Hoare Logic for Parallel Programs

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Abstract

In the following theories a formalization of the Owicki-Gries and the rely-guarantee methods is presented. These methods are widely used for correctness proofs of parallel imperative programs with shared variables. We define syntax, semantics and proof rules in Isabelle/HOL. The proof rules also provide for programs parameterized in the number of parallel components. Their correctness w.r.t. the semantics is proven. Completeness proofs for both methods are extended to the new case of parameterized programs. (These proofs have not been formalized in Isabelle. They can be found in [1].) Using this formalizations we verify several non-trivial examples for parameterized and non-parameterized programs. For the automatic generation of verification conditions with the Owicki-Gries method we define a tactic based on the proof rules. The most involved examples are the verification of two garbage-collection algorithms, the second one parameterized in the number of mutators.

For excellent descriptions of this work see [2, 4, 1, 3].

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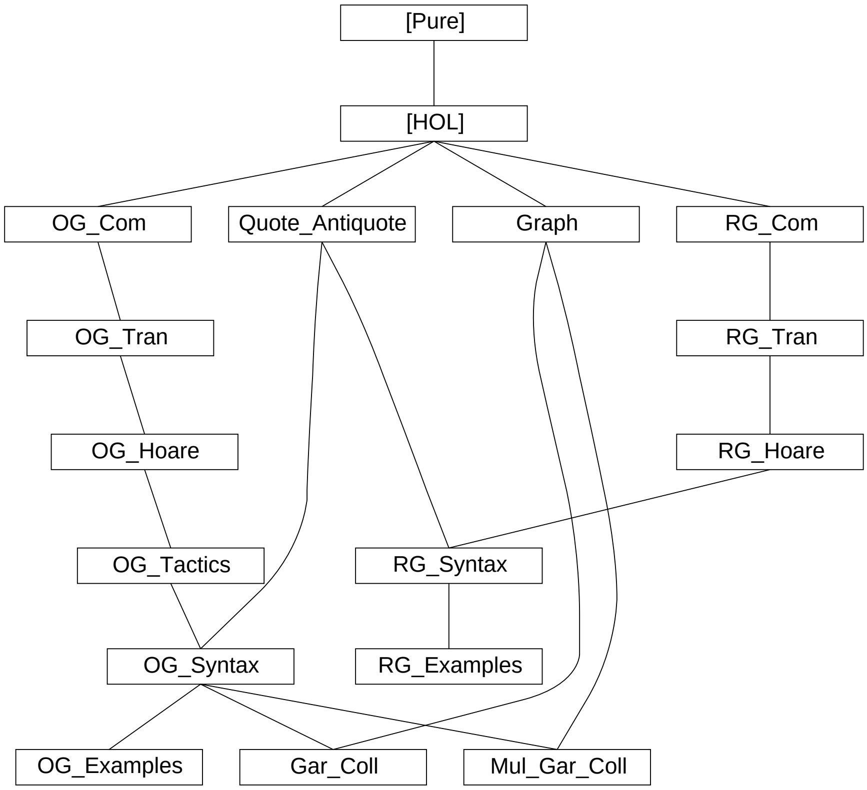
[Pure]
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Chapter 1

The Owicki-Gries Method

1.1 Abstract Syntax

theory *OG-Com* **imports** *Main* **begin**

Type abbreviations for boolean expressions and assertions:

types
  'a bexp = 'a set
  'a assn = 'a set

The syntax of commands is defined by two mutually recursive datatypes: *'a ann-com* for annotated commands and *'a com* for non-annotated commands.

datatype *'a ann-com* =
    AnnBasic ('a assn) ('a ⇒ 'a)
  | *AnnSeq ('a ann-com) ('a ann-com)*
  | *AnnCond1 ('a assn) ('a bexp) ('a ann-com) ('a ann-com)*
  | *AnnCond2 ('a assn) ('a bexp) ('a ann-com)*
  | *AnnWhile ('a assn) ('a bexp) ('a assn) ('a ann-com)*
  | *AnnAwait ('a assn) ('a bexp) ('a com)*
and *'a com* =
    Parallel ('a ann-com option × 'a assn) list
  | *Basic ('a ⇒ 'a)*
  | *Seq ('a com) ('a com)*
  | *Cond ('a bexp) ('a com) ('a com)*
  | *While ('a bexp) ('a assn) ('a com)*

The function *pre* extracts the precondition of an annotated command:

consts
  pre ::'a ann-com ⇒ 'a assn
primrec
  pre (AnnBasic r f) = r
  pre (AnnSeq c1 c2) = pre c1
  pre (AnnCond1 r b c1 c2) = r
  pre (AnnCond2 r b c) = r
  pre (AnnWhile r b i c) = r

pre (*AnnAwait r b c*) = *r*

Well-formedness predicate for atomic programs:

consts *atom-com* :: *'a com* ⇒ *bool*
primrec
atom-com (*Parallel Ts*) = *False*
atom-com (*Basic f*) = *True*
atom-com (*Seq c1 c2*) = (*atom-com c1* ∧ *atom-com c2*)
atom-com (*Cond b c1 c2*) = (*atom-com c1* ∧ *atom-com c2*)
atom-com (*While b i c*) = *atom-com c*

end

1.2 Operational Semantics

theory *OG-Tran* **imports** *OG-Com* **begin**

types
'a ann-com-op = (*'a ann-com*) *option*
'a ann-triple-op = (*'a ann-com-op* × *'a assn*)

consts *com* :: *'a ann-triple-op* ⇒ *'a ann-com-op*
primrec *com* (*c*, *q*) = *c*

consts *post* :: *'a ann-triple-op* ⇒ *'a assn*
primrec *post* (*c*, *q*) = *q*

constdefs
All-None :: *'a ann-triple-op list* ⇒ *bool*
All-None Ts ≡ ∀ (*c*, *q*) ∈ *set Ts*. *c* = *None*

1.2.1 The Transition Relation

inductive-set
ann-transition :: ((*'a ann-com-op* × *'a*) × (*'a ann-com-op* × *'a*)) *set*
and *transition* :: ((*'a com* × *'a*) × (*'a com* × *'a*)) *set*
and *ann-transition'* :: (*'a ann-com-op* × *'a*) ⇒ (*'a ann-com-op* × *'a*) ⇒ *bool*
(- −*1*→ -[*81*,*81*] *100*)
and *transition'* :: (*'a com* × *'a*) ⇒ (*'a com* × *'a*) ⇒ *bool*
(- −*P1*→ -[*81*,*81*] *100*)
and *transitions* :: (*'a com* × *'a*) ⇒ (*'a com* × *'a*) ⇒ *bool*
(- −*P*∗→ -[*81*,*81*] *100*)
where
con-0 −*1*→ *con-1* ≡ (*con-0*, *con-1*) ∈ *ann-transition*
| *con-0* −*P1*→ *con-1* ≡ (*con-0*, *con-1*) ∈ *transition*
| *con-0* −*P*∗→ *con-1* ≡ (*con-0*, *con-1*) ∈ *transition**

| *AnnBasic*: (*Some* (*AnnBasic r f*), *s*) −*1*→ (*None*, *f s*)

| *AnnSeq1*: (*Some c0*, *s*) *−1→* (*None*, *t*) ⟹
  (*Some* (*AnnSeq c0 c1*), *s*) *−1→* (*Some c1*, *t*)
| *AnnSeq2*: (*Some c0*, *s*) *−1→* (*Some c2*, *t*) ⟹
  (*Some* (*AnnSeq c0 c1*), *s*) *−1→* (*Some* (*AnnSeq c2 c1*), *t*)

| *AnnCond1T*: $s \in b$ ⟹ (*Some* (*AnnCond1 r b c1 c2*), *s*) *−1→* (*Some c1*, *s*)
| *AnnCond1F*: $s \notin b$ ⟹ (*Some* (*AnnCond1 r b c1 c2*), *s*) *−1→* (*Some c2*, *s*)

| *AnnCond2T*: $s \in b$ ⟹ (*Some* (*AnnCond2 r b c*), *s*) *−1→* (*Some c*, *s*)
| *AnnCond2F*: $s \notin b$ ⟹ (*Some* (*AnnCond2 r b c*), *s*) *−1→* (*None*, *s*)

| *AnnWhileF*: $s \notin b$ ⟹ (*Some* (*AnnWhile r b i c*), *s*) *−1→* (*None*, *s*)
| *AnnWhileT*: $s \in b$ ⟹ (*Some* (*AnnWhile r b i c*), *s*) *−1→*
  (*Some* (*AnnSeq c* (*AnnWhile i b i c*)), *s*)

| *AnnAwait*: ⟦ $s \in b$; *atom-com c*; (*c*, *s*) *−P∗→* (*Parallel* [], *t*) ⟧ ⟹
  (*Some* (*AnnAwait r b c*), *s*) *−1→* (*None*, *t*)

| *Parallel*: ⟦ *i*<*length Ts*; *Ts*!*i* = (*Some c*, *q*); (*Some c*, *s*) *−1→* (*r*, *t*) ⟧
  ⟹ (*Parallel Ts*, *s*) *−P1→* (*Parallel* (*Ts* [*i*:=(*r*, *q*)]), *t*)

| *Basic*: (*Basic f*, *s*) *−P1→* (*Parallel* [], *f s*)

| *Seq1*: *All-None Ts* ⟹ (*Seq* (*Parallel Ts*) *c*, *s*) *−P1→* (*c*, *s*)
| *Seq2*: (*c0*, *s*) *−P1→* (*c2*, *t*) ⟹ (*Seq c0 c1*, *s*) *−P1→* (*Seq c2 c1*, *t*)

| *CondT*: $s \in b$ ⟹ (*Cond b c1 c2*, *s*) *−P1→* (*c1*, *s*)
| *CondF*: $s \notin b$ ⟹ (*Cond b c1 c2*, *s*) *−P1→* (*c2*, *s*)

| *WhileF*: $s \notin b$ ⟹ (*While b i c*, *s*) *−P1→* (*Parallel* [], *s*)
| *WhileT*: $s \in b$ ⟹ (*While b i c*, *s*) *−P1→* (*Seq c* (*While b i c*), *s*)

monos *rtrancl-mono*

The corresponding syntax translations are:

abbreviation
ann-transition-n :: ('*a ann-com-op* × '*a*) ⇒ *nat* ⇒ ('*a ann-com-op* × '*a*)
  ⇒ *bool* (- *−-→* -[*81*,*81*] *100*) **where**
con-0 *−n→* *con-1* ≡ (*con-0*, *con-1*) ∈ *ann-transition*^*n*

abbreviation
ann-transitions :: ('*a ann-com-op* × '*a*) ⇒ ('*a ann-com-op* × '*a*) ⇒ *bool*
  (- *−∗→* -[*81*,*81*] *100*) **where**
con-0 *−∗→* *con-1* ≡ (*con-0*, *con-1*) ∈ *ann-transition**

abbreviation
transition-n :: ('*a com* × '*a*) ⇒ *nat* ⇒ ('*a com* × '*a*) ⇒ *bool*
  (- *−P-→* -[*81*,*81*,*81*] *100*) **where**
con-0 *−Pn→* *con-1* ≡ (*con-0*, *con-1*) ∈ *transition*^*n*

1.2.2 Definition of Semantics

constdefs
ann-sem :: $'a$ *ann-com* $\Rightarrow$ $'a$ $\Rightarrow$ $'a$ *set*
ann-sem c $\equiv$ $\lambda s.$ $\{t.$ (*Some c*, *s*) $-*\rightarrow$ (*None*, *t*)$\}$

ann-SEM :: $'a$ *ann-com* $\Rightarrow$ $'a$ *set* $\Rightarrow$ $'a$ *set*
ann-SEM c S $\equiv$ $\bigcup$ *ann-sem c* ‘ *S*

sem :: $'a$ *com* $\Rightarrow$ $'a$ $\Rightarrow$ $'a$ *set*
sem c $\equiv$ $\lambda s.$ $\{t.$ $\exists$ *Ts*. (*c*, *s*) $-P*\rightarrow$ (*Parallel Ts*, *t*) $\wedge$ *All-None Ts*$\}$

SEM :: $'a$ *com* $\Rightarrow$ $'a$ *set* $\Rightarrow$ $'a$ *set*
SEM c S $\equiv$ $\bigcup$ *sem c* ‘ *S*

syntax *-Omega* :: $'a$ *com*   (Ω *63*)
translations Ω $\rightleftharpoons$ *While UNIV UNIV* (*Basic id*)

consts *fwhile* :: $'a$ *bexp* $\Rightarrow$ $'a$ *com* $\Rightarrow$ *nat* $\Rightarrow$ $'a$ *com*
primrec
fwhile b c 0 = Ω
fwhile b c (*Suc n*) = *Cond b* (*Seq c* (*fwhile b c n*)) (*Basic id*)

Proofs

declare *ann-transition-transition.intros* [*intro*]
inductive-cases *transition-cases*:
(*Parallel T*,*s*) $-P1\rightarrow$ *t*
(*Basic f*, *s*) $-P1\rightarrow$ *t*
(*Seq c1 c2*, *s*) $-P1\rightarrow$ *t*
(*Cond b c1 c2*, *s*) $-P1\rightarrow$ *t*
(*While b i c*, *s*) $-P1\rightarrow$ *t*

lemma *Parallel-empty-lemma* [*rule-format* (*no-asm*)]:
(*Parallel* [],*s*) $-Pn\rightarrow$ (*Parallel Ts*,*t*) $\longrightarrow$ *Ts*=[] $\wedge$ *n=0* $\wedge$ *s=t*
⟨*proof*⟩

lemma *Parallel-AllNone-lemma* [*rule-format* (*no-asm*)]:
All-None Ss $\longrightarrow$ (*Parallel Ss*,*s*) $-Pn\rightarrow$ (*Parallel Ts*,*t*) $\longrightarrow$ *Ts=Ss* $\wedge$ *n=0* $\wedge$ *s=t*
⟨*proof*⟩

lemma *Parallel-AllNone*: *All-None Ts* $\Longrightarrow$ (*SEM* (*Parallel Ts*) *X*) = *X*
⟨*proof*⟩

lemma *Parallel-empty*: *Ts*=[] $\Longrightarrow$ (*SEM* (*Parallel Ts*) *X*) = *X*
⟨*proof*⟩

Set of lemmas from Apt and Olderog "Verification of sequential and concurrent programs", page 63.

lemma *L3-5i*: *X*$\subseteq$*Y* $\Longrightarrow$ *SEM c X* $\subseteq$ *SEM c Y*

$\langle proof \rangle$

lemma *L3-5ii-lemma1*:
$[\![$ *(c1, s1) −P∗→ (Parallel Ts, s2); All-None Ts;*
(c2, s2) −P∗→ (Parallel Ss, s3); All-None Ss $]\!]$
$\Longrightarrow$ *(Seq c1 c2, s1) −P∗→ (Parallel Ss, s3)*
$\langle proof \rangle$

lemma *L3-5ii-lemma2 [rule-format (no-asm)]*:
$\forall$ *c1 c2 s t. (Seq c1 c2, s) −Pn→ (Parallel Ts, t)* $\longrightarrow$
(All-None Ts) $\longrightarrow$ ($\exists$ *y m Rs. (c1,s) −P∗→ (Parallel Rs, y)* $\wedge$
(All-None Rs) $\wedge$ *(c2, y) −Pm→ (Parallel Ts, t)* $\wedge$ $m \leq n$)
$\langle proof \rangle$

lemma *L3-5ii-lemma3*:
$[\![$*(Seq c1 c2,s) −P∗→ (Parallel Ts,t); All-None Ts*$]\!]$ $\Longrightarrow$
($\exists$ *y Rs. (c1,s) −P∗→ (Parallel Rs,y)* $\wedge$ *All-None Rs*
$\wedge$ *(c2,y) −P∗→ (Parallel Ts,t)*)
$\langle proof \rangle$

lemma *L3-5ii*: *SEM (Seq c1 c2) X = SEM c2 (SEM c1 X)*
$\langle proof \rangle$

lemma *L3-5iii*: *SEM (Seq (Seq c1 c2) c3) X = SEM (Seq c1 (Seq c2 c3)) X*
$\langle proof \rangle$

lemma *L3-5iv*:
SEM (Cond b c1 c2) X = (SEM c1 (X $\cap$ *b)) Un (SEM c2 (X* $\cap$ *(−b)))*
$\langle proof \rangle$

lemma *L3-5v-lemma1[rule-format]*:
(S,s) −Pn→ (T,t) $\longrightarrow$ $S{=}\Omega$ $\longrightarrow$ ($\neg$($\exists$ *Rs. T=(Parallel Rs)* $\wedge$ *All-None Rs*))
$\langle proof \rangle$

lemma *L3-5v-lemma2*: $[\![$(Ω, *s) −P∗→ (Parallel Ts, t); All-None Ts* $]\!]$ $\Longrightarrow$ *False*
$\langle proof \rangle$

lemma *L3-5v-lemma3*: *SEM* (Ω) *S* = {}
$\langle proof \rangle$

lemma *L3-5v-lemma4 [rule-format]*:
$\forall$ *s. (While b i c, s) −Pn→ (Parallel Ts, t)* $\longrightarrow$ *All-None Ts* $\longrightarrow$
($\exists$ *k. (fwhile b c k, s) −P∗→ (Parallel Ts, t)*)
$\langle proof \rangle$

lemma *L3-5v-lemma5 [rule-format]*:
$\forall$ *s. (fwhile b c k, s) −P∗→ (Parallel Ts, t)* $\longrightarrow$ *All-None Ts* $\longrightarrow$
(While b i c, s) −P∗→ (Parallel Ts,t)

⟨*proof*⟩

lemma *L3-5v*: *SEM* (*While b i c*) = ($\lambda x.$ ($\bigcup k.$ *SEM* (*fwhile b c k*) *x*))
⟨*proof*⟩

1.3 Validity of Correctness Formulas

constdefs
com-validity :: *'a assn* ⇒ *'a com* ⇒ *'a assn* ⇒ *bool* ((*3*‖= -// -//-) [*90*,*55*,*90*] *50*)
‖= *p c q* ≡ *SEM c p* ⊆ *q*

ann-com-validity :: *'a ann-com* ⇒ *'a assn* ⇒ *bool* (⊨ - - [*60*,*90*] *45*)
⊨ *c q* ≡ *ann-SEM c* (*pre c*) ⊆ *q*

end

1.4 The Proof System

theory *OG-Hoare* **imports** *OG-Tran* **begin**

consts *assertions* :: *'a ann-com* ⇒ (*'a assn*) *set*
primrec
assertions (*AnnBasic r f*) = {*r*}
assertions (*AnnSeq c1 c2*) = *assertions c1* ∪ *assertions c2*
assertions (*AnnCond1 r b c1 c2*) = {*r*} ∪ *assertions c1* ∪ *assertions c2*
assertions (*AnnCond2 r b c*) = {*r*} ∪ *assertions c*
assertions (*AnnWhile r b i c*) = {*r*, *i*} ∪ *assertions c*
assertions (*AnnAwait r b c*) = {*r*}

consts *atomics* :: *'a ann-com* ⇒ (*'a assn* × *'a com*) *set*
primrec
atomics (*AnnBasic r f*) = {(*r*, *Basic f*)}
atomics (*AnnSeq c1 c2*) = *atomics c1* ∪ *atomics c2*
atomics (*AnnCond1 r b c1 c2*) = *atomics c1* ∪ *atomics c2*
atomics (*AnnCond2 r b c*) = *atomics c*
atomics (*AnnWhile r b i c*) = *atomics c*
atomics (*AnnAwait r b c*) = {(*r* ∩ *b*, *c*)}

consts *com* :: *'a ann-triple-op* ⇒ *'a ann-com-op*
primrec *com* (*c*, *q*) = *c*

consts *post* :: *'a ann-triple-op* ⇒ *'a assn*
primrec *post* (*c*, *q*) = *q*

constdefs *interfree-aux* :: (*'a ann-com-op* × *'a assn* × *'a ann-com-op*) ⇒ *bool*
interfree-aux ≡ λ(*co*, *q*, *co'*). *co'*= *None* ∨
(∀ (*r*,*a*) ∈ *atomics* (*the co'*). ‖= (*q* ∩ *r*) *a q* ∧

$(co = None \lor (\forall p \in assertions\ (the\ co).\ \|= (p \cap r)\ a\ p)))$

constdefs $interfree :: (('a\ ann\text{-}triple\text{-}op)\ list) \Rightarrow bool$
$interfree\ Ts \equiv \forall i\ j.\ i < length\ Ts \land j < length\ Ts \land i \neq j \longrightarrow$
$interfree\text{-}aux\ (com\ (Ts!i),\ post\ (Ts!i),\ com\ (Ts!j))$

inductive
$oghoare :: 'a\ assn \Rightarrow 'a\ com \Rightarrow 'a\ assn \Rightarrow bool\ \ ((3\|-\ \text{-}//\text{-}//\text{-})\ [90,55,90]\ 50)$
and $ann\text{-}hoare :: 'a\ ann\text{-}com \Rightarrow 'a\ assn \Rightarrow bool\ \ ((2\vdash\ \text{-}//\ \text{-})\ [60,90]\ 45)$
where
$AnnBasic: r \subseteq \{s.\ f\ s \in q\} \Longrightarrow \vdash (AnnBasic\ r\ f)\ q$

$|\ AnnSeq: \ \ [\![\vdash c0\ pre\ c1; \vdash c1\ q\ ]\!] \Longrightarrow \vdash (AnnSeq\ c0\ c1)\ q$

$|\ AnnCond1: [\![r \cap b \subseteq pre\ c1; \vdash c1\ q;\ r \cap -b \subseteq pre\ c2; \vdash c2\ q]\!]$
$\Longrightarrow \vdash (AnnCond1\ r\ b\ c1\ c2)\ q$
$|\ AnnCond2: [\![r \cap b \subseteq pre\ c; \vdash c\ q;\ r \cap -b \subseteq q\ ]\!] \Longrightarrow \vdash (AnnCond2\ r\ b\ c)\ q$

$|\ AnnWhile: [\![r \subseteq i;\ i \cap b \subseteq pre\ c; \vdash c\ i;\ i \cap -b \subseteq q\ ]\!]$
$\Longrightarrow \vdash (AnnWhile\ r\ b\ i\ c)\ q$

$|\ AnnAwait: \ \ [\![atom\text{-}com\ c;\ \|-\ (r \cap b)\ c\ q\ ]\!] \Longrightarrow \vdash (AnnAwait\ r\ b\ c)\ q$

$|\ AnnConseq: [\![\vdash c\ q;\ q \subseteq q'\ ]\!] \Longrightarrow \vdash c\ q'$

$|\ Parallel: [\![\forall i{<}length\ Ts.\ \exists c\ q.\ Ts!i = (Some\ c,\ q) \land \vdash c\ q;\ interfree\ Ts\ ]\!]$
$\Longrightarrow \|-\ (\bigcap i{\in}\{i.\ i{<}length\ Ts\}.\ pre(the(com(Ts!i))))$
$Parallel\ Ts$
$(\bigcap i{\in}\{i.\ i{<}length\ Ts\}.\ post(Ts!i))$

