UP-cring*) *deg-one* [*simp*]:
  deg R $\mathbf{1}_P$ = *0*
proof (*rule le-anti-sym*)
  show *deg R* $\mathbf{1}_P$ <= *0* **by** (*rule deg-aboveI*) *simp-all*
next
  show *0* <= *deg R* $\mathbf{1}_P$ **by** (*rule deg-belowI*) *simp-all*
qed

lemma (**in** *UP-cring*) *deg-uminus* [*simp*]:
  assumes *R*: $p \in$ *carrier P* **shows** *deg R* ($\ominus_P$ *p*) = *deg R p*
proof (*rule le-anti-sym*)
  show *deg R* ($\ominus_P$ *p*) <= *deg R p* **by** (*simp add*: *deg-aboveI deg-aboveD R*)
next
  show *deg R p* <= *deg R* ($\ominus_P$ *p*)
    by (*simp add*: *deg-belowI lcoeff-nonzero-deg*
      inj-on-iff [*OF R.a-inv-inj*, *of* - $\mathbf{0}$, *simplified*] *R*)
qed

lemma (**in** *UP-domain*) *deg-smult-ring*:
  [| $a \in$ *carrier R*; $p \in$ *carrier P* |] ==>
  deg R (*a* $\odot_P$ *p*) <= (*if a* = $\mathbf{0}$ *then 0 else deg R p*)
  by (*cases a* = $\mathbf{0}$) (*simp add*: *deg-aboveI deg-aboveD*)+

lemma (**in** *UP-domain*) *deg-smult* [*simp*]:
  assumes *R*: $a \in$ *carrier R* $p \in$ *carrier P*
  shows *deg R* (*a* $\odot_P$ *p*) = (*if a* = $\mathbf{0}$ *then 0 else deg R p*)
proof (*rule le-anti-sym*)
  show *deg R* (*a* $\odot_P$ *p*) <= (*if a* = $\mathbf{0}$ *then 0 else deg R p*)
    using *R* **by** (*rule deg-smult-ring*)
next
  show (*if a* = $\mathbf{0}$ *then 0 else deg R p*) <= *deg R* (*a* $\odot_P$ *p*)
  proof (*cases a* = $\mathbf{0}$)
  qed (*simp*, *simp add*: *deg-belowI lcoeff-nonzero-deg integral-iff R*)

qed

lemma (**in** *UP-cring*) *deg-mult-cring*:
  assumes *R*: $p \in$ *carrier P* $q \in$ *carrier P*
  shows *deg R* ($p \otimes_P q$) <= *deg R p* + *deg R q*
proof (*rule deg-aboveI*)
  fix *m*
  assume *boundm*: *deg R p* + *deg R q* < *m*
  {
    fix *k i*
    assume *boundk*: *deg R p* + *deg R q* < *k*
    then have *coeff P p i* $\otimes$ *coeff P q* ($k - i$) = **0**
    proof (*cases deg R p* < *i*)
      case *True* **then show** *?thesis* **by** (*simp add*: *deg-aboveD R*)
    next
      case *False* **with** *boundk* **have** *deg R q* < $k - i$ **by** *arith*
      then show *?thesis* **by** (*simp add*: *deg-aboveD R*)
    qed
  }
  with *boundm R* **show** *coeff P* ($p \otimes_P q$) *m* = **0** **by** *simp*
qed (*simp add*: *R*)

lemma (**in** *UP-domain*) *deg-mult* [*simp*]:
  [| *p* ~= $\mathbf{0}_P$; *q* ~= $\mathbf{0}_P$; $p \in$ *carrier P*; $q \in$ *carrier P* |] ==>
  deg R ($p \otimes_P q$) = *deg R p* + *deg R q*
proof (*rule le-anti-sym*)
  assume $p \in$ *carrier P* $q \in$ *carrier P*
  then show *deg R* ($p \otimes_P q$) <= *deg R p* + *deg R q* **by** (*rule deg-mult-cring*)
next
  let *?s* = (%*i*. *coeff P p i* $\otimes$ *coeff P q* (*deg R p* + *deg R q* − *i*))
  assume *R*: $p \in$ *carrier P* $q \in$ *carrier P* **and** *nz*: *p* ~= $\mathbf{0}_P$ *q* ~= $\mathbf{0}_P$
  have *less-add-diff*: !!(*k*::*nat*) *n m*. *k* < *n* ==> *m* < *n* + *m* − *k* **by** *arith*
  show *deg R p* + *deg R q* <= *deg R* ($p \otimes_P q$)
  proof (*rule deg-belowI*, *simp add*: *R*)
    have ($\bigoplus i \in$ {.. *deg R p* + *deg R q*}. *?s i*)
      = ($\bigoplus i \in$ {..< *deg R p*} $\cup$ {*deg R p* .. *deg R p* + *deg R q*}. *?s i*)
      by (*simp only*: *ivl-disj-un-one*)
    also have ... = ($\bigoplus i \in$ {*deg R p* .. *deg R p* + *deg R q*}. *?s i*)
      by (*simp cong*: *R.finsum-cong add*: *R.finsum-Un-disjoint ivl-disj-int-one*
        deg-aboveD less-add-diff R Pi-def)
    also have ...= ($\bigoplus i \in$ {*deg R p*} $\cup$ {*deg R p* <.. *deg R p* + *deg R q*}. *?s i*)
      by (*simp only*: *ivl-disj-un-singleton*)
    also have ... = *coeff P p* (*deg R p*) $\otimes$ *coeff P q* (*deg R q*)
      by (*simp cong*: *R.finsum-cong*
        add: *ivl-disj-int-singleton deg-aboveD R Pi-def*)
    finally have ($\bigoplus i \in$ {.. *deg R p* + *deg R q*}. *?s i*)
      = *coeff P p* (*deg R p*) $\otimes$ *coeff P q* (*deg R q*) **.**
    with *nz* **show** ($\bigoplus i \in$ {.. *deg R p* + *deg R q*}. *?s i*) ~= **0**
      by (*simp add*: *integral-iff lcoeff-nonzero R*)

```
    qed (simp add: R)
  qed

lemma (in UP-cring) coeff-finsum:
  assumes fin: finite A
  shows p ∈ A -> carrier P ==>
    coeff P (finsum P p A) k = (⨁ i ∈ A. coeff P (p i) k)
  using fin by induct (auto simp: Pi-def)

lemma (in UP-cring) up-repr:
  assumes R: p ∈ carrier P
  shows (⨁ P i ∈ {..deg R p}. monom P (coeff P p i) i) = p
proof (rule up-eqI)
  let ?s = (%i. monom P (coeff P p i) i)
  fix k
  from R have RR: !!i. (if i = k then coeff P p i else 0) ∈ carrier R
    by simp
  show coeff P (⨁ P i ∈ {..deg R p}. ?s i) k = coeff P p k
  proof (cases k <= deg R p)
    case True
    hence coeff P (⨁ P i ∈ {..deg R p}. ?s i) k =
          coeff P (⨁ P i ∈ {..k} ∪ {k<..deg R p}. ?s i) k
      by (simp only: ivl-disj-un-one)
    also from True
    have ... = coeff P (⨁ P i ∈ {..k}. ?s i) k
      by (simp cong: R.finsum-cong add: R.finsum-Un-disjoint
        ivl-disj-int-one order-less-imp-not-eq2 coeff-finsum R RR Pi-def)
    also
    have ... = coeff P (⨁ P i ∈ {..<k} ∪ {k}. ?s i) k
      by (simp only: ivl-disj-un-singleton)
    also have ... = coeff P p k
      by (simp cong: R.finsum-cong
        add: ivl-disj-int-singleton coeff-finsum deg-aboveD R RR Pi-def)
    finally show ?thesis .
  next
    case False
    hence coeff P (⨁ P i ∈ {..deg R p}. ?s i) k =
          coeff P (⨁ P i ∈ {..<deg R p} ∪ {deg R p}. ?s i) k
      by (simp only: ivl-disj-un-singleton)
    also from False have ... = coeff P p k
      by (simp cong: R.finsum-cong
        add: ivl-disj-int-singleton coeff-finsum deg-aboveD R Pi-def)
    finally show ?thesis .
  qed
qed (simp-all add: R Pi-def)

lemma (in UP-cring) up-repr-le:
  [| deg R p <= n; p ∈ carrier P |] ==>
  (⨁ P i ∈ {..n}. monom P (coeff P p i) i) = p
```

proof –
let *?s* = (%*i*. *monom P* (*coeff P p i*) *i*)
assume *R*: *p* ∈ *carrier P* **and** *deg R p* <= *n*
then have *finsum P ?s* {..*n*} = *finsum P ?s* ({..*deg R p*} ∪ {*deg R p*<..*n*})
by (*simp only*: *ivl-disj-un-one*)
also have ... = *finsum P ?s* {..*deg R p*}
by (*simp cong*: *P.finsum-cong add*: *P.finsum-Un-disjoint ivl-disj-int-one*
deg-aboveD R Pi-def)
also have ... = *p* **using** *R* **by** (*rule up-repr*)
finally show *?thesis* **.**
qed

11.7 Polynomials over Integral Domains

lemma *domainI*:
assumes *cring*: *cring R*
and *one-not-zero*: *one R* ~= *zero R*
and *integral*: !!*a b*. [| *mult R a b* = *zero R*; *a* ∈ *carrier R*;
b ∈ *carrier R* |] ==> *a* = *zero R* | *b* = *zero R*
shows *domain R*
by (*auto intro*!: *domain.intro domain-axioms.intro cring.axioms prems*
del: *disjCI*)

lemma (**in** *UP-domain*) *UP-one-not-zero*:
$\mathbf{1}_P$ ~= $\mathbf{0}_P$
proof
assume $\mathbf{1}_P = \mathbf{0}_P$
hence *coeff P* $\mathbf{1}_P$ *0* = (*coeff P* $\mathbf{0}_P$ *0*) **by** *simp*
hence **1** = **0** **by** *simp*
with *one-not-zero* **show** *False* **by** *contradiction*
qed

lemma (**in** *UP-domain*) *UP-integral*:
[| $p \otimes_P q = \mathbf{0}_P$; *p* ∈ *carrier P*; *q* ∈ *carrier P* |] ==> $p = \mathbf{0}_P$ | $q = \mathbf{0}_P$
proof –
fix *p q*
assume *pq*: $p \otimes_P q = \mathbf{0}_P$ **and** *R*: *p* ∈ *carrier P q* ∈ *carrier P*
show $p = \mathbf{0}_P$ | $q = \mathbf{0}_P$
proof (*rule classical*)
assume *c*: ~ ($p = \mathbf{0}_P$ | $q = \mathbf{0}_P$)
with *R* **have** *deg R p* + *deg R q* = *deg R* ($p \otimes_P q$) **by** *simp*
also from *pq* **have** ... = *0* **by** *simp*
finally have *deg R p* + *deg R q* = *0* **.**
then have *f1*: *deg R p* = *0* & *deg R q* = *0* **by** *simp*
from *f1 R* **have** *p* = ($\bigoplus_P$ *i* ∈ {..*0*}. *monom P* (*coeff P p i*) *i*)
by (*simp only*: *up-repr-le*)
also from *R* **have** ... = *monom P* (*coeff P p 0*) *0* **by** *simp*
finally have *p*: *p* = *monom P* (*coeff P p 0*) *0* **.**
from *f1 R* **have** *q* = ($\bigoplus_P$ *i* ∈ {..*0*}. *monom P* (*coeff P q i*) *i*)

by (*simp only*: *up-repr-le*)
also from *R* **have** ... = *monom P* (*coeff P q 0*) *0* **by** *simp*
finally have *q*: *q* = *monom P* (*coeff P q 0*) *0* **.**
from *R* **have** *coeff P p 0* ⊗ *coeff P q 0* = *coeff P* (*p* $\otimes_P$ *q*) *0* **by** *simp*
also from *pq* **have** ... = **0** **by** *simp*
finally have *coeff P p 0* ⊗ *coeff P q 0* = **0** **.**
with *R* **have** *coeff P p 0* = **0** | *coeff P q 0* = **0**
by (*simp add*: *R.integral-iff*)
with *p q* **show** *p* = $\mathbf{0}_P$ | *q* = $\mathbf{0}_P$ **by** *fastsimp*
qed
qed

theorem (**in** *UP-domain*) *UP-domain*:
domain P
by (*auto intro*!: *domainI UP-cring UP-one-not-zero UP-integral del*: *disjCI*)

Interpretation of theorems from *domain*.

interpretation *UP-domain* < *domain P*
by *intro-locales* (*rule domain.axioms UP-domain*)+

11.8 The Evaluation Homomorphism and Universal Property

theorem (**in** *cring*) *diagonal-sum*:
[| *f* ∈ {..*n* + *m*::*nat*} −> *carrier R*; *g* ∈ {..*n* + *m*} −> *carrier R* |] ==>
(⨁ *k* ∈ {..*n* + *m*}. ⨁ *i* ∈ {..*k*}. *f i* ⊗ *g* (*k* − *i*)) =
(⨁ *k* ∈ {..*n* + *m*}. ⨁ *i* ∈ {..*n* + *m* − *k*}. *f k* ⊗ *g i*)
proof −
assume *Rf*: *f* ∈ {..*n* + *m*} −> *carrier R* **and** *Rg*: *g* ∈ {..*n* + *m*} −> *carrier R*
{
fix *j*
have *j* <= *n* + *m* ==>
(⨁ *k* ∈ {..*j*}. ⨁ *i* ∈ {..*k*}. *f i* ⊗ *g* (*k* − *i*)) =
(⨁ *k* ∈ {..*j*}. ⨁ *i* ∈ {..*j* − *k*}. *f k* ⊗ *g i*)
proof (*induct j*)
case *0* **from** *Rf Rg* **show** *?case* **by** (*simp add*: *Pi-def*)
next
case (*Suc j*)
have *R6*: !!*i k*. [| *k* <= *j*; *i* <= *Suc j* − *k* |] ==> *g i* ∈ *carrier R*
using *Suc* **by** (*auto intro*!: *funcset-mem* [*OF Rg*])
have *R8*: !!*i k*. [| *k* <= *Suc j*; *i* <= *k* |] ==> *g* (*k* − *i*) ∈ *carrier R*
using *Suc* **by** (*auto intro*!: *funcset-mem* [*OF Rg*])
have *R9*: !!*i k*. [| *k* <= *Suc j* |] ==> *f k* ∈ *carrier R*
using *Suc* **by** (*auto intro*!: *funcset-mem* [*OF Rf*])
have *R10*: !!*i k*. [| *k* <= *Suc j*; *i* <= *Suc j* − *k* |] ==> *g i* ∈ *carrier R*
using *Suc* **by** (*auto intro*!: *funcset-mem* [*OF Rg*])
have *R11*: *g 0* ∈ *carrier R*
using *Suc* **by** (*auto intro*!: *funcset-mem* [*OF Rg*])
from *Suc* **show** *?case*

```
      by (simp cong: finsum-cong add: Suc-diff-le a-ac
        Pi-def R6 R8 R9 R10 R11)
    qed
  }
  then show ?thesis by fast
qed

lemma (in abelian-monoid) boundD-carrier:
  [| bound 0 n f; n < m |] ==> f m ∈ carrier G
  by auto

theorem (in cring) cauchy-product:
  assumes bf: bound 0 n f and bg: bound 0 m g
    and Rf: f ∈ {..n} -> carrier R and Rg: g ∈ {..m} -> carrier R
  shows (⨁ k ∈ {..n + m}. ⨁ i ∈ {..k}. f i ⊗ g (k - i)) =
    (⨁ i ∈ {..n}. f i) ⊗ (⨁ i ∈ {..m}. g i)
proof -
  have f: !!x. f x ∈ carrier R
  proof -
    fix x
    show f x ∈ carrier R
      using Rf bf boundD-carrier by (cases x <= n) (auto simp: Pi-def)
  qed
  have g: !!x. g x ∈ carrier R
  proof -
    fix x
    show g x ∈ carrier R
      using Rg bg boundD-carrier by (cases x <= m) (auto simp: Pi-def)
  qed
  from f g have (⨁ k ∈ {..n + m}. ⨁ i ∈ {..k}. f i ⊗ g (k - i)) =
      (⨁ k ∈ {..n + m}. ⨁ i ∈ {..n + m - k}. f k ⊗ g i)
    by (simp add: diagonal-sum Pi-def)
  also have ... = (⨁ k ∈ {..n} ∪ {n<..n + m}. ⨁ i ∈ {..n + m - k}. f k ⊗ g i)
    by (simp only: ivl-disj-un-one)
  also from f g have ... = (⨁ k ∈ {..n}. ⨁ i ∈ {..n + m - k}. f k ⊗ g i)
    by (simp cong: finsum-cong
      add: bound.bound [OF bf] finsum-Un-disjoint ivl-disj-int-one Pi-def)
  also from f g
  have ... = (⨁ k ∈ {..n}. ⨁ i ∈ {..m} ∪ {m<..n + m - k}. f k ⊗ g i)
    by (simp cong: finsum-cong add: ivl-disj-un-one le-add-diff Pi-def)
  also from f g have ... = (⨁ k ∈ {..n}. ⨁ i ∈ {..m}. f k ⊗ g i)
    by (simp cong: finsum-cong
      add: bound.bound [OF bg] finsum-Un-disjoint ivl-disj-int-one Pi-def)
  also from f g have ... = (⨁ i ∈ {..n}. f i) ⊗ (⨁ i ∈ {..m}. g i)
    by (simp add: finsum-ldistr diagonal-sum Pi-def,
      simp cong: finsum-cong add: finsum-rdistr Pi-def)
  finally show ?thesis .
qed
```

lemma (**in** *UP-cring*) *const-ring-hom*:
  (%*a*. *monom P a 0*) $\in$ *ring-hom R P*
  by (*auto intro*!: *ring-hom-memI intro*: *up-eqI simp*: *monom-mult-is-smult*)

constdefs (**structure** *S*)
  eval :: [('*a*, '*m*) *ring-scheme*, ('*b*, '*n*) *ring-scheme*,
    '*a* => '*b*, '*b*, *nat* => '*a*] => '*b*
  eval R S phi s == $\lambda p \in$ *carrier* (*UP R*).
    $\bigoplus i \in \{..deg\ R\ p\}.\ phi\ (coeff\ (UP\ R)\ p\ i) \otimes s\ (\hat{\ })\ i$

lemma (**in** *UP*) *eval-on-carrier*:
  fixes *S* (**structure**)
  shows $p \in$ *carrier P* ==>
  $eval\ R\ S\ phi\ s\ p = (\bigoplus_S i \in \{..deg\ R\ p\}.\ phi\ (coeff\ P\ p\ i) \otimes_S s\ (\hat{\ })_S\ i)$
  by (*unfold eval-def*, *fold P-def*) *simp*

lemma (**in** *UP*) *eval-extensional*:
  eval R S phi p $\in$ *extensional* (*carrier P*)
  by (*unfold eval-def*, *fold P-def*) *simp*

The universal property of the polynomial ring

locale *UP-pre-univ-prop* = *ring-hom-cring R S h* + *UP-cring R P*

locale *UP-univ-prop* = *UP-pre-univ-prop* +
  fixes *s* **and** *Eval*
  assumes *indet-img-carrier* [*simp*, *intro*]: $s \in$ *carrier S*
  defines *Eval-def*: *Eval* == *eval R S h s*

theorem (**in** *UP-pre-univ-prop*) *eval-ring-hom*:
  assumes *S*: $s \in$ *carrier S*
  shows *eval R S h s* $\in$ *ring-hom P S*
proof (*rule ring-hom-memI*)
  fix *p*
  assume *R*: $p \in$ *carrier P*
  then show *eval R S h s p* $\in$ *carrier S*
    by (*simp only*: *eval-on-carrier*) (*simp add*: *S Pi-def*)
next
  fix *p q*
  assume *R*: $p \in$ *carrier P* $q \in$ *carrier P*
  then show $eval\ R\ S\ h\ s\ (p \otimes_P q) = eval\ R\ S\ h\ s\ p \otimes_S eval\ R\ S\ h\ s\ q$
  proof (*simp only*: *eval-on-carrier UP-mult-closed*)
    from *R S* **have**
      $(\bigoplus_S i \in \{..deg\ R\ (p \otimes_P q)\}.\ h\ (coeff\ P\ (p \otimes_P q)\ i) \otimes_S s\ (\hat{\ })_S\ i) =$
      $(\bigoplus_S i \in \{..deg\ R\ (p \otimes_P q)\} \cup \{deg\ R\ (p \otimes_P q)<..deg\ R\ p + deg\ R\ q\}.$
        $h\ (coeff\ P\ (p \otimes_P q)\ i) \otimes_S s\ (\hat{\ })_S\ i)$
      by (*simp cong*: *S.finsum-cong*
        add: *deg-aboveD S.finsum-Un-disjoint ivl-disj-int-one Pi-def*
        del: *coeff-mult*)

also from R **have** ... =
$(\bigoplus_S i \in \{..deg\ R\ p + deg\ R\ q\}.\ h\ (coeff\ P\ (p \otimes_P q)\ i) \otimes_S s\ (\hat{\ })_S\ i)$
by (*simp only*: *ivl-disj-un-one deg-mult-cring*)
also from $R\ S$ **have** ... =
$(\bigoplus_S i \in \{..deg\ R\ p + deg\ R\ q\}.$
$\bigoplus_S k \in \{..i\}.$
$h\ (coeff\ P\ p\ k) \otimes_S h\ (coeff\ P\ q\ (i - k)) \otimes_S$
$(s\ (\hat{\ })_S\ k \otimes_S s\ (\hat{\ })_S\ (i - k)))$
by (*simp cong*: *S.finsum-cong add*: *S.nat-pow-mult Pi-def*
S.m-ac S.finsum-rdistr)
also from $R\ S$ **have** ... =
$(\bigoplus_S i{\in}\{..deg\ R\ p\}.\ h\ (coeff\ P\ p\ i) \otimes_S s\ (\hat{\ })_S\ i) \otimes_S$
$(\bigoplus_S i{\in}\{..deg\ R\ q\}.\ h\ (coeff\ P\ q\ i) \otimes_S s\ (\hat{\ })_S\ i)$
by (*simp add*: *S.cauchy-product* [*THEN sym*] *bound.intro deg-aboveD S.m-ac*
Pi-def)
finally show
$(\bigoplus_S i \in \{..deg\ R\ (p \otimes_P q)\}.\ h\ (coeff\ P\ (p \otimes_P q)\ i) \otimes_S s\ (\hat{\ })_S\ i) =$
$(\bigoplus_S i \in \{..deg\ R\ p\}.\ h\ (coeff\ P\ p\ i) \otimes_S s\ (\hat{\ })_S\ i) \otimes_S$
$(\bigoplus_S i \in \{..deg\ R\ q\}.\ h\ (coeff\ P\ q\ i) \otimes_S s\ (\hat{\ })_S\ i)$ **.**
qed
next
fix $p\ q$
assume R: $p \in carrier\ P\ q \in carrier\ P$
then show $eval\ R\ S\ h\ s\ (p \oplus_P q) = eval\ R\ S\ h\ s\ p \oplus_S eval\ R\ S\ h\ s\ q$
proof (*simp only*: *eval-on-carrier P.a-closed*)
from $S\ R$ **have**
$(\bigoplus_S i{\in}\{..deg\ R\ (p \oplus_P q)\}.\ h\ (coeff\ P\ (p \oplus_P q)\ i) \otimes_S s\ (\hat{\ })_S\ i) =$
$(\bigoplus_S i{\in}\{..deg\ R\ (p \oplus_P q)\} \cup \{deg\ R\ (p \oplus_P q){<}..max\ (deg\ R\ p)\ (deg\ R\ q)\}.$
$h\ (coeff\ P\ (p \oplus_P q)\ i) \otimes_S s\ (\hat{\ })_S\ i)$
by (*simp cong*: *S.finsum-cong*
add: *deg-aboveD S.finsum-Un-disjoint ivl-disj-int-one Pi-def*
del: *coeff-add*)
also from R **have** ... =
$(\bigoplus_S i \in \{..max\ (deg\ R\ p)\ (deg\ R\ q)\}.$
$h\ (coeff\ P\ (p \oplus_P q)\ i) \otimes_S s\ (\hat{\ })_S\ i)$
by (*simp add*: *ivl-disj-un-one*)
also from $R\ S$ **have** ... =
$(\bigoplus_S i{\in}\{..max\ (deg\ R\ p)\ (deg\ R\ q)\}.\ h\ (coeff\ P\ p\ i) \otimes_S s\ (\hat{\ })_S\ i) \oplus_S$
$(\bigoplus_S i{\in}\{..max\ (deg\ R\ p)\ (deg\ R\ q)\}.\ h\ (coeff\ P\ q\ i) \otimes_S s\ (\hat{\ })_S\ i)$
by (*simp cong*: *S.finsum-cong*
add: *S.l-distr deg-aboveD ivl-disj-int-one Pi-def*)
also have ... =
$(\bigoplus_S i \in \{..deg\ R\ p\} \cup \{deg\ R\ p{<}..max\ (deg\ R\ p)\ (deg\ R\ q)\}.$
$h\ (coeff\ P\ p\ i) \otimes_S s\ (\hat{\ })_S\ i) \oplus_S$
$(\bigoplus_S i \in \{..deg\ R\ q\} \cup \{deg\ R\ q{<}..max\ (deg\ R\ p)\ (deg\ R\ q)\}.$
$h\ (coeff\ P\ q\ i) \otimes_S s\ (\hat{\ })_S\ i)$
by (*simp only*: *ivl-disj-un-one le-maxI1 le-maxI2*)
also from $R\ S$ **have** ... =
$(\bigoplus_S i \in \{..deg\ R\ p\}.\ h\ (coeff\ P\ p\ i) \otimes_S s\ (\hat{\ })_S\ i) \oplus_S$

$(\bigoplus_S\ i \in \{..deg\ R\ q\}.\ h\ (coeff\ P\ q\ i) \otimes_S s\ (\hat{}\,)_S\ i)$
by (*simp cong*: *S.finsum-cong*
add: *deg-aboveD S.finsum-Un-disjoint ivl-disj-int-one Pi-def*)
finally show
$(\bigoplus_S i \in \{..deg\ R\ (p \oplus_P q)\}.\ h\ (coeff\ P\ (p \oplus_P q)\ i) \otimes_S s\ (\hat{}\,)_S\ i) =$
$(\bigoplus_S i \in \{..deg\ R\ p\}.\ h\ (coeff\ P\ p\ i) \otimes_S s\ (\hat{}\,)_S\ i) \oplus_S$
$(\bigoplus_S i \in \{..deg\ R\ q\}.\ h\ (coeff\ P\ q\ i) \otimes_S s\ (\hat{}\,)_S\ i)$ **.**
qed
next
show *eval R S h s* $\mathbf{1}_P = \mathbf{1}_S$
by (*simp only*: *eval-on-carrier UP-one-closed*) *simp*
qed

Interpretation of ring homomorphism lemmas.

interpretation *UP-univ-prop* < *ring-hom-cring P S Eval*
apply (*unfold Eval-def*)
apply *intro-locales*
apply (*rule ring-hom-cring.axioms*)
apply (*rule ring-hom-cring.intro*)
apply *unfold-locales*
apply (*rule eval-ring-hom*)
apply *rule*
done

Further properties of the evaluation homomorphism.

The following lemma could be proved in *UP-cring* with the additional assumption that h is closed.

lemma (**in** *UP-pre-univ-prop*) *eval-const*:
$[|\ s \in carrier\ S;\ r \in carrier\ R\ |] ==> eval\ R\ S\ h\ s\ (monom\ P\ r\ 0) = h\ r$
by (*simp only*: *eval-on-carrier monom-closed*) *simp*

The following proof is complicated by the fact that in arbitrary rings one might have $\mathbf{1}_R = \mathbf{0}_R$.

lemma (**in** *UP-pre-univ-prop*) *eval-monom1*:
assumes *S*: $s \in carrier\ S$
shows *eval R S h s* (*monom P* $\mathbf{1}$ *1*) = *s*
proof (*simp only*: *eval-on-carrier monom-closed R.one-closed*)
from *S* **have**
$(\bigoplus_S\ i{\in}\{..deg\ R\ (monom\ P\ \mathbf{1}\ 1)\}.\ h\ (coeff\ P\ (monom\ P\ \mathbf{1}\ 1)\ i) \otimes_S s\ (\hat{}\,)_S\ i)$
$=$
$(\bigoplus_S\ i{\in}\{..deg\ R\ (monom\ P\ \mathbf{1}\ 1)\} \cup \{deg\ R\ (monom\ P\ \mathbf{1}\ 1){<}..1\}.$
$h\ (coeff\ P\ (monom\ P\ \mathbf{1}\ 1)\ i) \otimes_S s\ (\hat{}\,)_S\ i)$
by (*simp cong*: *S.finsum-cong del*: *coeff-monom*
add: *deg-aboveD S.finsum-Un-disjoint ivl-disj-int-one Pi-def*)
also have ... =
$(\bigoplus_S\ i \in \{..1\}.\ h\ (coeff\ P\ (monom\ P\ \mathbf{1}\ 1)\ i) \otimes_S s\ (\hat{}\,)_S\ i)$
by (*simp only*: *ivl-disj-un-one deg-monom-le R.one-closed*)
also have ... = *s*

proof (*cases* $s = \mathbf{0}_S$)
  case *True* **then show** *?thesis* **by** (*simp add*: *Pi-def*)
next
  case *False* **then show** *?thesis* **by** (*simp add*: *S Pi-def*)
qed
finally show ($\bigoplus_S i \in \{..deg\ R\ (monom\ P\ \mathbf{1}\ 1)\}.$
  $h\ (coeff\ P\ (monom\ P\ \mathbf{1}\ 1)\ i) \otimes_S s\ (\hat{\ })_S\ i) = s$ **.**
qed

lemma (**in** *UP-cring*) *monom-pow*:
  assumes *R*: $a \in carrier\ R$
  shows $(monom\ P\ a\ n)\ (\hat{\ })_P\ m = monom\ P\ (a\ (\hat{\ })\ m)\ (n * m)$
proof (*induct m*)
  case *0* **from** *R* **show** *?case* **by** *simp*
next
  case *Suc* **with** *R* **show** *?case*
    by (*simp del*: *monom-mult add*: *monom-mult* [*THEN sym*] *add-commute*)
qed

lemma (**in** *ring-hom-cring*) *hom-pow* [*simp*]:
  $x \in carrier\ R \Longrightarrow h\ (x\ (\hat{\ })\ n) = h\ x\ (\hat{\ })_S\ (n{::}nat)$
  by (*induct n*) *simp-all*

lemma (**in** *UP-univ-prop*) *Eval-monom*:
  $r \in carrier\ R \Longrightarrow Eval\ (monom\ P\ r\ n) = h\ r \otimes_S s\ (\hat{\ })_S\ n$
proof −
  assume *R*: $r \in carrier\ R$
  from *R* **have** $Eval\ (monom\ P\ r\ n) = Eval\ (monom\ P\ r\ 0 \otimes_P (monom\ P\ \mathbf{1}\ 1)$
$(\hat{\ })_P\ n)$
    by (*simp del*: *monom-mult add*: *monom-mult* [*THEN sym*] *monom-pow*)
  also
  from *R* *eval-monom1* [**where** $s = s$, *folded Eval-def*]
  have $... = h\ r \otimes_S s\ (\hat{\ })_S\ n$
    by (*simp add*: *eval-const* [**where** $s = s$, *folded Eval-def*])
  finally show *?thesis* **.**
qed

lemma (**in** *UP-pre-univ-prop*) *eval-monom*:
  assumes *R*: $r \in carrier\ R$ **and** *S*: $s \in carrier\ S$
  shows $eval\ R\ S\ h\ s\ (monom\ P\ r\ n) = h\ r \otimes_S s\ (\hat{\ })_S\ n$
proof −
  interpret *UP-univ-prop* [*R S h P s* -]
    using ⟨*UP-pre-univ-prop R S h*⟩ *P-def R S*
    by (*auto intro*: *UP-univ-prop.intro UP-univ-prop-axioms.intro*)
  from *R*
  show *?thesis* **by** (*rule Eval-monom*)
qed

lemma (**in** *UP-univ-prop*) *Eval-smult*:

[| $r \in$ *carrier R*; $p \in$ *carrier P* |] ==> *Eval* $(r \odot_P p) = h\ r \otimes_S$ *Eval p*
proof –
  assume *R*: $r \in$ *carrier R* **and** *P*: $p \in$ *carrier P*
  then show *?thesis*
    by (*simp add*: *monom-mult-is-smult* [*THEN sym*]
      eval-const [**where** $s = s$, *folded Eval-def*])
qed

lemma *ring-hom-cringI*:
  assumes *cring R*
    and *cring S*
    and $h \in$ *ring-hom R S*
  shows *ring-hom-cring R S h*
  by (*fast intro*: *ring-hom-cring.intro ring-hom-cring-axioms.intro*
    cring.axioms prems)

lemma (**in** *UP-pre-univ-prop*) *UP-hom-unique*:
  includes *ring-hom-cring P S Phi*
  assumes *Phi*: *Phi* (*monom P* **1** (*Suc 0*)) = *s*
      !!*r*. $r \in$ *carrier R* ==> *Phi* (*monom P r 0*) = *h r*
  includes *ring-hom-cring P S Psi*
  assumes *Psi*: *Psi* (*monom P* **1** (*Suc 0*)) = *s*
      !!*r*. $r \in$ *carrier R* ==> *Psi* (*monom P r 0*) = *h r*
    and *P*: $p \in$ *carrier P* **and** *S*: $s \in$ *carrier S*
  shows *Phi p* = *Psi p*
proof –
  have *Phi p* =
      Phi ($\bigoplus_P$ $i \in \{..deg\ R\ p\}$. *monom P* (*coeff P p i*) *0* $\otimes_P$ *monom P* **1** *1* $(\hat{\ })_P$
i)
   by (*simp add*: *up-repr P monom-mult* [*THEN sym*] *monom-pow del*: *monom-mult*)
  also
  have ... =
      Psi ($\bigoplus_P$ $i\in\{..deg\ R\ p\}$. *monom P* (*coeff P p i*) *0* $\otimes_P$ *monom P* **1** *1* $(\hat{\ })_P$ *i*)
    by (*simp add*: *Phi Psi P Pi-def comp-def*)
  also have ... = *Psi p*
   by (*simp add*: *up-repr P monom-mult* [*THEN sym*] *monom-pow del*: *monom-mult*)
  finally show *?thesis* **.**
qed

lemma (**in** *UP-pre-univ-prop*) *ring-homD*:
  assumes *Phi*: $Phi \in$ *ring-hom P S*
  shows *ring-hom-cring P S Phi*
proof (*rule ring-hom-cring.intro*)
  show *ring-hom-cring-axioms P S Phi*
  by (*rule ring-hom-cring-axioms.intro*) (*rule Phi*)
qed *unfold-locales*

theorem (**in** *UP-pre-univ-prop*) *UP-universal-property*:
  assumes *S*: $s \in$ *carrier S*

shows *EX*! *Phi*. *Phi* ∈ *ring-hom P S* ∩ *extensional* (*carrier P*) &
Phi (*monom P* **1** *1*) = *s* &
(*ALL r* : *carrier R*. *Phi* (*monom P r 0*) = *h r*)
using *S eval-monom1*
apply (*auto intro*: *eval-ring-hom eval-const eval-extensional*)
apply (*rule extensionalityI*)
apply (*auto intro*: *UP-hom-unique ring-homD*)
done

11.9 Sample Application of Evaluation Homomorphism

lemma *UP-pre-univ-propI*:
assumes *cring R*
and *cring S*
and *h* ∈ *ring-hom R S*
shows *UP-pre-univ-prop R S h*
using *assms*
by (*auto intro*!: *UP-pre-univ-prop.intro ring-hom-cring.intro*
ring-hom-cring-axioms.intro UP-cring.intro)

constdefs
INTEG :: *int ring*
INTEG == (| *carrier* = *UNIV*, *mult* = *op* ∗, *one* = *1*, *zero* = *0*, *add* = *op* +
|)

lemma *INTEG-cring*:
cring INTEG
by (*unfold INTEG-def*) (*auto intro*!: *cringI abelian-groupI comm-monoidI*
zadd-zminus-inverse2 zadd-zmult-distrib)

lemma *INTEG-id-eval*:
UP-pre-univ-prop INTEG INTEG id
by (*fast intro*: *UP-pre-univ-propI INTEG-cring id-ring-hom*)

Interpretation now enables to import all theorems and lemmas valid in the context of homomorphisms between *INTEG* and *UP INTEG* globally.

interpretation *INTEG*: *UP-pre-univ-prop* [*INTEG INTEG id*]
apply *simp*
using *INTEG-id-eval*
apply *simp*
done

lemma *INTEG-closed* [*intro*, *simp*]:
z ∈ *carrier INTEG*
by (*unfold INTEG-def*) *simp*

lemma *INTEG-mult* [*simp*]:
mult INTEG z w = *z* ∗ *w*
by (*unfold INTEG-def*) *simp*

lemma *INTEG-pow* [*simp*]:
pow INTEG z n = *z* ^ *n*
by (*induct n*) (*simp-all add*: *INTEG-def nat-pow-def*)

lemma *eval INTEG INTEG id 10* (*monom* (*UP INTEG*) *5 2*) = *500*
by (*simp add*: *INTEG.eval-monom*)

end

theory *AbelCoset*
imports *Coset Ring*
begin

12 More Lifting from Groups to Abelian Groups

12.1 Definitions

Hiding <+> from *Sum-Type* until I come up with better syntax here

hide *const Plus*

constdefs (**structure** *G*)
a-r-coset :: [-, ′*a set*, ′*a*] ⇒ ′*a set* (**infixl** +>ı *60*)
a-r-coset G ≡ *r-coset* (|*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*|)

a-l-coset :: [-, ′*a*, ′*a set*] ⇒ ′*a set* (**infixl** <+ı *60*)
a-l-coset G ≡ *l-coset* (|*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*|)

A-RCOSETS :: [-, ′*a set*] ⇒ (′*a set*)*set* (*a*′*-rcosets*ı - [*81*] *80*)
A-RCOSETS G H ≡ *RCOSETS* (|*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*|) *H*

set-add :: [-, ′*a set* ,′*a set*] ⇒ ′*a set* (**infixl** <+>ı *60*)
set-add G ≡ *set-mult* (|*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*|)

A-SET-INV :: [-,′*a set*] ⇒ ′*a set* (*a*′*-set*′*-inv*ı - [*81*] *80*)
A-SET-INV G H ≡ *SET-INV* (|*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*|) *H*

constdefs (**structure** *G*)
a-r-congruent :: [(′*a*,′*b*)*ring-scheme*, ′*a set*] ⇒ (′*a*∗′*a*)*set*
(*racong*ı -)
a-r-congruent G ≡ *r-congruent* (|*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*|)

constdefs

A-FactGroup :: [('*a*,'*b*) *ring-scheme*, '*a set*] ⇒ ('*a set*) *monoid*
(**infixl** *A'-Mod 65*)
— Actually defined for groups rather than monoids
A-FactGroup G H ≡ *FactGroup* (|*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*|) *H*

constdefs
a-kernel :: ('*a*, '*m*) *ring-scheme* ⇒ ('*b*, '*n*) *ring-scheme* ⇒
('*a* ⇒ '*b*) ⇒ '*a set*
— the kernel of a homomorphism (additive)
a-kernel G H h ≡ *kernel* (|*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*|)
(|*carrier* = *carrier H*, *mult* = *add H*, *one* = *zero H*|) *h*

locale *abelian-group-hom* = *abelian-group G* + *abelian-group H* + *var h* +
assumes *a-group-hom*: *group-hom* (| *carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G* |)
(| *carrier* = *carrier H*, *mult* = *add H*, *one* = *zero H* |) *h*

lemmas *a-r-coset-defs* =
a-r-coset-def r-coset-def

lemma *a-r-coset-def'*:
includes *struct G*
shows $H +> a \equiv \bigcup h\in H.\ \{h \oplus a\}$
unfolding *a-r-coset-defs*
by *simp*

lemmas *a-l-coset-defs* =
a-l-coset-def l-coset-def

lemma *a-l-coset-def'*:
includes *struct G*
shows $a <+ H \equiv \bigcup h\in H.\ \{a \oplus h\}$
unfolding *a-l-coset-defs*
by *simp*

lemmas *A-RCOSETS-defs* =
A-RCOSETS-def RCOSETS-def

lemma *A-RCOSETS-def'*:
includes *struct G*
shows $a\text{-}rcosets\ H \equiv \bigcup a\in carrier\ G.\ \{H +> a\}$
unfolding *A-RCOSETS-defs*
by (*fold a-r-coset-def*, *simp*)

lemmas *set-add-defs* =
set-add-def set-mult-def

lemma *set-add-def'*:

includes *struct G*
shows $H <+> K \equiv \bigcup h{\in}H.\ \bigcup k{\in}K.\ \{h \oplus k\}$
unfolding *set-add-defs*
by *simp*

lemmas *A-SET-INV-defs* =
A-SET-INV-def SET-INV-def

lemma *A-SET-INV-def'*:
includes *struct G*
shows $a\text{-}set\text{-}inv\ H \equiv \bigcup h{\in}H.\ \{\ominus\ h\}$
unfolding *A-SET-INV-defs*
by (*fold a-inv-def*)