$|\ Basic: \ \ \|-\ \{s.\ f\ s \in q\}\ (Basic\ f)\ q$

$|\ Seq: \ \ [\![\|-\ p\ c1\ r;\ \|-\ r\ c2\ q\ ]\!] \Longrightarrow \|-\ p\ (Seq\ c1\ c2)\ q$

$|\ Cond: \ \ [\![\|-\ (p \cap b)\ c1\ q;\ \|-\ (p \cap -b)\ c2\ q\ ]\!] \Longrightarrow \|-\ p\ (Cond\ b\ c1\ c2)\ q$

$|\ While: \ [\![\|-\ (p \cap b)\ c\ p\ ]\!] \Longrightarrow \|-\ p\ (While\ b\ i\ c)\ (p \cap -b)$

$|\ Conseq: [\![p' \subseteq p;\ \|-\ p\ c\ q\ ;\ q \subseteq q'\ ]\!] \Longrightarrow \|-\ p'\ c\ q'$

1.5 Soundness

lemmas *[cong del] = if-weak-cong*

lemmas *ann-hoare-induct = oghoare-ann-hoare.induct [THEN conjunct2]*
lemmas *oghoare-induct = oghoare-ann-hoare.induct [THEN conjunct1]*

lemmas *AnnBasic = oghoare-ann-hoare.AnnBasic*

lemmas *AnnSeq* = *oghoare-ann-hoare.AnnSeq*
lemmas *AnnCond1* = *oghoare-ann-hoare.AnnCond1*
lemmas *AnnCond2* = *oghoare-ann-hoare.AnnCond2*
lemmas *AnnWhile* = *oghoare-ann-hoare.AnnWhile*
lemmas *AnnAwait* = *oghoare-ann-hoare.AnnAwait*
lemmas *AnnConseq* = *oghoare-ann-hoare.AnnConseq*

lemmas *Parallel* = *oghoare-ann-hoare.Parallel*
lemmas *Basic* = *oghoare-ann-hoare.Basic*
lemmas *Seq* = *oghoare-ann-hoare.Seq*
lemmas *Cond* = *oghoare-ann-hoare.Cond*
lemmas *While* = *oghoare-ann-hoare.While*
lemmas *Conseq* = *oghoare-ann-hoare.Conseq*

1.5.1 Soundness of the System for Atomic Programs

lemma *Basic-ntran* [*rule-format*]:
$(Basic\ f,\ s)\ -Pn\rightarrow\ (Parallel\ Ts,\ t) \longrightarrow All\text{-}None\ Ts \longrightarrow t = f\ s$
⟨*proof*⟩

lemma *SEM-fwhile*: $SEM\ S\ (p \cap b) \subseteq p \Longrightarrow SEM\ (fwhile\ b\ S\ k)\ p \subseteq (p \cap -b)$
⟨*proof*⟩

lemma *atom-hoare-sound* [*rule-format*]:
$\|\!-\ p\ c\ q \longrightarrow atom\text{-}com(c) \longrightarrow \|\!=\ p\ c\ q$
⟨*proof*⟩

1.5.2 Soundness of the System for Component Programs

inductive-cases *ann-transition-cases*:
$(None,s)\ -1\rightarrow\ (c',\ s')$
$(Some\ (AnnBasic\ r\ f),s)\ -1\rightarrow\ (c',\ s')$
$(Some\ (AnnSeq\ c1\ c2),\ s)\ -1\rightarrow\ (c',\ s')$
$(Some\ (AnnCond1\ r\ b\ c1\ c2),\ s)\ -1\rightarrow\ (c',\ s')$
$(Some\ (AnnCond2\ r\ b\ c),\ s)\ -1\rightarrow\ (c',\ s')$
$(Some\ (AnnWhile\ r\ b\ I\ c),\ s)\ -1\rightarrow\ (c',\ s')$
$(Some\ (AnnAwait\ r\ b\ c),s)\ -1\rightarrow\ (c',\ s')$

Strong Soundness for Component Programs:

lemma *ann-hoare-case-analysis* [*rule-format*]: $\vdash C\ q' \longrightarrow$
$((\forall r\,f.\ C = AnnBasic\ r\ f \longrightarrow (\exists q.\ r \subseteq \{s.\ f\ s \in q\} \wedge q \subseteq q')) \wedge$
$(\forall c0\ c1.\ C = AnnSeq\ c0\ c1 \longrightarrow (\exists q.\ q \subseteq q' \wedge \vdash c0\ pre\ c1 \wedge \vdash c1\ q)) \wedge$
$(\forall r\ b\ c1\ c2.\ C = AnnCond1\ r\ b\ c1\ c2 \longrightarrow (\exists q.\ q \subseteq q' \wedge$
$r \cap b \subseteq pre\ c1 \wedge \vdash c1\ q \wedge r \cap -b \subseteq pre\ c2 \wedge \vdash c2\ q)) \wedge$
$(\forall r\ b\ c.\ C = AnnCond2\ r\ b\ c \longrightarrow$
$(\exists q.\ q \subseteq q' \wedge r \cap b \subseteq pre\ c \wedge \vdash c\ q \wedge r \cap -b \subseteq q)) \wedge$
$(\forall r\ i\ b\ c.\ C = AnnWhile\ r\ b\ i\ c \longrightarrow$
$(\exists q.\ q \subseteq q' \wedge r \subseteq i \wedge i \cap b \subseteq pre\ c \wedge \vdash c\ i \wedge i \cap -b \subseteq q)) \wedge$
$(\forall r\ b\ c.\ C = AnnAwait\ r\ b\ c \longrightarrow (\exists q.\ q \subseteq q' \wedge \|\!-\ (r \cap b)\ c\ q)))$

⟨proof⟩

lemma *Help*: $(transition \cap \{(x,y).\ True\}) = (transition)$
⟨proof⟩

lemma *Strong-Soundness-aux-aux* [*rule-format*]:
$(co,\ s)\ -1\rightarrow\ (co',\ t) \longrightarrow (\forall\, c.\ co = Some\ c \longrightarrow s\in pre\ c \longrightarrow$
$(\forall\, q.\ \vdash c\ q \longrightarrow (if\ co' = None\ then\ t\in q\ else\ t \in pre(the\ co') \wedge \vdash (the\ co')\ q\ )))$
⟨proof⟩

lemma *Strong-Soundness-aux*: $[\![\ (Some\ c,\ s)\ -*\rightarrow\ (co,\ t);\ s \in pre\ c;\ \vdash c\ q\ ]\!]$
$\Longrightarrow if\ co = None\ then\ t \in q\ else\ t \in pre\ (the\ co) \wedge \vdash (the\ co)\ q$
⟨proof⟩

lemma *Strong-Soundness*: $[\![\ (Some\ c,\ s)-*\rightarrow(co,\ t);\ s \in pre\ c;\ \vdash c\ q\ ]\!]$
$\Longrightarrow if\ co = None\ then\ t\in q\ else\ t \in pre\ (the\ co)$
⟨proof⟩

lemma *ann-hoare-sound*: $\vdash c\ q \ \Longrightarrow \models c\ q$
⟨proof⟩

1.5.3 Soundness of the System for Parallel Programs

lemma *Parallel-length-post-P1*: $(Parallel\ Ts,s)\ -P1\rightarrow\ (R',\ t) \Longrightarrow$
$(\exists\, Rs.\ R' = (Parallel\ Rs) \wedge (length\ Rs) = (length\ Ts) \wedge$
$(\forall\, i.\ i<length\ Ts \longrightarrow post(Rs\ !\ i) = post(Ts\ !\ i)))$
⟨proof⟩

lemma *Parallel-length-post-PStar*: $(Parallel\ Ts,s)\ -P*\rightarrow\ (R',t) \Longrightarrow$
$(\exists\, Rs.\ R' = (Parallel\ Rs) \wedge (length\ Rs) = (length\ Ts) \wedge$
$(\forall\, i.\ i<length\ Ts \longrightarrow post(Ts\ !\ i) = post(Rs\ !\ i)))$
⟨proof⟩

lemma *assertions-lemma*: $pre\ c \in assertions\ c$
⟨proof⟩

lemma *interfree-aux1* [*rule-format*]:
$(c,s)\ -1\rightarrow\ (r,t) \ \longrightarrow (interfree\text{-}aux(c1,\ q1,\ c) \longrightarrow interfree\text{-}aux(c1,\ q1,\ r))$
⟨proof⟩

lemma *interfree-aux2* [*rule-format*]:
$(c,s)\ -1\rightarrow\ (r,t) \longrightarrow (interfree\text{-}aux(c,\ q,\ a) \ \longrightarrow interfree\text{-}aux(r,\ q,\ a)\ )$
⟨proof⟩

lemma *interfree-lemma*: $[\![\ (Some\ c,\ s)\ -1\rightarrow\ (r,\ t);interfree\ Ts\ ;\ i<length\ Ts;$
$Ts!i = (Some\ c,\ q)\ ]\!] \Longrightarrow interfree\ (Ts[i:=\ (r,\ q)])$
⟨proof⟩

Strong Soundness Theorem for Parallel Programs:

lemma *Parallel-Strong-Soundness-Seq-aux*:
$[\![$*interfree Ts*; i<*length Ts*; *com*(*Ts* ! *i*) = *Some*(*AnnSeq c0 c1*) $]\!]$
$\Longrightarrow$ *interfree* (*Ts*[*i*:=(*Some c0*, *pre c1*)])
⟨*proof*⟩

lemma *Parallel-Strong-Soundness-Seq* [*rule-format* (*no-asm*)]:
$[\![$ ∀ *i*<*length Ts*. (*if com*(*Ts*!*i*) = *None then b* ∈ *post*(*Ts*!*i*)
else b ∈ *pre*(*the*(*com*(*Ts*!*i*))) ∧ ⊢ *the*(*com*(*Ts*!*i*)) *post*(*Ts*!*i*));
com(*Ts* ! *i*) = *Some*(*AnnSeq c0 c1*); *i*<*length Ts*; *interfree Ts* $]\!] \Longrightarrow$
(∀ *ia*<*length Ts*. (*if com*(*Ts*[*i*:=(*Some c0*, *pre c1*)]! *ia*) = *None*
then b ∈ *post*(*Ts*[*i*:=(*Some c0*, *pre c1*)]! *ia*)
else b ∈ *pre*(*the*(*com*(*Ts*[*i*:=(*Some c0*, *pre c1*)]! *ia*))) ∧
⊢ *the*(*com*(*Ts*[*i*:=(*Some c0*, *pre c1*)]! *ia*)) *post*(*Ts*[*i*:=(*Some c0*, *pre c1*)]! *ia*)))
∧ *interfree* (*Ts*[*i*:= (*Some c0*, *pre c1*)])
⟨*proof*⟩

lemma *Parallel-Strong-Soundness-aux-aux* [*rule-format*]:
(*Some c*, *b*) −*1*→ (*co*, *t*) ⟶
(∀ *Ts*. *i*<*length Ts* ⟶ *com*(*Ts* ! *i*) = *Some c* ⟶
(∀ *i*<*length Ts*. (*if com*(*Ts* ! *i*) = *None then b*∈*post*(*Ts*!*i*)
else b∈*pre*(*the*(*com*(*Ts*!*i*))) ∧ ⊢ *the*(*com*(*Ts*!*i*)) *post*(*Ts*!*i*))) ⟶
interfree Ts ⟶
(∀ *j*. *j*<*length Ts* ∧ *i*≠*j* ⟶ (*if com*(*Ts*!*j*) = *None then t*∈*post*(*Ts*!*j*)
else t∈*pre*(*the*(*com*(*Ts*!*j*))) ∧ ⊢ *the*(*com*(*Ts*!*j*)) *post*(*Ts*!*j*))))
⟨*proof*⟩

lemma *Parallel-Strong-Soundness-aux* [*rule-format*]:
$[\![$(*Ts*′,*s*) −*P*∗→ (*Rs*′,*t*); *Ts*′ = (*Parallel Ts*); *interfree Ts*;
∀ *i*. *i*<*length Ts* ⟶ (∃ *c q*. (*Ts* ! *i*) = (*Some c*, *q*) ∧ *s*∈(*pre c*) ∧ ⊢ *c q*) $]\!] \Longrightarrow$
∀ *Rs*. *Rs*′ = (*Parallel Rs*) ⟶ (∀ *j*. *j*<*length Rs* ⟶
(*if com*(*Rs* ! *j*) = *None then t*∈*post*(*Ts* ! *j*)
else t∈*pre*(*the*(*com*(*Rs* ! *j*))) ∧ ⊢ *the*(*com*(*Rs* ! *j*)) *post*(*Ts* ! *j*))) ∧ *interfree Rs*
⟨*proof*⟩

lemma *Parallel-Strong-Soundness*:
$[\![$(*Parallel Ts*, *s*) −*P*∗→ (*Parallel Rs*, *t*); *interfree Ts*; *j*<*length Rs*;
∀ *i*. *i*<*length Ts* ⟶ (∃ *c q*. *Ts* ! *i* = (*Some c*, *q*) ∧ *s*∈*pre c* ∧ ⊢ *c q*) $]\!] \Longrightarrow$
if com(*Rs* ! *j*) = *None then t*∈*post*(*Ts* ! *j*) *else t*∈*pre* (*the*(*com*(*Rs* ! *j*)))
⟨*proof*⟩

lemma *oghoare-sound* [*rule-format*]: ‖− *p c q* ⟶ ‖= *p c q*
⟨*proof*⟩

end

1.6 Generation of Verification Conditions

theory *OG-Tactics* **imports** *OG-Hoare*
begin

lemmas *ann-hoare-intros=AnnBasic AnnSeq AnnCond1 AnnCond2 AnnWhile AnnAwait AnnConseq*
lemmas *oghoare-intros=Parallel Basic Seq Cond While Conseq*

lemma *ParallelConseqRule*:
$[\![\ p \subseteq (\bigcap i{\in}\{i.\ i{<}length\ Ts\}.\ pre(the(com(Ts\ !\ i))));$
$\Vert{-}\ (\bigcap i{\in}\{i.\ i{<}length\ Ts\}.\ pre(the(com(Ts\ !\ i))))$
$(Parallel\ Ts)$
$(\bigcap i{\in}\{i.\ i{<}length\ Ts\}.\ post(Ts\ !\ i));$
$(\bigcap i{\in}\{i.\ i{<}length\ Ts\}.\ post(Ts\ !\ i)) \subseteq q\ ]\!]$
$\Longrightarrow \Vert{-}\ p\ (Parallel\ Ts)\ q$
⟨*proof*⟩

lemma *SkipRule*: $p \subseteq q \Longrightarrow \Vert{-}\ p\ (Basic\ id)\ q$
⟨*proof*⟩

lemma *BasicRule*: $p \subseteq \{s.\ (f\ s){\in}q\} \Longrightarrow \Vert{-}\ p\ (Basic\ f)\ q$
⟨*proof*⟩

lemma *SeqRule*: $[\![\ \Vert{-}\ p\ c1\ r;\ \Vert{-}\ r\ c2\ q\ ]\!] \Longrightarrow \Vert{-}\ p\ (Seq\ c1\ c2)\ q$
⟨*proof*⟩

lemma *CondRule*:
$[\![\ p \subseteq \{s.\ (s{\in}b \longrightarrow s{\in}w) \wedge (s{\notin}b \longrightarrow s{\in}w')\};\ \Vert{-}\ w\ c1\ q;\ \Vert{-}\ w'\ c2\ q\ ]\!]$
$\Longrightarrow \Vert{-}\ p\ (Cond\ b\ c1\ c2)\ q$
⟨*proof*⟩

lemma *WhileRule*: $[\![\ p \subseteq i;\ \Vert{-}\ (i \cap b)\ c\ i\ ;\ (i \cap (-b)) \subseteq q\ ]\!]$
$\Longrightarrow \Vert{-}\ p\ (While\ b\ i\ c)\ q$
⟨*proof*⟩

Three new proof rules for special instances of the *AnnBasic* and the *AnnAwait* commands when the transformation performed on the state is the identity, and for an *AnnAwait* command where the boolean condition is $\{s.\ True\}$:

lemma *AnnatomRule*:
$[\![\ atom\text{-}com(c);\ \Vert{-}\ r\ c\ q\ ]\!] \Longrightarrow \vdash (AnnAwait\ r\ \{s.\ True\}\ c)\ q$
⟨*proof*⟩

lemma *AnnskipRule*:
$r \subseteq q \Longrightarrow \vdash (AnnBasic\ r\ id)\ q$
⟨*proof*⟩

lemma *AnnwaitRule*:
$[\![\ (r \cap b) \subseteq q\ ]\!] \Longrightarrow \vdash (AnnAwait\ r\ b\ (Basic\ id))\ q$
⟨*proof*⟩

Lemmata to avoid using the definition of *map-ann-hoare*, *interfree-aux*, *interfree-swap* and *interfree* by splitting it into different cases:

lemma *interfree-aux-rule1*: $interfree\text{-}aux(co,\ q,\ None)$
⟨*proof*⟩

lemma *interfree-aux-rule2*:
$\forall (R,r)\in(atomics\ a).\ \Vert\!-\ (q \cap R)\ r\ q \Longrightarrow interfree\text{-}aux(None,\ q,\ Some\ a)$
⟨*proof*⟩

lemma *interfree-aux-rule3*:
$(\forall (R,\ r)\in(atomics\ a).\ \Vert\!-\ (q \cap R)\ r\ q \wedge (\forall p\in(assertions\ c).\ \Vert\!-\ (p \cap R)\ r\ p))$
$\Longrightarrow interfree\text{-}aux(Some\ c,\ q,\ Some\ a)$
⟨*proof*⟩

lemma *AnnBasic-assertions*:
$[\![interfree\text{-}aux(None,\ r,\ Some\ a);\ interfree\text{-}aux(None,\ q,\ Some\ a)]\!] \Longrightarrow$
$interfree\text{-}aux(Some\ (AnnBasic\ r\ f),\ q,\ Some\ a)$
⟨*proof*⟩

lemma *AnnSeq-assertions*:
$[\![\ interfree\text{-}aux(Some\ c1,\ q,\ Some\ a);\ interfree\text{-}aux(Some\ c2,\ q,\ Some\ a)]\!]\Longrightarrow$
$interfree\text{-}aux(Some\ (AnnSeq\ c1\ c2),\ q,\ Some\ a)$
⟨*proof*⟩

lemma *AnnCond1-assertions*:
$[\![\ interfree\text{-}aux(None,\ r,\ Some\ a);\ interfree\text{-}aux(Some\ c1,\ q,\ Some\ a);$
$interfree\text{-}aux(Some\ c2,\ q,\ Some\ a)]\!]\Longrightarrow$
$interfree\text{-}aux(Some(AnnCond1\ r\ b\ c1\ c2),\ q,\ Some\ a)$
⟨*proof*⟩

lemma *AnnCond2-assertions*:
$[\![\ interfree\text{-}aux(None,\ r,\ Some\ a);\ interfree\text{-}aux(Some\ c,\ q,\ Some\ a)]\!]\Longrightarrow$
$interfree\text{-}aux(Some\ (AnnCond2\ r\ b\ c),\ q,\ Some\ a)$
⟨*proof*⟩

lemma *AnnWhile-assertions*:
$[\![\ interfree\text{-}aux(None,\ r,\ Some\ a);\ interfree\text{-}aux(None,\ i,\ Some\ a);$
$interfree\text{-}aux(Some\ c,\ q,\ Some\ a)]\!]\Longrightarrow$
$interfree\text{-}aux(Some\ (AnnWhile\ r\ b\ i\ c),\ q,\ Some\ a)$
⟨*proof*⟩

lemma *AnnAwait-assertions*:
$[\![\ interfree\text{-}aux(None,\ r,\ Some\ a);\ interfree\text{-}aux(None,\ q,\ Some\ a)]\!]\Longrightarrow$
$interfree\text{-}aux(Some\ (AnnAwait\ r\ b\ c),\ q,\ Some\ a)$
⟨*proof*⟩

lemma *AnnBasic-atomics*:
$\Vert\!-\ (q \cap r)\ (Basic\ f)\ q \Longrightarrow interfree\text{-}aux(None,\ q,\ Some\ (AnnBasic\ r\ f))$
⟨*proof*⟩

lemma *AnnSeq-atomics*:
$\llbracket$ *interfree-aux*(*Any*, *q*, *Some a1*); *interfree-aux*(*Any*, *q*, *Some a2*)$\rrbracket \Longrightarrow$
interfree-aux(*Any*, *q*, *Some* (*AnnSeq a1 a2*))
⟨*proof*⟩

lemma *AnnCond1-atomics*:
$\llbracket$ *interfree-aux*(*Any*, *q*, *Some a1*); *interfree-aux*(*Any*, *q*, *Some a2*)$\rrbracket \Longrightarrow$
interfree-aux(*Any*, *q*, *Some* (*AnnCond1 r b a1 a2*))
⟨*proof*⟩

lemma *AnnCond2-atomics*:
interfree-aux (*Any*, *q*, *Some a*)$\Longrightarrow$ *interfree-aux*(*Any*, *q*, *Some* (*AnnCond2 r b a*))
⟨*proof*⟩

lemma *AnnWhile-atomics*: *interfree-aux* (*Any*, *q*, *Some a*)
$\Longrightarrow$ *interfree-aux*(*Any*, *q*, *Some* (*AnnWhile r b i a*))
⟨*proof*⟩

lemma *Annatom-atomics*:
$\|-$ $(q \cap r)$ *a q* $\Longrightarrow$ *interfree-aux* (*None*, *q*, *Some* (*AnnAwait r* $\{x.\ True\}$ *a*))
⟨*proof*⟩

lemma *AnnAwait-atomics*:
$\|-$ $(q \cap (r \cap b))$ *a q* $\Longrightarrow$ *interfree-aux* (*None*, *q*, *Some* (*AnnAwait r b a*))
⟨*proof*⟩

constdefs
interfree-swap :: ($'a$ *ann-triple-op* $*$ ($'a$ *ann-triple-op*) *list*) $\Rightarrow$ *bool*
interfree-swap == $\lambda(x,\ xs).\ \forall y{\in}set\ xs.$ *interfree-aux* (*com x*, *post x*, *com y*)
$\wedge$ *interfree-aux*(*com y*, *post y*, *com x*)

lemma *interfree-swap-Empty*: *interfree-swap* (*x*, [])
⟨*proof*⟩

lemma *interfree-swap-List*:
$\llbracket$ *interfree-aux* (*com x*, *post x*, *com y*);
interfree-aux (*com y*, *post y* ,*com x*); *interfree-swap* (*x*, *xs*) $\rrbracket$
$\Longrightarrow$ *interfree-swap* (*x*, *y#xs*)
⟨*proof*⟩

lemma *interfree-swap-Map*: $\forall k.\ i{\le}k \wedge k{<}j \longrightarrow$ *interfree-aux* (*com x*, *post x*, *c k*)
$\wedge$ *interfree-aux* (*c k*, *Q k*, *com x*)
$\Longrightarrow$ *interfree-swap* (*x*, *map* ($\lambda k.$ (*c k*, *Q k*)) [*i*..<*j*])
⟨*proof*⟩

lemma *interfree-Empty*: *interfree* []
⟨*proof*⟩

lemma *interfree-List*:
$[\![$ *interfree-swap*$(x,\ xs)$; *interfree* xs $]\!] \Longrightarrow$ *interfree* $(x\#xs)$
⟨*proof*⟩

lemma *interfree-Map*:
$(\forall\, i\ j.\ a{\le}i \wedge i{<}b \wedge a{\le}j \wedge j{<}b \ \wedge i{\neq}j \longrightarrow$ *interfree-aux* $(c\ i,\ Q\ i,\ c\ j))$
$\Longrightarrow$ *interfree* $(map\ (\lambda k.\ (c\ k,\ Q\ k))\ [a..{<}b])$
⟨*proof*⟩

constdefs *map-ann-hoare* :: $(('a\ ann\text{-}com\text{-}op * 'a\ assn)\ list) \Rightarrow bool\ \ ([\vdash]\ \text{-}\ [0]\ 45)$
$[\vdash]\ Ts == (\forall\, i{<}length\ Ts.\ \exists\, c\ q.\ Ts!i{=}(Some\ c,\ q) \wedge \vdash\ c\ q)$

lemma *MapAnnEmpty*: $[\vdash]\ []$
⟨*proof*⟩

lemma *MapAnnList*: $[\![\ \vdash\ c\ q\ ;\ [\vdash]\ xs\ ]\!] \Longrightarrow [\vdash]\ (Some\ c,q)\#xs$
⟨*proof*⟩

lemma *MapAnnMap*:
$\forall\, k.\ i{\le}k \wedge k{<}j \longrightarrow \vdash (c\ k)\ (Q\ k) \Longrightarrow [\vdash]\ map\ (\lambda k.\ (Some\ (c\ k),\ Q\ k))\ [i..{<}j]$
⟨*proof*⟩

lemma *ParallelRule*:$[\![\ [\vdash]\ Ts\ ;\ interfree\ Ts\ ]\!]$
$\Longrightarrow \Vdash\ (\bigcap i{\in}\{i.\ i{<}length\ Ts\}.\ pre(the(com(Ts!i))))$
Parallel Ts
$(\bigcap i{\in}\{i.\ i{<}length\ Ts\}.\ post(Ts!i))$
⟨*proof*⟩