12.2 Cosets

lemma (**in** *abelian-group*) *a-coset-add-assoc*:
$[|\ M \subseteq carrier\ G;\ g \in carrier\ G;\ h \in carrier\ G\ |]$
$==> (M +> g) +> h = M +> (g \oplus h)$
by (*rule group.coset-mult-assoc* [*OF a-group*,
folded a-r-coset-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-group*) *a-coset-add-zero* [*simp*]:
$M \subseteq carrier\ G ==> M +> \mathbf{0} = M$
by (*rule group.coset-mult-one* [*OF a-group*,
folded a-r-coset-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-group*) *a-coset-add-inv1*:
$[|\ M +> (x \oplus (\ominus\ y)) = M;\ \ x \in carrier\ G\ ;\ y \in carrier\ G;$
$M \subseteq carrier\ G\ |] ==> M +> x = M +> y$
by (*rule group.coset-mult-inv1* [*OF a-group*,
folded a-r-coset-def a-inv-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-group*) *a-coset-add-inv2*:
$[|\ M +> x = M +> y;\ \ x \in carrier\ G;\ \ y \in carrier\ G;\ \ M \subseteq carrier\ G\ |]$
$==> M +> (x \oplus (\ominus\ y)) = M$
by (*rule group.coset-mult-inv2* [*OF a-group*,
folded a-r-coset-def a-inv-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-group*) *a-coset-join1*:
$[|\ H +> x = H;\ \ x \in carrier\ G;\ \ subgroup\ H\ (|carrier = carrier\ G,\ mult = add\ G,\ one = zero\ G|)\ |] ==> x \in H$
by (*rule group.coset-join1* [*OF a-group*,
folded a-r-coset-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-group*) *a-solve-equation*:
$[\![subgroup\ H\ (|carrier = carrier\ G,\ mult = add\ G,\ one = zero\ G|);\ x \in H;\ y \in H]\!] \Longrightarrow \exists\, h{\in}H.\ y = h \oplus x$
by (*rule group.solve-equation* [*OF a-group*,

folded a-r-coset-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-group*) *a-repr-independence*:
$[\![y \in H +> x;\ \ x \in carrier\ G;\ subgroup\ H\ (\!|carrier = carrier\ G,\ mult = add\ G,\ one = zero\ G|\!)\ ]\!] \Longrightarrow H +> x = H +> y$
by (*rule group.repr-independence* [*OF a-group*,
folded a-r-coset-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-group*) *a-coset-join2*:
$[\![x \in carrier\ G;\ \ subgroup\ H\ (\!|carrier = carrier\ G,\ mult = add\ G,\ one = zero\ G|\!);\ x{\in}H]\!] \Longrightarrow H +> x = H$
by (*rule group.coset-join2* [*OF a-group*,
folded a-r-coset-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-monoid*) *a-r-coset-subset-G*:
$[|\ H \subseteq carrier\ G;\ x \in carrier\ G\ |] ==> H +> x \subseteq carrier\ G$
by (*rule monoid.r-coset-subset-G* [*OF a-monoid*,
folded a-r-coset-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-group*) *a-rcosI*:
$[|\ h \in H;\ H \subseteq carrier\ G;\ x \in carrier\ G|] ==> h \oplus x \in H +> x$
by (*rule group.rcosI* [*OF a-group*,
folded a-r-coset-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-group*) *a-rcosetsI*:
$[\![H \subseteq carrier\ G;\ x \in carrier\ G]\!] \Longrightarrow H +> x \in a\text{-}rcosets\ H$
by (*rule group.rcosetsI* [*OF a-group*,
folded a-r-coset-def A-RCOSETS-def, *simplified monoid-record-simps*])

Really needed?

lemma (**in** *abelian-group*) *a-transpose-inv*:
$[|\ x \oplus y = z;\ \ x \in carrier\ G;\ \ y \in carrier\ G;\ \ z \in carrier\ G\ |]$
$==> (\ominus\ x) \oplus z = y$
by (*rule group.transpose-inv* [*OF a-group*,
folded a-r-coset-def a-inv-def, *simplified monoid-record-simps*])

12.3 Subgroups

locale *additive-subgroup* = *var H* + *struct G* +
assumes *a-subgroup*: $subgroup\ H\ (\!|carrier = carrier\ G,\ mult = add\ G,\ one = zero\ G|\!)$

lemma (**in** *additive-subgroup*) *is-additive-subgroup*:
shows *additive-subgroup H G*
by *fact*

lemma *additive-subgroupI*:
includes *struct G*
assumes *a-subgroup*: $subgroup\ H\ (\!|carrier = carrier\ G,\ mult = add\ G,\ one =$

zero G⦈
  shows *additive-subgroup H G*
by (*rule additive-subgroup.intro*) (*rule a-subgroup*)

lemma (**in** *additive-subgroup*) *a-subset*:
  $H \subseteq carrier\ G$
by (*rule subgroup.subset*[*OF a-subgroup*,
  simplified monoid-record-simps])

lemma (**in** *additive-subgroup*) *a-closed* [*intro*, *simp*]:
  $[\![x \in H;\ y \in H]\!] \Longrightarrow x \oplus y \in H$
by (*rule subgroup.m-closed*[*OF a-subgroup*,
  simplified monoid-record-simps])

lemma (**in** *additive-subgroup*) *zero-closed* [*simp*]:
  $\mathbf{0} \in H$
by (*rule subgroup.one-closed*[*OF a-subgroup*,
  simplified monoid-record-simps])

lemma (**in** *additive-subgroup*) *a-inv-closed* [*intro*,*simp*]:
  $x \in H \Longrightarrow \ominus\ x \in H$
by (*rule subgroup.m-inv-closed*[*OF a-subgroup*,
  folded a-inv-def, *simplified monoid-record-simps*])

12.4 Normal additive subgroups

12.4.1 Definition of *abelian-subgroup*

Every subgroup of an *abelian-group* is normal

locale *abelian-subgroup* = *additive-subgroup H G* + *abelian-group G* +
  assumes *a-normal*: *normal H* (|*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*|)

lemma (**in** *abelian-subgroup*) *is-abelian-subgroup*:
  shows *abelian-subgroup H G*
by *fact*

lemma *abelian-subgroupI*:
  assumes *a-normal*: *normal H* (|*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*|)
    and *a-comm*: !!*x y*. [| $x \in carrier\ G$; $y \in carrier\ G$ |] ==> $x \oplus_G y = y \oplus_G x$
  shows *abelian-subgroup H G*
proof −
  interpret *normal* [*H* (|*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*|)]
  by (*rule a-normal*)

  show *abelian-subgroup H G*
  by (*unfold-locales*, *simp add*: *a-comm*)

qed

lemma *abelian-subgroupI2*:
 includes *struct G*
 assumes *a-comm-group*: *comm-group* (|*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*|)
 and *a-subgroup*: *subgroup H* (|*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*|)
 shows *abelian-subgroup H G*
proof −
 interpret *comm-group* [(|*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*|)]
 by (*rule a-comm-group*)
 interpret *subgroup* [*H* (|*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*|)]
 by (*rule a-subgroup*)

 show *abelian-subgroup H G*
 apply *unfold-locales*
 proof (*simp add*: *r-coset-def l-coset-def*, *clarsimp*)
 fix *x*
 assume *xcarr*: $x \in carrier\ G$
 from *a-subgroup*
 have *Hcarr*: $H \subseteq carrier\ G$ **by** (*unfold subgroup-def*, *simp*)
 from *xcarr Hcarr*
 show $(\bigcup h{\in}H.\ \{h \oplus_G x\}) = (\bigcup h{\in}H.\ \{x \oplus_G h\})$
 using *m-comm*[*simplified*]
 by *fast*
 qed
qed

lemma *abelian-subgroupI3*:
 includes *struct G*
 assumes *asg*: *additive-subgroup H G*
 and *ag*: *abelian-group G*
 shows *abelian-subgroup H G*
apply (*rule abelian-subgroupI2*)
apply (*rule abelian-group.a-comm-group*[*OF ag*])
apply (*rule additive-subgroup.a-subgroup*[*OF asg*])
done

lemma (**in** *abelian-subgroup*) *a-coset-eq*:
 $(\forall x \in carrier\ G.\ H +> x = x <+ H)$
by (*rule normal.coset-eq*[*OF a-normal*,
 folded a-r-coset-def a-l-coset-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-subgroup*) *a-inv-op-closed1*:
 shows $[\![x \in carrier\ G;\ h \in H]\!] \Longrightarrow (\ominus\ x) \oplus h \oplus x \in H$
by (*rule normal.inv-op-closed1* [*OF a-normal*,
 folded a-inv-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-subgroup*) *a-inv-op-closed2*:
shows $[\![x \in \mathit{carrier}\ G;\ h \in H]\!] \Longrightarrow x \oplus h \oplus (\ominus\ x) \in H$
by (*rule normal.inv-op-closed2* [*OF a-normal*,
folded a-inv-def, *simplified monoid-record-simps*])

Alternative characterization of normal subgroups

lemma (**in** *abelian-group*) *a-normal-inv-iff*:
$(N \lhd (\!|\mathit{carrier} = \mathit{carrier}\ G,\ \mathit{mult} = \mathit{add}\ G,\ \mathit{one} = \mathit{zero}\ G|\!)) =$
$(\mathit{subgroup}\ N\ (\!|\mathit{carrier} = \mathit{carrier}\ G,\ \mathit{mult} = \mathit{add}\ G,\ \mathit{one} = \mathit{zero}\ G|\!)\ \&\ (\forall\, x \in \mathit{carrier}\ G.\ \forall\, h \in N.\ x \oplus h \oplus (\ominus\ x) \in N))$
(**is** - = *?rhs*)
by (*rule group.normal-inv-iff* [*OF a-group*,
folded a-inv-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-group*) *a-lcos-m-assoc*:
$[|\ M \subseteq \mathit{carrier}\ G;\ g \in \mathit{carrier}\ G;\ h \in \mathit{carrier}\ G\ |]$
$==> g <\!+ (h <\!+ M) = (g \oplus h) <\!+ M$
by (*rule group.lcos-m-assoc* [*OF a-group*,
folded a-l-coset-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-group*) *a-lcos-mult-one*:
$M \subseteq \mathit{carrier}\ G ==> \mathbf{0} <\!+ M = M$
by (*rule group.lcos-mult-one* [*OF a-group*,
folded a-l-coset-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-group*) *a-l-coset-subset-G*:
$[|\ H \subseteq \mathit{carrier}\ G;\ x \in \mathit{carrier}\ G\ |] ==> x <\!+ H \subseteq \mathit{carrier}\ G$
by (*rule group.l-coset-subset-G* [*OF a-group*,
folded a-l-coset-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-group*) *a-l-coset-swap*:
$[\![y \in x <\!+ H;\ \ x \in \mathit{carrier}\ G;\ \ \mathit{subgroup}\ H\ (\!|\mathit{carrier} = \mathit{carrier}\ G,\ \mathit{mult} = \mathit{add}\ G,\ \mathit{one} = \mathit{zero}\ G|\!)]\!] \Longrightarrow x \in y <\!+ H$
by (*rule group.l-coset-swap* [*OF a-group*,
folded a-l-coset-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-group*) *a-l-coset-carrier*:
$[|\ y \in x <\!+ H;\ \ x \in \mathit{carrier}\ G;\ \ \mathit{subgroup}\ H\ (\!|\mathit{carrier} = \mathit{carrier}\ G,\ \mathit{mult} = \mathit{add}\ G,\ \mathit{one} = \mathit{zero}\ G|\!)\ |] ==> y \in \mathit{carrier}\ G$
by (*rule group.l-coset-carrier* [*OF a-group*,
folded a-l-coset-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-group*) *a-l-repr-imp-subset*:
assumes *y*: $y \in x <\!+ H$ **and** *x*: $x \in \mathit{carrier}\ G$ **and** *sb*: $\mathit{subgroup}\ H\ (\!|\mathit{carrier} = \mathit{carrier}\ G,\ \mathit{mult} = \mathit{add}\ G,\ \mathit{one} = \mathit{zero}\ G|\!)$
shows $y <\!+ H \subseteq x <\!+ H$
apply (*rule group.l-repr-imp-subset* [*OF a-group*,

folded a-l-coset-def, *simplified monoid-record-simps*])
apply (*rule y*)
apply (*rule x*)
apply (*rule sb*)
done

lemma (**in** *abelian-group*) *a-l-repr-independence*:
assumes *y*: $y \in x <+ H$ **and** *x*: $x \in$ *carrier G* **and** *sb*: *subgroup H* $(\!|$*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*$|\!)$
shows $x <+ H = y <+ H$
apply (*rule group.l-repr-independence* [*OF a-group*,
folded a-l-coset-def, *simplified monoid-record-simps*])
apply (*rule y*)
apply (*rule x*)
apply (*rule sb*)
done

lemma (**in** *abelian-group*) *setadd-subset-G*:
$[\![H \subseteq carrier\ G;\ K \subseteq carrier\ G]\!] \Longrightarrow H <+> K \subseteq carrier\ G$
by (*rule group.setmult-subset-G* [*OF a-group*,
folded set-add-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-group*) *subgroup-add-id*: *subgroup H* $(\!|$*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*$|\!) \Longrightarrow H <+> H = H$
by (*rule group.subgroup-mult-id* [*OF a-group*,
folded set-add-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-subgroup*) *a-rcos-inv*:
assumes *x*: $x \in carrier\ G$
shows $a\text{-}set\text{-}inv\ (H +> x) = H +> (\ominus\ x)$
by (*rule normal.rcos-inv* [*OF a-normal*,
folded a-r-coset-def a-inv-def A-SET-INV-def, *simplified monoid-record-simps*]) (*rule x*)

lemma (**in** *abelian-group*) *a-setmult-rcos-assoc*:
$[\![H \subseteq carrier\ G;\ K \subseteq carrier\ G;\ x \in carrier\ G]\!]$
$\Longrightarrow H <+> (K +> x) = (H <+> K) +> x$
by (*rule group.setmult-rcos-assoc* [*OF a-group*,
folded set-add-def a-r-coset-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-group*) *a-rcos-assoc-lcos*:
$[\![H \subseteq carrier\ G;\ K \subseteq carrier\ G;\ x \in carrier\ G]\!]$
$\Longrightarrow (H +> x) <+> K = H <+> (x <+ K)$
by (*rule group.rcos-assoc-lcos* [*OF a-group*,
folded set-add-def a-r-coset-def a-l-coset-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-subgroup*) *a-rcos-sum*:
$[\![x \in carrier\ G;\ y \in carrier\ G]\!]$
$\Longrightarrow (H +> x) <+> (H +> y) = H +> (x \oplus y)$

by (*rule normal.rcos-sum* [*OF a-normal*,
folded set-add-def a-r-coset-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-subgroup*) *rcosets-add-eq*:
$M \in$ *a-rcosets* $H \Longrightarrow H$ <+> $M = M$
— generalizes *subgroup-mult-id*
by (*rule normal.rcosets-mult-eq* [*OF a-normal*,
folded set-add-def A-RCOSETS-def, *simplified monoid-record-simps*])

12.5 Congruence Relation

lemma (**in** *abelian-subgroup*) *a-equiv-rcong*:
shows *equiv* (*carrier G*) (*racong H*)
by (*rule subgroup.equiv-rcong* [*OF a-subgroup a-group*,
folded a-r-congruent-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-subgroup*) *a-l-coset-eq-rcong*:
assumes *a*: $a \in$ *carrier G*
shows a <+ H = *racong H* ‘‘ $\{a\}$
by (*rule subgroup.l-coset-eq-rcong* [*OF a-subgroup a-group*,
folded a-r-congruent-def a-l-coset-def, *simplified monoid-record-simps*]) (*rule a*)

lemma (**in** *abelian-subgroup*) *a-rcos-equation*:
shows
$[\![ha \oplus a = h \oplus b;\ a \in carrier\ G;\ \ b \in carrier\ G;$
$h \in H;\ \ ha \in H;\ \ hb \in H]\!]$
$\Longrightarrow hb \oplus a \in (\bigcup h{\in}H.\ \{h \oplus b\})$
by (*rule group.rcos-equation* [*OF a-group a-subgroup*,
folded a-r-congruent-def a-l-coset-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-subgroup*) *a-rcos-disjoint*:
shows $[\![a \in$ *a-rcosets* $H;\ b \in$ *a-rcosets* $H;\ a{\neq}b]\!] \Longrightarrow a \cap b = \{\}$
by (*rule group.rcos-disjoint* [*OF a-group a-subgroup*,
folded A-RCOSETS-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-subgroup*) *a-rcos-self*:
shows $x \in$ *carrier G* $\Longrightarrow x \in H$ +> x
by (*rule group.rcos-self* [*OF a-group a-subgroup*,
folded a-r-coset-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-subgroup*) *a-rcosets-part-G*:
shows $\bigcup$(*a-rcosets H*) = *carrier G*
by (*rule group.rcosets-part-G* [*OF a-group a-subgroup*,
folded A-RCOSETS-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-subgroup*) *a-cosets-finite*:
$[\![c \in$ *a-rcosets* $H;\ \ H \subseteq$ *carrier G*; *finite* (*carrier G*)$]\!] \Longrightarrow$ *finite c*
by (*rule group.cosets-finite* [*OF a-group*,
folded A-RCOSETS-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-group*) *a-card-cosets-equal*:
$[\![c \in \textit{a-rcosets } H;\ \ H \subseteq \textit{carrier } G;\ \textit{finite}(\textit{carrier } G)]\!]$
$\Longrightarrow \textit{card } c = \textit{card } H$
by (*rule group.card-cosets-equal* [*OF a-group*,
folded A-RCOSETS-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-group*) *rcosets-subset-PowG*:
$\textit{additive-subgroup } H\ G \Longrightarrow \textit{a-rcosets } H \subseteq \textit{Pow}(\textit{carrier } G)$
by (*rule group.rcosets-subset-PowG* [*OF a-group*,
folded A-RCOSETS-def, *simplified monoid-record-simps*],
rule additive-subgroup.a-subgroup)

theorem (**in** *abelian-group*) *a-lagrange*:
$[\![\textit{finite}(\textit{carrier } G);\ \textit{additive-subgroup } H\ G]\!]$
$\Longrightarrow \textit{card}(\textit{a-rcosets } H) * \textit{card}(H) = \textit{order}(G)$
by (*rule group.lagrange* [*OF a-group*,
folded A-RCOSETS-def, *simplified monoid-record-simps order-def*, *folded order-def*])
(*fast intro*!: *additive-subgroup.a-subgroup*)+

12.6 Factorization

lemmas *A-FactGroup-defs* = *A-FactGroup-def FactGroup-def*

lemma *A-FactGroup-def'*:
includes *struct G*
shows $G\ \textit{A-Mod}\ H \equiv (\!|\textit{carrier} = \textit{a-rcosets}_G\ H,\ \textit{mult} = \textit{set-add } G,\ \textit{one} = H|\!)$
unfolding *A-FactGroup-defs*
by (*fold A-RCOSETS-def set-add-def*)

lemma (**in** *abelian-subgroup*) *a-setmult-closed*:
$[\![K1 \in \textit{a-rcosets } H;\ K2 \in \textit{a-rcosets } H]\!] \Longrightarrow K1 <+> K2 \in \textit{a-rcosets } H$
by (*rule normal.setmult-closed* [*OF a-normal*,
folded A-RCOSETS-def set-add-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-subgroup*) *a-setinv-closed*:
$K \in \textit{a-rcosets } H \Longrightarrow \textit{a-set-inv } K \in \textit{a-rcosets } H$
by (*rule normal.setinv-closed* [*OF a-normal*,
folded A-RCOSETS-def A-SET-INV-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-subgroup*) *a-rcosets-assoc*:
$[\![M1 \in \textit{a-rcosets } H;\ M2 \in \textit{a-rcosets } H;\ M3 \in \textit{a-rcosets } H]\!]$
$\Longrightarrow M1 <+> M2 <+> M3 = M1 <+> (M2 <+> M3)$
by (*rule normal.rcosets-assoc* [*OF a-normal*,
folded A-RCOSETS-def set-add-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-subgroup*) *a-subgroup-in-rcosets*:
$H \in \textit{a-rcosets } H$

by (*rule subgroup.subgroup-in-rcosets* [*OF a-subgroup a-group*,
folded A-RCOSETS-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-subgroup*) *a-rcosets-inv-mult-group-eq*:
$M \in$ *a-rcosets* $H \Longrightarrow$ *a-set-inv* M <+> $M = H$
by (*rule normal.rcosets-inv-mult-group-eq* [*OF a-normal*,
folded A-RCOSETS-def A-SET-INV-def set-add-def, *simplified monoid-record-simps*])

theorem (**in** *abelian-subgroup*) *a-factorgroup-is-group*:
group (G *A-Mod* H)
by (*rule normal.factorgroup-is-group* [*OF a-normal*,
folded A-FactGroup-def, *simplified monoid-record-simps*])

Since the Factorization is based on an *abelian* subgroup, is results in a commutative group

theorem (**in** *abelian-subgroup*) *a-factorgroup-is-comm-group*:
comm-group (G *A-Mod* H)
apply (*intro comm-group.intro comm-monoid.intro*) **prefer** *3*
apply (*rule a-factorgroup-is-group*)
apply (*rule group.axioms*[*OF a-factorgroup-is-group*])
apply (*rule comm-monoid-axioms.intro*)
apply (*unfold A-FactGroup-def FactGroup-def RCOSETS-def*, *fold set-add-def a-r-coset-def*, *clarsimp*)
apply (*simp add*: *a-rcos-sum a-comm*)
done

lemma *add-A-FactGroup* [*simp*]: $X \otimes_{(G\ A\text{-}Mod\ H)} X' = X$ <+>$_G$ X'
by (*simp add*: *A-FactGroup-def set-add-def*)

lemma (**in** *abelian-subgroup*) *a-inv-FactGroup*:
$X \in$ *carrier* (G *A-Mod* H) $\Longrightarrow$ $inv_{G\ A\text{-}Mod\ H}\ X =$ *a-set-inv* X
by (*rule normal.inv-FactGroup* [*OF a-normal*,
folded A-FactGroup-def A-SET-INV-def, *simplified monoid-record-simps*])

The coset map is a homomorphism from G to the quotient group G *Mod* H

lemma (**in** *abelian-subgroup*) *a-r-coset-hom-A-Mod*:
(λa. H +> a) $\in$ *hom* (|*carrier* = *carrier* G, *mult* = *add* G, *one* = *zero* G|) (G *A-Mod* H)
by (*rule normal.r-coset-hom-Mod* [*OF a-normal*,
folded A-FactGroup-def a-r-coset-def, *simplified monoid-record-simps*])

The isomorphism theorems have been omitted from lifting, at least for now

12.7 The First Isomorphism Theorem

The quotient by the kernel of a homomorphism is isomorphic to the range of that homomorphism.

lemmas *a-kernel-defs* =

a-kernel-def kernel-def

lemma *a-kernel-def'*:
a-kernel R S h $\equiv$ {*x* $\in$ *carrier R*. *h x* = $\mathbf{0}_S$}
by (*rule a-kernel-def*[*unfolded kernel-def*, *simplified ring-record-simps*])

12.8 Homomorphisms

lemma *abelian-group-homI*:
includes *abelian-group G*
includes *abelian-group H*
assumes *a-group-hom*: *group-hom* (| *carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G* |)
(| *carrier* = *carrier H*, *mult* = *add H*, *one* = *zero H* |) *h*
shows *abelian-group-hom G H h*
apply (*intro abelian-group-hom.intro abelian-group-hom-axioms.intro*)
apply (*rule G.abelian-group-axioms*)
apply (*rule H.abelian-group-axioms*)
apply (*rule a-group-hom*)
done

lemma (**in** *abelian-group-hom*) *is-abelian-group-hom*:
abelian-group-hom G H h
by (*unfold-locales*)

lemma (**in** *abelian-group-hom*) *hom-add* [*simp*]:
[| *x* : *carrier G*; *y* : *carrier G* |]
==> *h* ($x \oplus_G y$) = *h x* $\oplus_H$ *h y*
by (*rule group-hom.hom-mult*[*OF a-group-hom*,
simplified ring-record-simps])

lemma (**in** *abelian-group-hom*) *hom-closed* [*simp*]:
x $\in$ *carrier G* $\Longrightarrow$ *h x* $\in$ *carrier H*
by (*rule group-hom.hom-closed*[*OF a-group-hom*,
simplified ring-record-simps])

lemma (**in** *abelian-group-hom*) *zero-closed* [*simp*]:
h $\mathbf{0}$ $\in$ *carrier H*
by (*rule group-hom.one-closed*[*OF a-group-hom*,
simplified ring-record-simps])

lemma (**in** *abelian-group-hom*) *hom-zero* [*simp*]:
h $\mathbf{0} = \mathbf{0}_H$
by (*rule group-hom.hom-one*[*OF a-group-hom*,
simplified ring-record-simps])

lemma (**in** *abelian-group-hom*) *a-inv-closed* [*simp*]:
x $\in$ *carrier G* ==> *h* ($\ominus x$) $\in$ *carrier H*
by (*rule group-hom.inv-closed*[*OF a-group-hom*,

folded a-inv-def, *simplified ring-record-simps*])

lemma (**in** *abelian-group-hom*) *hom-a-inv* [*simp*]:
$x \in carrier\ G \implies h\ (\ominus x) = \ominus_H\ (h\ x)$
by (*rule group-hom.hom-inv*[*OF a-group-hom*,
folded a-inv-def, *simplified ring-record-simps*])

lemma (**in** *abelian-group-hom*) *additive-subgroup-a-kernel*:
additive-subgroup (*a-kernel G H h*) *G*
apply (*rule additive-subgroup.intro*)
apply (*rule group-hom.subgroup-kernel*[*OF a-group-hom*,
folded a-kernel-def, *simplified ring-record-simps*])
done

The kernel of a homomorphism is an abelian subgroup

lemma (**in** *abelian-group-hom*) *abelian-subgroup-a-kernel*:
abelian-subgroup (*a-kernel G H h*) *G*
apply (*rule abelian-subgroupI*)
apply (*rule group-hom.normal-kernel*[*OF a-group-hom*,
folded a-kernel-def, *simplified ring-record-simps*])
apply (*simp add*: *G.a-comm*)
done

lemma (**in** *abelian-group-hom*) *A-FactGroup-nonempty*:
assumes *X*: $X \in carrier\ (G\ A\text{-}Mod\ a\text{-}kernel\ G\ H\ h)$
shows $X \neq \{\}$
by (*rule group-hom.FactGroup-nonempty*[*OF a-group-hom*,
folded a-kernel-def A-FactGroup-def, *simplified ring-record-simps*]) (*rule X*)

lemma (**in** *abelian-group-hom*) *FactGroup-contents-mem*:
assumes *X*: $X \in carrier\ (G\ A\text{-}Mod\ (a\text{-}kernel\ G\ H\ h))$
shows *contents* (*h*‘*X*) $\in$ *carrier H*
by (*rule group-hom.FactGroup-contents-mem*[*OF a-group-hom*,
folded a-kernel-def A-FactGroup-def, *simplified ring-record-simps*]) (*rule X*)

lemma (**in** *abelian-group-hom*) *A-FactGroup-hom*:
(λX. *contents* (*h*‘*X*)) $\in$ *hom* (*G A-Mod* (*a-kernel G H h*))
(|*carrier* = *carrier H*, *mult* = *add H*, *one* = *zero H*|)
by (*rule group-hom.FactGroup-hom*[*OF a-group-hom*,
folded a-kernel-def A-FactGroup-def, *simplified ring-record-simps*])

lemma (**in** *abelian-group-hom*) *A-FactGroup-inj-on*:
inj-on (λX. *contents* (*h* ‘ *X*)) (*carrier* (*G A-Mod a-kernel G H h*))
by (*rule group-hom.FactGroup-inj-on*[*OF a-group-hom*,
folded a-kernel-def A-FactGroup-def, *simplified ring-record-simps*])

If the homomorphism h is onto H, then so is the homomorphism from the quotient group

lemma (**in** *abelian-group-hom*) *A-FactGroup-onto*:

assumes h: h ` $carrier\ G = carrier\ H$
shows $(\lambda X.\ contents\ (h\ `\ X))$ ` $carrier\ (G\ A\text{-}Mod\ a\text{-}kernel\ G\ H\ h) = carrier\ H$
by (*rule group-hom.FactGroup-onto*[*OF a-group-hom*,
folded a-kernel-def A-FactGroup-def, *simplified ring-record-simps*]) (*rule h*)

If h is a homomorphism from G onto H, then the quotient group G *Mod kernel* $G\ H\ h$ is isomorphic to H.

theorem (**in** *abelian-group-hom*) *A-FactGroup-iso*:
h ` $carrier\ G = carrier\ H$
$\Longrightarrow (\lambda X.\ contents\ (h`X)) \in (G\ A\text{-}Mod\ (a\text{-}kernel\ G\ H\ h)) \cong$
$(|\ carrier = carrier\ H,\ mult = add\ H,\ one = zero\ H\ |)$
by (*rule group-hom.FactGroup-iso*[*OF a-group-hom*,
folded a-kernel-def A-FactGroup-def, *simplified ring-record-simps*])

13 Lemmas Lifted from CosetExt.thy

Not eveything from `CosetExt.thy` is lifted here.

13.1 General Lemmas from `AlgebraExt.thy`

lemma (**in** *additive-subgroup*) *a-Hcarr* [*simp*]:
assumes hH: $h \in H$
shows $h \in carrier\ G$
by (*rule subgroup.mem-carrier* [*OF a-subgroup*,
simplified monoid-record-simps]) (*rule hH*)

13.2 Lemmas for Right Cosets

lemma (**in** *abelian-subgroup*) *a-elemrcos-carrier*:
assumes *acarr*: $a \in carrier\ G$
and a': $a' \in H \mathbin{+>} a$
shows $a' \in carrier\ G$
by (*rule subgroup.elemrcos-carrier* [*OF a-subgroup a-group*,
folded a-r-coset-def, *simplified monoid-record-simps*]) (*rule acarr*, *rule* a')

lemma (**in** *abelian-subgroup*) *a-rcos-const*:
assumes hH: $h \in H$
shows $H \mathbin{+>} h = H$
by (*rule subgroup.rcos-const* [*OF a-subgroup a-group*,
folded a-r-coset-def, *simplified monoid-record-simps*]) (*rule hH*)

lemma (**in** *abelian-subgroup*) *a-rcos-module-imp*:
assumes *xcarr*: $x \in carrier\ G$
and $x'cos$: $x' \in H \mathbin{+>} x$
shows $(x' \oplus \ominus x) \in H$
by (*rule subgroup.rcos-module-imp* [*OF a-subgroup a-group*,
folded a-r-coset-def a-inv-def, *simplified monoid-record-simps*]) (*rule xcarr*, *rule* $x'cos$)

lemma (**in** *abelian-subgroup*) *a-rcos-module-rev*:
assumes $x \in$ *carrier G* $x' \in$ *carrier G*
and $(x' \oplus \ominus x) \in H$
shows $x' \in H \mathrel{+>} x$
using *assms*
by (*rule subgroup.rcos-module-rev* [*OF a-subgroup a-group*,
folded a-r-coset-def a-inv-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-subgroup*) *a-rcos-module*:
assumes $x \in$ *carrier G* $x' \in$ *carrier G*
shows $(x' \in H \mathrel{+>} x) = (x' \oplus \ominus x \in H)$
using *assms*
by (*rule subgroup.rcos-module* [*OF a-subgroup a-group*,
folded a-r-coset-def a-inv-def, *simplified monoid-record-simps*])

— variant
lemma (**in** *abelian-subgroup*) *a-rcos-module-minus*:
includes *ring G*
assumes *carr*: $x \in$ *carrier G* $x' \in$ *carrier G*
shows $(x' \in H \mathrel{+>} x) = (x' \ominus x \in H)$
proof −
from *carr*
have $(x' \in H \mathrel{+>} x) = (x' \oplus \ominus x \in H)$ **by** (*rule a-rcos-module*)
with *carr*
show $(x' \in H \mathrel{+>} x) = (x' \ominus x \in H)$
by (*simp add*: *minus-eq*)
qed

lemma (**in** *abelian-subgroup*) *a-repr-independence'*:
assumes *y*: $y \in H \mathrel{+>} x$
and *xcarr*: $x \in$ *carrier G*
shows $H \mathrel{+>} x = H \mathrel{+>} y$
apply (*rule a-repr-independence*)
apply (*rule y*)
apply (*rule xcarr*)
apply (*rule a-subgroup*)
done

lemma (**in** *abelian-subgroup*) *a-repr-independenceD*:
assumes *ycarr*: $y \in$ *carrier G*
and *repr*: $H \mathrel{+>} x = H \mathrel{+>} y$
shows $y \in H \mathrel{+>} x$
by (*rule group.repr-independenceD* [*OF a-group a-subgroup*,
folded a-r-coset-def, *simplified monoid-record-simps*]) (*rule ycarr*, *rule repr*)

13.3 Lemmas for the Set of Right Cosets

lemma (**in** *abelian-subgroup*) *a-rcosets-carrier*:

X ∈ *a-rcosets H* ⟹ *X* ⊆ *carrier G*
by (*rule subgroup.rcosets-carrier* [*OF a-subgroup a-group*,
folded A-RCOSETS-def, *simplified monoid-record-simps*])

13.4 Addition of Subgroups

lemma (**in** *abelian-monoid*) *set-add-closed*:
assumes *Acarr*: *A* ⊆ *carrier G*
and *Bcarr*: *B* ⊆ *carrier G*
shows *A* <+> *B* ⊆ *carrier G*
by (*rule monoid.set-mult-closed* [*OF a-monoid*,
folded set-add-def, *simplified monoid-record-simps*]) (*rule Acarr*, *rule Bcarr*)

lemma (**in** *abelian-group*) *add-additive-subgroups*:
assumes *subH*: *additive-subgroup H G*
and *subK*: *additive-subgroup K G*
shows *additive-subgroup* (*H* <+> *K*) *G*
apply (*rule additive-subgroup.intro*)
apply (*unfold set-add-def*)
apply (*intro comm-group.mult-subgroups*)
apply (*rule a-comm-group*)
apply (*rule additive-subgroup.a-subgroup*[*OF subH*])
apply (*rule additive-subgroup.a-subgroup*[*OF subK*])
done

end

theory *Ideal*
imports *Ring AbelCoset*
begin

14 Ideals

14.1 General definition

locale *ideal* = *additive-subgroup I R* + *ring R* +
assumes *I-l-closed*: ⟦*a* ∈ *I*; *x* ∈ *carrier R*⟧ ⟹ *x* ⊗ *a* ∈ *I*
and *I-r-closed*: ⟦*a* ∈ *I*; *x* ∈ *carrier R*⟧ ⟹ *a* ⊗ *x* ∈ *I*

interpretation *ideal* ⊆ *abelian-subgroup I R*
apply (*intro abelian-subgroupI3 abelian-group.intro*)
apply (*rule ideal.axioms*, *rule ideal-axioms*)
apply (*rule abelian-group.axioms*, *rule ring.axioms*, *rule ideal.axioms*, *rule ideal-axioms*)
apply (*rule abelian-group.axioms*, *rule ring.axioms*, *rule ideal.axioms*, *rule ideal-axioms*)
done

lemma (**in** *ideal*) *is-ideal*:

ideal I R
by *fact*

lemma *idealI*:
includes *ring*
assumes *a-subgroup*: *subgroup I* $(\!|$*carrier* = *carrier R*, *mult* = *add R*, *one* = *zero R*$|\!)$
and *I-l-closed*: $\bigwedge a\ x.\ [\![a \in I;\ x \in carrier\ R]\!] \Longrightarrow x \otimes a \in I$
and *I-r-closed*: $\bigwedge a\ x.\ [\![a \in I;\ x \in carrier\ R]\!] \Longrightarrow a \otimes x \in I$
shows *ideal I R*
apply (*intro ideal.intro ideal-axioms.intro additive-subgroupI*)
apply (*rule a-subgroup*)
apply (*rule is-ring*)
apply (*erule* (*1*) *I-l-closed*)
apply (*erule* (*1*) *I-r-closed*)
done

14.2 Ideals Generated by a Subset of *carrier R*

constdefs (**structure** *R*)
genideal :: (*'a*, *'b*) *ring-scheme* ⇒ *'a set* ⇒ *'a set* (*Idl*ı - [*80*] *79*)
genideal R S ≡ *Inter* {*I*. *ideal I R* ∧ *S* ⊆ *I*}

14.3 Principal Ideals

locale *principalideal* = *ideal* +
assumes *generate*: ∃ *i* ∈ *carrier R*. *I* = *Idl* {*i*}

lemma (**in** *principalideal*) *is-principalideal*:
shows *principalideal I R*
by *fact*

lemma *principalidealI*:
includes *ideal*
assumes *generate*: ∃ *i* ∈ *carrier R*. *I* = *Idl* {*i*}
shows *principalideal I R*
by (*intro principalideal.intro principalideal-axioms.intro*) (*rule is-ideal*, *rule generate*)

14.4 Maximal Ideals

locale *maximalideal* = *ideal* +
assumes *I-notcarr*: *carrier R* ≠ *I*
and *I-maximal*: $[\![ideal\ J\ R;\ I \subseteq J;\ J \subseteq carrier\ R]\!] \Longrightarrow J = I \lor J = carrier\ R$

lemma (**in** *maximalideal*) *is-maximalideal*:
shows *maximalideal I R*
by *fact*

lemma *maximalidealI*:
includes *ideal*
assumes *I-notcarr*: *carrier* $R \neq I$
and *I-maximal*: $\bigwedge J$. $[\![$*ideal* J R; $I \subseteq J$; $J \subseteq$ *carrier* $R]\!] \Longrightarrow J = I \vee J =$ *carrier* R
shows *maximalideal* I R
by (*intro maximalideal.intro maximalideal-axioms.intro*)
(*rule is-ideal*, *rule I-notcarr*, *rule I-maximal*)

14.5 Prime Ideals

locale *primeideal* = *ideal* + *cring* +
assumes *I-notcarr*: *carrier* $R \neq I$
and *I-prime*: $[\![a \in$ *carrier* R; $b \in$ *carrier* R; $a \otimes b \in I]\!] \Longrightarrow a \in I \vee b \in I$

lemma (**in** *primeideal*) *is-primeideal*:
shows *primeideal* I R
by *fact*

lemma *primeidealI*:
includes *ideal*
includes *cring*
assumes *I-notcarr*: *carrier* $R \neq I$
and *I-prime*: $\bigwedge a$ b. $[\![a \in$ *carrier* R; $b \in$ *carrier* R; $a \otimes b \in I]\!] \Longrightarrow a \in I \vee b \in I$
shows *primeideal* I R
by (*intro primeideal.intro primeideal-axioms.intro*)
(*rule is-ideal*, *rule is-cring*, *rule I-notcarr*, *rule I-prime*)

lemma *primeidealI2*:
includes *additive-subgroup* I R
includes *cring*
assumes *I-l-closed*: $\bigwedge a$ x. $[\![a \in I$; $x \in$ *carrier* $R]\!] \Longrightarrow x \otimes a \in I$
and *I-r-closed*: $\bigwedge a$ x. $[\![a \in I$; $x \in$ *carrier* $R]\!] \Longrightarrow a \otimes x \in I$
and *I-notcarr*: *carrier* $R \neq I$
and *I-prime*: $\bigwedge a$ b. $[\![a \in$ *carrier* R; $b \in$ *carrier* R; $a \otimes b \in I]\!] \Longrightarrow a \in I \vee b \in I$
shows *primeideal* I R
apply (*intro-locales*)
apply (*intro ideal-axioms.intro*)
apply (*erule* (*1*) *I-l-closed*)
apply (*erule* (*1*) *I-r-closed*)
apply (*intro primeideal-axioms.intro*)
apply (*rule I-notcarr*)
apply (*erule* (*2*) *I-prime*)
done