The following are some useful lemmas and simplification tactics to control which theorems are used to simplify at each moment, so that the original input does not suffer any unexpected transformation.

lemma *Compl-Collect*: $-(Collect\ b) = \{x.\ \neg(b\ x)\}$
⟨*proof*⟩
lemma *list-length*: $length\ []{=}0 \wedge length\ (x\#xs) = Suc(length\ xs)$
⟨*proof*⟩
lemma *list-lemmas*: $length\ []{=}0 \wedge length\ (x\#xs) = Suc(length\ xs)$
$\wedge\ (x\#xs)\ !\ 0{=}x \wedge (x\#xs)\ !\ Suc\ n = xs\ !\ n$
⟨*proof*⟩
lemma *le-Suc-eq-insert*: $\{i.\ i\ {<}Suc\ n\} = insert\ n\ \{i.\ i{<}\ n\}$
⟨*proof*⟩
lemmas *primrecdef-list* = *pre.simps assertions.simps atomics.simps atom-com.simps*
lemmas *my-simp-list* = *list-lemmas fst-conv snd-conv*
not-less0 refl le-Suc-eq-insert Suc-not-Zero Zero-not-Suc Suc-Suc-eq
Collect-mem-eq ball-simps option.simps primrecdef-list
lemmas *ParallelConseq-list* = *INTER-def Collect-conj-eq length-map length-upt*
length-append list-length

⟨*ML*⟩

The following tactic applies *tac* to each conjunct in a subgoal of the form $A \Longrightarrow a1 \wedge a2 \wedge .. \wedge an$ returning n subgoals, one for each conjunct:

⟨*ML*⟩

Tactic for the generation of the verification conditions

The tactic basically uses two subtactics:

HoareRuleTac is called at the level of parallel programs, it uses the ParallelTac to solve parallel composition of programs. This verification has two parts, namely, (1) all component programs are correct and (2) they are interference free. *HoareRuleTac* is also called at the level of atomic regions, i.e. ⟨ ⟩ and *AWAIT b THEN - END*, and at each interference freedom test.

AnnHoareRuleTac is for component programs which are annotated programs and so, there are not unknown assertions (no need to use the parameter precond, see NOTE).

NOTE: precond(::bool) informs if the subgoal has the form $\|-$ *?p c q*, in this case we have precond=False and the generated verification condition would have the form *?p* $\subseteq \ldots$ which can be solved by *rtac subset-refl*, if True we proceed to simplify it using the simplification tactics above.

⟨*ML*⟩

The final tactic is given the name *oghoare*:

⟨*ML*⟩

Notice that the tactic for parallel programs *oghoare-tac* is initially invoked with the value *true* for the parameter *precond*.

Parts of the tactic can be also individually used to generate the verification conditions for annotated sequential programs and to generate verification conditions out of interference freedom tests:

⟨*ML*⟩

The so defined ML tactics are then "exported" to be used in Isabelle proofs.

⟨*ML*⟩

Tactics useful for dealing with the generated verification conditions:

⟨*ML*⟩

end

1.7 Concrete Syntax

theory *Quote-Antiquote* **imports** *Main* **begin**

syntax
  -quote :: $'b \Rightarrow ('a \Rightarrow 'b)$ ((«-») [0] 1000)
  -antiquote :: $('a \Rightarrow 'b) \Rightarrow 'b$ (´- [1000] 1000)
  -Assert :: $'a \Rightarrow 'a$ *set* ((.{-}.) [0] 1000)

syntax (*xsymbols*)
  -Assert :: $'a \Rightarrow 'a$ *set* ((⦃-⦄) [0] 1000)

translations
  .{b}. ⇀ *Collect* «b»

⟨*ML*⟩

end
theory *OG-Syntax*
imports *OG-Tactics Quote-Antiquote*
begin

Syntax for commands and for assertions and boolean expressions in commands *com* and annotated commands *ann-com*.

syntax
  -Assign :: *idt* ⇒ $'b \Rightarrow 'a$ *com* ((´- :=/ -) [70, 65] 61)
  -AnnAssign :: $'a$ *assn* ⇒ *idt* ⇒ $'b \Rightarrow 'a$ *com* ((- ´- :=/ -) [90,70,65] 61)

translations
  ´ x := a ⇀ *Basic* «´ (*-update-name* x (*K-record* a))»
  r ´ x := a ⇀ *AnnBasic* r «´ (*-update-name* x (*K-record* a))»

syntax
  -AnnSkip :: $'a$ *assn* ⇒ $'a$ *ann-com* (-//SKIP [90] 63)
  -AnnSeq :: $'a$ *ann-com* ⇒ $'a$ *ann-com* ⇒ $'a$ *ann-com* (-;;/ - [60,61] 60)

  -AnnCond1 :: $'a$ *assn* ⇒ $'a$ *bexp* ⇒ $'a$ *ann-com* ⇒ $'a$ *ann-com* ⇒ $'a$ *ann-com*
    (- //IF - /THEN - /ELSE - /FI [90,0,0,0] 61)
  -AnnCond2 :: $'a$ *assn* ⇒ $'a$ *bexp* ⇒ $'a$ *ann-com* ⇒ $'a$ *ann-com*
    (- //IF - /THEN - /FI [90,0,0] 61)
  -AnnWhile :: $'a$ *assn* ⇒ $'a$ *bexp* ⇒ $'a$ *assn* ⇒ $'a$ *ann-com* ⇒ $'a$ *ann-com*
    (- //WHILE - /INV - //DO -//OD [90,0,0,0] 61)
  -AnnAwait :: $'a$ *assn* ⇒ $'a$ *bexp* ⇒ $'a$ *com* ⇒ $'a$ *ann-com*
    (- //AWAIT - /THEN /- /END [90,0,0] 61)
  -AnnAtom :: $'a$ *assn* ⇒ $'a$ *com* ⇒ $'a$ *ann-com* (-//⟨-⟩ [90,0] 61)
  -AnnWait :: $'a$ *assn* ⇒ $'a$ *bexp* ⇒ $'a$ *ann-com* (-//WAIT - END [90,0] 61)

  -Skip :: $'a$ *com* (SKIP 63)
  -Seq :: $'a$ *com* ⇒ $'a$ *com* ⇒ $'a$ *com* (-,,/ - [55, 56] 55)

-Cond :: *'a bexp* ⇒ *'a com* ⇒ *'a com* ⇒ *'a com*
((*0IF -/ THEN -/ ELSE -/ FI*) [*0*, *0*, *0*] *61*)
-Cond2 :: *'a bexp* ⇒ *'a com* ⇒ *'a com* (*IF - THEN - FI* [*0*,*0*] *56*)
-While-inv :: *'a bexp* ⇒ *'a assn* ⇒ *'a com* ⇒ *'a com*
((*0WHILE -/ INV - //DO - /OD*) [*0*, *0*, *0*] *61*)
-While :: *'a bexp* ⇒ *'a com* ⇒ *'a com*
((*0WHILE - //DO - /OD*) [*0*, *0*] *61*)

translations
SKIP ⇌ *Basic id*
c-1,, *c-2* ⇌ *Seq c-1 c-2*

IF b THEN c1 ELSE c2 FI ⇀ *Cond* .{*b*}. *c1 c2*
IF b THEN c FI ⇌ *IF b THEN c ELSE SKIP FI*
WHILE b INV i DO c OD ⇀ *While* .{*b*}. *i c*
WHILE b DO c OD ⇌ *WHILE b INV arbitrary DO c OD*

r SKIP ⇌ *AnnBasic r id*
c-1;; *c-2* ⇌ *AnnSeq c-1 c-2*
r IF b THEN c1 ELSE c2 FI ⇀ *AnnCond1 r* .{*b*}. *c1 c2*
r IF b THEN c FI ⇀ *AnnCond2 r* .{*b*}. *c*
r WHILE b INV i DO c OD ⇀ *AnnWhile r* .{*b*}. *i c*
r AWAIT b THEN c END ⇀ *AnnAwait r* .{*b*}. *c*
r ⟨*c*⟩ ⇌ *r AWAIT True THEN c END*
r WAIT b END ⇌ *r AWAIT b THEN SKIP END*

nonterminals
prgs

syntax
-PAR :: *prgs* ⇒ *'a* (*COBEGIN//-//COEND* [*57*] *56*)
-prg :: [*'a*, *'a*] ⇒ *prgs* (*-//-* [*60*, *90*] *57*)
-prgs :: [*'a*, *'a*, *prgs*] ⇒ *prgs* (*-//-//‖//-* [*60*,*90*,*57*] *57*)

-prg-scheme :: [*'a*, *'a*, *'a*, *'a*, *'a*] ⇒ *prgs*
(*SCHEME* [- ≤ - < -] *-// -* [*0*,*0*,*0*,*60*, *90*] *57*)

translations
-prg c q ⇌ [(*Some c*, *q*)]
-prgs c q ps ⇌ (*Some c*, *q*) # *ps*
-PAR ps ⇌ *Parallel ps*

-prg-scheme j i k c q ⇌ *map* (λ*i*. (*Some c*, *q*)) [*j*..<*k*]

⟨*ML*⟩

end

1.8 Examples

theory *OG-Examples* **imports** *OG-Syntax* **begin**

1.8.1 Mutual Exclusion

Peterson's Algorithm I

Eike Best. "Semantics of Sequential and Parallel Programs", page 217.

record *Petersons-mutex-1* =
pr1 :: *nat*
pr2 :: *nat*
in1 :: *bool*
in2 :: *bool*
hold :: *nat*

lemma *Petersons-mutex-1*:
```
  ||- .{´pr1=0 ∧ ¬´in1 ∧ ´pr2=0 ∧ ¬´in2 }.
  COBEGIN .{´pr1=0 ∧ ¬´in1}.
  WHILE True INV .{´pr1=0 ∧ ¬´in1}.
  DO
  .{´pr1=0 ∧ ¬´in1}. ⟨ ´in1:=True,,´pr1:=1 ⟩;;
  .{´pr1=1 ∧ ´in1}.  ⟨ ´hold:=1,,´pr1:=2 ⟩;;
  .{´pr1=2 ∧ ´in1 ∧ (´hold=1 ∨ ´hold=2 ∧ ´pr2=2)}.
  AWAIT (¬´in2 ∨ ¬(´hold=1)) THEN ´pr1:=3 END;;
  .{´pr1=3 ∧ ´in1 ∧ (´hold=1 ∨ ´hold=2 ∧ ´pr2=2)}.
   ⟨´in1:=False,,´pr1:=0⟩
  OD .{´pr1=0 ∧ ¬´in1}.
  ||
  .{´pr2=0 ∧ ¬´in2}.
  WHILE True INV .{´pr2=0 ∧ ¬´in2}.
  DO
  .{´pr2=0 ∧ ¬´in2}. ⟨ ´in2:=True,,´pr2:=1 ⟩;;
  .{´pr2=1 ∧ ´in2}. ⟨  ´hold:=2,,´pr2:=2 ⟩;;
  .{´pr2=2 ∧ ´in2 ∧ (´hold=2 ∨ (´hold=1 ∧ ´pr1=2))}.
  AWAIT (¬´in1 ∨ ¬(´hold=2)) THEN ´pr2:=3  END;;
  .{´pr2=3 ∧ ´in2 ∧ (´hold=2 ∨ (´hold=1 ∧ ´pr1=2))}.
    ⟨´in2:=False,,´pr2:=0⟩
  OD .{´pr2=0 ∧ ¬´in2}.
  COEND
  .{´pr1=0 ∧ ¬´in1 ∧ ´pr2=0 ∧ ¬´in2}.
⟨proof⟩
```

Peterson's Algorithm II: A Busy Wait Solution

Apt and Olderog. "Verification of sequential and concurrent Programs", page 282.

record *Busy-wait-mutex* =
flag1 :: *bool*

```
 flag2 :: bool
 turn  :: nat
 after1 :: bool
 after2 :: bool
```

lemma *Busy-wait-mutex*:

```
 ∥−  .{True}.
  ´flag1:=False,, ´flag2:=False,,
  COBEGIN .{¬´flag1}.
        WHILE True
        INV .{¬´flag1}.
        DO .{¬´flag1}. ⟨ ´flag1:=True,,´after1:=False ⟩;;
           .{´flag1 ∧ ¬´after1}. ⟨ ´turn:=1,,´after1:=True ⟩;;
           .{´flag1 ∧ ´after1 ∧ (´turn=1 ∨ ´turn=2)}.
            WHILE ¬(´flag2 ⟶ ´turn=2)
            INV .{´flag1 ∧ ´after1 ∧ (´turn=1 ∨ ´turn=2)}.
            DO .{´flag1 ∧ ´after1 ∧ (´turn=1 ∨ ´turn=2)}. SKIP OD;;
           .{´flag1 ∧ ´after1 ∧ (´flag2 ∧ ´after2 ⟶ ´turn=2)}.
            ´flag1:=False
        OD
       .{False}.
  ∥
     .{¬´flag2}.
        WHILE True
        INV .{¬´flag2}.
        DO .{¬´flag2}. ⟨ ´flag2:=True,,´after2:=False ⟩;;
           .{´flag2 ∧ ¬´after2}. ⟨ ´turn:=2,,´after2:=True ⟩;;
           .{´flag2 ∧ ´after2 ∧ (´turn=1 ∨ ´turn=2)}.
            WHILE ¬(´flag1 ⟶ ´turn=1)
            INV .{´flag2 ∧ ´after2 ∧ (´turn=1 ∨ ´turn=2)}.
            DO .{´flag2 ∧ ´after2 ∧ (´turn=1 ∨ ´turn=2)}. SKIP OD;;
           .{´flag2 ∧ ´after2 ∧ (´flag1 ∧ ´after1 ⟶ ´turn=1)}.
            ´flag2:=False
        OD
       .{False}.
  COEND
  .{False}.
⟨proof⟩
```

Peterson's Algorithm III: A Solution using Semaphores

record *Semaphores-mutex* =

```
 out :: bool
 who :: nat
```

lemma *Semaphores-mutex*:

```
 ∥− .{i≠j}.
  ´out:=True ,,
  COBEGIN .{i≠j}.
```

WHILE True INV .{i≠j}.
DO .{i≠j}. AWAIT ´out THEN ´out:=False,, ´who:=i END;;
.{¬´out ∧ ´who=i ∧ i≠j}. ´out:=True OD
.{False}.
‖
.{i≠j}.
WHILE True INV .{i≠j}.
DO .{i≠j}. AWAIT ´out THEN ´out:=False,,´who:=j END;;
.{¬´out ∧ ´who=j ∧ i≠j}. ´out:=True OD
.{False}.
COEND
.{False}.
⟨proof⟩

Peterson's Algorithm III: Parameterized version:

lemma *Semaphores-parameterized-mutex*:
0<n ⟹ ‖− .{True}.
´out:=True ,,
COBEGIN
SCHEME [0≤ i< n]
.{True}.
WHILE True INV .{True}.
DO .{True}. AWAIT ´out THEN ´out:=False,, ´who:=i END;;
.{¬´out ∧ ´who=i}. ´out:=True OD
.{False}.
COEND
.{False}.
⟨proof⟩

The Ticket Algorithm

record *Ticket-mutex* =
num :: nat
nextv :: nat
turn :: nat list
index :: nat

lemma *Ticket-mutex*:
⟦ 0<n; I=«n=length ´turn ∧ 0<´nextv ∧ (∀ k l. k<n ∧ l<n ∧ k≠l
⟶ ´turn!k < ´num ∧ (´turn!k =0 ∨ ´turn!k≠´turn!l))» ⟧
⟹ ‖− .{n=length ´turn}.
´index:= 0,,
WHILE ´index < n INV .{n=length ´turn ∧ (∀ i<´index. ´turn!i=0)}.
DO ´turn:= ´turn[´index:=0],, ´index:=´index +1 OD,,
´num:=1 ,, ´nextv:=1 ,,
COBEGIN
SCHEME [0≤ i< n]
.{´I}.
WHILE True INV .{´I}.

$DO\ .\{{}^\acute{}I\}.\ \langle\ {}^\acute{}turn := {}^\acute{}turn[i:={}^\acute{}num],,\ {}^\acute{}num:={}^\acute{}num+1\ \rangle;;$
$.\{{}^\acute{}I\}.\ WAIT\ \ {}^\acute{}turn!i={}^\acute{}nextv\ END;;$
$.\{{}^\acute{}I \wedge {}^\acute{}turn!i={}^\acute{}nextv\}.\ {}^\acute{}nextv:={}^\acute{}nextv+1$
OD
$.\{False\}.$
$COEND$
$.\{False\}.$
$\langle proof\rangle$

1.8.2 Parallel Zero Search

Synchronized Zero Search. Zero-6

Apt and Olderog. "Verification of sequential and concurrent Programs" page 294:

record *Zero-search* =
turn :: *nat*
found :: *bool*
x :: *nat*
y :: *nat*

lemma *Zero-search*:
$[\![I1= \ll a\leq{}^\acute{}x \wedge ({}^\acute{}found \longrightarrow (a<{}^\acute{}x \wedge f({}^\acute{}x)=0) \vee ({}^\acute{}y\leq a \wedge f({}^\acute{}y)=0))$
$\wedge (\neg{}^\acute{}found \wedge a<{}^\acute{}\ x \longrightarrow f({}^\acute{}x)\neq 0) \gg ;$
$I2= \ll{}^\acute{}y\leq a+1 \wedge ({}^\acute{}found \longrightarrow (a<{}^\acute{}x \wedge f({}^\acute{}x)=0) \vee ({}^\acute{}y\leq a \wedge f({}^\acute{}y)=0))$
$\wedge (\neg{}^\acute{}found \wedge {}^\acute{}y\leq a \longrightarrow f({}^\acute{}y)\neq 0) \gg]\!] \Longrightarrow$
$\|-\ .\{\exists\ u.\ f(u)=0\}.$
${}^\acute{}turn:=1,,\ {}^\acute{}found:= False,,$
${}^\acute{}x:=a,,\ {}^\acute{}y:=a+1\ ,,$
$COBEGIN\ .\{{}^\acute{}I1\}.$
$WHILE\ \neg{}^\acute{}found$
$INV\ .\{{}^\acute{}I1\}.$
$DO\ .\{a\leq{}^\acute{}x \wedge ({}^\acute{}found \longrightarrow {}^\acute{}y\leq a \wedge f({}^\acute{}y)=0) \wedge (a<{}^\acute{}x \longrightarrow f({}^\acute{}x)\neq 0)\}.$
$WAIT\ \ {}^\acute{}turn=1\ END;;$
$.\{a\leq{}^\acute{}x \wedge ({}^\acute{}found \longrightarrow {}^\acute{}y\leq a \wedge f({}^\acute{}y)=0) \wedge (a<{}^\acute{}x \longrightarrow f({}^\acute{}x)\neq 0)\}.$
${}^\acute{}turn:=2;;$
$.\{a\leq{}^\acute{}x \wedge ({}^\acute{}found \longrightarrow {}^\acute{}y\leq a \wedge f({}^\acute{}y)=0) \wedge (a<{}^\acute{}x \longrightarrow f({}^\acute{}x)\neq 0)\}.$
$\langle\ {}^\acute{}x:={}^\acute{}x+1,,$
$IF\ f({}^\acute{}x)=0\ THEN\ \ {}^\acute{}found:=True\ ELSE\ SKIP\ FI\rangle$
$OD;;$
$.\{{}^\acute{}I1\ \wedge {}^\acute{}found\}.$
${}^\acute{}turn:=2$
$.\{{}^\acute{}I1 \wedge {}^\acute{}found\}.$
$\|$
$.\{{}^\acute{}I2\}.$
$WHILE\ \neg{}^\acute{}found$
$INV\ .\{{}^\acute{}I2\}.$
$DO\ .\{{}^\acute{}y\leq a+1 \wedge ({}^\acute{}found \longrightarrow a<{}^\acute{}x \wedge f({}^\acute{}x)=0) \wedge ({}^\acute{}y\leq a \longrightarrow f({}^\acute{}y)\neq 0)\}.$
$WAIT\ \ {}^\acute{}turn=2\ END;;$

$.\{\acute{}y{\le}a{+}1 \wedge (\acute{}found \longrightarrow a{<}\acute{}x \wedge f(\acute{}x){=}0) \wedge (\acute{}y{\le}a \longrightarrow f(\acute{}y){\neq}0)\}.$
$\acute{}turn{:=}1;;$
$.\{\acute{}y{\le}a{+}1 \wedge (\acute{}found \longrightarrow a{<}\acute{}x \wedge f(\acute{}x){=}0) \wedge (\acute{}y{\le}a \longrightarrow f(\acute{}y){\neq}0)\}.$
$\langle\ \acute{}y{:=}(\acute{}y - 1),,$
$IF\ f(\acute{}y){=}0\ THEN\ \acute{}found{:=}True\ ELSE\ SKIP\ FI\rangle$
$OD;;$
$.\{\acute{}I2 \wedge \acute{}found\}.$
$\acute{}turn{:=}1$
$.\{\acute{}I2 \wedge \acute{}found\}.$
$COEND$
$.\{f(\acute{}x){=}0 \vee f(\acute{}y){=}0\}.$
$\langle proof\rangle$

Easier Version: without AWAIT. Apt and Olderog. page 256:

lemma *Zero-Search-2*:
$[\![I1{=}\ll a{\le}\acute{}x \wedge (\acute{}found \longrightarrow (a{<}\acute{}x \wedge f(\acute{}x){=}0) \vee (\acute{}y{\le}a \wedge f(\acute{}y){=}0))$
$\wedge (\neg\acute{}found \wedge a{<}\acute{}x \longrightarrow f(\acute{}x){\neq}0)\gg;$
$I2{=}\ll\acute{}y{\le}a{+}1 \wedge (\acute{}found \longrightarrow (a{<}\acute{}x \wedge f(\acute{}x){=}0) \vee (\acute{}y{\le}a \wedge f(\acute{}y){=}0))$
$\wedge (\neg\acute{}found \wedge \acute{}y{\le}a \longrightarrow f(\acute{}y){\neq}0)\gg]\!] \Longrightarrow$
$\Vert{-}\ .\{\exists u.\ f(u){=}0\}.$
$\acute{}found{:=}\ False,,$
$\acute{}x{:=}a,,\ \acute{}y{:=}a{+}1,,$
$COBEGIN\ .\{\acute{}I1\}.$
$WHILE\ \neg\acute{}found$
$INV\ .\{\acute{}I1\}.$
$DO\ .\{a{\le}\acute{}x \wedge (\acute{}found \longrightarrow \acute{}y{\le}a \wedge f(\acute{}y){=}0) \wedge (a{<}\acute{}x \longrightarrow f(\acute{}x){\neq}0)\}.$
$\langle\ \acute{}x{:=}\acute{}x{+}1,,IF\ f(\acute{}x){=}0\ THEN\ \acute{}found{:=}True\ ELSE\ SKIP\ FI\rangle$
OD
$.\{\acute{}I1 \wedge \acute{}found\}.$
$\Vert$
$.\{\acute{}I2\}.$
$WHILE\ \neg\acute{}found$
$INV\ .\{\acute{}I2\}.$
$DO\ .\{\acute{}y{\le}a{+}1 \wedge (\acute{}found \longrightarrow a{<}\acute{}x \wedge f(\acute{}x){=}0) \wedge (\acute{}y{\le}a \longrightarrow f(\acute{}y){\neq}0)\}.$
$\langle\ \acute{}y{:=}(\acute{}y - 1),,IF\ f(\acute{}y){=}0\ THEN\ \acute{}found{:=}True\ ELSE\ SKIP\ FI\rangle$
OD
$.\{\acute{}I2 \wedge \acute{}found\}.$
$COEND$
$.\{f(\acute{}x){=}0 \vee f(\acute{}y){=}0\}.$
$\langle proof\rangle$

1.8.3 Producer/Consumer

Previous lemmas

lemma *nat-lemma2*: $[\![\ b = m{*}(n{::}nat) + t;\ a = s{*}n + u;\ t{=}u;\ b{-}a < n\ ]\!] \Longrightarrow m \le s$
$\langle proof\rangle$

lemma *mod-lemma*: $[\![\ (c{::}nat) \le a;\ a < b;\ b - c < n\ ]\!] \Longrightarrow b\ mod\ n \neq a\ mod\ n$

⟨*proof*⟩

Producer/Consumer Algorithm

record *Producer-consumer* =

```
  ins :: nat
  outs :: nat
  li :: nat
  lj :: nat
  vx :: nat
  vy :: nat
  buffer :: nat list
  b :: nat list
```

The whole proof takes aprox. 4 minutes.

lemma *Producer-consumer*:

```
 ⟦INIT= «0<length a ∧ 0<length ´buffer ∧ length ´b=length a» ;
   I= «(∀ k<´ins. ´outs≤k ⟶ (a ! k) = ´buffer ! (k mod (length ´buffer))) ∧
           ´outs≤´ins ∧ ´ins−´outs≤length ´buffer» ;
   I1= «´I ∧ ´li≤length a» ;
   p1= «´I1 ∧ ´li=´ins» ;
   I2 = «´I ∧ (∀ k<´lj. (a ! k)=(´b ! k)) ∧ ´lj≤length a» ;
   p2 = «´I2 ∧ ´lj=´outs» ⟧ ⟹
 ‖− .{´INIT}.
´ins:=0,, ´outs:=0,, ´li:=0,, ´lj:=0,,
COBEGIN .{´p1 ∧ ´INIT}.
  WHILE ´li <length a
    INV .{´p1 ∧ ´INIT}.
  DO .{´p1 ∧ ´INIT ∧ ´li<length a}.
       ´vx:= (a ! ´li);;
     .{´p1 ∧ ´INIT ∧ ´li<length a ∧ ´vx=(a ! ´li)}.
       WAIT ´ins−´outs < length ´buffer END;;
     .{´p1 ∧ ´INIT ∧ ´li<length a ∧ ´vx=(a ! ´li)
       ∧ ´ins−´outs < length ´buffer}.
      ´buffer:=(list-update ´buffer (´ins mod (length ´buffer)) ´vx);;
     .{´p1 ∧ ´INIT ∧ ´li<length a
       ∧ (a ! ´li)=(´buffer ! (´ins mod (length ´buffer)))
       ∧ ´ins−´outs <length ´buffer}.
      ´ins:=´ins+1;;
     .{´I1 ∧ ´INIT ∧ (´li+1)=´ins ∧ ´li<length a}.
      ´li:=´li+1
  OD
 .{´p1 ∧ ´INIT ∧ ´li=length a}.
 ‖
 .{´p2 ∧ ´INIT}.
  WHILE ´lj < length a
    INV .{´p2 ∧ ´INIT}.
  DO .{´p2 ∧ ´lj<length a ∧ ´INIT}.
       WAIT ´outs<´ins END;;
```

$.\{´p2 \wedge ´lj<length\ a \wedge ´outs<´ins \wedge ´INIT\}.$
$´vy:=(´buffer\ !\ (´outs\ mod\ (length\ ´buffer)));;$
$.\{´p2 \wedge ´lj<length\ a \wedge ´outs<´ins \wedge ´vy=(a\ !\ ´lj) \wedge ´INIT\}.$
$´outs:=´outs+1;;$
$.\{´I2 \wedge (´lj+1)=´outs \wedge ´lj<length\ a \wedge ´vy=(a\ !\ ´lj) \wedge ´INIT\}.$
$´b:=(list\text{-}update\ ´b\ ´lj\ ´vy);;$
$.\{´I2 \wedge (´lj+1)=´outs \wedge ´lj<length\ a \wedge (a\ !\ ´lj)=(´b\ !\ ´lj) \wedge ´INIT\}.$
$´lj:=´lj+1$
OD
$.\{´p2 \wedge ´lj=length\ a \wedge ´INIT\}.$
COEND
$.\{\ \forall k<length\ a.\ (a\ !\ k)=(´b\ !\ k)\}.$
⟨*proof*⟩

1.8.4 Parameterized Examples

Set Elements of an Array to Zero

record *Example1* =
$a :: nat \Rightarrow nat$

lemma *Example1*:
$\|-\ .\{True\}.$
$COBEGIN\ SCHEME\ [0\leq i<n]\ .\{True\}.\ ´a:=´a\ (i:=0)\ .\{´a\ i=0\}.\ COEND$
$.\{\forall i < n.\ ´a\ i = 0\}.$
⟨*proof*⟩

Same example with lists as auxiliary variables.

record *Example1-list* =
$A :: nat\ list$
lemma *Example1-list*:
$\|-\ .\{n < length\ ´A\}.$
COBEGIN
$SCHEME\ [0\leq i<n]\ .\{n < length\ ´A\}.\ ´A:=´A[i:=0]\ .\{´A!i=0\}.$
COEND
$.\{\forall i < n.\ ´A!i = 0\}.$
⟨*proof*⟩

Increment a Variable in Parallel

First some lemmas about summation properties.

lemma *Example2-lemma2-aux*: $!!b.\ j<n \Longrightarrow$
$(\sum i=0..<n.\ (b\ i::nat)) =$
$(\sum i=0..<j.\ b\ i) + b\ j + (\sum i=0..<n-(Suc\ j)\ .\ b\ (Suc\ j + i))$
⟨*proof*⟩

lemma *Example2-lemma2-aux2*:
$!!b.\ j\leq\ s \Longrightarrow (\sum i::nat=0..<j.\ (b\ (s:=t))\ i) = (\sum i=0..<j.\ b\ i)$
⟨*proof*⟩

lemma *Example2-lemma2*:
$!!b.\ [\![j<n;\ b\ j=0]\!] \Longrightarrow Suc\ (\sum i::nat=0..<n.\ b\ i)=(\sum i=0..<n.\ (b\ (j := Suc\ 0))\ i)$
$\langle proof \rangle$

record *Example2* =
$c :: nat \Rightarrow nat$
$x :: nat$

lemma *Example-2*: $0<n \Longrightarrow$
$\|-\ .\{\acute{}x=0 \wedge (\sum i=0..<n.\ \acute{}c\ i)=0\}.$
COBEGIN
SCHEME $[0\le i<n]$
$.\{\acute{}x=(\sum i=0..<n.\ \acute{}c\ i) \wedge \acute{}c\ i=0\}.$
$\langle\ \acute{}x:=\acute{}x+(Suc\ 0),,\ \acute{}c:=\acute{}c\ (i:=(Suc\ 0))\ \rangle$
$.\{\acute{}x=(\sum i=0..<n.\ \acute{}c\ i) \wedge \acute{}c\ i=(Suc\ 0)\}.$
COEND
$.\{\acute{}x=n\}.$
$\langle proof \rangle$

end

Chapter 2

Case Study: Single and Multi-Mutator Garbage Collection Algorithms

2.1 Formalization of the Memory

theory *Graph* **imports** *Main* **begin**

datatype $node = Black \mid White$

types
$nodes = node\ list$
$edge\ = nat \times nat$
$edges = edge\ list$

consts $Roots :: nat\ set$

constdefs
$Proper\text{-}Roots :: nodes \Rightarrow bool$
$Proper\text{-}Roots\ M \equiv Roots \neq \{\} \wedge Roots \subseteq \{i.\ i<length\ M\}$

$Proper\text{-}Edges :: (nodes \times edges) \Rightarrow bool$
$Proper\text{-}Edges \equiv (\lambda(M,E).\ \forall i<length\ E.\ fst(E!i)<length\ M \wedge snd(E!i)<length\ M)$

$BtoW :: (edge \times nodes) \Rightarrow bool$
$BtoW \equiv (\lambda(e,M).\ (M!fst\ e)=Black \wedge (M!snd\ e)\neq Black)$

$Blacks :: nodes \Rightarrow nat\ set$
$Blacks\ M \equiv \{i.\ i<length\ M \wedge M!i=Black\}$

$Reach :: edges \Rightarrow nat\ set$
$Reach\ E \equiv \{x.\ (\exists path.\ 1<length\ path \wedge path!(length\ path - 1)\in Roots \wedge x=path!0$

$\wedge$ ($\forall\, i<length\ path\ -\ 1.\ (\exists\, j<length\ E.\ E!j=(path!(i+1),\ path!i)))$)
$\vee\ x\in Roots\}$

Reach: the set of reachable nodes is the set of Roots together with the nodes reachable from some Root by a path represented by a list of nodes (at least two since we traverse at least one edge), where two consecutive nodes correspond to an edge in E.

2.1.1 Proofs about Graphs

lemmas *Graph-defs*= *Blacks-def Proper-Roots-def Proper-Edges-def BtoW-def*
declare *Graph-defs* [*simp*]

Graph 1

lemma *Graph1-aux* [*rule-format*]:
$[\![\ Roots\subseteq Blacks\ M;\ \forall\, i<length\ E.\ \neg BtoW(E!i,M)]\!]$
$\Longrightarrow 1<\ length\ path \longrightarrow (path!(length\ path\ -\ 1))\in Roots \longrightarrow$
$(\forall\, i<length\ path\ -\ 1.\ (\exists\, j.\ j\ <\ length\ E\ \wedge\ E!j=(path!(Suc\ i),\ path!i)))$
$\longrightarrow M!(path!0)\ =\ Black$
$\langle proof\rangle$

lemma *Graph1*:
$[\![Roots\subseteq Blacks\ M;\ Proper\text{-}Edges(M,\ E);\ \forall\, i<length\ E.\ \neg BtoW(E!i,M)\ ]\!]$
$\Longrightarrow Reach\ E\subseteq Blacks\ M$
$\langle proof\rangle$

Graph 2

lemma *Ex-first-occurrence* [*rule-format*]:
$P\ (n::nat) \longrightarrow (\exists\, m.\ P\ m\ \wedge\ (\forall\, i.\ i<m \longrightarrow \neg\ P\ i))$
$\langle proof\rangle$

lemma *Compl-lemma*: $(n::nat)\leq l \Longrightarrow (\exists\, m.\ m\leq l\ \wedge\ n=l\ -\ m)$
$\langle proof\rangle$

lemma *Ex-last-occurrence*:
$[\![P\ (n::nat);\ n\leq l]\!] \Longrightarrow (\exists\, m.\ P\ (l\ -\ m)\ \wedge\ (\forall\, i.\ i<m \longrightarrow \neg P\ (l\ -\ i)))$
$\langle proof\rangle$

lemma *Graph2*:
$[\![T\ \in\ Reach\ E;\ R<length\ E]\!] \Longrightarrow T\ \in\ Reach\ (E[R:=(fst(E!R),\ T)])$
$\langle proof\rangle$

Graph 3

lemma *Graph3*:
$[\![\ T\in Reach\ E;\ R<length\ E\ ]\!] \Longrightarrow Reach(E[R:=(fst(E!R),T)])\ \subseteq\ Reach\ E$
$\langle proof\rangle$

Graph 4

lemma *Graph4*:
$[\![T \in \mathit{Reach}\ E;\ \mathit{Roots}{\subseteq}\mathit{Blacks}\ M;\ I{\leq}\mathit{length}\ E;\ T{<}\mathit{length}\ M;\ R{<}\mathit{length}\ E;$
$\forall\, i{<}I.\ \neg \mathit{BtoW}(E!i,M);\ R{<}I;\ M!\mathit{fst}(E!R){=}\mathit{Black};\ M!T{\neq}\mathit{Black}]\!] \Longrightarrow$
$(\exists\, r.\ I{\leq}r \wedge r{<}\mathit{length}\ E \wedge \mathit{BtoW}(E[R{:=}(\mathit{fst}(E!R),T)]!r,M))$
⟨*proof*⟩

Graph 5

lemma *Graph5*:
$[\![\ T \in \mathit{Reach}\ E\ ;\ \mathit{Roots} \subseteq \mathit{Blacks}\ M;\ \forall\, i{<}R.\ \neg \mathit{BtoW}(E!i,M);\ T{<}\mathit{length}\ M;$
$R{<}\mathit{length}\ E;\ M!\mathit{fst}(E!R){=}\mathit{Black};\ M!\mathit{snd}(E!R){=}\mathit{Black};\ M!T \neq \mathit{Black}]\!]$
$\Longrightarrow (\exists\, r.\ R{<}r \wedge r{<}\mathit{length}\ E \wedge \mathit{BtoW}(E[R{:=}(\mathit{fst}(E!R),T)]!r,M))$
⟨*proof*⟩

Other lemmas about graphs

lemma *Graph6*:
$[\![\mathit{Proper\text{-}Edges}(M,E);\ R{<}\mathit{length}\ E\ ;\ T{<}\mathit{length}\ M]\!] \Longrightarrow \mathit{Proper\text{-}Edges}(M,E[R{:=}(\mathit{fst}(E!R),T)])$
⟨*proof*⟩

lemma *Graph7*:
$[\![\mathit{Proper\text{-}Edges}(M,E)]\!] \Longrightarrow \mathit{Proper\text{-}Edges}(M[T{:=}a],E)$
⟨*proof*⟩

lemma *Graph8*:
$[\![\mathit{Proper\text{-}Roots}(M)]\!] \Longrightarrow \mathit{Proper\text{-}Roots}(M[T{:=}a])$
⟨*proof*⟩

Some specific lemmata for the verification of garbage collection algorithms.

lemma *Graph9*: $j{<}\mathit{length}\ M \Longrightarrow \mathit{Blacks}\ M{\subseteq}\mathit{Blacks}\ (M[j := \mathit{Black}])$
⟨*proof*⟩

lemma *Graph10* [*rule-format* (*no-asm*)]: $\forall\, i.\ M!i{=}a \longrightarrow M[i{:=}a]{=}M$
⟨*proof*⟩

lemma *Graph11* [*rule-format* (*no-asm*)]:
$[\![\ M!j{\neq}\mathit{Black};j{<}\mathit{length}\ M]\!] \Longrightarrow \mathit{Blacks}\ M \subset \mathit{Blacks}\ (M[j := \mathit{Black}])$
⟨*proof*⟩

lemma *Graph12*: $[\![a{\subseteq}\mathit{Blacks}\ M;j{<}\mathit{length}\ M]\!] \Longrightarrow a{\subseteq}\mathit{Blacks}\ (M[j := \mathit{Black}])$
⟨*proof*⟩

lemma *Graph13*: $[\![a{\subset}\ \mathit{Blacks}\ M;j{<}\mathit{length}\ M]\!] \Longrightarrow a \subset \mathit{Blacks}\ (M[j := \mathit{Black}])$
⟨*proof*⟩

declare *Graph-defs* [*simp del*]

end

2.2 The Single Mutator Case

theory *Gar-Coll* **imports** *Graph OG-Syntax* **begin**

declare *psubsetE* [*rule del*]

Declaration of variables:

record *gar-coll-state* =
 M :: *nodes*
 E :: *edges*
 bc :: *nat set*
 obc :: *nat set*
 Ma :: *nodes*
 ind :: *nat*
 k :: *nat*
 z :: *bool*

2.2.1 The Mutator

The mutator first redirects an arbitrary edge R from an arbitrary accessible node towards an arbitrary accessible node T. It then colors the new target T black.

We declare the arbitrarily selected node and edge as constants:

consts R :: *nat* T :: *nat*

The following predicate states, given a list of nodes m and a list of edges e, the conditions under which the selected edge R and node T are valid:

constdefs
 Mut-init :: *gar-coll-state* $\Rightarrow$ *bool*
 Mut-init $\equiv \ll T \in \mathit{Reach}\ {}^{\acute{}}E \wedge R < \mathit{length}\ {}^{\acute{}}E \wedge T < \mathit{length}\ {}^{\acute{}}M \gg$

For the mutator we consider two modules, one for each action. An auxiliary variable ´z is set to false if the mutator has already redirected an edge but has not yet colored the new target.

constdefs
 Redirect-Edge :: *gar-coll-state ann-com*
 Redirect-Edge $\equiv .\{{}^{\acute{}}\mathit{Mut\text{-}init} \wedge {}^{\acute{}}z\}.\ \langle {}^{\acute{}}E{:=}{}^{\acute{}}E[R{:=}(\mathit{fst}({}^{\acute{}}E!R),\ T)],,\ {}^{\acute{}}z{:=}\ (\neg {}^{\acute{}}z)\rangle$

 Color-Target :: *gar-coll-state ann-com*
 Color-Target $\equiv .\{{}^{\acute{}}\mathit{Mut\text{-}init} \wedge \neg {}^{\acute{}}z\}.\ \langle {}^{\acute{}}M{:=}{}^{\acute{}}M[T{:=}\mathit{Black}],,\ {}^{\acute{}}z{:=}\ (\neg {}^{\acute{}}z)\rangle$

 Mutator :: *gar-coll-state ann-com*
 Mutator $\equiv$
 $.\{{}^{\acute{}}\mathit{Mut\text{-}init} \wedge {}^{\acute{}}z\}.$

$WHILE\ True\ INV\ .\{´Mut\text{-}init \wedge ´z\}.$
$DO\ \ Redirect\text{-}Edge\ ;;\ Color\text{-}Target\ \ OD$

Correctness of the mutator

lemmas *mutator-defs* = *Mut-init-def Redirect-Edge-def Color-Target-def*

lemma *Redirect-Edge*:
$\vdash Redirect\text{-}Edge\ pre(Color\text{-}Target)$
⟨*proof*⟩

lemma *Color-Target*:
$\vdash Color\text{-}Target\ .\{´Mut\text{-}init \wedge ´z\}.$
⟨*proof*⟩

lemma *Mutator*:
$\vdash Mutator\ .\{False\}.$
⟨*proof*⟩

2.2.2 The Collector

A constant *M-init* is used to give *´Ma* a suitable first value, defined as a list of nodes where only the *Roots* are black.

consts *M-init* :: *nodes*

constdefs
$Proper\text{-}M\text{-}init :: gar\text{-}coll\text{-}state \Rightarrow bool$
$Proper\text{-}M\text{-}init \equiv \ll Blacks\ M\text{-}init{=}Roots \wedge length\ M\text{-}init{=}length\ ´M \gg$

$Proper :: gar\text{-}coll\text{-}state \Rightarrow bool$
$Proper \equiv \ll Proper\text{-}Roots\ ´M \wedge Proper\text{-}Edges(´M,\ ´E) \wedge ´Proper\text{-}M\text{-}init \gg$

$Safe :: gar\text{-}coll\text{-}state \Rightarrow bool$
$Safe \equiv \ll Reach\ ´E \subseteq Blacks\ ´M \gg$

lemmas *collector-defs* = *Proper-M-init-def Proper-def Safe-def*

Blackening the roots

constdefs
$Blacken\text{-}Roots ::\ gar\text{-}coll\text{-}state\ ann\text{-}com$
$Blacken\text{-}Roots \equiv$
$.\{´Proper\}.$
$´ind{:=}0;;$
$.\{´Proper \wedge ´ind{=}0\}.$
$WHILE\ ´ind{<}length\ ´M$
$\ INV\ .\{´Proper \wedge (\forall i{<}´ind.\ i \in Roots \longrightarrow ´M!i{=}Black) \wedge ´ind{\le}length\ ´M\}.$
$DO\ .\{´Proper \wedge (\forall i{<}´ind.\ i \in Roots \longrightarrow ´M!i{=}Black) \wedge ´ind{<}length\ ´M\}.$
$\ IF\ ´ind{\in}Roots\ THEN$

.{´*Proper* ∧ (∀ *i*<´*ind*. *i* ∈ *Roots* ⟶ ´*M*!*i*=*Black*) ∧ ´*ind*<*length* ´*M* ∧ ´*ind*∈*Roots*}.
´*M*:=´*M*[´*ind*:=*Black*] *FI*;;
.{´*Proper* ∧ (∀ *i*<´*ind*+*1*. *i* ∈ *Roots* ⟶ ´*M*!*i*=*Black*) ∧ ´*ind*<*length* ´*M*}.
´*ind*:=´*ind*+*1*
OD

lemma *Blacken-Roots*:
⊢ *Blacken-Roots* .{´*Proper* ∧ *Roots*⊆*Blacks* ´*M*}.
⟨*proof*⟩

Propagating black

constdefs
PBInv :: *gar-coll-state* ⇒ *nat* ⇒ *bool*
PBInv ≡ ≪ λ*ind*. ´*obc* < *Blacks* ´*M* ∨ (∀ *i* <*ind*. ¬*BtoW* (´*E*!*i*, ´*M*) ∨
(¬´*z* ∧ *i*=*R* ∧ (*snd*(´*E*!*R*)) = *T* ∧ (∃ *r*. *ind* ≤ *r* ∧ *r* < *length* ´*E* ∧ *BtoW*(´*E*!*r*,´*M*))))≫

constdefs
Propagate-Black-aux :: *gar-coll-state ann-com*
Propagate-Black-aux ≡
.{´*Proper* ∧ *Roots*⊆*Blacks* ´*M* ∧ ´*obc*⊆*Blacks* ´*M* ∧ ´*bc*⊆*Blacks* ´*M*}.
´*ind*:=*0*;;
.{´*Proper* ∧ *Roots*⊆*Blacks* ´*M* ∧ ´*obc*⊆*Blacks* ´*M* ∧ ´*bc*⊆*Blacks* ´*M* ∧ ´*ind*=*0*}.