15 Properties of Ideals

15.1 Special Ideals

```
lemma (in ring) zeroideal:
  shows ideal {0} R
apply (intro idealI subgroup.intro)
      apply (rule is-ring)
     apply simp+
  apply (fold a-inv-def, simp)
 apply simp+
done

lemma (in ring) oneideal:
  shows ideal (carrier R) R
apply (intro idealI  subgroup.intro)
      apply (rule is-ring)
     apply simp+
  apply (fold a-inv-def, simp)
 apply simp+
done

lemma (in domain) zeroprimeideal:
 shows primeideal {0} R
apply (intro primeidealI)
   apply (rule zeroideal)
  apply (rule domain.axioms, rule domain-axioms)
 defer 1
 apply (simp add: integral)
proof (rule ccontr, simp)
  assume carrier R = {0}
  from this have 1 = 0 by (rule one-zeroI)
  from this and one-not-zero
      show False by simp
qed
```

15.2 General Ideal Properies

```
lemma (in ideal) one-imp-carrier:
  assumes I-one-closed: 1 ∈ I
  shows I = carrier R
apply (rule)
apply (rule)
apply (rule a-Hcarr, simp)
proof
  fix x
  assume xcarr: x ∈ carrier R
  from I-one-closed and this
      have x ⊗ 1 ∈ I by (intro I-l-closed)
  from this and xcarr
```

show $x \in I$ **by** *simp*
qed

lemma (**in** *ideal*) *Icarr*:
assumes *iI*: $i \in I$
shows $i \in$ *carrier R*
using *iI* **by** (*rule a-Hcarr*)

15.3 Intersection of Ideals

Intersection of two ideals The intersection of any two ideals is again an ideal in R

lemma (**in** *ring*) *i-intersect*:
includes *ideal I R*
includes *ideal J R*
shows *ideal* $(I \cap J)$ R
apply (*intro idealI subgroup.intro*)
apply (*rule is-ring*)
apply (*force simp add*: *a-subset*)
apply (*simp add*: *a-inv-def* [*symmetric*])
apply *simp*
apply (*simp add*: *a-inv-def* [*symmetric*])
apply (*clarsimp*, *rule*)
apply (*fast intro*: *ideal.I-l-closed ideal.intro prems*)+
apply (*clarsimp*, *rule*)
apply (*fast intro*: *ideal.I-r-closed ideal.intro prems*)+
done

15.3.1 Intersection of a Set of Ideals

The intersection of any Number of Ideals is again an Ideal in R

lemma (**in** *ring*) *i-Intersect*:
assumes *Sideals*: $\bigwedge I.\ I \in S \Longrightarrow$ *ideal I R*
and *notempty*: $S \neq \{\}$
shows *ideal* (*Inter S*) *R*
apply (*unfold-locales*)
apply (*simp-all add*: *Inter-def INTER-def*)
apply (*rule*, *simp*) **defer** *1*
apply *rule* **defer** *1*
apply *rule* **defer** *1*
apply (*fold a-inv-def*, *rule*) **defer** *1*
apply *rule* **defer** *1*
apply *rule* **defer** *1*
proof −
fix x
assume $\forall I \in S.\ x \in I$
hence *xI*: $\bigwedge I.\ I \in S \Longrightarrow x \in I$ **by** *simp*

from *notempty* **have** $\exists\, I0.\ I0 \in S$ **by** *blast*
from *this* **obtain** *I0* **where** *I0S*: $I0 \in S$ **by** *auto*

interpret *ideal* [*I0 R*] **by** (*rule Sideals*[*OF I0S*])

from *xI*[*OF I0S*] **have** $x \in I0$ **.**
from *this* **and** *a-subset* **show** $x \in carrier\ R$ **by** *fast*
next
fix *x y*
assume $\forall\, I{\in}S.\ x \in I$
hence *xI*: $\bigwedge I.\ I \in S \Longrightarrow x \in I$ **by** *simp*
assume $\forall\, I{\in}S.\ y \in I$
hence *yI*: $\bigwedge I.\ I \in S \Longrightarrow y \in I$ **by** *simp*

fix *J*
assume *JS*: $J \in S$
interpret *ideal* [*J R*] **by** (*rule Sideals*[*OF JS*])
from *xI*[*OF JS*] **and** *yI*[*OF JS*]
show $x \oplus y \in J$ **by** (*rule a-closed*)
next
fix *J*
assume *JS*: $J \in S$
interpret *ideal* [*J R*] **by** (*rule Sideals*[*OF JS*])
show $\mathbf{0} \in J$ **by** *simp*
next
fix *x*
assume $\forall\, I{\in}S.\ x \in I$
hence *xI*: $\bigwedge I.\ I \in S \Longrightarrow x \in I$ **by** *simp*

fix *J*
assume *JS*: $J \in S$
interpret *ideal* [*J R*] **by** (*rule Sideals*[*OF JS*])

from *xI*[*OF JS*]
show $\ominus\ x \in J$ **by** (*rule a-inv-closed*)
next
fix *x y*
assume $\forall\, I{\in}S.\ x \in I$
hence *xI*: $\bigwedge I.\ I \in S \Longrightarrow x \in I$ **by** *simp*
assume *ycarr*: $y \in carrier\ R$

fix *J*
assume *JS*: $J \in S$
interpret *ideal* [*J R*] **by** (*rule Sideals*[*OF JS*])

from *xI*[*OF JS*] **and** *ycarr*
show $y \otimes x \in J$ **by** (*rule I-l-closed*)
next
fix *x y*

```
  assume ∀ I∈S. x ∈ I
  hence xI: ⋀I. I ∈ S ⟹ x ∈ I by simp
  assume ycarr: y ∈ carrier R

  fix J
  assume JS: J ∈ S
  interpret ideal [J R] by (rule Sideals[OF JS])

  from xI[OF JS] and ycarr
      show x ⊗ y ∈ J by (rule I-r-closed)
qed
```

15.4 Addition of Ideals

```
lemma (in ring) add-ideals:
  assumes idealI: ideal I R
      and idealJ: ideal J R
  shows ideal (I <+> J) R
apply (rule ideal.intro)
  apply (rule add-additive-subgroups)
   apply (intro ideal.axioms[OF idealI])
  apply (intro ideal.axioms[OF idealJ])
 apply (rule is-ring)
apply (rule ideal-axioms.intro)
 apply (simp add: set-add-defs, clarsimp) defer 1
 apply (simp add: set-add-defs, clarsimp) defer 1
proof −
  fix x i j
  assume xcarr: x ∈ carrier R
     and iI: i ∈ I
     and jJ: j ∈ J
  from xcarr ideal.Icarr[OF idealI iI] ideal.Icarr[OF idealJ jJ]
      have c: (i ⊕ j) ⊗ x = (i ⊗ x) ⊕ (j ⊗ x) by algebra
  from xcarr and iI
      have a: i ⊗ x ∈ I by (simp add: ideal.I-r-closed[OF idealI])
  from xcarr and jJ
      have b: j ⊗ x ∈ J by (simp add: ideal.I-r-closed[OF idealJ])
  from a b c
      show ∃ ha∈I. ∃ ka∈J. (i ⊕ j) ⊗ x = ha ⊕ ka by fast
next
  fix x i j
  assume xcarr: x ∈ carrier R
     and iI: i ∈ I
     and jJ: j ∈ J
  from xcarr ideal.Icarr[OF idealI iI] ideal.Icarr[OF idealJ jJ]
      have c: x ⊗ (i ⊕ j) = (x ⊗ i) ⊕ (x ⊗ j) by algebra
  from xcarr and iI
      have a: x ⊗ i ∈ I by (simp add: ideal.I-l-closed[OF idealI])
  from xcarr and jJ
```

have b: $x \otimes j \in J$ **by** $(simp\ add: ideal.I\text{-}l\text{-}closed[OF\ idealJ])$
from $a\ b\ c$
show $\exists ha{\in}I.\ \exists ka{\in}J.\ x \otimes (i \oplus j) = ha \oplus ka$ **by** *fast*
qed

15.5 Ideals generated by a subset of *carrier R*

15.5.1 Generation of Ideals in General Rings

genideal generates an ideal

lemma (**in** *ring*) *genideal-ideal*:
assumes *Scarr*: $S \subseteq carrier\ R$
shows $ideal\ (Idl\ S)\ R$
unfolding *genideal-def*
proof (*rule i-Intersect*, *fast*, *simp*)
from *oneideal* **and** *Scarr*
show $\exists I.\ ideal\ I\ R \wedge S \leq I$ **by** *fast*
qed

lemma (**in** *ring*) *genideal-self*:
assumes $S \subseteq carrier\ R$
shows $S \subseteq Idl\ S$
unfolding *genideal-def*
by *fast*

lemma (**in** *ring*) *genideal-self'*:
assumes *carr*: $i \in carrier\ R$
shows $i \in Idl\ \{i\}$
proof −
from *carr*
have $\{i\} \subseteq Idl\ \{i\}$ **by** (*fast intro*!: *genideal-self*)
thus $i \in Idl\ \{i\}$ **by** *fast*
qed

genideal generates the minimal ideal

lemma (**in** *ring*) *genideal-minimal*:
assumes a: $ideal\ I\ R$
and b: $S \subseteq I$
shows $Idl\ S \subseteq I$
unfolding *genideal-def*
by (*rule*, *elim InterD*, *simp add*: $a\ b$)

Generated ideals and subsets

lemma (**in** *ring*) *Idl-subset-ideal*:
assumes *Iideal*: $ideal\ I\ R$
and *Hcarr*: $H \subseteq carrier\ R$
shows $(Idl\ H \subseteq I) = (H \subseteq I)$
proof
assume a: $Idl\ H \subseteq I$

from *Hcarr* **have** $H \subseteq$ *Idl* H **by** (*rule genideal-self*)
from *this* **and** *a*
  show $H \subseteq I$ **by** *simp*
next
fix x
assume *HI*: $H \subseteq I$

from *Iideal* **and** *HI*
  have $I \in \{I.$ *ideal* I $R \wedge H \subseteq I\}$ **by** *fast*
from *this*
  show *Idl* $H \subseteq I$
  unfolding *genideal-def*
  by *fast*
qed

lemma (**in** *ring*) *subset-Idl-subset*:
  assumes *Icarr*: $I \subseteq$ *carrier* R
    and *HI*: $H \subseteq I$
  shows *Idl* $H \subseteq$ *Idl* I
proof −
  from *HI* **and** *genideal-self*[*OF Icarr*]
    have *HIdlI*: $H \subseteq$ *Idl* I **by** *fast*

  from *Icarr*
    have *Iideal*: *ideal* (*Idl* I) R **by** (*rule genideal-ideal*)
  from *HI* **and** *Icarr*
    have $H \subseteq$ *carrier* R **by** *fast*
  from *Iideal* **and** *this*
    have ($H \subseteq$ *Idl* I) = (*Idl* $H \subseteq$ *Idl* I)
    by (*rule Idl-subset-ideal*[*symmetric*])

  from *HIdlI* **and** *this*
    show *Idl* $H \subseteq$ *Idl* I **by** *simp*
qed

lemma (**in** *ring*) *Idl-subset-ideal′*:
  assumes *acarr*: $a \in$ *carrier* R **and** *bcarr*: $b \in$ *carrier* R
  shows (*Idl* $\{a\} \subseteq$ *Idl* $\{b\}$) = ($a \in$ *Idl* $\{b\}$)
apply (*subst Idl-subset-ideal*[*OF genideal-ideal*[*of* $\{b\}$], *of* $\{a\}$])
  apply (*fast intro*: *bcarr*, *fast intro*: *acarr*)
apply *fast*
done

lemma (**in** *ring*) *genideal-zero*:
  Idl $\{\mathbf{0}\} = \{\mathbf{0}\}$
apply *rule*
 apply (*rule genideal-minimal*[*OF zeroideal*], *simp*)
apply (*simp add*: *genideal-self′*)
done

lemma (**in** *ring*) *genideal-one*:
  Idl $\{\mathbf{1}\}$ = *carrier R*
proof −
  interpret *ideal* [*Idl* $\{\mathbf{1}\}$ *R*] **by** (*rule genideal-ideal*, *fast intro*: *one-closed*)
  show *Idl* $\{\mathbf{1}\}$ = *carrier R*
  apply (*rule*, *rule a-subset*)
  apply (*simp add*: *one-imp-carrier genideal-self′*)
  done
qed

15.5.2 Generation of Principal Ideals in Commutative Rings

constdefs (**structure** *R*)
  cgenideal :: ($'a$, $'b$) *monoid-scheme* $\Rightarrow$ $'a$ $\Rightarrow$ $'a$ *set* (*PIdl*ı - [*80*] *79*)
  cgenideal R a $\equiv \{ x \otimes a \mid x.\ x \in$ *carrier R* $\}$

genhideal (?) really generates an ideal

lemma (**in** *cring*) *cgenideal-ideal*:
  assumes *acarr*: $a \in$ *carrier R*
  shows *ideal* (*PIdl a*) *R*
apply (*unfold cgenideal-def*)
apply (*rule idealI*[*OF is-ring*])
   apply (*rule subgroup.intro*)
      apply (*simp-all add*: *monoid-record-simps*)
      apply (*blast intro*: *acarr m-closed*)
      apply *clarsimp* **defer** *1*
      defer *1*
      apply (*fold a-inv-def*, *clarsimp*) **defer** *1*
      apply *clarsimp* **defer** *1*
      apply *clarsimp* **defer** *1*
proof −
  fix *x y*
  assume *xcarr*: $x \in$ *carrier R*
    and *ycarr*: $y \in$ *carrier R*
  note *carr* = *acarr xcarr ycarr*

  from *carr*
      have $x \otimes a \oplus y \otimes a = (x \oplus y) \otimes a$ **by** (*simp add*: *l-distr*)
  from *this* **and** *carr*
      show $\exists z.\ x \otimes a \oplus y \otimes a = z \otimes a \land z \in$ *carrier R* **by** *fast*
next
  from *l-null*[*OF acarr*, *symmetric*] **and** *zero-closed*
      show $\exists x.\ \mathbf{0} = x \otimes a \land x \in$ *carrier R* **by** *fast*
next
  fix *x*
  assume *xcarr*: $x \in$ *carrier R*
  note *carr* = *acarr xcarr*

from *carr*
have $\ominus$ $(x \otimes a) = (\ominus\ x) \otimes a$ **by** (*simp add*: *l-minus*)
from *this* **and** *carr*
show $\exists z.\ \ominus\ (x \otimes a) = z \otimes a \wedge z \in$ *carrier R* **by** *fast*
next
fix *x y*
assume *xcarr*: $x \in$ *carrier R*
and *ycarr*: $y \in$ *carrier R*
note *carr* = *acarr xcarr ycarr*

from *carr*
have $y \otimes a \otimes x = (y \otimes x) \otimes a$ **by** (*simp add*: *m-assoc*, *simp add*: *m-comm*)
from *this* **and** *carr*
show $\exists z.\ y \otimes a \otimes x = z \otimes a \wedge z \in$ *carrier R* **by** *fast*
next
fix *x y*
assume *xcarr*: $x \in$ *carrier R*
and *ycarr*: $y \in$ *carrier R*
note *carr* = *acarr xcarr ycarr*

from *carr*
have $x \otimes (y \otimes a) = (x \otimes y) \otimes a$ **by** (*simp add*: *m-assoc*)
from *this* **and** *carr*
show $\exists z.\ x \otimes (y \otimes a) = z \otimes a \wedge z \in$ *carrier R* **by** *fast*
qed

lemma (**in** *ring*) *cgenideal-self*:
assumes *icarr*: $i \in$ *carrier R*
shows $i \in$ *PIdl i*
unfolding *cgenideal-def*
proof *simp*
from *icarr*
have $i = \mathbf{1} \otimes i$ **by** *simp*
from *this* **and** *icarr*
show $\exists x.\ i = x \otimes i \wedge x \in$ *carrier R* **by** *fast*
qed

cgenideal is minimal

lemma (**in** *ring*) *cgenideal-minimal*:
includes *ideal J R*
assumes *aJ*: $a \in J$
shows *PIdl a* $\subseteq J$
unfolding *cgenideal-def*
apply *rule*
apply *clarify*
using *aJ*
apply (*erule I-l-closed*)
done

```
lemma (in cring) cgenideal-eq-genideal:
  assumes icarr: i ∈ carrier R
  shows PIdl i = Idl {i}
apply rule
 apply (intro cgenideal-minimal)
  apply (rule genideal-ideal, fast intro: icarr)
 apply (rule genideal-self', fast intro: icarr)
apply (intro genideal-minimal)
 apply (rule cgenideal-ideal [OF icarr])
apply (simp, rule cgenideal-self [OF icarr])
done

lemma (in cring) cgenideal-eq-rcos:
 PIdl i = carrier R #> i
unfolding cgenideal-def r-coset-def
by fast

lemma (in cring) cgenideal-is-principalideal:
  assumes icarr: i ∈ carrier R
  shows principalideal (PIdl i) R
apply (rule principalidealI)
apply (rule cgenideal-ideal [OF icarr])
proof −
  from icarr
      have PIdl i = Idl {i} by (rule cgenideal-eq-genideal)
  from icarr and this
      show ∃ i'∈carrier R. PIdl i = Idl {i'} by fast
qed
```

15.6 Union of Ideals

```
lemma (in ring) union-genideal:
  assumes idealI: ideal I R
      and idealJ: ideal J R
  shows Idl (I ∪ J) = I <+> J
apply rule
 apply (rule ring.genideal-minimal)
   apply (rule R.is-ring)
  apply (rule add-ideals[OF idealI idealJ])
 apply (rule)
 apply (simp add: set-add-defs) apply (elim disjE) defer 1 defer 1
 apply (rule) apply (simp add: set-add-defs genideal-def) apply clarsimp defer
1
proof −
  fix x
  assume xI: x ∈ I
  have ZJ: 0 ∈ J
      by (intro additive-subgroup.zero-closed, rule ideal.axioms[OF idealJ])
  from ideal.Icarr[OF idealI xI]
```

have $x = x \oplus \mathbf{0}$ **by** *algebra*
from *xI* **and** *ZJ* **and** *this*
show $\exists h{\in}I.\ \exists k{\in}J.\ x = h \oplus k$ **by** *fast*
next
fix x
assume *xJ*: $x \in J$
have *ZI*: $\mathbf{0} \in I$
by (*intro additive-subgroup.zero-closed*, *rule ideal.axioms*[*OF idealI*])
from *ideal.Icarr*[*OF idealJ xJ*]
have $x = \mathbf{0} \oplus x$ **by** *algebra*
from *ZI* **and** *xJ* **and** *this*
show $\exists h{\in}I.\ \exists k{\in}J.\ x = h \oplus k$ **by** *fast*
next
fix $i\ j\ K$
assume *iI*: $i \in I$
and *jJ*: $j \in J$
and *idealK*: *ideal K R*
and *IK*: $I \subseteq K$
and *JK*: $J \subseteq K$
from *iI* **and** *IK*
have *iK*: $i \in K$ **by** *fast*
from *jJ* **and** *JK*
have *jK*: $j \in K$ **by** *fast*
from *iK* **and** *jK*
show $i \oplus j \in K$ **by** (*intro additive-subgroup.a-closed*) (*rule ideal.axioms*[*OF idealK*])
qed

15.7 Properties of Principal Ideals

0 generates the zero ideal

lemma (**in** *ring*) *zero-genideal*:
shows *Idl* $\{\mathbf{0}\} = \{\mathbf{0}\}$
apply *rule*
apply (*simp add*: *genideal-minimal zeroideal*)
apply (*fast intro*!: *genideal-self*)
done

1 generates the unit ideal

lemma (**in** *ring*) *one-genideal*:
shows *Idl* $\{\mathbf{1}\}$ = *carrier R*
proof −
have $\mathbf{1} \in$ *Idl* $\{\mathbf{1}\}$ **by** (*simp add*: *genideal-self'*)
thus *Idl* $\{\mathbf{1}\}$ = *carrier R* **by** (*intro ideal.one-imp-carrier*, *fast intro*: *genideal-ideal*)
qed

The zero ideal is a principal ideal

corollary (**in** *ring*) *zeropideal*:

shows *principalideal* $\{\mathbf{0}\}$ *R*
apply (*rule principalidealI*)
apply (*rule zeroideal*)
apply (*blast intro*!: *zero-closed zero-genideal*[*symmetric*])
done

The unit ideal is a principal ideal

corollary (**in** *ring*) *onepideal*:
shows *principalideal* (*carrier R*) *R*
apply (*rule principalidealI*)
apply (*rule oneideal*)
apply (*blast intro*!: *one-closed one-genideal*[*symmetric*])
done

Every principal ideal is a right coset of the carrier

lemma (**in** *principalideal*) *rcos-generate*:
includes *cring*
shows $\exists x \in I.\ I = carrier\ R\ \#{>}\ x$
proof −
from *generate*
obtain *i*
where *icarr*: $i \in carrier\ R$
and *I1*: $I = Idl\ \{i\}$
by *fast+*

from *icarr* **and** *genideal-self*[*of* $\{i\}$]
have $i \in Idl\ \{i\}$ **by** *fast*
hence *iI*: $i \in I$ **by** (*simp add*: *I1*)

from *I1 icarr*
have *I2*: $I = PIdl\ i$ **by** (*simp add*: *cgenideal-eq-genideal*)

have $PIdl\ i = carrier\ R\ \#{>}\ i$
unfolding *cgenideal-def r-coset-def*
by *fast*

from *I2* **and** *this*
have $I = carrier\ R\ \#{>}\ i$ **by** *simp*

from *iI* **and** *this*
show $\exists x \in I.\ I = carrier\ R\ \#{>}\ x$ **by** *fast*
qed

15.8 Prime Ideals

lemma (**in** *ideal*) *primeidealCD*:
includes *cring*
assumes *notprime*: $\neg$ *primeideal I R*

shows *carrier R* $= I \lor (\exists\, a\ b.\ a \in$ *carrier R* $\land\ b \in$ *carrier R* $\land\ a \otimes b \in I \land a \notin I \land b \notin I)$
proof (*rule ccontr*, *clarsimp*)
assume *InR*: *carrier R* $\neq I$
and $\forall\, a.\ a \in$ *carrier R* $\longrightarrow (\forall\, b.\ a \otimes b \in I \longrightarrow b \in$ *carrier R* $\longrightarrow a \in I \lor b \in I)$
hence *I-prime*: $\bigwedge a\ b.\ [\![a \in$ *carrier R*; $b \in$ *carrier R*; $a \otimes b \in I]\!] \Longrightarrow a \in I \lor b \in I$ **by** *simp*
have *primeideal I R*
apply (*rule primeideal.intro* [*OF is-ideal is-cring*])
apply (*rule primeideal-axioms.intro*)
apply (*rule InR*)
apply (*erule* (*2*) *I-prime*)
done
from *this* **and** *notprime*
show *False* **by** *simp*
qed

lemma (**in** *ideal*) *primeidealCE*:
includes *cring*
assumes *notprime*: $\neg$ *primeideal I R*
obtains *carrier R* $= I$
| $\exists\, a\ b.\ a \in$ *carrier R* $\land\ b \in$ *carrier R* $\land\ a \otimes b \in I \land a \notin I \land b \notin I$
using *primeidealCD* [*OF R.is-cring notprime*] **by** *blast*

If $\{\mathbf{0}\}$ is a prime ideal of a commutative ring, the ring is a domain

lemma (**in** *cring*) *zeroprimeideal-domainI*:
assumes *pi*: *primeideal* $\{\mathbf{0}\}$ *R*
shows *domain R*
apply (*rule domain.intro*, *rule is-cring*)
apply (*rule domain-axioms.intro*)
proof (*rule ccontr*, *simp*)
interpret *primeideal* [$\{\mathbf{0}\}$ *R*] **by** (*rule pi*)
assume $\mathbf{1} = \mathbf{0}$
hence *carrier R* $= \{\mathbf{0}\}$ **by** (*rule one-zeroD*)
from *this*[*symmetric*] **and** *I-notcarr*
show *False* **by** *simp*
next
interpret *primeideal* [$\{\mathbf{0}\}$ *R*] **by** (*rule pi*)
fix *a b*
assume *ab*: $a \otimes b = \mathbf{0}$
and *carr*: $a \in$ *carrier R* $b \in$ *carrier R*
from *ab*
have *abI*: $a \otimes b \in \{\mathbf{0}\}$ **by** *fast*
from *carr* **and** *this*
have $a \in \{\mathbf{0}\} \lor b \in \{\mathbf{0}\}$ **by** (*rule I-prime*)
thus $a = \mathbf{0} \lor b = \mathbf{0}$ **by** *simp*
qed

corollary (**in** *cring*) *domain-eq-zeroprimeideal*:
shows *domain R* = *primeideal* {**0**} *R*
apply *rule*
apply (*erule domain.zeroprimeideal*)
apply (*erule zeroprimeideal-domainI*)
done

15.9 Maximal Ideals

lemma (**in** *ideal*) *helper-I-closed*:
assumes *carr*: $a \in carrier\ R$ $x \in carrier\ R$ $y \in carrier\ R$
and *axI*: $a \otimes x \in I$
shows $a \otimes (x \otimes y) \in I$
proof −
from *axI* **and** *carr*
have $(a \otimes x) \otimes y \in I$ **by** (*simp add*: *I-r-closed*)
also from *carr*
have $(a \otimes x) \otimes y = a \otimes (x \otimes y)$ **by** (*simp add*: *m-assoc*)
finally
show $a \otimes (x \otimes y) \in I$ **.**
qed

lemma (**in** *ideal*) *helper-max-prime*:
includes *cring*
assumes *acarr*: $a \in carrier\ R$
shows *ideal* $\{x \in carrier\ R.\ a \otimes x \in I\}$ *R*
apply (*rule idealI*)
apply (*rule cring.axioms*[*OF is-cring*])
apply (*rule subgroup.intro*)
apply (*simp*, *fast*)
apply *clarsimp* **apply** (*simp add*: *r-distr acarr*)
apply (*simp add*: *acarr*)
apply (*simp add*: *a-inv-def*[*symmetric*], *clarify*) **defer** *1*
apply *clarsimp* **defer** *1*
apply (*fast intro*!: *helper-I-closed acarr*)
proof −
fix *x*
assume *xcarr*: $x \in carrier\ R$
and *ax*: $a \otimes x \in I$
from *ax* **and** *acarr xcarr*
have $\ominus(a \otimes x) \in I$ **by** *simp*
also from *acarr xcarr*
have $\ominus(a \otimes x) = a \otimes (\ominus x)$ **by** *algebra*
finally
show $a \otimes (\ominus x) \in I$ **.**
from *acarr*
have $a \otimes \mathbf{0} = \mathbf{0}$ **by** *simp*
next
fix *x y*

```
  assume xcarr: x ∈ carrier R
     and ycarr: y ∈ carrier R
     and ayI: a ⊗ y ∈ I
  from ayI and acarr xcarr ycarr
      have a ⊗ (y ⊗ x) ∈ I by (simp add: helper-I-closed)
  moreover from xcarr ycarr
      have y ⊗ x = x ⊗ y by (simp add: m-comm)
  ultimately
      show a ⊗ (x ⊗ y) ∈ I by simp
qed
```

In a cring every maximal ideal is prime

```
lemma (in cring) maximalideal-is-prime:
  includes maximalideal
  shows primeideal I R
apply (rule ccontr)
apply (rule primeidealCE)
   apply (rule is-cring)
  apply assumption
 apply (simp add: I-notcarr)
proof −
  assume ∃ a b. a ∈ carrier R ∧ b ∈ carrier R ∧ a ⊗ b ∈ I ∧ a ∉ I ∧ b ∉ I
  from this
      obtain a b
        where acarr: a ∈ carrier R
        and bcarr: b ∈ carrier R
        and abI: a ⊗ b ∈ I
        and anI: a ∉ I
        and bnI: b ∉ I
      by fast
  def J ≡ {x∈carrier R. a ⊗ x ∈ I}

  from R.is-cring and acarr
  have idealJ: ideal J R unfolding J-def by (rule helper-max-prime)

  have IsubJ: I ⊆ J
  proof
    fix x
    assume xI: x ∈ I
    from this and acarr
    have a ⊗ x ∈ I by (intro I-l-closed)
    from xI[THEN a-Hcarr] this
    show x ∈ J unfolding J-def by fast
  qed

  from abI and acarr bcarr
      have b ∈ J unfolding J-def by fast
  from bnI and this
      have JnI: J ≠ I by fast
```

from *acarr*
  have $a = a \otimes \mathbf{1}$ **by** *algebra*
from *this* **and** *anI*
  have $a \otimes \mathbf{1} \notin I$ **by** *simp*
from *one-closed* **and** *this*
  have $\mathbf{1} \notin J$ **unfolding** *J-def* **by** *fast*
hence *Jncarr*: $J \neq$ *carrier R* **by** *fast*

interpret *ideal* [*J R*] **by** (*rule idealJ*)

have $J = I \lor J =$ *carrier R*
  apply (*intro I-maximal*)
  apply (*rule idealJ*)
  apply (*rule IsubJ*)
  apply (*rule a-subset*)
  done

from *this* **and** *JnI* **and** *Jncarr*
  show *False* **by** *simp*
qed

15.10 Derived Theorems Involving Ideals

— A non-zero cring that has only the two trivial ideals is a field
lemma (**in** *cring*) *trivialideals-fieldI*:
  assumes *carrnzero*: *carrier R* $\neq \{\mathbf{0}\}$
    and *haveideals*: $\{I.$ *ideal I R*$\} = \{\{\mathbf{0}\}$, *carrier R*$\}$
  shows *field R*
apply (*rule cring-fieldI*)
apply (*rule*, *rule*, *rule*)
apply (*erule Units-closed*)
defer *1*
  apply *rule*
defer *1*
proof (*rule ccontr*, *simp*)
  assume *zUnit*: $\mathbf{0} \in$ *Units R*
  hence *a*: $\mathbf{0} \otimes$ *inv* $\mathbf{0} = \mathbf{1}$ **by** (*rule Units-r-inv*)
  from *zUnit*
    have $\mathbf{0} \otimes$ *inv* $\mathbf{0} = \mathbf{0}$ **by** (*intro l-null*, *rule Units-inv-closed*)
  from *a*[*symmetric*] **and** *this*
    have $\mathbf{1} = \mathbf{0}$ **by** *simp*
  hence *carrier R* $= \{\mathbf{0}\}$ **by** (*rule one-zeroD*)
  from *this* **and** *carrnzero*
    show *False* **by** *simp*
next
  fix *x*
  assume *xcarr'*: $x \in$ *carrier R* $- \{\mathbf{0}\}$
  hence *xcarr*: $x \in$ *carrier R* **by** *fast*
  from *xcarr'*

have *xnZ*: $x \neq \mathbf{0}$ **by** *fast*
from *xcarr*
have *xIdl*: *ideal* (*PIdl* *x*) *R* **by** (*intro cgenideal-ideal*, *fast*)

from *xcarr*
have $x \in$ *PIdl* *x* **by** (*intro cgenideal-self*, *fast*)
from *this* **and** *xnZ*
have *PIdl* $x \neq \{\mathbf{0}\}$ **by** *fast*
from *haveideals* **and** *this*
have *PIdl* $x =$ *carrier R*
by (*blast intro*!: *xIdl*)
hence $\mathbf{1} \in$ *PIdl* *x* **by** *simp*
hence $\exists\, y.\ \mathbf{1} = y \otimes x \wedge y \in$ *carrier R* **unfolding** *cgenideal-def* **by** *blast*
from *this*
obtain *y*
where *ycarr*: $y \in$ *carrier R*
and *ylinv*: $\mathbf{1} = y \otimes x$
by *fast*+
from *ylinv* **and** *xcarr ycarr*
have *yrinv*: $\mathbf{1} = x \otimes y$ **by** (*simp add*: *m-comm*)
from *ycarr* **and** *ylinv*[*symmetric*] **and** *yrinv*[*symmetric*]
have $\exists\, y \in$ *carrier R*. $y \otimes x = \mathbf{1} \wedge x \otimes y = \mathbf{1}$ **by** *fast*
from *this* **and** *xcarr*
show $x \in$ *Units R*
unfolding *Units-def*
by *fast*
qed

lemma (**in** *field*) *all-ideals*:
shows $\{I.$ *ideal I R*$\} = \{\{\mathbf{0}\},$ *carrier R*$\}$
apply (*rule*, *rule*)
proof −
fix *I*
assume *a*: $I \in \{I.$ *ideal I R*$\}$
with *this*
interpret *ideal* [*I R*] **by** *simp*

show $I \in \{\{\mathbf{0}\},$ *carrier R*$\}$
proof (*cases* $\exists\, a.\ a \in I - \{\mathbf{0}\}$)
assume $\exists\, a.\ a \in I - \{\mathbf{0}\}$
from *this*
obtain *a*
where *aI*: $a \in I$
and *anZ*: $a \neq \mathbf{0}$
by *fast*+
from *aI*[*THEN a-Hcarr*] *anZ*
have *aUnit*: $a \in$ *Units R* **by** (*simp add*: *field-Units*)
hence *a*: $a \otimes$ *inv* $a = \mathbf{1}$ **by** (*rule Units-r-inv*)
from *aI* **and** *aUnit*

```
      have a ⊗ inv a ∈ I by (simp add: I-r-closed)
    hence oneI: 1 ∈ I by (simp add: a[symmetric])

    have carrier R ⊆ I
    proof
      fix x
      assume xcarr: x ∈ carrier R
      from oneI and this
          have 1 ⊗ x ∈ I by (rule I-r-closed)
      from this and xcarr
          show x ∈ I by simp
    qed
    from this and a-subset
        have I = carrier R by fast
    thus I ∈ {{0}, carrier R} by fast
  next
    assume ¬ (∃ a. a ∈ I − {0})
    hence IZ: ⋀a. a ∈ I ⟹ a = 0 by simp

    have a: I ⊆ {0}
    proof
      fix x
      assume x ∈ I
      hence x = 0 by (rule IZ)
      thus x ∈ {0} by fast
    qed

    have 0 ∈ I by simp
    hence {0} ⊆ I by fast

    from this and a
        have I = {0} by fast
    thus I ∈ {{0}, carrier R} by fast
  qed
qed (simp add: zeroideal oneideal)

— Jacobson Theorem 2.2
lemma (in cring) trivialideals-eq-field:
  assumes carrnzero: carrier R ≠ {0}
  shows ({I. ideal I R} = {{0}, carrier R}) = field R
by (fast intro!: trivialideals-fieldI[OF carrnzero] field.all-ideals)
```

Like zeroprimeideal for domains

```
lemma (in field) zeromaximalideal:
  maximalideal {0} R
apply (rule maximalidealI)
  apply (rule zeroideal)
proof−
  from one-not-zero
```

have $\mathbf{1} \notin \{\mathbf{0}\}$ **by** *simp*
from *this* **and** *one-closed*
show *carrier* $R \neq \{\mathbf{0}\}$ **by** *fast*
next
fix J
assume *Jideal*: *ideal* J R
hence $J \in \{I.\ ideal\ I\ R\}$
by *fast*

from *this* **and** *all-ideals*
show $J = \{\mathbf{0}\} \vee J = carrier\ R$ **by** *simp*
qed

lemma (**in** *cring*) *zeromaximalideal-fieldI*:
assumes *zeromax*: *maximalideal* $\{\mathbf{0}\}$ R
shows *field* R
apply (*rule trivialideals-fieldI*, *rule maximalideal.I-notcarr*[*OF zeromax*])
apply *rule* **apply** *clarsimp* **defer** *1*
apply (*simp add*: *zeroideal oneideal*)
proof −
fix J
assume *Jn0*: $J \neq \{\mathbf{0}\}$
and *idealJ*: *ideal* J R
interpret *ideal* [J R] **by** (*rule idealJ*)
have $\{\mathbf{0}\} \subseteq J$ **by** (*rule ccontr*, *simp*)
from *zeromax* **and** *idealJ* **and** *this* **and** *a-subset*
have $J = \{\mathbf{0}\} \vee J = carrier\ R$ **by** (*rule maximalideal.I-maximal*)
from *this* **and** *Jn0*
show $J = carrier\ R$ **by** *simp*
qed

lemma (**in** *cring*) *zeromaximalideal-eq-field*:
maximalideal $\{\mathbf{0}\}$ R = *field* R
apply *rule*
apply (*erule zeromaximalideal-fieldI*)
apply (*erule field.zeromaximalideal*)
done

end

theory *RingHom*
imports *Ideal*
begin

16 Homomorphisms of Non-Commutative Rings

Lifting existing lemmas in a *ring-hom-ring* locale

locale *ring-hom-ring* = *ring R* + *ring S* + *var h* +
  assumes *homh*: $h \in$ *ring-hom R S*
  notes *hom-mult* [*simp*] = *ring-hom-mult* [*OF homh*]
    and *hom-one* [*simp*] = *ring-hom-one* [*OF homh*]

interpretation *ring-hom-cring* $\subseteq$ *ring-hom-ring*
  by (*unfold-locales*, *rule homh*)

interpretation *ring-hom-ring* $\subseteq$ *abelian-group-hom R S*
apply (*rule abelian-group-homI*)
  apply (*rule R.is-abelian-group*)
 apply (*rule S.is-abelian-group*)
apply (*intro group-hom.intro group-hom-axioms.intro*)
  apply (*rule R.a-group*)
 apply (*rule S.a-group*)
apply (*insert homh*, *unfold hom-def ring-hom-def*)
apply *simp*
done

lemma (**in** *ring-hom-ring*) *is-ring-hom-ring*:
  includes *struct R* + *struct S*
  shows *ring-hom-ring R S h*
by *fact*

lemma *ring-hom-ringI*:
  includes *ring R* + *ring S*
  assumes
      hom-closed: !!x. $x \in$ *carrier R* ==> $h\ x \in$ *carrier S*
    and *compatible-mult*: !!$x\ y$. [| x : *carrier R*; y : *carrier R* |] ==> $h\ (x \otimes y)$ = $h\ x \otimes_S h\ y$
    and *compatible-add*: !!$x\ y$. [| x : *carrier R*; y : *carrier R* |] ==> $h\ (x \oplus y)$ = $h\ x \oplus_S h\ y$
    and *compatible-one*: $h\ \mathbf{1} = \mathbf{1}_S$
  shows *ring-hom-ring R S h*
apply *unfold-locales*
apply (*unfold ring-hom-def*, *safe*)
   apply (*simp add*: *hom-closed Pi-def*)
  apply (*erule* (*1*) *compatible-mult*)
 apply (*erule* (*1*) *compatible-add*)
apply (*rule compatible-one*)
done

lemma *ring-hom-ringI2*:
  includes *ring R* + *ring S*
  assumes *h*: $h \in$ *ring-hom R S*
  shows *ring-hom-ring R S h*

apply (*intro ring-hom-ring.intro ring-hom-ring-axioms.intro*)
apply (*rule R.is-ring*)
apply (*rule S.is-ring*)
apply (*rule h*)
done

lemma *ring-hom-ringI3*:
includes *abelian-group-hom R S + ring R + ring S*
assumes *compatible-mult*: !!*x y*. [| *x* : *carrier R*; *y* : *carrier R* |] ==> *h* ($x \otimes y$) = *h* $x \otimes_S h\ y$
and *compatible-one*: $h\ \mathbf{1} = \mathbf{1}_S$
shows *ring-hom-ring R S h*
apply (*intro ring-hom-ring.intro ring-hom-ring-axioms.intro*, *rule R.is-ring*, *rule S.is-ring*)
apply (*insert group-hom.homh*[*OF a-group-hom*])
apply (*unfold hom-def ring-hom-def*, *simp*)
apply *safe*
apply (*erule* (*1*) *compatible-mult*)
apply (*rule compatible-one*)
done

lemma *ring-hom-cringI*:
includes *ring-hom-ring R S h + cring R + cring S*
shows *ring-hom-cring R S h*
by (*intro ring-hom-cring.intro ring-hom-cring-axioms.intro*)
(*rule R.is-cring*, *rule S.is-cring*, *rule homh*)