WHILE ´*ind*<*length* ´*E*
INV .{´*Proper* ∧ *Roots*⊆*Blacks* ´*M* ∧ ´*obc*⊆*Blacks* ´*M* ∧ ´*bc*⊆*Blacks* ´*M*
∧ ´*PBInv* ´*ind* ∧ ´*ind*≤*length* ´*E*}.
DO .{´*Proper* ∧ *Roots*⊆*Blacks* ´*M* ∧ ´*obc*⊆*Blacks* ´*M* ∧ ´*bc*⊆*Blacks* ´*M*
∧ ´*PBInv* ´*ind* ∧ ´*ind*<*length* ´*E*}.
IF ´*M*!(*fst* (´*E*!´*ind*)) = *Black THEN*
.{´*Proper* ∧ *Roots*⊆*Blacks* ´*M* ∧ ´*obc*⊆*Blacks* ´*M* ∧ ´*bc*⊆*Blacks* ´*M*
∧ ´*PBInv* ´*ind* ∧ ´*ind*<*length* ´*E* ∧ ´*M*!*fst*(´*E*!´*ind*)=*Black*}.
´*M*:=´*M*[*snd*(´*E*!´*ind*):=*Black*];;
.{´*Proper* ∧ *Roots*⊆*Blacks* ´*M* ∧ ´*obc*⊆*Blacks* ´*M* ∧ ´*bc*⊆*Blacks* ´*M*
∧ ´*PBInv* (´*ind* + *1*) ∧ ´*ind*<*length* ´*E*}.
´*ind*:=´*ind*+*1*
FI
OD

lemma *Propagate-Black-aux*:
⊢ *Propagate-Black-aux*
.{´*Proper* ∧ *Roots*⊆*Blacks* ´*M* ∧ ´*obc*⊆*Blacks* ´*M* ∧ ´*bc*⊆*Blacks* ´*M*
∧ (´*obc* < *Blacks* ´*M* ∨ ´*Safe*)}.
⟨*proof*⟩

Refining propagating black

constdefs
Auxk :: *gar-coll-state* ⇒ *bool*

```
  Auxk ≡ «´k<length ´M ∧ (´M!´k≠Black ∨ ¬BtoW(´E!´ind, ´M) ∨
          ´obc<Blacks ´M ∨ (¬´z ∧ ´ind=R ∧ snd(´E!R)=T
          ∧ (∃r. ´ind<r ∧ r<length ´E ∧ BtoW(´E!r, ´M))))»
```

constdefs

```
  Propagate-Black :: gar-coll-state ann-com
  Propagate-Black ≡
  .{´Proper ∧ Roots⊆Blacks ´M ∧ ´obc⊆Blacks ´M ∧ ´bc⊆Blacks ´M}.
  ´ind:=0;;
  .{´Proper ∧ Roots⊆Blacks ´M ∧ ´obc⊆Blacks ´M ∧ ´bc⊆Blacks ´M ∧ ´ind=0}.
  WHILE ´ind<length ´E
   INV .{´Proper ∧ Roots⊆Blacks ´M ∧ ´obc⊆Blacks ´M ∧ ´bc⊆Blacks ´M
        ∧ ´PBInv ´ind ∧ ´ind≤length ´E}.
  DO .{´Proper ∧ Roots⊆Blacks ´M ∧ ´obc⊆Blacks ´M ∧ ´bc⊆Blacks ´M
      ∧ ´PBInv ´ind ∧ ´ind<length ´E}.
   IF (´M!(fst (´E!´ind)))=Black THEN
    .{´Proper ∧ Roots⊆Blacks ´M ∧ ´obc⊆Blacks ´M ∧ ´bc⊆Blacks ´M
     ∧ ´PBInv ´ind ∧ ´ind<length ´E ∧ (´M!fst(´E!´ind))=Black}.
     ´k:=(snd(´E!´ind));;
    .{´Proper ∧ Roots⊆Blacks ´M ∧ ´obc⊆Blacks ´M ∧ ´bc⊆Blacks ´M
     ∧ ´PBInv ´ind ∧ ´ind<length ´E ∧ (´M!fst(´E!´ind))=Black
     ∧ ´Auxk}.
     ⟨´M:=´M[´k:=Black],, ´ind:=´ind+1⟩
   ELSE .{´Proper ∧ Roots⊆Blacks ´M ∧ ´obc⊆Blacks ´M ∧ ´bc⊆Blacks ´M
         ∧ ´PBInv ´ind ∧ ´ind<length ´E}.
        ⟨IF (´M!(fst (´E!´ind)))≠Black THEN ´ind:=´ind+1 FI⟩
   FI
  OD
```

lemma *Propagate-Black*:

```
 ⊢ Propagate-Black
  .{´Proper ∧ Roots⊆Blacks ´M ∧ ´obc⊆Blacks ´M ∧ ´bc⊆Blacks ´M
   ∧ ( ´obc < Blacks ´M ∨ ´Safe)}.
⟨proof⟩
```

Counting black nodes

constdefs

```
  CountInv :: gar-coll-state ⇒ nat ⇒ bool
  CountInv ≡ « λind. {i. i<ind ∧ ´Ma!i=Black}⊆´bc »
```

constdefs

```
  Count :: gar-coll-state ann-com
  Count ≡
  .{´Proper ∧ Roots⊆Blacks ´M
   ∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M
   ∧ length ´Ma=length ´M ∧ (´obc < Blacks ´Ma ∨ ´Safe) ∧ ´bc={}}.
  ´ind:=0;;
  .{´Proper ∧ Roots⊆Blacks ´M
```

$\wedge$ *´obc*$\subseteq$*Blacks ´Ma* $\wedge$ *Blacks ´Ma*$\subseteq$*Blacks ´M* $\wedge$ *´bc*$\subseteq$*Blacks ´M*
$\wedge$ *length ´Ma*=*length ´M* $\wedge$ (*´obc* < *Blacks ´Ma* $\vee$ *´Safe*) $\wedge$ *´bc*={}
$\wedge$ *´ind*=*0*}.
WHILE ´ind<*length ´M*
INV .{*´Proper* $\wedge$ *Roots*$\subseteq$*Blacks ´M*
$\wedge$ *´obc*$\subseteq$*Blacks ´Ma* $\wedge$ *Blacks ´Ma*$\subseteq$*Blacks ´M* $\wedge$ *´bc*$\subseteq$*Blacks ´M*
$\wedge$ *length ´Ma*=*length ´M* $\wedge$ *´CountInv ´ind*
$\wedge$ (*´obc* < *Blacks ´Ma* $\vee$ *´Safe*) $\wedge$ *´ind*$\leq$*length ´M*}.
DO .{*´Proper* $\wedge$ *Roots*$\subseteq$*Blacks ´M*
$\wedge$ *´obc*$\subseteq$*Blacks ´Ma* $\wedge$ *Blacks ´Ma*$\subseteq$*Blacks ´M* $\wedge$ *´bc*$\subseteq$*Blacks ´M*
$\wedge$ *length ´Ma*=*length ´M* $\wedge$ *´CountInv ´ind*
$\wedge$ (*´obc* < *Blacks ´Ma* $\vee$ *´Safe*) $\wedge$ *´ind*<*length ´M*}.
IF ´M!*´ind*=*Black*
THEN .{*´Proper* $\wedge$ *Roots*$\subseteq$*Blacks ´M*
$\wedge$ *´obc*$\subseteq$*Blacks ´Ma* $\wedge$ *Blacks ´Ma*$\subseteq$*Blacks ´M* $\wedge$ *´bc*$\subseteq$*Blacks ´M*
$\wedge$ *length ´Ma*=*length ´M* $\wedge$ *´CountInv ´ind*
$\wedge$ (*´obc* < *Blacks ´Ma* $\vee$ *´Safe*) $\wedge$ *´ind*<*length ´M* $\wedge$ *´M*!*´ind*=*Black*}.
´bc:=*insert ´ind ´bc*
FI;;
.{*´Proper* $\wedge$ *Roots*$\subseteq$*Blacks ´M*
$\wedge$ *´obc*$\subseteq$*Blacks ´Ma* $\wedge$ *Blacks ´Ma*$\subseteq$*Blacks ´M* $\wedge$ *´bc*$\subseteq$*Blacks ´M*
$\wedge$ *length ´Ma*=*length ´M* $\wedge$ *´CountInv* (*´ind*+*1*)
$\wedge$ (*´obc* < *Blacks ´Ma* $\vee$ *´Safe*) $\wedge$ *´ind*<*length ´M*}.
´ind:=*´ind*+*1*
OD

lemma *Count*:
$\vdash$ *Count*
.{*´Proper* $\wedge$ *Roots*$\subseteq$*Blacks ´M*
$\wedge$ *´obc*$\subseteq$*Blacks ´Ma* $\wedge$ *Blacks ´Ma*$\subseteq$*´bc* $\wedge$ *´bc*$\subseteq$*Blacks ´M* $\wedge$ *length ´Ma*=*length ´M*
$\wedge$ (*´obc* < *Blacks ´Ma* $\vee$ *´Safe*)}.
$\langle$*proof*$\rangle$

Appending garbage nodes to the free list

consts *Append-to-free* :: *nat* $\times$ *edges* $\Rightarrow$ *edges*

axioms
Append-to-free0: *length* (*Append-to-free* (*i*, *e*)) = *length e*
Append-to-free1: *Proper-Edges* (*m*, *e*)
$\Longrightarrow$ *Proper-Edges* (*m*, *Append-to-free*(*i*, *e*))
Append-to-free2: *i* $\notin$ *Reach e*
$\Longrightarrow$ *n* $\in$ *Reach* (*Append-to-free*(*i*, *e*)) = (*n* = *i* $\vee$ *n* $\in$ *Reach e*)

constdefs
AppendInv :: *gar-coll-state* $\Rightarrow$ *nat* $\Rightarrow$ *bool*
AppendInv $\equiv$ «λ*ind*. $\forall$ *i*<*length ´M*. *ind*$\leq$*i* $\longrightarrow$ *i*$\in$*Reach ´E* $\longrightarrow$ *´M*!*i*=*Black*»

constdefs
  Append :: *gar-coll-state ann-com*
  Append ≡
  .{´*Proper* ∧ *Roots*⊆*Blacks* ´*M* ∧ ´*Safe*}.
  ´*ind*:=*0*;;
  .{´*Proper* ∧ *Roots*⊆*Blacks* ´*M* ∧ ´*Safe* ∧ ´*ind*=*0*}.
    WHILE ´*ind*<*length* ´*M*
      INV .{´*Proper* ∧ ´*AppendInv* ´*ind* ∧ ´*ind*≤*length* ´*M*}.
    DO .{´*Proper* ∧ ´*AppendInv* ´*ind* ∧ ´*ind*<*length* ´*M*}.
       IF ´*M*!´*ind*=*Black* *THEN*
          .{´*Proper* ∧ ´*AppendInv* ´*ind* ∧ ´*ind*<*length* ´*M* ∧ ´*M*!´*ind*=*Black*}.
          ´*M*:=´*M*[´*ind*:=*White*]
      ELSE .{´*Proper* ∧ ´*AppendInv* ´*ind* ∧ ´*ind*<*length* ´*M* ∧ ´*ind*∉*Reach* ´*E*}.
              ´*E*:=*Append-to-free*(´*ind*,´*E*)
       FI;;
     .{´*Proper* ∧ ´*AppendInv* (´*ind*+*1*) ∧ ´*ind*<*length* ´*M*}.
       ´*ind*:=´*ind*+*1*
    OD

lemma *Append*:
  ⊢ *Append* .{´*Proper*}.
⟨*proof*⟩

Correctness of the Collector

constdefs
  Collector :: *gar-coll-state ann-com*
  Collector ≡
.{´*Proper*}.
 WHILE *True* *INV* .{´*Proper*}.
 DO
  Blacken-Roots;;
  .{´*Proper* ∧ *Roots*⊆*Blacks* ´*M*}.
   ´*obc*:={};;
  .{´*Proper* ∧ *Roots*⊆*Blacks* ´*M* ∧ ´*obc*={}}.
   ´*bc*:=*Roots*;;
  .{´*Proper* ∧ *Roots*⊆*Blacks* ´*M* ∧ ´*obc*={} ∧ ´*bc*=*Roots*}.
   ´*Ma*:=*M-init*;;
  .{´*Proper* ∧ *Roots*⊆*Blacks* ´*M* ∧ ´*obc*={} ∧ ´*bc*=*Roots* ∧ ´*Ma*=*M-init*}.
   WHILE ´*obc*≠´*bc*
     INV .{´*Proper* ∧ *Roots*⊆*Blacks* ´*M*
          ∧ ´*obc*⊆*Blacks* ´*Ma* ∧ *Blacks* ´*Ma*⊆´*bc* ∧ ´*bc*⊆*Blacks* ´*M*
          ∧ *length* ´*Ma*=*length* ´*M* ∧ (´*obc* < *Blacks* ´*Ma* ∨ ´*Safe*)}.
   DO .{´*Proper* ∧ *Roots*⊆*Blacks* ´*M* ∧ ´*bc*⊆*Blacks* ´*M*}.
       ´*obc*:=´*bc*;;
       Propagate-Black;;
      .{´*Proper* ∧ *Roots*⊆*Blacks* ´*M* ∧ ´*obc*⊆*Blacks* ´*M* ∧ ´*bc*⊆*Blacks* ´*M*
       ∧ (´*obc* < *Blacks* ´*M* ∨ ´*Safe*)}.
       ´*Ma*:=´*M*;;

.{´*Proper* ∧ *Roots*⊆*Blacks* ´*M* ∧ ´*obc*⊆*Blacks* ´*Ma*
∧ *Blacks* ´*Ma*⊆*Blacks* ´*M* ∧ ´*bc*⊆*Blacks* ´*M* ∧ *length* ´*Ma*=*length* ´*M*
∧ (´*obc* < *Blacks* ´*Ma* ∨ ´*Safe*)}.
´*bc*:={};;
Count
OD;;
Append
OD

lemma *Collector*:
⊢ *Collector* .{*False*}.
⟨*proof*⟩

2.2.3 Interference Freedom

lemmas *modules* = *Redirect-Edge-def Color-Target-def Blacken-Roots-def Propagate-Black-def Count-def Append-def*
lemmas *Invariants* = *PBInv-def Auxk-def CountInv-def AppendInv-def*
lemmas *abbrev* = *collector-defs mutator-defs Invariants*

lemma *interfree-Blacken-Roots-Redirect-Edge*:
interfree-aux (*Some Blacken-Roots*, {}, *Some Redirect-Edge*)
⟨*proof*⟩

lemma *interfree-Redirect-Edge-Blacken-Roots*:
interfree-aux (*Some Redirect-Edge*, {}, *Some Blacken-Roots*)
⟨*proof*⟩

lemma *interfree-Blacken-Roots-Color-Target*:
interfree-aux (*Some Blacken-Roots*, {}, *Some Color-Target*)
⟨*proof*⟩

lemma *interfree-Color-Target-Blacken-Roots*:
interfree-aux (*Some Color-Target*, {}, *Some Blacken-Roots*)
⟨*proof*⟩

lemma *interfree-Propagate-Black-Redirect-Edge*:
interfree-aux (*Some Propagate-Black*, {}, *Some Redirect-Edge*)
⟨*proof*⟩

lemma *interfree-Redirect-Edge-Propagate-Black*:
interfree-aux (*Some Redirect-Edge*, {}, *Some Propagate-Black*)
⟨*proof*⟩

lemma *interfree-Propagate-Black-Color-Target*:
interfree-aux (*Some Propagate-Black*, {}, *Some Color-Target*)
⟨*proof*⟩

lemma *interfree-Color-Target-Propagate-Black*:

interfree-aux (*Some Color-Target*, {}, *Some Propagate-Black*)
⟨*proof*⟩

lemma *interfree-Count-Redirect-Edge*:
interfree-aux (*Some Count*, {}, *Some Redirect-Edge*)
⟨*proof*⟩

lemma *interfree-Redirect-Edge-Count*:
interfree-aux (*Some Redirect-Edge*, {}, *Some Count*)
⟨*proof*⟩

lemma *interfree-Count-Color-Target*:
interfree-aux (*Some Count*, {}, *Some Color-Target*)
⟨*proof*⟩

lemma *interfree-Color-Target-Count*:
interfree-aux (*Some Color-Target*, {}, *Some Count*)
⟨*proof*⟩

lemma *interfree-Append-Redirect-Edge*:
interfree-aux (*Some Append*, {}, *Some Redirect-Edge*)
⟨*proof*⟩

lemma *interfree-Redirect-Edge-Append*:
interfree-aux (*Some Redirect-Edge*, {}, *Some Append*)
⟨*proof*⟩

lemma *interfree-Append-Color-Target*:
interfree-aux (*Some Append*, {}, *Some Color-Target*)
⟨*proof*⟩

lemma *interfree-Color-Target-Append*:
interfree-aux (*Some Color-Target*, {}, *Some Append*)
⟨*proof*⟩

lemmas *collector-mutator-interfree* =
interfree-Blacken-Roots-Redirect-Edge interfree-Blacken-Roots-Color-Target
interfree-Propagate-Black-Redirect-Edge interfree-Propagate-Black-Color-Target
interfree-Count-Redirect-Edge interfree-Count-Color-Target
interfree-Append-Redirect-Edge interfree-Append-Color-Target
interfree-Redirect-Edge-Blacken-Roots interfree-Color-Target-Blacken-Roots
interfree-Redirect-Edge-Propagate-Black interfree-Color-Target-Propagate-Black
interfree-Redirect-Edge-Count interfree-Color-Target-Count
interfree-Redirect-Edge-Append interfree-Color-Target-Append

Interference freedom Collector-Mutator

lemma *interfree-Collector-Mutator*:
interfree-aux (*Some Collector*, {}, *Some Mutator*)

⟨*proof*⟩

Interference freedom Mutator-Collector

lemma *interfree-Mutator-Collector*:
interfree-aux (*Some Mutator*, {}, *Some Collector*)
⟨*proof*⟩

The Garbage Collection algorithm

In total there are 289 verification conditions.

lemma *Gar-Coll*:
  ‖− .{´*Proper* ∧ ´*Mut-init* ∧ ´*z*}.
  COBEGIN
   Collector
  .{*False*}.
 ‖
   Mutator
  .{*False*}.
 COEND
  .{*False*}.
⟨*proof*⟩

end

2.3 The Multi-Mutator Case

theory *Mul-Gar-Coll* **imports** *Graph OG-Syntax* **begin**

The full theory takes aprox. 18 minutes.

record *mut* =
  Z :: *bool*
  R :: *nat*
  T :: *nat*

Declaration of variables:

record *mul-gar-coll-state* =
  M :: *nodes*
  E :: *edges*
  bc :: *nat set*
  obc :: *nat set*
  Ma :: *nodes*
  ind :: *nat*
  k :: *nat*
  q :: *nat*
  l :: *nat*
  Muts :: *mut list*

2.3.1 The Mutators

constdefs

$Mul\text{-}mut\text{-}init$ :: $mul\text{-}gar\text{-}coll\text{-}state \Rightarrow nat \Rightarrow bool$
$Mul\text{-}mut\text{-}init \equiv \ll \lambda n.\ n{=}length\ {}'Muts \wedge (\forall\, i{<}n.\ R\ ({}'Muts!i){<}length\ {}'E$
$\wedge\ T\ ({}'Muts!i){<}length\ {}'M) \gg$

$Mul\text{-}Redirect\text{-}Edge$ :: $nat \Rightarrow nat \Rightarrow mul\text{-}gar\text{-}coll\text{-}state\ ann\text{-}com$
$Mul\text{-}Redirect\text{-}Edge\ j\ n \equiv$
$.\{{}'Mul\text{-}mut\text{-}init\ n \wedge Z\ ({}'Muts!j)\}.$
$\langle IF\ T({}'Muts!j) \in Reach\ {}'E\ THEN$
${}'E{:=}\ {}'E[R\ ({}'Muts!j){:=}\ (fst\ ({}'E!R({}'Muts!j)),\ T\ ({}'Muts!j))]\ FI,,$
${}'Muts{:=}\ {}'Muts[j{:=}\ ({}'Muts!j)\ (\!|Z{:=}False|\!)]\rangle$

$Mul\text{-}Color\text{-}Target$ :: $nat \Rightarrow nat \Rightarrow mul\text{-}gar\text{-}coll\text{-}state\ ann\text{-}com$
$Mul\text{-}Color\text{-}Target\ j\ n \equiv$
$.\{{}'Mul\text{-}mut\text{-}init\ n \wedge \neg\ Z\ ({}'Muts!j)\}.$
$\langle {}'M{:=}{}'M[T\ ({}'Muts!j){:=}Black],,\ {}'Muts{:=}{}'Muts[j{:=}\ ({}'Muts!j)\ (\!|Z{:=}True|\!)]\rangle$