16.1 The kernel of a ring homomorphism

— the kernel of a ring homomorphism is an ideal
lemma (**in** *ring-hom-ring*) *kernel-is-ideal*:
shows *ideal* (*a-kernel R S h*) *R*
apply (*rule idealI*)
apply (*rule R.is-ring*)
apply (*rule additive-subgroup.a-subgroup*[*OF additive-subgroup-a-kernel*])
apply (*unfold a-kernel-def′*, *simp+*)
done

Elements of the kernel are mapped to zero

lemma (**in** *abelian-group-hom*) *kernel-zero* [*simp*]:
$i \in$ *a-kernel R S h* $\Longrightarrow h\ i = \mathbf{0}_S$
by (*simp add*: *a-kernel-defs*)

16.2 Cosets

Cosets of the kernel correspond to the elements of the image of the homomorphism

lemma (**in** *ring-hom-ring*) *rcos-imp-homeq*:
assumes *acarr*: $a \in$ *carrier R*

    and *xrcos*: $x \in$ *a-kernel R S h* +> *a*
  shows $h\ x = h\ a$
proof −
  interpret *ideal* [*a-kernel R S h R*] **by** (*rule kernel-is-ideal*)

  from *xrcos*
    have $\exists\, i \in$ *a-kernel R S h*. $x = i \oplus a$ **by** (*simp add*: *a-r-coset-defs*)
  from *this* **obtain** *i*
    where *iker*: $i \in$ *a-kernel R S h*
      and *x*: $x = i \oplus a$
    by *fast*+
  note *carr* = *acarr iker*[*THEN a-Hcarr*]

  from *x*
    have $h\ x = h\ (i \oplus a)$ **by** *simp*
  also from *carr*
    have $\ldots = h\ i \oplus_S h\ a$ **by** *simp*
  also from *iker*
    have $\ldots = \mathbf{0}_S \oplus_S h\ a$ **by** *simp*
  also from *carr*
    have $\ldots = h\ a$ **by** *simp*
  finally
    show $h\ x = h\ a$ **.**
qed

lemma (**in** *ring-hom-ring*) *homeq-imp-rcos*:
  assumes *acarr*: $a \in$ *carrier R*
    and *xcarr*: $x \in$ *carrier R*
    and *hx*: $h\ x = h\ a$
  shows $x \in$ *a-kernel R S h* +> *a*
proof −
  interpret *ideal* [*a-kernel R S h R*] **by** (*rule kernel-is-ideal*)

  note *carr* = *acarr xcarr*
  note *hcarr* = *acarr*[*THEN hom-closed*] *xcarr*[*THEN hom-closed*]

  from *hx* **and** *hcarr*
    have *a*: $h\ x \oplus_S \ominus_S h\ a = \mathbf{0}_S$ **by** *algebra*
  from *carr*
    have $h\ x \oplus_S \ominus_S h\ a = h\ (x \oplus \ominus a)$ **by** *simp*
  from *a* **and** *this*
    have *b*: $h\ (x \oplus \ominus a) = \mathbf{0}_S$ **by** *simp*

  from *carr* **have** $x \oplus \ominus a \in$ *carrier R* **by** *simp*
  from *this* **and** *b*
    have $x \oplus \ominus a \in$ *a-kernel R S h*
    unfolding *a-kernel-def′*
    by *fast*

from *this* **and** *carr*
show $x \in$ *a-kernel R S h* $+>$ *a* **by** (*simp add*: *a-rcos-module-rev*)
qed

corollary (**in** *ring-hom-ring*) *rcos-eq-homeq*:
assumes *acarr*: $a \in$ *carrier R*
shows (*a-kernel R S h*) $+>$ $a = \{x \in$ *carrier R*. $h\ x = h\ a\}$
apply *rule* **defer** *1*
apply *clarsimp* **defer** *1*
proof
interpret *ideal* [*a-kernel R S h R*] **by** (*rule kernel-is-ideal*)

fix *x*
assume *xrcos*: $x \in$ *a-kernel R S h* $+>$ *a*
from *acarr* **and** *this*
have *xcarr*: $x \in$ *carrier R*
by (*rule a-elemrcos-carrier*)

from *xrcos*
have $h\ x = h\ a$ **by** (*rule rcos-imp-homeq*[*OF acarr*])
from *xcarr* **and** *this*
show $x \in \{x \in$ *carrier R*. $h\ x = h\ a\}$ **by** *fast*
next
interpret *ideal* [*a-kernel R S h R*] **by** (*rule kernel-is-ideal*)

fix *x*
assume *xcarr*: $x \in$ *carrier R*
and *hx*: $h\ x = h\ a$
from *acarr xcarr hx*
show $x \in$ *a-kernel R S h* $+>$ *a* **by** (*rule homeq-imp-rcos*)
qed

end

17 QuotRing: Quotient Rings

theory *QuotRing*
imports *RingHom*
begin

17.1 Multiplication on Cosets

constdefs (**structure** *R*)
rcoset-mult :: [(*'a*, -) *ring-scheme*, *'a set*, *'a set*, *'a set*] $\Rightarrow$ *'a set*
([*mod* -:] - $\bigotimes\imath$ - [*81*,*81*,*81*] *80*)
rcoset-mult R I A B $\equiv \bigcup a \in A.\ \bigcup b \in B.\ I +> (a \otimes b)$

rcoset-mult fulfils the properties required by congruences

lemma (**in** *ideal*) *rcoset-mult-add*:
$[\![x \in carrier\ R;\ y \in carrier\ R]\!] \Longrightarrow [mod\ I:]\ (I\ {+}{>}\ x) \bigotimes (I\ {+}{>}\ y) = I\ {+}{>}\ (x \otimes y)$
apply *rule*
apply (*rule*, *simp add*: *rcoset-mult-def*, *clarsimp*)
defer *1*
apply (*rule*, *simp add*: *rcoset-mult-def*)
defer *1*
proof −
fix $z\ x'\ y'$
assume *carr*: $x \in carrier\ R\ y \in carrier\ R$
and *x'rcos*: $x' \in I\ {+}{>}\ x$
and *y'rcos*: $y' \in I\ {+}{>}\ y$
and *zrcos*: $z \in I\ {+}{>}\ x' \otimes y'$

from *x'rcos*
have $\exists h \in I.\ x' = h \oplus x$ **by** (*simp add*: *a-r-coset-def r-coset-def*)
from *this* **obtain** *hx*
where *hxI*: $hx \in I$
and *x'*: $x' = hx \oplus x$
by *fast+*

from *y'rcos*
have $\exists h \in I.\ y' = h \oplus y$ **by** (*simp add*: *a-r-coset-def r-coset-def*)
from *this*
obtain *hy*
where *hyI*: $hy \in I$
and *y'*: $y' = hy \oplus y$
by *fast+*

from *zrcos*
have $\exists h \in I.\ z = h \oplus (x' \otimes y')$ **by** (*simp add*: *a-r-coset-def r-coset-def*)
from *this*
obtain *hz*
where *hzI*: $hz \in I$
and *z*: $z = hz \oplus (x' \otimes y')$
by *fast+*

note *carr* = *carr hxI*[*THEN a-Hcarr*] *hyI*[*THEN a-Hcarr*] *hzI*[*THEN a-Hcarr*]

from *z* **have** $z = hz \oplus (x' \otimes y')$ **.**
also from $x'\ y'$
have $\ldots = hz \oplus ((hx \oplus x) \otimes (hy \oplus y))$ **by** *simp*
also from *carr*
have $\ldots = (hz \oplus (hx \otimes (hy \oplus y)) \oplus x \otimes hy) \oplus x \otimes y$ **by** *algebra*
finally
have *z2*: $z = (hz \oplus (hx \otimes (hy \oplus y)) \oplus x \otimes hy) \oplus x \otimes y$ **.**

from *hxI hyI hzI carr*

have $hz \oplus (hx \otimes (hy \oplus y)) \oplus x \otimes hy \in I$ **by** (*simp add*: *I-l-closed I-r-closed*)

from *this* **and** *z2*
have $\exists h{\in}I.\ z = h \oplus x \otimes y$ **by** *fast*
thus $z \in I +> x \otimes y$ **by** (*simp add*: *a-r-coset-def r-coset-def*)
next
fix z
assume *xcarr*: $x \in$ *carrier* R
and *ycarr*: $y \in$ *carrier* R
and *zrcos*: $z \in I +> x \otimes y$
from *xcarr*
have *xself*: $x \in I +> x$ **by** (*intro a-rcos-self*)
from *ycarr*
have *yself*: $y \in I +> y$ **by** (*intro a-rcos-self*)

from *xself* **and** *yself* **and** *zrcos*
show $\exists a{\in}I +> x.\ \exists b{\in}I +> y.\ z \in I +> a \otimes b$ **by** *fast*
qed

17.2 Quotient Ring Definition

constdefs (**structure** R)
FactRing :: [('a,'b) *ring-scheme*, 'a *set*] ⇒ ('a *set*) *ring*
(**infixl** *Quot 65*)
FactRing R I ≡
(|*carrier* = *a-rcosets* I, *mult* = *rcoset-mult* R I, *one* = $(I +> \mathbf{1})$, *zero* = I, *add* = *set-add* R|)

17.3 Factorization over General Ideals

The quotient is a ring

lemma (**in** *ideal*) *quotient-is-ring*:
shows *ring* (R *Quot* I)
apply (*rule ringI*)
— abelian group
apply (*rule comm-group-abelian-groupI*)
apply (*simp add*: *FactRing-def*)
apply (*rule a-factorgroup-is-comm-group*[*unfolded A-FactGroup-def'*])
— mult monoid
apply (*rule monoidI*)
apply (*simp-all add*: *FactRing-def A-RCOSETS-def RCOSETS-def a-r-coset-def*[*symmetric*])
— mult closed
apply (*clarify*)
apply (*simp add*: *rcoset-mult-add*, *fast*)
— mult *one-closed*
apply (*force intro*: *one-closed*)
— mult assoc
apply *clarify*

```
   apply (simp add: rcoset-mult-add m-assoc)
  — mult one
  apply clarify
  apply (simp add: rcoset-mult-add l-one)
 apply clarify
 apply (simp add: rcoset-mult-add r-one)
— distr
apply clarify
apply (simp add: rcoset-mult-add a-rcos-sum l-distr)
apply clarify
apply (simp add: rcoset-mult-add a-rcos-sum r-distr)
done
```

This is a ring homomorphism

```
lemma (in ideal) rcos-ring-hom:
  (op +> I) ∈ ring-hom R (R Quot I)
apply (rule ring-hom-memI)
   apply (simp add: FactRing-def a-rcosetsI[OF a-subset])
  apply (simp add: FactRing-def rcoset-mult-add)
 apply (simp add: FactRing-def a-rcos-sum)
apply (simp add: FactRing-def)
done

lemma (in ideal) rcos-ring-hom-ring:
  ring-hom-ring R (R Quot I) (op +> I)
apply (rule ring-hom-ringI)
     apply (rule is-ring, rule quotient-is-ring)
   apply (simp add: FactRing-def a-rcosetsI[OF a-subset])
  apply (simp add: FactRing-def rcoset-mult-add)
 apply (simp add: FactRing-def a-rcos-sum)
apply (simp add: FactRing-def)
done
```

The quotient of a cring is also commutative

```
lemma (in ideal) quotient-is-cring:
  includes cring
  shows cring (R Quot I)
apply (intro cring.intro comm-monoid.intro comm-monoid-axioms.intro)
  apply (rule quotient-is-ring)
 apply (rule ring.axioms[OF quotient-is-ring])
apply (simp add: FactRing-def A-RCOSETS-defs a-r-coset-def[symmetric])
apply clarify
apply (simp add: rcoset-mult-add m-comm)
done
```

Cosets as a ring homomorphism on crings

```
lemma (in ideal) rcos-ring-hom-cring:
  includes cring
  shows ring-hom-cring R (R Quot I) (op +> I)
```

apply (*rule ring-hom-cringI*)
  apply (*rule rcos-ring-hom-ring*)
 apply (*rule R.is-cring*)
apply (*rule quotient-is-cring*)
apply (*rule R.is-cring*)
done

17.4 Factorization over Prime Ideals

The quotient ring generated by a prime ideal is a domain

lemma (**in** *primeideal*) *quotient-is-domain*:
  shows *domain* (*R Quot I*)
apply (*rule domain.intro*)
 apply (*rule quotient-is-cring*, *rule is-cring*)
apply (*rule domain-axioms.intro*)
 apply (*simp add*: *FactRing-def*) **defer** *1*
 apply (*simp add*: *FactRing-def A-RCOSETS-defs a-r-coset-def*[*symmetric*], *clarify*)
 apply (*simp add*: *rcoset-mult-add*) **defer** *1*
proof (*rule ccontr*, *clarsimp*)
  assume $I +> \mathbf{1} = I$
  hence $\mathbf{1} \in I$ **by** (*simp only*: *a-coset-join1 one-closed a-subgroup*)
  hence *carrier R* $\subseteq I$ **by** (*subst one-imp-carrier*, *simp*, *fast*)
  from *this* **and** *a-subset*
    have $I =$ *carrier R* **by** *fast*
  from *this* **and** *I-notcarr*
    show *False* **by** *fast*
next
  fix x y
  assume *carr*: $x \in$ *carrier R* $y \in$ *carrier R*
    and *a*: $I +> x \otimes y = I$
    and *b*: $I +> y \neq I$

  have *ynI*: $y \notin I$
  proof (*rule ccontr*, *simp*)
    assume $y \in I$
    hence $I +> y = I$ **by** (*rule a-rcos-const*)
    from *this* **and** *b*
      show *False* **by** *simp*
  qed

  from *carr*
    have $x \otimes y \in I +> x \otimes y$ **by** (*simp add*: *a-rcos-self*)
  from *this*
    have *xyI*: $x \otimes y \in I$ **by** (*simp add*: *a*)

  from *xyI* **and** *carr*
    have *xI*: $x \in I \vee y \in I$ **by** (*simp add*: *I-prime*)
  from *this* **and** *ynI*

    have $x \in I$ **by** *fast*
  thus I +> $x = I$ **by** (*rule a-rcos-const*)
qed

Generating right cosets of a prime ideal is a homomorphism on commutative rings

lemma (**in** *primeideal*) *rcos-ring-hom-cring*:
  shows *ring-hom-cring R* (*R Quot I*) (*op* +> *I*)
by (*rule rcos-ring-hom-cring*, *rule is-cring*)

17.5 Factorization over Maximal Ideals

In a commutative ring, the quotient ring over a maximal ideal is a field. The proof follows "W. Adkins, S. Weintraub: Algebra – An Approach via Module Theory"

lemma (**in** *maximalideal*) *quotient-is-field*:
  includes *cring*
  shows *field* (*R Quot I*)
apply (*intro cring.cring-fieldI2*)
  apply (*rule quotient-is-cring*, *rule is-cring*)
 defer *1*
 apply (*simp add*: *FactRing-def A-RCOSETS-defs a-r-coset-def*[*symmetric*], *clarsimp*)
 apply (*simp add*: *rcoset-mult-add*) **defer** *1*
proof (*rule ccontr*, *simp*)
  — Quotient is not empty
  assume $\mathbf{0}_{R\ Quot\ I} = \mathbf{1}_{R\ Quot\ I}$
  hence *II1*: $I = I$ +> **1** **by** (*simp add*: *FactRing-def*)
  from *a-rcos-self*[*OF one-closed*]
  have $\mathbf{1} \in I$ **by** (*simp add*: *II1*[*symmetric*])
  hence I = *carrier R* **by** (*rule one-imp-carrier*)
  from *this* **and** *I-notcarr*
  show *False* **by** *simp*
next
  — Existence of Inverse
  fix *a*
  assume *IanI*: I +> $a \neq I$
    and *acarr*: $a \in$ *carrier R*

  — Helper ideal J
  def $J \equiv$ (*carrier R* #> *a*) <+> I :: *'a set*
  have *idealJ*: *ideal J R*
    apply (*unfold J-def*, *rule add-ideals*)
    apply (*simp only*: *cgenideal-eq-rcos*[*symmetric*], *rule cgenideal-ideal*, *rule acarr*)
    apply (*rule is-ideal*)
    done

  — Showing J not smaller than I

```
  have IinJ: I ⊆ J
  proof (rule, simp add: J-def r-coset-def set-add-defs)
    fix x
    assume xI: x ∈ I
    have Zcarr: 0 ∈ carrier R by fast
    from xI[THEN a-Hcarr] acarr
    have x = 0 ⊗ a ⊕ x by algebra

    from Zcarr and xI and this
    show ∃xa∈carrier R. ∃k∈I. x = xa ⊗ a ⊕ k by fast
  qed

  — Showing J ≠ I
  have anI: a ∉ I
  proof (rule ccontr, simp)
    assume a ∈ I
    hence I +> a = I by (rule a-rcos-const)
    from this and IanI
    show False by simp
  qed

  have aJ: a ∈ J
  proof (simp add: J-def r-coset-def set-add-defs)
    from acarr
    have a = 1 ⊗ a ⊕ 0 by algebra
      from one-closed and additive-subgroup.zero-closed[OF is-additive-subgroup]
and this
    show ∃x∈carrier R. ∃k∈I. a = x ⊗ a ⊕ k by fast
  qed

  from aJ and anI
  have JnI: J ≠ I by fast

  — Deducing J = carrier R because I is maximal
  from idealJ and IinJ
  have J = I ∨ J = carrier R
  proof (rule I-maximal, unfold J-def)
    have carrier R #> a ⊆ carrier R
      using subset-refl acarr
      by (rule r-coset-subset-G)
    from this and a-subset
    show carrier R #> a <+> I ⊆ carrier R by (rule set-add-closed)
  qed

  from this and JnI
  have Jcarr: J = carrier R by simp

  — Calculating an inverse for a
  from one-closed[folded Jcarr]
```

```
  have ∃r∈carrier R. ∃i∈I. 1 = r ⊗ a ⊕ i
    by (simp add: J-def r-coset-def set-add-defs)
  from this
  obtain r i
    where rcarr: r ∈ carrier R
      and iI: i ∈ I
      and one: 1 = r ⊗ a ⊕ i
    by fast
  from one and rcarr and acarr and iI[THEN a-Hcarr]
  have rai1: a ⊗ r = ⊖i ⊕ 1 by algebra

  — Lifting to cosets
  from iI
  have ⊖i ⊕ 1 ∈ I +> 1
    by (intro a-rcosI, simp, intro a-subset, simp)
  from this and rai1
  have a ⊗ r ∈ I +> 1 by simp
  from this have I +> 1 = I +> a ⊗ r
    by (rule a-repr-independence, simp) (rule a-subgroup)

  from rcarr and this[symmetric]
  show ∃r∈carrier R. I +> a ⊗ r = I +> 1 by fast
qed

end



theory IntRing
imports QuotRing IntDef
begin
```

18 The Ring of Integers

18.1 Some properties of *int*

```
lemma dvds-imp-absеq:
  [[l dvd k; k dvd l]] ⟹ abs l = abs (k::int)
apply (subst abs-split, rule conjI)
 apply (clarsimp, subst abs-split, rule conjI)
  apply (clarsimp)
  apply (cases k=0, simp)
  apply (cases l=0, simp)
  apply (simp add: zdvd-anti-sym)
 apply clarsimp
 apply (cases k=0, simp)
 apply (simp add: zdvd-anti-sym)
apply (clarsimp, subst abs-split, rule conjI)
 apply (clarsimp)
```

apply (*cases* $l=0$, *simp*)
apply (*simp add*: *zdvd-anti-sym*)
apply (*clarsimp*)
apply (*subgoal-tac* $-l = -k$, *simp*)
apply (*intro zdvd-anti-sym*, *simp*+)
done

lemma *abseq-imp-dvd*:
assumes *a-lk*: *abs* l = *abs* (k::*int*)
shows l *dvd* k
proof −
from *a-lk*
have *nat* (*abs* l) = *nat* (*abs* k) **by** *simp*
hence *nat* (*abs* l) *dvd* *nat* (*abs* k) **by** *simp*
hence *int* (*nat* (*abs* l)) *dvd* k **by** (*subst int-dvd-iff*)
hence *abs* l *dvd* k **by** *simp*
thus l *dvd* k
apply (*unfold dvd-def*, *cases* $l<0$)
defer *1* **apply** *clarsimp*
proof (*clarsimp*)
fix k
assume *l0*: $l < 0$
have $-(l * k) = l * (-k)$ **by** *simp*
thus $\exists ka. -(l * k) = l * ka$ **by** *fast*
qed
qed

lemma *dvds-eq-abseq*:
(l *dvd* k $\wedge$ k *dvd* l) = (*abs* l = *abs* (k::*int*))
apply *rule*
apply (*simp add*: *dvds-imp-abseq*)
apply (*rule conjI*)
apply (*simp add*: *abseq-imp-dvd*)+
done

18.2 The Set of Integers as Algebraic Structure

18.2.1 Definition of $\mathcal{Z}$

constdefs
int-ring :: *int ring* ($\mathcal{Z}$)
int-ring $\equiv$ (| *carrier* = *UNIV*, *mult* = *op* $*$, *one* = *1*, *zero* = *0*, *add* = *op* + |)

lemma *int-Zcarr* [*intro*!, *simp*]:
$k \in$ *carrier* $\mathcal{Z}$
by (*simp add*: *int-ring-def*)

lemma *int-is-cring*:
cring $\mathcal{Z}$
unfolding *int-ring-def*

apply (*rule cringI*)
  apply (*rule abelian-groupI*, *simp-all*)
  defer *1*
  apply (*rule comm-monoidI*, *simp-all*)
 apply (*rule zadd-zmult-distrib*)
apply (*fast intro*: *zadd-zminus-inverse2*)
done

18.2.2 Interpretations

Since definitions of derived operations are global, their interpretation needs to be done as early as possible — that is, with as few assumptions as possible.

interpretation *int*: *monoid* [$\mathcal{Z}$]
  where *carrier* $\mathcal{Z}$ = *UNIV*
    and *mult* $\mathcal{Z}$ x y = $x * y$
    and *one* $\mathcal{Z}$ = *1*
    and *pow* $\mathcal{Z}$ x n = $x\hat{}n$
proof −
  — Specification
  show *monoid* $\mathcal{Z}$ **by** (*unfold-locales*) (*auto simp*: *int-ring-def*)
  then interpret *int*: *monoid* [$\mathcal{Z}$] **.**

  — Carrier
  show *carrier* $\mathcal{Z}$ = *UNIV* **by** (*simp add*: *int-ring-def*)

  — Operations
  { fix x y **show** *mult* $\mathcal{Z}$ x y = $x * y$ **by** (*simp add*: *int-ring-def*) **}**
  note *mult* = *this*
  show *one*: *one* $\mathcal{Z}$ = *1* **by** (*simp add*: *int-ring-def*)
  show *pow* $\mathcal{Z}$ x n = $x\hat{}n$ **by** (*induct n*) (*simp*, *simp add*: *int-ring-def*)+
qed

interpretation *int*: *comm-monoid* [$\mathcal{Z}$]
  where *finprod* $\mathcal{Z}$ f A = (*if finite A then setprod f A else arbitrary*)
proof −
  — Specification
  show *comm-monoid* $\mathcal{Z}$ **by** (*unfold-locales*) (*auto simp*: *int-ring-def*)
  then interpret *int*: *comm-monoid* [$\mathcal{Z}$] **.**

  — Operations
  { fix x y **have** *mult* $\mathcal{Z}$ x y = $x * y$ **by** (*simp add*: *int-ring-def*) **}**
  note *mult* = *this*
  have *one*: *one* $\mathcal{Z}$ = *1* **by** (*simp add*: *int-ring-def*)
  show *finprod* $\mathcal{Z}$ f A = (*if finite A then setprod f A else arbitrary*)
  proof (*cases finite A*)
    case *True* **then show** *?thesis* **proof** *induct*
      case *empty* **show** *?case* **by** (*simp add*: *one*)
    next
      case *insert* **then show** *?case* **by** (*simp add*: *Pi-def mult*)

```
    qed
  next
    case False then show ?thesis by (simp add: finprod-def)
  qed
qed

interpretation int: abelian-monoid [Z]
  where zero Z = 0
    and add Z x y = x + y
    and finsum Z f A = (if finite A then setsum f A else arbitrary)
proof -
  — Specification
  show abelian-monoid Z by (unfold-locales) (auto simp: int-ring-def)
  then interpret int: abelian-monoid [Z] .

  — Operations
  { fix x y show add Z x y = x + y by (simp add: int-ring-def) }
  note add = this
  show zero: zero Z = 0 by (simp add: int-ring-def)
  show finsum Z f A = (if finite A then setsum f A else arbitrary)
  proof (cases finite A)
    case True then show ?thesis proof induct
      case empty show ?case by (simp add: zero)
    next
      case insert then show ?case by (simp add: Pi-def add)
    qed
  next
    case False then show ?thesis by (simp add: finsum-def finprod-def)
  qed
qed

interpretation int: abelian-group [Z]
  where a-inv Z x = - x
    and a-minus Z x y = x - y
proof -
  — Specification
  show abelian-group Z
  proof (rule abelian-groupI)
    show !!x. x ∈ carrier Z ==> EX y : carrier Z. y ⊕_Z x = 𝟎_Z
      by (simp add: int-ring-def) arith
  qed (auto simp: int-ring-def)
  then interpret int: abelian-group [Z] .

  — Operations
  { fix x y have add Z x y = x + y by (simp add: int-ring-def) }
  note add = this
  have zero: zero Z = 0 by (simp add: int-ring-def)
  { fix x
    have add Z (-x) x = zero Z by (simp add: add zero)
```

```
    then show a-inv Z x = - x by (simp add: int.minus-equality) }
  note a-inv = this
  show a-minus Z x y = x - y by (simp add: int.minus-eq add a-inv)
qed

interpretation int: domain [Z]
  by (unfold-locales) (auto simp: int-ring-def left-distrib right-distrib)
```

Removal of occurrences of *UNIV* in interpretation result — experimental.

```
lemma UNIV:
  x ∈ UNIV = True
  A ⊆ UNIV = True
  (ALL x : UNIV. P x) = (ALL x. P x)
  (EX x : UNIV. P x) = (EX x. P x)
  (True --> Q) = Q
  (True ==> PROP R) == PROP R
  by simp-all

interpretation int [unfolded UNIV]:
  partial-order [(| carrier = UNIV::int set, le = op ≤ |)]
  where carrier (| carrier = UNIV::int set, le = op ≤ |) = UNIV
    and le (| carrier = UNIV::int set, le = op ≤ |) x y = (x ≤ y)
    and lless (| carrier = UNIV::int set, le = op ≤ |) x y = (x < y)
proof -
  show partial-order (| carrier = UNIV::int set, le = op ≤ |)
    by unfold-locales simp-all
  show carrier (| carrier = UNIV::int set, le = op ≤ |) = UNIV
    by simp
  show le (| carrier = UNIV::int set, le = op ≤ |) x y = (x ≤ y)
    by simp
  show lless (| carrier = UNIV::int set, le = op ≤ |) x y = (x < y)
    by (simp add: lless-def) auto
qed

interpretation int [unfolded UNIV]:
  lattice [(| carrier = UNIV::int set, le = op ≤ |)]
  where join (| carrier = UNIV::int set, le = op ≤ |) x y = max x y
    and meet (| carrier = UNIV::int set, le = op ≤ |) x y = min x y
proof -
  let ?Z = (| carrier = UNIV::int set, le = op ≤ |)
  show lattice ?Z
    apply unfold-locales
    apply (simp add: least-def Upper-def)
    apply arith
    apply (simp add: greatest-def Lower-def)
    apply arith
    done
  then interpret int: lattice [?Z] .
  show join ?Z x y = max x y
```

apply (*rule int.joinI*)
apply (*simp-all add*: *least-def Upper-def*)
apply *arith*
done
show *meet ?Z x y* = *min x y*
apply (*rule int.meetI*)
apply (*simp-all add*: *greatest-def Lower-def*)
apply *arith*
done
qed

interpretation *int* [*unfolded UNIV*]:
total-order [(| *carrier* = *UNIV*::*int set*, *le* = *op* ≤ |)]
by *unfold-locales clarsimp*

18.2.3 Generated Ideals of $\mathcal{Z}$

lemma *int-Idl*:
$Idl_{\mathcal{Z}}$ {*a*} = {*x* ∗ *a* | *x*. *True*}
apply (*subst int.cgenideal-eq-genideal*[*symmetric*]) **apply** (*simp add*: *int-ring-def*)
apply (*simp add*: *cgenideal-def int-ring-def*)
done

lemma *multiples-principalideal*:
principalideal {*x* ∗ *a* | *x*. *True* } $\mathcal{Z}$
apply (*subst int-Idl*[*symmetric*], *rule principalidealI*)
apply (*rule int.genideal-ideal*, *simp*)
apply *fast*
done

lemma *prime-primeideal*:
assumes *prime*: *prime* (*nat p*)
shows *primeideal* ($Idl_{\mathcal{Z}}$ {*p*}) $\mathcal{Z}$
apply (*rule primeidealI*)
apply (*rule int.genideal-ideal*, *simp*)
apply (*rule int-is-cring*)
apply (*simp add*: *int.cgenideal-eq-genideal*[*symmetric*] *cgenideal-def*)
apply (*simp add*: *int-ring-def*)
apply *clarsimp* **defer** *1*
apply (*simp add*: *int.cgenideal-eq-genideal*[*symmetric*] *cgenideal-def*)
apply (*simp add*: *int-ring-def*)
apply (*elim exE*)
proof −
fix *a b x*

from *prime*
have *ppos*: *0* <= *p* **by** (*simp add*: *prime-def*)
have *unnat*: !!*x*. *nat p dvd nat* (*abs x*) ==> *p dvd x*
proof −

```
    fix x
    assume nat p dvd nat (abs x)
    hence int (nat p) dvd x by (simp add: int-dvd-iff[symmetric])
    thus p dvd x by (simp add: ppos)
  qed


  assume a * b = x * p
  hence p dvd a * b by simp
  hence nat p dvd nat (abs (a * b))
  apply (subst nat-dvd-iff, clarsimp)
  apply (rule conjI, clarsimp, simp add: zabs-def)
  proof (clarsimp)
    assume a: ~ 0 <= p
    from prime
        have 0 < p by (simp add: prime-def)
    from a and this
        have False by simp
    thus nat (abs (a * b)) = 0 ..
  qed
  hence nat p dvd (nat (abs a) * nat (abs b)) by (simp add: nat-abs-mult-distrib)
  hence nat p dvd nat (abs a) | nat p dvd nat (abs b) by (rule prime-dvd-mult[OF
prime])
  hence p dvd a | p dvd b by (fast intro: unnat)
  thus (EX x. a = x * p) | (EX x. b = x * p)
  proof
    assume p dvd a
    hence EX x. a = p * x by (simp add: dvd-def)
    from this obtain x
        where a = p * x by fast
    hence a = x * p by simp
    hence EX x. a = x * p by simp
    thus (EX x. a = x * p) | (EX x. b = x * p) ..
  next
    assume p dvd b
    hence EX x. b = p * x by (simp add: dvd-def)
    from this obtain x
        where b = p * x by fast
    hence b = x * p by simp
    hence EX x. b = x * p by simp
    thus (EX x. a = x * p) | (EX x. b = x * p) ..
  qed
next
  assume UNIV = {uu. EX x. uu = x * p}
  from this obtain x
      where 1 = x * p by fast
  from this [symmetric]
      have p * x = 1 by (subst zmult-commute)
  hence |p * x| = 1 by simp
```

hence $|p| = 1$ **by** (*rule abs-zmult-eq-1*)
from *this* **and** *prime*
show *False* **by** (*simp add*: *prime-def*)
qed

18.2.4 Ideals and Divisibility

lemma *int-Idl-subset-ideal*:
$Idl_{\mathcal{Z}}\ \{k\} \subseteq Idl_{\mathcal{Z}}\ \{l\} = (k \in Idl_{\mathcal{Z}}\ \{l\})$
by (*rule int.Idl-subset-ideal′*, *simp+*)

lemma *Idl-subset-eq-dvd*:
$(Idl_{\mathcal{Z}}\ \{k\} \subseteq Idl_{\mathcal{Z}}\ \{l\}) = (l\ dvd\ k)$
apply (*subst int-Idl-subset-ideal*, *subst int-Idl*, *simp*)
apply (*rule*, *clarify*)
apply (*simp add*: *dvd-def*, *clarify*)
apply (*simp add*: *int.m-comm*)
done

lemma *dvds-eq-Idl*:
$(l\ dvd\ k \wedge k\ dvd\ l) = (Idl_{\mathcal{Z}}\ \{k\} = Idl_{\mathcal{Z}}\ \{l\})$
proof −
have a: $l\ dvd\ k = (Idl_{\mathcal{Z}}\ \{k\} \subseteq Idl_{\mathcal{Z}}\ \{l\})$ **by** (*rule Idl-subset-eq-dvd*[*symmetric*])
have b: $k\ dvd\ l = (Idl_{\mathcal{Z}}\ \{l\} \subseteq Idl_{\mathcal{Z}}\ \{k\})$ **by** (*rule Idl-subset-eq-dvd*[*symmetric*])

have $(l\ dvd\ k \wedge k\ dvd\ l) = ((Idl_{\mathcal{Z}}\ \{k\} \subseteq Idl_{\mathcal{Z}}\ \{l\}) \wedge (Idl_{\mathcal{Z}}\ \{l\} \subseteq Idl_{\mathcal{Z}}\ \{k\}))$
by (*subst a*, *subst b*, *simp*)
also have $((Idl_{\mathcal{Z}}\ \{k\} \subseteq Idl_{\mathcal{Z}}\ \{l\}) \wedge (Idl_{\mathcal{Z}}\ \{l\} \subseteq Idl_{\mathcal{Z}}\ \{k\})) = (Idl_{\mathcal{Z}}\ \{k\} = Idl_{\mathcal{Z}}\ \{l\})$ **by** (*rule*, *fast+*)
finally
show *?thesis* **.**
qed

lemma *Idl-eq-abs*:
$(Idl_{\mathcal{Z}}\ \{k\} = Idl_{\mathcal{Z}}\ \{l\}) = (abs\ l = abs\ k)$
apply (*subst dvds-eq-abseq*[*symmetric*])
apply (*rule dvds-eq-Idl*[*symmetric*])
done

18.2.5 Ideals and the Modulus

constdefs
ZMod :: *int* => *int* => *int set*
$ZMod\ k\ r == (Idl_{\mathcal{Z}}\ \{k\}) +>_{\mathcal{Z}} r$

lemmas *ZMod-defs* =
ZMod-def genideal-def

lemma *rcos-zfact*:
assumes *kIl*: $k \in ZMod\ l\ r$

```
  shows EX x. k = x * l + r
proof -
  from kIl[unfolded ZMod-def]
    have ∃xl∈Idl_Z {l}. k = xl + r by (simp add: a-r-coset-defs int-ring-def)
  from this obtain xl
    where xl: xl ∈ Idl_Z {l}
    and k: k = xl + r
    by auto
  from xl obtain x
    where xl = x * l
    by (simp add: int-Idl, fast)
  from k and this
    have k = x * l + r by simp
  thus ∃x. k = x * l + r ..
qed

lemma ZMod-imp-zmod:
  assumes zmods: ZMod m a = ZMod m b
  shows a mod m = b mod m
proof -
  interpret ideal [Idl_Z {m} Z] by (rule int.genideal-ideal, fast)
  from zmods
    have b ∈ ZMod m a
    unfolding ZMod-def
    by (simp add: a-repr-independenceD)
  from this
    have EX x. b = x * m + a by (rule rcos-zfact)
  from this obtain x
    where b = x * m + a
    by fast

  hence b mod m = (x * m + a) mod m by simp
  also
    have ... = ((x * m) mod m) + (a mod m) by (simp add: zmod-zadd1-eq)
  also
    have ... = a mod m by simp
  finally
    have b mod m = a mod m .
  thus a mod m = b mod m ..
qed

lemma ZMod-mod:
  shows ZMod m a = ZMod m (a mod m)
proof -
  interpret ideal [Idl_Z {m} Z] by (rule int.genideal-ideal, fast)
  show ?thesis
    unfolding ZMod-def
  apply (rule a-repr-independence'[symmetric])
  apply (simp add: int-Idl a-r-coset-defs)
```

apply (*simp add*: *int-ring-def*)
proof −
have $a = m * (a \; div \; m) + (a \; mod \; m)$ **by** (*simp add*: *zmod-zdiv-equality*)
hence $a = (a \; div \; m) * m + (a \; mod \; m)$ **by** *simp*
thus $\exists h. (\exists x. h = x * m) \wedge a = h + a \; mod \; m$ **by** *fast*
qed *simp*
qed

lemma *zmod-imp-ZMod*:
assumes *modeq*: $a \; mod \; m = b \; mod \; m$
shows *ZMod* $m \; a$ = *ZMod* $m \; b$
proof −
have *ZMod* $m \; a$ = *ZMod* $m \; (a \; mod \; m)$ **by** (*rule ZMod-mod*)
also have ... = *ZMod* $m \; (b \; mod \; m)$ **by** (*simp add*: *modeq*[*symmetric*])
also have ... = *ZMod* $m \; b$ **by** (*rule ZMod-mod*[*symmetric*])
finally show *?thesis* **.**
qed

corollary *ZMod-eq-mod*:
shows (*ZMod* $m \; a$ = *ZMod* $m \; b$) = ($a \; mod \; m = b \; mod \; m$)
by (*rule*, *erule ZMod-imp-zmod*, *erule zmod-imp-ZMod*)

18.2.6 Factorization

constdefs
ZFact :: *int* ⇒ *int set ring*
ZFact k == $\mathcal{Z}$ *Quot* (*Idl*$_{\mathcal{Z}}$ $\{k\}$)

lemmas *ZFact-defs* = *ZFact-def FactRing-def*

lemma *ZFact-is-cring*:
shows *cring* (*ZFact* k)
apply (*unfold ZFact-def*)
apply (*rule ideal.quotient-is-cring*)
apply (*intro ring.genideal-ideal*)
apply (*simp add*: *cring.axioms*[*OF int-is-cring*] *ring.intro*)
apply *simp*
apply (*rule int-is-cring*)
done

lemma *ZFact-zero*:
carrier (*ZFact 0*) = $(\bigcup a. \{\{a\}\})$
apply (*insert int.genideal-zero*)
apply (*simp add*: *ZFact-defs A-RCOSETS-defs r-coset-def int-ring-def ring-record-simps*)
done

lemma *ZFact-one*:
carrier (*ZFact 1*) = {*UNIV*}
apply (*simp only*: *ZFact-defs A-RCOSETS-defs r-coset-def int-ring-def ring-record-simps*)

apply (*subst int.genideal-one*[*unfolded int-ring-def*, *simplified ring-record-simps*])
apply (*rule*, *rule*, *clarsimp*)
 apply (*rule*, *rule*, *clarsimp*)
 apply (*rule*, *clarsimp*, *arith*)
apply (*rule*, *clarsimp*)
apply (*rule exI*[*of - 0*], *clarsimp*)
done

lemma *ZFact-prime-is-domain*:
  assumes *pprime*: *prime* (*nat p*)
  shows *domain* (*ZFact p*)
apply (*unfold ZFact-def*)
apply (*rule primeideal.quotient-is-domain*)
apply (*rule prime-primeideal*[*OF pprime*])
done

end

References

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[3] F. Kammüller and L. C. Paulson. A formal proof of sylow's theorem: An experiment in abstract algebra with Isabelle HOL. *J. Automated Reasoning*, (23):235–264, 1999.