$Mul\text{-}Mutator$ :: $nat \Rightarrow nat \Rightarrow\ mul\text{-}gar\text{-}coll\text{-}state\ ann\text{-}com$
$Mul\text{-}Mutator\ j\ n \equiv$
$.\{{}'Mul\text{-}mut\text{-}init\ n \wedge Z\ ({}'Muts!j)\}.$
$WHILE\ True$
$INV\ .\{{}'Mul\text{-}mut\text{-}init\ n \wedge Z\ ({}'Muts!j)\}.$
$DO\ Mul\text{-}Redirect\text{-}Edge\ j\ n\ ;;$
$Mul\text{-}Color\text{-}Target\ j\ n$
OD

lemmas *mul-mutator-defs* = *Mul-mut-init-def Mul-Redirect-Edge-def Mul-Color-Target-def*

Correctness of the proof outline of one mutator

lemma *Mul-Redirect-Edge*: $0 \le j \wedge j{<}n \Longrightarrow$
$\vdash Mul\text{-}Redirect\text{-}Edge\ j\ n$
$pre(Mul\text{-}Color\text{-}Target\ j\ n)$
$\langle proof \rangle$

lemma *Mul-Color-Target*: $0 \le j \wedge j{<}n \Longrightarrow$
$\vdash\ Mul\text{-}Color\text{-}Target\ j\ n$
$.\{{}'Mul\text{-}mut\text{-}init\ n \wedge Z\ ({}'Muts!j)\}.$
$\langle proof \rangle$

lemma *Mul-Mutator*: $0 \le j \wedge j{<}n \Longrightarrow$
$\vdash Mul\text{-}Mutator\ j\ n\ .\{False\}.$
$\langle proof \rangle$

Interference freedom between mutators

lemma *Mul-interfree-Redirect-Edge-Redirect-Edge*:
$[\![0 \le i;\ i{<}n;\ 0 \le j;\ j{<}n;\ i \neq j]\!] \Longrightarrow$

interfree-aux (*Some* (*Mul-Redirect-Edge* *i* *n*),{}, *Some*(*Mul-Redirect-Edge* *j* *n*))
⟨*proof*⟩

lemma *Mul-interfree-Redirect-Edge-Color-Target*:
$[\![0 \le i;\ i<n;\ 0 \le j;\ j<n;\ i \ne j]\!] \Longrightarrow$
interfree-aux (*Some*(*Mul-Redirect-Edge* *i* *n*),{},*Some*(*Mul-Color-Target* *j* *n*))
⟨*proof*⟩

lemma *Mul-interfree-Color-Target-Redirect-Edge*:
$[\![0 \le i;\ i<n;\ 0 \le j;\ j<n;\ i \ne j]\!] \Longrightarrow$
interfree-aux (*Some*(*Mul-Color-Target* *i* *n*),{},*Some*(*Mul-Redirect-Edge* *j* *n*))
⟨*proof*⟩

lemma *Mul-interfree-Color-Target-Color-Target*:
$[\![0 \le i;\ i<n;\ 0 \le j;\ j<n;\ i \ne j]\!] \Longrightarrow$
interfree-aux (*Some*(*Mul-Color-Target* *i* *n*),{},*Some*(*Mul-Color-Target* *j* *n*))
⟨*proof*⟩

lemmas *mul-mutator-interfree* =
Mul-interfree-Redirect-Edge-Redirect-Edge Mul-interfree-Redirect-Edge-Color-Target
Mul-interfree-Color-Target-Redirect-Edge Mul-interfree-Color-Target-Color-Target

lemma *Mul-interfree-Mutator-Mutator*: $[\![i < n;\ j < n;\ i \neq j]\!] \Longrightarrow$
interfree-aux (*Some* (*Mul-Mutator* *i* *n*), {}, *Some* (*Mul-Mutator* *j* *n*))
⟨*proof*⟩

Modular Parameterized Mutators

lemma *Mul-Parameterized-Mutators*: $0<n \Longrightarrow$
‖− .{´*Mul-mut-init* *n* ∧ (∀ *i*<*n*. *Z* (´*Muts*!*i*))}.
COBEGIN
SCHEME [*0*≤ *j*< *n*]
Mul-Mutator *j* *n*
.{*False*}.
COEND
.{*False*}.
⟨*proof*⟩

2.3.2 The Collector

constdefs
Queue :: *mul-gar-coll-state* ⇒ *nat*
Queue ≡ « *length* (*filter* (λ*i*. ¬ *Z* *i* ∧ ´*M*!(*T* *i*) ≠ *Black*) ´*Muts*) »

consts *M-init* :: *nodes*

constdefs
Proper-M-init :: *mul-gar-coll-state* ⇒ *bool*
Proper-M-init ≡ « *Blacks* *M-init*=*Roots* ∧ *length* *M-init*=*length* ´*M* »

Mul-Proper :: *mul-gar-coll-state* ⇒ *nat* ⇒ *bool*
Mul-Proper ≡ ≪ λ*n*. *Proper-Roots* ´*M* ∧ *Proper-Edges* (´*M*, ´*E*) ∧ ´*Proper-M-init* ∧ *n*=*length* ´*Muts* ≫

Safe :: *mul-gar-coll-state* ⇒ *bool*
Safe ≡ ≪ *Reach* ´*E* ⊆ *Blacks* ´*M* ≫

lemmas *mul-collector-defs* = *Proper-M-init-def Mul-Proper-def Safe-def*

Blackening Roots

constdefs
Mul-Blacken-Roots :: *nat* ⇒ *mul-gar-coll-state ann-com*
Mul-Blacken-Roots n ≡
.{´*Mul-Proper n*}.
´*ind*:=*0*;;
.{´*Mul-Proper n* ∧ ´*ind*=*0*}.
WHILE ´*ind*<*length* ´*M*
INV .{´*Mul-Proper n* ∧ (∀ *i*<´*ind*. *i*∈*Roots* ⟶ ´*M*!*i*=*Black*) ∧ ´*ind*≤*length* ´*M*}.
DO .{´*Mul-Proper n* ∧ (∀ *i*<´*ind*. *i*∈*Roots* ⟶ ´*M*!*i*=*Black*) ∧ ´*ind*<*length* ´*M*}.
IF ´*ind*∈*Roots THEN*
.{´*Mul-Proper n* ∧ (∀ *i*<´*ind*. *i*∈*Roots* ⟶ ´*M*!*i*=*Black*) ∧ ´*ind*<*length* ´*M* ∧ ´*ind*∈*Roots*}.
´*M*:=´*M*[´*ind*:=*Black*] *FI*;;
.{´*Mul-Proper n* ∧ (∀ *i*<´*ind*+*1*. *i*∈*Roots* ⟶ ´*M*!*i*=*Black*) ∧ ´*ind*<*length* ´*M*}.
´*ind*:=´*ind*+*1*
OD

lemma *Mul-Blacken-Roots*:
⊢ *Mul-Blacken-Roots n*
.{´*Mul-Proper n* ∧ *Roots* ⊆ *Blacks* ´*M*}.
⟨*proof*⟩

Propagating Black

constdefs
Mul-PBInv :: *mul-gar-coll-state* ⇒ *bool*
Mul-PBInv ≡ ≪´*Safe* ∨ ´*obc*⊂*Blacks* ´*M* ∨ ´*l*<´*Queue*
∨ (∀ *i*<´*ind*. ¬*BtoW*(´*E*!*i*,´*M*)) ∧ ´*l*≤´*Queue*≫

Mul-Auxk :: *mul-gar-coll-state* ⇒ *bool*
Mul-Auxk ≡ ≪´*l*<´*Queue* ∨ ´*M*!´*k*≠*Black* ∨ ¬*BtoW*(´*E*!´*ind*, ´*M*) ∨ ´*obc*⊂*Blacks* ´*M*≫

constdefs
Mul-Propagate-Black :: *nat* ⇒ *mul-gar-coll-state ann-com*
Mul-Propagate-Black n ≡

.{´Mul-Proper n ∧ Roots⊆Blacks ´M ∧ ´obc⊆Blacks ´M ∧ ´bc⊆Blacks ´M
∧ (´Safe ∨ ´l≤´Queue ∨ ´obc⊂Blacks ´M)}.
´ind:=0;;
.{´Mul-Proper n ∧ Roots⊆Blacks ´M
∧ ´obc⊆Blacks ´M ∧ Blacks ´M⊆Blacks ´M ∧ ´bc⊆Blacks ´M
∧ (´Safe ∨ ´l≤´Queue ∨ ´obc⊂Blacks ´M) ∧ ´ind=0}.
WHILE ´ind<length ´E
INV .{´Mul-Proper n ∧ Roots⊆Blacks ´M
∧ ´obc⊆Blacks ´M ∧ ´bc⊆Blacks ´M
∧ ´Mul-PBInv ∧ ´ind≤length ´E}.
DO .{´Mul-Proper n ∧ Roots⊆Blacks ´M
∧ ´obc⊆Blacks ´M ∧ ´bc⊆Blacks ´M
∧ ´Mul-PBInv ∧ ´ind<length ´E}.
IF ´M!(fst (´E!´ind))=Black THEN
.{´Mul-Proper n ∧ Roots⊆Blacks ´M
∧ ´obc⊆Blacks ´M ∧ ´bc⊆Blacks ´M
∧ ´Mul-PBInv ∧ (´M!fst(´E!´ind))=Black ∧ ´ind<length ´E}.
´k:=snd(´E!´ind);;
.{´Mul-Proper n ∧ Roots⊆Blacks ´M
∧ ´obc⊆Blacks ´M ∧ ´bc⊆Blacks ´M
∧ (´Safe ∨ ´obc⊂Blacks ´M ∨ ´l<´Queue ∨ (∀ i<´ind. ¬BtoW(´E!i,´M))
∧ ´l≤´Queue ∧ ´Mul-Auxk) ∧ ´k<length ´M ∧ ´M!fst(´E!´ind)=Black
∧ ´ind<length ´E}.
⟨´M:=´M[´k:=Black],,´ind:=´ind+1⟩
ELSE .{´Mul-Proper n ∧ Roots⊆Blacks ´M
∧ ´obc⊆Blacks ´M ∧ ´bc⊆Blacks ´M
∧ ´Mul-PBInv ∧ ´ind<length ´E}.
⟨IF ´M!(fst (´E!´ind))≠Black THEN ´ind:=´ind+1 FI⟩ FI
OD

lemma *Mul-Propagate-Black*:
⊢ Mul-Propagate-Black n
.{´Mul-Proper n ∧ Roots⊆Blacks ´M ∧ ´obc⊆Blacks ´M ∧ ´bc⊆Blacks ´M
∧ (´Safe ∨ ´obc⊂Blacks ´M ∨ ´l<´Queue ∧ (´l≤´Queue ∨ ´obc⊂Blacks
´M))}.
⟨proof⟩

Counting Black Nodes

constdefs
Mul-CountInv :: mul-gar-coll-state ⇒ nat ⇒ bool
Mul-CountInv ≡ « λind. {i. i<ind ∧ ´Ma!i=Black}⊆´bc »

Mul-Count :: nat ⇒ mul-gar-coll-state ann-com
Mul-Count n ≡
.{´Mul-Proper n ∧ Roots⊆Blacks ´M
∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M
∧ length ´Ma=length ´M
∧ (´Safe ∨ ´obc⊂Blacks ´Ma ∨ ´l<´q ∧ (´q≤´Queue ∨ ´obc⊂Blacks ´M))

$\wedge$ ´q<n+1 $\wedge$ ´bc={}}.
´ind:=0;;
.{´Mul-$Proper$ n $\wedge$ $Roots$⊆$Blacks$ ´M
$\wedge$ ´obc⊆$Blacks$ ´Ma $\wedge$ $Blacks$ ´Ma⊆$Blacks$ ´M $\wedge$ ´bc⊆$Blacks$ ´M
$\wedge$ $length$ ´Ma=$length$ ´M
$\wedge$ (´$Safe$ $\vee$ ´obc⊂$Blacks$ ´Ma $\vee$ ´l<´q $\wedge$ (´q≤´$Queue$ $\vee$ ´obc⊂$Blacks$ ´M))
$\wedge$ ´q<n+1 $\wedge$ ´bc={} $\wedge$ ´ind=0}.
WHILE ´ind<$length$ ´M
INV .{´Mul-$Proper$ n $\wedge$ $Roots$⊆$Blacks$ ´M
$\wedge$ ´obc⊆$Blacks$ ´Ma $\wedge$ $Blacks$ ´Ma⊆$Blacks$ ´M $\wedge$ ´bc⊆$Blacks$ ´M
$\wedge$ $length$ ´Ma=$length$ ´M $\wedge$ ´Mul-$CountInv$ ´ind
$\wedge$ (´$Safe$ $\vee$ ´obc⊂$Blacks$ ´Ma $\vee$ ´l<´q $\wedge$ (´q≤´$Queue$ $\vee$ ´obc⊂$Blacks$
´M))
$\wedge$ ´q<n+1 $\wedge$ ´ind≤$length$ ´M}.
DO .{´Mul-$Proper$ n $\wedge$ $Roots$⊆$Blacks$ ´M
$\wedge$ ´obc⊆$Blacks$ ´Ma $\wedge$ $Blacks$ ´Ma⊆$Blacks$ ´M $\wedge$ ´bc⊆$Blacks$ ´M
$\wedge$ $length$ ´Ma=$length$ ´M $\wedge$ ´Mul-$CountInv$ ´ind
$\wedge$ (´$Safe$ $\vee$ ´obc⊂$Blacks$ ´Ma $\vee$ ´l<´q $\wedge$ (´q≤´$Queue$ $\vee$ ´obc⊂$Blacks$ ´M))
$\wedge$ ´q<n+1 $\wedge$ ´ind<$length$ ´M}.
IF ´M!´ind=$Black$
THEN .{´Mul-$Proper$ n $\wedge$ $Roots$⊆$Blacks$ ´M
$\wedge$ ´obc⊆$Blacks$ ´Ma $\wedge$ $Blacks$ ´Ma⊆$Blacks$ ´M $\wedge$ ´bc⊆$Blacks$ ´M
$\wedge$ $length$ ´Ma=$length$ ´M $\wedge$ ´Mul-$CountInv$ ´ind
$\wedge$ (´$Safe$ $\vee$ ´obc⊂$Blacks$ ´Ma $\vee$ ´l<´q $\wedge$ (´q≤´$Queue$ $\vee$ ´obc⊂$Blacks$
´M))
$\wedge$ ´q<n+1 $\wedge$ ´ind<$length$ ´M $\wedge$ ´M!´ind=$Black$}.
´bc:=$insert$ ´ind ´bc
FI;;
.{´Mul-$Proper$ n $\wedge$ $Roots$⊆$Blacks$ ´M
$\wedge$ ´obc⊆$Blacks$ ´Ma $\wedge$ $Blacks$ ´Ma⊆$Blacks$ ´M $\wedge$ ´bc⊆$Blacks$ ´M
$\wedge$ $length$ ´Ma=$length$ ´M $\wedge$ ´Mul-$CountInv$ (´ind+1)
$\wedge$ (´$Safe$ $\vee$ ´obc⊂$Blacks$ ´Ma $\vee$ ´l<´q $\wedge$ (´q≤´$Queue$ $\vee$ ´obc⊂$Blacks$ ´M))
$\wedge$ ´q<n+1 $\wedge$ ´ind<$length$ ´M}.
´ind:=´ind+1
OD

lemma *Mul-Count*:
⊢ *Mul-Count n*
.{´Mul-$Proper$ n $\wedge$ $Roots$⊆$Blacks$ ´M
$\wedge$ ´obc⊆$Blacks$ ´Ma $\wedge$ $Blacks$ ´Ma⊆$Blacks$ ´M $\wedge$ ´bc⊆$Blacks$ ´M
$\wedge$ $length$ ´Ma=$length$ ´M $\wedge$ $Blacks$ ´Ma⊆´bc
$\wedge$ (´$Safe$ $\vee$ ´obc⊂$Blacks$ ´Ma $\vee$ ´l<´q $\wedge$ (´q≤´$Queue$ $\vee$ ´obc⊂$Blacks$ ´M))
$\wedge$ ´q<n+1}.
⟨*proof*⟩

Appending garbage nodes to the free list

consts *Append-to-free* :: *nat* × *edges* ⇒ *edges*

axioms

$Append\text{-}to\text{-}free0$: $length\ (Append\text{-}to\text{-}free\ (i,\ e)) = length\ e$

$Append\text{-}to\text{-}free1$: $Proper\text{-}Edges\ (m,\ e)$
$\Longrightarrow Proper\text{-}Edges\ (m,\ Append\text{-}to\text{-}free(i,\ e))$

$Append\text{-}to\text{-}free2$: $i \notin Reach\ e$
$\Longrightarrow n \in Reach\ (Append\text{-}to\text{-}free(i,\ e)) = (\ n = i \lor n \in Reach\ e)$

constdefs

$Mul\text{-}AppendInv :: mul\text{-}gar\text{-}coll\text{-}state \Rightarrow nat \Rightarrow bool$

$Mul\text{-}AppendInv \equiv \ll \lambda ind.\ (\forall\, i.\ ind{\le}i \longrightarrow i{<}length\ {\acute{}}M \longrightarrow i{\in}Reach\ {\acute{}}E \longrightarrow {\acute{}}M!i{=}Black)\gg$

$Mul\text{-}Append :: nat \Rightarrow\ mul\text{-}gar\text{-}coll\text{-}state\ ann\text{-}com$
$Mul\text{-}Append\ n \equiv$
$.\{{\acute{}}Mul\text{-}Proper\ n \land Roots{\subseteq}Blacks\ {\acute{}}M \land {\acute{}}Safe\}.$
${\acute{}}ind{:=}0;;$
$.\{{\acute{}}Mul\text{-}Proper\ n \land Roots{\subseteq}Blacks\ {\acute{}}M \land {\acute{}}Safe \land {\acute{}}ind{=}0\}.$
$WHILE\ {\acute{}}ind{<}length\ {\acute{}}M$
$INV\ .\{{\acute{}}Mul\text{-}Proper\ n \land {\acute{}}Mul\text{-}AppendInv\ {\acute{}}ind \land {\acute{}}ind{\le}length\ {\acute{}}M\}.$
$DO\ .\{{\acute{}}Mul\text{-}Proper\ n \land {\acute{}}Mul\text{-}AppendInv\ {\acute{}}ind \land {\acute{}}ind{<}length\ {\acute{}}M\}.$
$IF\ \ {\acute{}}M!{\acute{}}ind{=}Black\ THEN$
$.\{{\acute{}}Mul\text{-}Proper\ n \land {\acute{}}Mul\text{-}AppendInv\ {\acute{}}ind \land {\acute{}}ind{<}length\ {\acute{}}M \land {\acute{}}M!{\acute{}}ind{=}Black\}.$

${\acute{}}M{:=}{\acute{}}M[{\acute{}}ind{:=}White]$
$ELSE$
$.\{{\acute{}}Mul\text{-}Proper\ n \land {\acute{}}Mul\text{-}AppendInv\ {\acute{}}ind \land {\acute{}}ind{<}length\ {\acute{}}M \land {\acute{}}ind{\notin}Reach\ {\acute{}}E\}.$
${\acute{}}E{:=}Append\text{-}to\text{-}free({\acute{}}ind,{\acute{}}E)$
$FI;;$
$.\{{\acute{}}Mul\text{-}Proper\ n \land {\acute{}}Mul\text{-}AppendInv\ ({\acute{}}ind{+}1) \land {\acute{}}ind{<}length\ {\acute{}}M\}.$
${\acute{}}ind{:=}{\acute{}}ind{+}1$
OD

lemma *Mul-Append*:
$\vdash Mul\text{-}Append\ n$
$.\{{\acute{}}Mul\text{-}Proper\ n\}.$
$\langle proof\rangle$

Collector

constdefs

$Mul\text{-}Collector :: nat \Rightarrow\ mul\text{-}gar\text{-}coll\text{-}state\ ann\text{-}com$
$Mul\text{-}Collector\ n \equiv$
$.\{{\acute{}}Mul\text{-}Proper\ n\}.$
$WHILE\ True\ INV\ .\{{\acute{}}Mul\text{-}Proper\ n\}.$
DO
$Mul\text{-}Blacken\text{-}Roots\ n\ ;;$
$.\{{\acute{}}Mul\text{-}Proper\ n \land Roots{\subseteq}Blacks\ {\acute{}}M\}.$
${\acute{}}obc{:=}\{\};;$

```
.{´Mul-Proper n ∧ Roots⊆Blacks ´M ∧ ´obc={}}.
 ´bc:=Roots;;
.{´Mul-Proper n ∧ Roots⊆Blacks ´M ∧ ´obc={} ∧ ´bc=Roots}.
 ´l:=0;;
.{´Mul-Proper n ∧ Roots⊆Blacks ´M ∧ ´obc={} ∧ ´bc=Roots ∧ ´l=0}.
 WHILE ´l<n+1
   INV .{´Mul-Proper n ∧ Roots⊆Blacks ´M ∧ ´bc⊆Blacks ´M ∧
        (´Safe ∨ (´l≤´Queue ∨ ´bc⊂Blacks ´M) ∧ ´l<n+1)}.
 DO .{´Mul-Proper n ∧ Roots⊆Blacks ´M ∧ ´bc⊆Blacks ´M
      ∧ (´Safe ∨ ´l≤´Queue ∨ ´bc⊂Blacks ´M)}.
    ´obc:=´bc;;
    Mul-Propagate-Black n;;
    .{´Mul-Proper n ∧ Roots⊆Blacks ´M
      ∧ ´obc⊆Blacks ´M ∧ ´bc⊆Blacks ´M
      ∧ (´Safe ∨ ´obc⊂Blacks ´M ∨ ´l<´Queue
      ∧ (´l≤´Queue ∨ ´obc⊂Blacks ´M))}.
    ´bc:={};;
    .{´Mul-Proper n ∧ Roots⊆Blacks ´M
      ∧ ´obc⊆Blacks ´M ∧ ´bc⊆Blacks ´M
      ∧ (´Safe ∨ ´obc⊂Blacks ´M ∨ ´l<´Queue
      ∧ (´l≤´Queue ∨ ´obc⊂Blacks ´M)) ∧ ´bc={}}.
      ⟨ ´Ma:=´M,, ´q:=´Queue ⟩;;
    Mul-Count n;;
    .{´Mul-Proper n ∧ Roots⊆Blacks ´M
      ∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M
      ∧ length ´Ma=length ´M ∧ Blacks ´Ma⊆´bc
      ∧ (´Safe ∨ ´obc⊂Blacks ´Ma ∨ ´l<´q ∧ (´q≤´Queue ∨ ´obc⊂Blacks ´M))
      ∧ ´q<n+1}.
    IF ´obc=´bc THEN
    .{´Mul-Proper n ∧ Roots⊆Blacks ´M
      ∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M
      ∧ length ´Ma=length ´M ∧ Blacks ´Ma⊆´bc
      ∧ (´Safe ∨ ´obc⊂Blacks ´Ma ∨ ´l<´q ∧ (´q≤´Queue ∨ ´obc⊂Blacks ´M))
      ∧ ´q<n+1 ∧ ´obc=´bc}.
    ´l:=´l+1
    ELSE .{´Mul-Proper n ∧ Roots⊆Blacks ´M
         ∧ ´obc⊆Blacks ´Ma ∧ Blacks ´Ma⊆Blacks ´M ∧ ´bc⊆Blacks ´M
         ∧ length ´Ma=length ´M ∧ Blacks ´Ma⊆´bc
           ∧ (´Safe ∨ ´obc⊂Blacks ´Ma ∨ ´l<´q ∧ (´q≤´Queue ∨ ´obc⊂Blacks
´M))
         ∧ ´q<n+1 ∧ ´obc≠´bc}.
        ´l:=0 FI
 OD;;
 Mul-Append n
OD
```

lemmas *mul-modules = Mul-Redirect-Edge-def Mul-Color-Target-def Mul-Blacken-Roots-def Mul-Propagate-Black-def Mul-Count-def Mul-Append-def*

lemma *Mul-Collector*:
$\vdash$ *Mul-Collector n*
.{*False*}.
⟨*proof*⟩

2.3.3 Interference Freedom

lemma *le-length-filter-update*[*rule-format*]:
$\forall i.\ (\neg P\ (list!i) \vee P\ j) \wedge i<length\ list$
$\longrightarrow length(filter\ P\ list) \leq length(filter\ P\ (list[i:=j]))$
⟨*proof*⟩

lemma *less-length-filter-update* [*rule-format*]:
$\forall i.\ P\ j \wedge \neg(P\ (list!i)) \wedge i<length\ list$
$\longrightarrow length(filter\ P\ list) < length(filter\ P\ (list[i:=j]))$
⟨*proof*⟩

lemma *Mul-interfree-Blacken-Roots-Redirect-Edge*: $[\![0\leq j;\ j<n]\!] \Longrightarrow$
interfree-aux (*Some*(*Mul-Blacken-Roots n*),{},*Some*(*Mul-Redirect-Edge j n*))
⟨*proof*⟩

lemma *Mul-interfree-Redirect-Edge-Blacken-Roots*: $[\![0\leq j;\ j<n]\!]\Longrightarrow$
interfree-aux (*Some*(*Mul-Redirect-Edge j n*),{},*Some* (*Mul-Blacken-Roots n*))
⟨*proof*⟩

lemma *Mul-interfree-Blacken-Roots-Color-Target*: $[\![0\leq j;\ j<n]\!]\Longrightarrow$
interfree-aux (*Some*(*Mul-Blacken-Roots n*),{},*Some* (*Mul-Color-Target j n*))
⟨*proof*⟩

lemma *Mul-interfree-Color-Target-Blacken-Roots*: $[\![0\leq j;\ j<n]\!]\Longrightarrow$
interfree-aux (*Some*(*Mul-Color-Target j n*),{},*Some* (*Mul-Blacken-Roots n*))
⟨*proof*⟩

lemma *Mul-interfree-Propagate-Black-Redirect-Edge*: $[\![0\leq j;\ j<n]\!]\Longrightarrow$
interfree-aux (*Some*(*Mul-Propagate-Black n*),{},*Some* (*Mul-Redirect-Edge j n*))
⟨*proof*⟩

lemma *Mul-interfree-Redirect-Edge-Propagate-Black*: $[\![0\leq j;\ j<n]\!]\Longrightarrow$
interfree-aux (*Some*(*Mul-Redirect-Edge j n*),{},*Some* (*Mul-Propagate-Black n*))
⟨*proof*⟩

lemma *Mul-interfree-Propagate-Black-Color-Target*: $[\![0\leq j;\ j<n]\!]\Longrightarrow$
interfree-aux (*Some*(*Mul-Propagate-Black n*),{},*Some* (*Mul-Color-Target j n*))
⟨*proof*⟩

lemma *Mul-interfree-Color-Target-Propagate-Black*: $[\![0\leq j;\ j<n]\!]\Longrightarrow$
interfree-aux (*Some*(*Mul-Color-Target j n*),{},*Some*(*Mul-Propagate-Black n*))
⟨*proof*⟩

lemma *Mul-interfree-Count-Redirect-Edge*: $[\![0 \leq j;\ j < n]\!] \Longrightarrow$
interfree-aux (*Some*(*Mul-Count n*),{},*Some*(*Mul-Redirect-Edge j n*))
⟨*proof*⟩

lemma *Mul-interfree-Redirect-Edge-Count*: $[\![0 \leq j;\ j < n]\!] \Longrightarrow$
interfree-aux (*Some*(*Mul-Redirect-Edge j n*),{},*Some*(*Mul-Count n*))
⟨*proof*⟩

lemma *Mul-interfree-Count-Color-Target*: $[\![0 \leq j;\ j < n]\!] \Longrightarrow$
interfree-aux (*Some*(*Mul-Count n*),{},*Some*(*Mul-Color-Target j n*))
⟨*proof*⟩

lemma *Mul-interfree-Color-Target-Count*: $[\![0 \leq j;\ j < n]\!] \Longrightarrow$
interfree-aux (*Some*(*Mul-Color-Target j n*),{}, *Some*(*Mul-Count n*))
⟨*proof*⟩

lemma *Mul-interfree-Append-Redirect-Edge*: $[\![0 \leq j;\ j < n]\!] \Longrightarrow$
interfree-aux (*Some*(*Mul-Append n*),{}, *Some*(*Mul-Redirect-Edge j n*))
⟨*proof*⟩

lemma *Mul-interfree-Redirect-Edge-Append*: $[\![0 \leq j;\ j < n]\!] \Longrightarrow$
interfree-aux (*Some*(*Mul-Redirect-Edge j n*),{},*Some*(*Mul-Append n*))
⟨*proof*⟩

lemma *Mul-interfree-Append-Color-Target*: $[\![0 \leq j;\ j < n]\!] \Longrightarrow$
interfree-aux (*Some*(*Mul-Append n*),{}, *Some*(*Mul-Color-Target j n*))
⟨*proof*⟩

lemma *Mul-interfree-Color-Target-Append*: $[\![0 \leq j;\ j < n]\!] \Longrightarrow$
interfree-aux (*Some*(*Mul-Color-Target j n*),{}, *Some*(*Mul-Append n*))
⟨*proof*⟩

Interference freedom Collector-Mutator

lemmas *mul-collector-mutator-interfree* =
Mul-interfree-Blacken-Roots-Redirect-Edge Mul-interfree-Blacken-Roots-Color-Target

Mul-interfree-Propagate-Black-Redirect-Edge Mul-interfree-Propagate-Black-Color-Target

Mul-interfree-Count-Redirect-Edge Mul-interfree-Count-Color-Target
Mul-interfree-Append-Redirect-Edge Mul-interfree-Append-Color-Target
Mul-interfree-Redirect-Edge-Blacken-Roots Mul-interfree-Color-Target-Blacken-Roots

Mul-interfree-Redirect-Edge-Propagate-Black Mul-interfree-Color-Target-Propagate-Black

Mul-interfree-Redirect-Edge-Count Mul-interfree-Color-Target-Count
Mul-interfree-Redirect-Edge-Append Mul-interfree-Color-Target-Append

lemma *Mul-interfree-Collector-Mutator*: $j<n \implies$
interfree-aux (*Some* (*Mul-Collector n*), {}, *Some* (*Mul-Mutator j n*))
⟨*proof*⟩

Interference freedom Mutator-Collector

lemma *Mul-interfree-Mutator-Collector*: $j < n \implies$
interfree-aux (*Some* (*Mul-Mutator j n*), {}, *Some* (*Mul-Collector n*))
⟨*proof*⟩

The Multi-Mutator Garbage Collection Algorithm

The total number of verification conditions is 328

lemma *Mul-Gar-Coll*:
$\Vert-$.{´*Mul-Proper n* ∧ ´*Mul-mut-init n* ∧ (∀ *i*<*n*. *Z* (´*Muts*!*i*))}.
COBEGIN
Mul-Collector n
.{*False*}.
$\Vert$
SCHEME [*0*≤ *j*< *n*]
Mul-Mutator j n
.{*False*}.
COEND
.{*False*}.
⟨*proof*⟩

end

Chapter 3

The Rely-Guarantee Method

3.1 Abstract Syntax

theory *RG-Com* **imports** *Main* **begin**

Semantics of assertions and boolean expressions (bexp) as sets of states. Syntax of commands *com* and parallel commands *par-com*.

types
  'a bexp = *'a set*

datatype *'a com* =
  Basic 'a ⇒*'a*
  | *Seq 'a com 'a com*
  | *Cond 'a bexp 'a com 'a com*
  | *While 'a bexp 'a com*
  | *Await 'a bexp 'a com*

types *'a par-com* = ((*'a com*) *option*) *list*

end

3.2 Operational Semantics

theory *RG-Tran*
imports *RG-Com*
begin

3.2.1 Semantics of Component Programs

Environment transitions

types *'a conf* = ((*'a com*) *option*) × *'a*

inductive-set
  etran :: (*'a conf* × *'a conf*) *set*

and *etran'* :: *'a conf* ⇒ *'a conf* ⇒ *bool* (- −*e*→ - [*81*,*81*] *80*)
where
P −*e*→ Q ≡ (P,Q) ∈ *etran*
| *Env*: $(P,\ s)$ −*e*→ $(P,\ t)$

lemma *etranE*: c −*e*→ c' ⟹ ($\bigwedge P\ s\ t.\ c = (P,\ s) \Longrightarrow c' = (P,\ t) \Longrightarrow Q) \Longrightarrow Q$
⟨*proof*⟩

Component transitions

inductive-set
ctran :: (*'a conf* × *'a conf*) *set*
and *ctran'* :: *'a conf* ⇒ *'a conf* ⇒ *bool* (- −*c*→ - [*81*,*81*] *80*)
and *ctrans* :: *'a conf* ⇒ *'a conf* ⇒ *bool* (- −*c*∗→ - [*81*,*81*] *80*)
where
P −*c*→ Q ≡ (P,Q) ∈ *ctran*
| P −*c*∗→ Q ≡ (P,Q) ∈ *ctran*^∗

| *Basic*: (*Some*(*Basic f*), *s*) −*c*→ (*None*, *f s*)

| *Seq1*: (*Some P0*, *s*) −*c*→ (*None*, *t*) ⟹ (*Some*(*Seq P0 P1*), *s*) −*c*→ (*Some P1*, *t*)

| *Seq2*: (*Some P0*, *s*) −*c*→ (*Some P2*, *t*) ⟹ (*Some*(*Seq P0 P1*), *s*) −*c*→ (*Some*(*Seq P2 P1*), *t*)

| *CondT*: *s*∈*b* ⟹ (*Some*(*Cond b P1 P2*), *s*) −*c*→ (*Some P1*, *s*)
| *CondF*: *s*∉*b* ⟹ (*Some*(*Cond b P1 P2*), *s*) −*c*→ (*Some P2*, *s*)

| *WhileF*: *s*∉*b* ⟹ (*Some*(*While b P*), *s*) −*c*→ (*None*, *s*)
| *WhileT*: *s*∈*b* ⟹ (*Some*(*While b P*), *s*) −*c*→ (*Some*(*Seq P* (*While b P*)), *s*)

| *Await*: ⟦*s*∈*b*; (*Some P*, *s*) −*c*∗→ (*None*, *t*)⟧ ⟹ (*Some*(*Await b P*), *s*) −*c*→ (*None*, *t*)

monos *rtrancl-mono*

3.2.2 Semantics of Parallel Programs

types *'a par-conf* = (*'a par-com*) × *'a*

inductive-set
par-etran :: (*'a par-conf* × *'a par-conf*) *set*
and *par-etran'* :: [*'a par-conf*,*'a par-conf*] ⇒ *bool* (- −*pe*→ - [*81*,*81*] *80*)
where
P −*pe*→ Q ≡ (P,Q) ∈ *par-etran*
| *ParEnv*: (*Ps*, *s*) −*pe*→ (*Ps*, *t*)

inductive-set
par-ctran :: (*'a par-conf* × *'a par-conf*) *set*

and $\mathit{par\text{-}ctran}'$ :: $['a\ \mathit{par\text{-}conf},'a\ \mathit{par\text{-}conf}] \Rightarrow \mathit{bool}$ $(\text{-}\ {-pc\rightarrow}\ \text{-}\ [81,81]\ 80)$
where
$P\ {-pc\rightarrow}\ Q \equiv (P,Q) \in \mathit{par\text{-}ctran}$
| $\mathit{ParComp}$: $[\![i<\mathit{length}\ Ps;\ (Ps!i,\ s)\ {-c\rightarrow}\ (r,\ t)]\!] \Longrightarrow (Ps,\ s)\ {-pc\rightarrow}\ (Ps[i:=r],\ t)$

lemma $\mathit{par\text{-}ctranE}$: $c\ {-pc\rightarrow}\ c' \Longrightarrow$
$(\bigwedge i\ Ps\ s\ r\ t.\ c = (Ps,\ s) \Longrightarrow c' = (Ps[i := r],\ t) \Longrightarrow i < \mathit{length}\ Ps \Longrightarrow$
$(Ps\ !\ i,\ s)\ {-c\rightarrow}\ (r,\ t) \Longrightarrow P) \Longrightarrow P$
$\langle \mathit{proof} \rangle$

3.2.3 Computations

Sequential computations

types $'a\ \mathit{confs} = ('a\ \mathit{conf})\ \mathit{list}$

inductive-set cptn :: $('a\ \mathit{confs})\ \mathit{set}$
where
$\mathit{CptnOne}$: $[(P,s)] \in \mathit{cptn}$
| $\mathit{CptnEnv}$: $(P,\ t)\#xs \in \mathit{cptn} \Longrightarrow (P,s)\#(P,t)\#xs \in \mathit{cptn}$
| $\mathit{CptnComp}$: $[\![(P,s)\ {-c\rightarrow}\ (Q,t);\ (Q,\ t)\#xs \in \mathit{cptn}\ ]\!] \Longrightarrow (P,s)\#(Q,t)\#xs \in \mathit{cptn}$

constdefs
cp :: $('a\ \mathit{com})\ \mathit{option} \Rightarrow 'a \Rightarrow ('a\ \mathit{confs})\ \mathit{set}$
$\mathit{cp}\ P\ s \equiv \{l.\ l!0=(P,s) \wedge l \in \mathit{cptn}\}$

Parallel computations

types $'a\ \mathit{par\text{-}confs} = ('a\ \mathit{par\text{-}conf})\ \mathit{list}$

inductive-set $\mathit{par\text{-}cptn}$ :: $('a\ \mathit{par\text{-}confs})\ \mathit{set}$
where
$\mathit{ParCptnOne}$: $[(P,s)] \in \mathit{par\text{-}cptn}$
| $\mathit{ParCptnEnv}$: $(P,t)\#xs \in \mathit{par\text{-}cptn} \Longrightarrow (P,s)\#(P,t)\#xs \in \mathit{par\text{-}cptn}$
| $\mathit{ParCptnComp}$: $[\![\ (P,s)\ {-pc\rightarrow}\ (Q,t);\ (Q,t)\#xs \in \mathit{par\text{-}cptn}\ ]\!] \Longrightarrow (P,s)\#(Q,t)\#xs$
$\in \mathit{par\text{-}cptn}$

constdefs
$\mathit{par\text{-}cp}$ :: $'a\ \mathit{par\text{-}com} \Rightarrow 'a \Rightarrow ('a\ \mathit{par\text{-}confs})\ \mathit{set}$
$\mathit{par\text{-}cp}\ P\ s \equiv \{l.\ l!0=(P,s) \wedge l \in \mathit{par\text{-}cptn}\}$

3.2.4 Modular Definition of Computation

constdefs
lift :: $'a\ \mathit{com} \Rightarrow 'a\ \mathit{conf} \Rightarrow 'a\ \mathit{conf}$
$\mathit{lift}\ Q \equiv \lambda(P,\ s).\ (\mathit{if}\ P=\mathit{None}\ \mathit{then}\ (\mathit{Some}\ Q,s)\ \mathit{else}\ (\mathit{Some}(\mathit{Seq}\ (\mathit{the}\ P)\ Q),\ s))$

inductive-set $\mathit{cptn\text{-}mod}$ :: $('a\ \mathit{confs})\ \mathit{set}$
where
$\mathit{CptnModOne}$: $[(P,\ s)] \in \mathit{cptn\text{-}mod}$

| *CptnModEnv*: $(P,\ t)\#xs \in cptn\text{-}mod \Longrightarrow (P,\ s)\#(P,\ t)\#xs \in cptn\text{-}mod$
| *CptnModNone*: $[\![(Some\ P,\ s)\ -c\rightarrow\ (None,\ t);\ (None,\ t)\#xs \in cptn\text{-}mod\ ]\!] \Longrightarrow (Some\ P,s)\#(None,\ t)\#xs \in cptn\text{-}mod$
| *CptnModCondT*: $[\![(Some\ P0,\ s)\#ys \in cptn\text{-}mod;\ s \in b\ ]\!] \Longrightarrow (Some(Cond\ b\ P0\ P1),\ s)\#(Some\ P0,\ s)\#ys \in cptn\text{-}mod$
| *CptnModCondF*: $[\![(Some\ P1,\ s)\#ys \in cptn\text{-}mod;\ s \notin b\ ]\!] \Longrightarrow (Some(Cond\ b\ P0\ P1),\ s)\#(Some\ P1,\ s)\#ys \in cptn\text{-}mod$
| *CptnModSeq1*: $[\![(Some\ P0,\ s)\#xs \in cptn\text{-}mod;\ zs{=}map\ (lift\ P1)\ xs\ ]\!]$
$\Longrightarrow (Some(Seq\ P0\ P1),\ s)\#zs \in cptn\text{-}mod$
| *CptnModSeq2*:
$[\![(Some\ P0,\ s)\#xs \in cptn\text{-}mod;\ fst(last\ ((Some\ P0,\ s)\#xs)) = None;$
$(Some\ P1,\ snd(last\ ((Some\ P0,\ s)\#xs)))\#ys \in cptn\text{-}mod;$
$zs{=}(map\ (lift\ P1)\ xs)@ys\ ]\!] \Longrightarrow (Some(Seq\ P0\ P1),\ s)\#zs \in cptn\text{-}mod$

| *CptnModWhile1*:
$[\![\ (Some\ P,\ s)\#xs \in cptn\text{-}mod;\ s \in b;\ zs{=}map\ (lift\ (While\ b\ P))\ xs\ ]\!]$
$\Longrightarrow (Some(While\ b\ P),\ s)\#(Some(Seq\ P\ (While\ b\ P)),\ s)\#zs \in cptn\text{-}mod$
| *CptnModWhile2*:
$[\![\ (Some\ P,\ s)\#xs \in cptn\text{-}mod;\ fst(last\ ((Some\ P,\ s)\#xs)){=}None;\ s \in b;$
$zs{=}(map\ (lift\ (While\ b\ P))\ xs)@ys;$
$(Some(While\ b\ P),\ snd(last\ ((Some\ P,\ s)\#xs)))\#ys \in cptn\text{-}mod]\!]$
$\Longrightarrow (Some(While\ b\ P),\ s)\#(Some(Seq\ P\ (While\ b\ P)),\ s)\#zs \in cptn\text{-}mod$

3.2.5 Equivalence of Both Definitions.

lemma *last-length*: $((a\#xs)!(length\ xs)){=}last\ (a\#xs)$
$\langle proof \rangle$

lemma *div-seq* [*rule-format*]: $list \in cptn\text{-}mod \Longrightarrow$
$(\forall s\ P\ Q\ zs.\ list{=}(Some\ (Seq\ P\ Q),\ s)\#zs \longrightarrow$
$(\exists xs.\ (Some\ P,\ s)\#xs \in cptn\text{-}mod\ \wedge (zs{=}(map\ (lift\ Q)\ xs) \vee$
$(\ fst(((Some\ P,\ s)\#xs)!length\ xs){=}None \wedge$
$(\exists ys.\ (Some\ Q,\ snd(((Some\ P,\ s)\#xs)!length\ xs))\#ys \in cptn\text{-}mod$
$\wedge\ zs{=}(map\ (lift\ (Q))\ xs)@ys)))))$
$\langle proof \rangle$

lemma *cptn-onlyif-cptn-mod-aux* [*rule-format*]:
$\forall s\ Q\ t\ xs.((Some\ a,\ s),\ Q,\ t) \in ctran \longrightarrow (Q,\ t)\ \#\ xs \in cptn\text{-}mod$
$\longrightarrow (Some\ a,\ s)\ \#\ (Q,\ t)\ \#\ xs \in cptn\text{-}mod$
$\langle proof \rangle$

lemma *cptn-onlyif-cptn-mod* [*rule-format*]: $c \in cptn \Longrightarrow c \in cptn\text{-}mod$
$\langle proof \rangle$

lemma *lift-is-cptn*: $c{\in}cptn \Longrightarrow map\ (lift\ P)\ c \in cptn$
$\langle proof \rangle$

lemma *cptn-append-is-cptn* [*rule-format*]:
$\forall b\ a.\ b\#c1{\in}cptn \longrightarrow\ a\#c2{\in}cptn \longrightarrow (b\#c1)!length\ c1{=}a \longrightarrow b\#c1@c2{\in}cptn$

$\langle proof \rangle$

lemma *last-lift*: $[\![xs{\neq}[\,]; fst(xs!(length\ xs - (Suc\ 0)))=None]\!]$
$\Longrightarrow fst((map\ (lift\ P)\ xs)!(length\ (map\ (lift\ P)\ xs)- (Suc\ 0)))=(Some\ P)$
$\langle proof \rangle$

lemma *last-fst* [*rule-format*]: $P((a\#x)!length\ x) \longrightarrow \neg P\ a \longrightarrow P\ (x!(length\ x - (Suc\ 0)))$
$\langle proof \rangle$

lemma *last-fst-esp*:
$fst(((Some\ a,s)\#xs)!(length\ xs))=None \Longrightarrow fst(xs!(length\ xs - (Suc\ 0)))=None$
$\langle proof \rangle$

lemma *last-snd*: $xs{\neq}[\,] \Longrightarrow$
$snd(((map\ (lift\ P)\ xs))!(length\ (map\ (lift\ P)\ xs) - (Suc\ 0)))=snd(xs!(length\ xs - (Suc\ 0)))$
$\langle proof \rangle$

lemma *Cons-lift*: $(Some\ (Seq\ P\ Q),\ s)\ \#\ (map\ (lift\ Q)\ xs) = map\ (lift\ Q)\ ((Some\ P,\ s)\ \#\ xs)$
$\langle proof \rangle$

lemma *Cons-lift-append*:
$(Some\ (Seq\ P\ Q),\ s)\ \#\ (map\ (lift\ Q)\ xs)\ @\ ys = map\ (lift\ Q)\ ((Some\ P,\ s)\ \#\ xs)@\ ys$
$\langle proof \rangle$

lemma *lift-nth*: $i{<}length\ xs \Longrightarrow map\ (lift\ Q)\ xs\ !\ i = lift\ Q\ \ (xs!\ i)$
$\langle proof \rangle$

lemma *snd-lift*: $i{<}\ length\ xs \Longrightarrow snd(lift\ Q\ (xs\ !\ i))= snd\ (xs\ !\ i)$
$\langle proof \rangle$

lemma *cptn-if-cptn-mod*: $c \in cptn\text{-}mod \Longrightarrow c \in cptn$
$\langle proof \rangle$

theorem *cptn-iff-cptn-mod*: $(c \in cptn) = (c \in cptn\text{-}mod)$
$\langle proof \rangle$

3.3 Validity of Correctness Formulas

3.3.1 Validity for Component Programs.

types $'a\ rgformula = 'a\ com \times 'a\ set \times ('a \times 'a)\ set \times ('a \times 'a)\ set \times 'a\ set$

constdefs
$assum :: ('a\ set \times ('a \times 'a)\ set) \Rightarrow ('a\ confs)\ set$
$assum \equiv \lambda(pre,\ rely).\ \{c.\ snd(c!0) \in pre \wedge (\forall\, i.\ Suc\ i{<}length\ c \longrightarrow$

$c!i\ -e\rightarrow\ c!(Suc\ i) \longrightarrow (snd(c!i),\ snd(c!Suc\ i)) \in rely)\}$

$comm :: (('a \times 'a)\ set \times 'a\ set) \Rightarrow ('a\ confs)\ set$
$comm \equiv \lambda(guar,\ post).\ \{c.\ (\forall i.\ Suc\ i<length\ c \longrightarrow$
$c!i\ -c\rightarrow\ c!(Suc\ i) \longrightarrow (snd(c!i),\ snd(c!Suc\ i)) \in guar) \wedge$
$(fst\ (last\ c) = None \longrightarrow snd\ (last\ c) \in post)\}$

$com\text{-}validity :: 'a\ com \Rightarrow 'a\ set \Rightarrow ('a \times 'a)\ set \Rightarrow ('a \times 'a)\ set \Rightarrow 'a\ set \Rightarrow bool$

$(\models\ \text{-}\ sat\ [\text{-},\ \text{-},\ \text{-},\ \text{-}]\ [60,0,0,0,0]\ 45)$
$\models P\ sat\ [pre,\ rely,\ guar,\ post] \equiv$
$\forall s.\ cp\ (Some\ P)\ s \cap assum(pre,\ rely) \subseteq comm(guar,\ post)$

3.3.2 Validity for Parallel Programs.

constdefs
$All\text{-}None :: (('a\ com)\ option)\ list \Rightarrow bool$
$All\text{-}None\ xs \equiv \forall c \in set\ xs.\ c=None$

$par\text{-}assum :: ('a\ set \times ('a \times 'a)\ set) \Rightarrow ('a\ par\text{-}confs)\ set$
$par\text{-}assum \equiv \lambda(pre,\ rely).\ \{c.\ snd(c!0) \in pre \wedge (\forall i.\ Suc\ i<length\ c \longrightarrow$
$c!i\ -pe\rightarrow\ c!Suc\ i \longrightarrow (snd(c!i),\ snd(c!Suc\ i)) \in rely)\}$

$par\text{-}comm :: (('a \times 'a)\ set \times 'a\ set) \Rightarrow ('a\ par\text{-}confs)\ set$
$par\text{-}comm \equiv \lambda(guar,\ post).\ \{c.\ (\forall i.\ Suc\ i<length\ c \longrightarrow$
$c!i\ -pc\rightarrow\ c!Suc\ i \longrightarrow (snd(c!i),\ snd(c!Suc\ i)) \in guar) \wedge$
$(All\text{-}None\ (fst\ (last\ c)) \longrightarrow snd(\ last\ c) \in post)\}$

$par\text{-}com\text{-}validity :: 'a\ \ par\text{-}com \Rightarrow 'a\ set \Rightarrow ('a \times 'a)\ set \Rightarrow ('a \times 'a)\ set$
$\Rightarrow 'a\ set \Rightarrow bool\ \ (\models\ \text{-}\ SAT\ [\text{-},\ \text{-},\ \text{-},\ \text{-}]\ [60,0,0,0,0]\ 45)$
$\models Ps\ SAT\ [pre,\ rely,\ guar,\ post] \equiv$
$\forall s.\ par\text{-}cp\ Ps\ s \cap par\text{-}assum(pre,\ rely) \subseteq par\text{-}comm(guar,\ post)$

3.3.3 Compositionality of the Semantics

Definition of the conjoin operator

constdefs
$same\text{-}length :: 'a\ par\text{-}confs \Rightarrow ('a\ confs)\ list \Rightarrow bool$
$same\text{-}length\ c\ clist \equiv (\forall i<length\ clist.\ length(clist!i)=length\ c)$

$same\text{-}state :: 'a\ par\text{-}confs \Rightarrow ('a\ confs)\ list \Rightarrow bool$
$same\text{-}state\ c\ clist \equiv (\forall i\ <length\ clist.\ \forall j<length\ c.\ snd(c!j) = snd((clist!i)!j))$

$same\text{-}program :: 'a\ par\text{-}confs \Rightarrow ('a\ confs)\ list \Rightarrow bool$
$same\text{-}program\ c\ clist \equiv (\forall j<length\ c.\ fst(c!j) = map\ (\lambda x.\ fst(nth\ x\ j))\ clist)$

$compat\text{-}label :: 'a\ par\text{-}confs \Rightarrow ('a\ confs)\ list \Rightarrow bool$
$compat\text{-}label\ c\ clist \equiv (\forall j.\ Suc\ j<length\ c \longrightarrow$
$(c!j\ -pc\rightarrow\ c!Suc\ j \wedge (\exists i<length\ clist.\ (clist!i)!j\ -c\rightarrow\ (clist!i)!\ Suc\ j \wedge$

$(\forall\, l<length\ clist.\ l\neq i \longrightarrow (clist!l)!j\ -e\rightarrow\ (clist!l)!\ Suc\ j))) \lor$
$(c!j\ -pe\rightarrow\ c!Suc\ j \land (\forall\, i<length\ clist.\ (clist!i)!j\ -e\rightarrow\ (clist!i)!\ Suc\ j)))$

conjoin :: *'a par-confs* ⇒ (*'a confs*) *list* ⇒ *bool* (- ∝ - [*65*,*65*] *64*)
c ∝ *clist* ≡ (*same-length c clist*) ∧ (*same-state c clist*) ∧ (*same-program c clist*) ∧ (*compat-label c clist*)

Some previous lemmas

lemma *list-eq-if* [*rule-format*]:
$\forall\, ys.\ xs=ys \longrightarrow (length\ xs = length\ ys) \longrightarrow (\forall\, i<length\ xs.\ xs!i=ys!i)$
⟨*proof*⟩

lemma *list-eq*: $(length\ xs = length\ ys \land (\forall\, i<length\ xs.\ xs!i=ys!i)) = (xs=ys)$
⟨*proof*⟩

lemma *nth-tl*: $[\![\ ys!0=a;\ ys\neq[]\ ]\!] \Longrightarrow ys=(a\#(tl\ ys))$
⟨*proof*⟩

lemma *nth-tl-if* [*rule-format*]: $ys\neq[] \longrightarrow ys!0=a \longrightarrow P\ ys \longrightarrow P\ (a\#(tl\ ys))$
⟨*proof*⟩

lemma *nth-tl-onlyif* [*rule-format*]: $ys\neq[] \longrightarrow ys!0=a \longrightarrow P\ (a\#(tl\ ys)) \longrightarrow P\ ys$
⟨*proof*⟩

lemma *seq-not-eq1*: *Seq c1 c2*≠*c1*
⟨*proof*⟩

lemma *seq-not-eq2*: *Seq c1 c2*≠*c2*
⟨*proof*⟩

lemma *if-not-eq1*: *Cond b c1 c2* ≠*c1*
⟨*proof*⟩

lemma *if-not-eq2*: *Cond b c1 c2*≠*c2*
⟨*proof*⟩

lemmas *seq-and-if-not-eq* [*simp*] = *seq-not-eq1 seq-not-eq2*
seq-not-eq1 [*THEN not-sym*] *seq-not-eq2* [*THEN not-sym*]
if-not-eq1 if-not-eq2 if-not-eq1 [*THEN not-sym*] *if-not-eq2* [*THEN not-sym*]

lemma *prog-not-eq-in-ctran-aux*:
assumes *c*: $(P,s)\ -c\rightarrow\ (Q,t)$
shows $P\neq Q$ ⟨*proof*⟩

lemma *prog-not-eq-in-ctran* [*simp*]: $\neg\ (P,s)\ -c\rightarrow\ (P,t)$
⟨*proof*⟩

lemma *prog-not-eq-in-par-ctran-aux* [*rule-format*]: $(P,s)\ -pc\rightarrow\ (Q,t) \Longrightarrow (P\neq Q)$

$\langle proof \rangle$

lemma *prog-not-eq-in-par-ctran* [*simp*]: $\neg\ (P,s)\ -pc\rightarrow\ (P,t)$
$\langle proof \rangle$

lemma *tl-in-cptn*: $[\![\ a\#xs\ \in cptn;\ xs\neq[]\ ]\!] \Longrightarrow xs\in cptn$
$\langle proof \rangle$

lemma *tl-zero*[*rule-format*]:
$P\ (ys!Suc\ j) \longrightarrow Suc\ j<length\ ys \longrightarrow ys\neq[] \longrightarrow P\ (tl(ys)!j)$
$\langle proof \rangle$

3.3.4 The Semantics is Compositional

lemma *aux-if* [*rule-format*]:
$\forall xs\ s\ clist.\ (length\ clist = length\ xs \wedge (\forall i<length\ xs.\ (xs!i,s)\#clist!i \in cptn)$
$\wedge ((xs,\ s)\#ys \propto map\ (\lambda i.\ (fst\ i,s)\#snd\ i)\ (zip\ xs\ clist))$
$\longrightarrow (xs,\ s)\#ys \in par\text{-}cptn)$
$\langle proof \rangle$

lemma *less-Suc-0* [*iff*]: $(n < Suc\ 0) = (n = 0)$
$\langle proof \rangle$

lemma *aux-onlyif* [*rule-format*]: $\forall xs\ s.\ (xs,\ s)\#ys \in par\text{-}cptn \longrightarrow$
$(\exists clist.\ (length\ clist = length\ xs) \wedge$
$(xs,\ s)\#ys \propto map\ (\lambda i.\ (fst\ i,s)\#(snd\ i))\ (zip\ xs\ clist) \wedge$
$(\forall i<length\ xs.\ (xs!i,s)\#(clist!i) \in cptn))$
$\langle proof \rangle$

lemma *one-iff-aux*: $xs\neq[] \Longrightarrow (\forall ys.\ ((xs,\ s)\#ys \in par\text{-}cptn) =$
$(\exists clist.\ length\ clist= length\ xs \wedge$
$((xs,\ s)\#ys \propto map\ (\lambda i.\ (fst\ i,s)\#(snd\ i))\ (zip\ xs\ clist)) \wedge$
$(\forall i<length\ xs.\ (xs!i,s)\#(clist!i) \in cptn))) =$
$(par\text{-}cp\ (xs)\ s = \{c.\ \exists clist.\ (length\ clist)=(length\ xs) \wedge$
$(\forall i<length\ clist.\ (clist!i) \in cp(xs!i)\ s) \wedge c \propto clist\})$
$\langle proof \rangle$

theorem *one*: $xs\neq[] \Longrightarrow$
$par\text{-}cp\ xs\ s = \{c.\ \exists clist.\ (length\ clist)=(length\ xs) \wedge$
$(\forall i<length\ clist.\ (clist!i) \in cp(xs!i)\ s) \wedge c \propto clist\}$
$\langle proof \rangle$

end

3.4 The Proof System

theory *RG-Hoare* **imports** *RG-Tran* **begin**

3.4.1 Proof System for Component Programs

declare *Un-subset-iff* [*iff del*]
declare *Cons-eq-map-conv*[*iff*]

constdefs
$stable :: 'a\ set \Rightarrow ('a \times 'a)\ set \Rightarrow bool$
$stable \equiv \lambda f\ g.\ (\forall x\ y.\ x \in f \longrightarrow (x,\ y) \in g \longrightarrow y \in f)$

inductive
$rghoare :: ['a\ com,\ 'a\ set,\ ('a \times 'a)\ set,\ ('a \times 'a)\ set,\ 'a\ set] \Rightarrow bool$
$(\vdash\ \text{-}\ sat\ [\text{-},\ \text{-},\ \text{-},\ \text{-}]\ [60,0,0,0,0]\ 45)$
where
$Basic: [\![\ pre \subseteq \{s.\ f\ s \in post\};\ \{(s,t).\ s \in pre \wedge (t{=}f\ s \vee t{=}s)\} \subseteq guar;$
$stable\ pre\ rely;\ stable\ post\ rely\ ]\!]$
$\Longrightarrow \vdash Basic\ f\ sat\ [pre,\ rely,\ guar,\ post]$

$|\ Seq: [\![\vdash P\ sat\ [pre,\ rely,\ guar,\ mid];\ \vdash Q\ sat\ [mid,\ rely,\ guar,\ post]\ ]\!]$
$\Longrightarrow \vdash Seq\ P\ Q\ sat\ [pre,\ rely,\ guar,\ post]$

$|\ Cond: [\![\ stable\ pre\ rely;\ \vdash P1\ sat\ [pre \cap b,\ rely,\ guar,\ post];$
$\vdash P2\ sat\ [pre \cap -b,\ rely,\ guar,\ post];\ \forall s.\ (s,s){\in}guar\ ]\!]$
$\Longrightarrow \vdash Cond\ b\ P1\ P2\ sat\ [pre,\ rely,\ guar,\ post]$

$|\ While: [\![\ stable\ pre\ rely;\ (pre \cap -b) \subseteq post;\ stable\ post\ rely;$
$\vdash P\ sat\ [pre \cap b,\ rely,\ guar,\ pre];\ \forall s.\ (s,s){\in}guar\ ]\!]$
$\Longrightarrow \vdash While\ b\ P\ sat\ [pre,\ rely,\ guar,\ post]$

$|\ Await: [\![\ stable\ pre\ rely;\ stable\ post\ rely;$
$\forall V.\ \vdash P\ sat\ [pre \cap b \cap \{V\},\ \{(s,\ t).\ s = t\},$
$UNIV,\ \{s.\ (V,\ s) \in guar\} \cap post]\ ]\!]$
$\Longrightarrow \vdash Await\ b\ P\ sat\ [pre,\ rely,\ guar,\ post]$

$|\ Conseq: [\![\ pre \subseteq pre';\ rely \subseteq rely';\ guar' \subseteq guar;\ post' \subseteq post;$
$\vdash P\ sat\ [pre',\ rely',\ guar',\ post']\ ]\!]$
$\Longrightarrow \vdash P\ sat\ [pre,\ rely,\ guar,\ post]$

constdefs
$Pre :: 'a\ rgformula \Rightarrow 'a\ set$
$Pre\ x \equiv fst(snd\ x)$
$Post :: 'a\ rgformula \Rightarrow 'a\ set$
$Post\ x \equiv snd(snd(snd(snd\ x)))$
$Rely :: 'a\ rgformula \Rightarrow ('a \times 'a)\ set$
$Rely\ x \equiv fst(snd(snd\ x))$
$Guar :: 'a\ rgformula \Rightarrow ('a \times 'a)\ set$
$Guar\ x \equiv fst(snd(snd(snd\ x)))$
$Com :: 'a\ rgformula \Rightarrow 'a\ com$
$Com\ x \equiv fst\ x$

3.4.2 Proof System for Parallel Programs

types $'a$ *par-rgformula* $= ('a$ *rgformula*$)$ *list* $\times$ $'a$ *set* $\times$ $('a \times 'a)$ *set* $\times$ $('a \times 'a)$ *set* $\times$ $'a$ *set*

inductive
par-rghoare :: $('a$ *rgformula*$)$ *list* $\Rightarrow$ $'a$ *set* $\Rightarrow$ $('a \times 'a)$ *set* $\Rightarrow$ $('a \times 'a)$ *set* $\Rightarrow$ $'a$ *set* $\Rightarrow$ *bool*
$(\vdash$ - SAT [-, -, -, -] $[60,0,0,0,0]$ $45)$
where
Parallel:
$[\![\ \forall i{<}length\ xs.\ rely \cup (\bigcup j{\in}\{j.\ j{<}length\ xs \wedge j{\neq}i\}.\ Guar(xs!j)) \subseteq Rely(xs!i);$
$(\bigcup j{\in}\{j.\ j{<}length\ xs\}.\ Guar(xs!j)) \subseteq guar;$
$pre \subseteq (\bigcap i{\in}\{i.\ i{<}length\ xs\}.\ Pre(xs!i));$
$(\bigcap i{\in}\{i.\ i{<}length\ xs\}.\ Post(xs!i)) \subseteq post;$
$\forall i{<}length\ xs.\ \vdash Com(xs!i)\ sat\ [Pre(xs!i),Rely(xs!i),Guar(xs!i),Post(xs!i)]\ ]\!]$
$\Longrightarrow\ \vdash xs\ SAT\ [pre,\ rely,\ guar,\ post]$

3.5 Soundness

Some previous lemmas

lemma *tl-of-assum-in-assum*:
$(P,\ s)\ \#\ (P,\ t)\ \#\ xs \in assum\ (pre,\ rely) \Longrightarrow stable\ pre\ rely$
$\Longrightarrow (P,\ t)\ \#\ xs \in assum\ (pre,\ rely)$
⟨*proof*⟩

lemma *etran-in-comm*:
$(P,\ t)\ \#\ xs \in comm(guar,\ post) \Longrightarrow (P,\ s)\ \#\ (P,\ t)\ \#\ xs \in comm(guar,\ post)$
⟨*proof*⟩

lemma *ctran-in-comm*:
$[\![(s,\ s) \in guar;\ (Q,\ s)\ \#\ xs \in comm(guar,\ post)]\!]$
$\Longrightarrow (P,\ s)\ \#\ (Q,\ s)\ \#\ xs \in comm(guar,\ post)$
⟨*proof*⟩

lemma *takecptn-is-cptn* [*rule-format*, *elim*!]:
$\forall j.\ c \in cptn \longrightarrow take\ (Suc\ j)\ c \in cptn$
⟨*proof*⟩

lemma *dropcptn-is-cptn* [*rule-format*,*elim*!]:
$\forall j{<}length\ c.\ c \in cptn \longrightarrow drop\ j\ c \in cptn$
⟨*proof*⟩

lemma *takepar-cptn-is-par-cptn* [*rule-format*,*elim*]:
$\forall j.\ c \in par\text{-}cptn \longrightarrow take\ (Suc\ j)\ c \in par\text{-}cptn$
⟨*proof*⟩

lemma *droppar-cptn-is-par-cptn* [*rule-format*]:
$\forall j{<}length\ c.\ c \in par\text{-}cptn \longrightarrow drop\ j\ c \in par\text{-}cptn$

$\langle proof \rangle$

lemma *tl-of-cptn-is-cptn*: $[\![x \# xs \in cptn;\ xs \neq []]\!] \Longrightarrow xs \ \in cptn$
$\langle proof \rangle$

lemma *not-ctran-None* [*rule-format*]:
$\forall s.\ (None,\ s)\#xs \in cptn \longrightarrow (\forall i<length\ xs.\ ((None,\ s)\#xs)!i\ -e\rightarrow\ xs!i)$
$\langle proof \rangle$

lemma *cptn-not-empty* [*simp*]:$[] \notin cptn$
$\langle proof \rangle$

lemma *etran-or-ctran* [*rule-format*]:
$\forall m\ i.\ x{\in}cptn \longrightarrow m \leq length\ x$
$\longrightarrow (\forall i.\ Suc\ i < m \longrightarrow \neg\ x!i\ -c\rightarrow\ x!Suc\ i) \longrightarrow Suc\ i < m$
$\longrightarrow x!i\ -e\rightarrow\ x!Suc\ i$
$\langle proof \rangle$

lemma *etran-or-ctran2* [*rule-format*]:
$\forall i.\ Suc\ i<length\ x \longrightarrow x{\in}cptn \longrightarrow (x!i\ -c\rightarrow\ x!Suc\ i \longrightarrow \neg\ x!i\ -e\rightarrow\ x!Suc\ i)$
$\vee\ (x!i\ -e\rightarrow\ x!Suc\ i \longrightarrow \neg\ x!i\ -c\rightarrow\ x!Suc\ i)$
$\langle proof \rangle$

lemma *etran-or-ctran2-disjI1*:
$[\![\ x{\in}cptn;\ Suc\ i<length\ x;\ x!i\ -c\rightarrow\ x!Suc\ i]\!] \Longrightarrow \neg\ x!i\ -e\rightarrow\ x!Suc\ i$
$\langle proof \rangle$

lemma *etran-or-ctran2-disjI2*:
$[\![\ x{\in}cptn;\ Suc\ i<length\ x;\ x!i\ -e\rightarrow\ x!Suc\ i]\!] \Longrightarrow \neg\ x!i\ -c\rightarrow\ x!Suc\ i$
$\langle proof \rangle$

lemma *not-ctran-None2* [*rule-format*]:
$[\![\ (None,\ s)\ \#\ xs\ {\in}cptn;\ i<length\ xs]\!] \Longrightarrow \neg\ ((None,\ s)\ \#\ xs)\ !\ i\ -c\rightarrow\ xs\ !\ i$
$\langle proof \rangle$

lemma *Ex-first-occurrence* [*rule-format*]: $P\ (n::nat) \longrightarrow (\exists m.\ P\ m \wedge (\forall i<m.\ \neg\ P\ i))$
$\langle proof \rangle$

lemma *stability* [*rule-format*]:
$\forall j\ k.\ x \in cptn \longrightarrow stable\ p\ rely \longrightarrow j{\leq}k \longrightarrow k<length\ x \longrightarrow snd(x!j){\in}p \ \longrightarrow$
$(\forall i.\ (Suc\ i)<length\ x \longrightarrow$
$(x!i\ -e\rightarrow\ x!(Suc\ i)) \longrightarrow (snd(x!i),\ snd(x!(Suc\ i))) \in rely) \longrightarrow$
$(\forall i.\ j{\leq}i \wedge i<k \longrightarrow x!i\ -e\rightarrow\ x!Suc\ i) \longrightarrow snd(x!k){\in}p \wedge fst(x!j){=}fst(x!k)$
$\langle proof \rangle$

3.5.1 Soundness of the System for Component Programs

Soundness of the Basic rule

lemma *unique-ctran-Basic* [*rule-format*]:
$\forall s\ i.\ x \in cptn \longrightarrow x\ !\ 0 = (Some\ (Basic\ f),\ s) \longrightarrow$
$Suc\ i{<}length\ x \longrightarrow x!i\ -c\rightarrow\ x!Suc\ i \longrightarrow$
$(\forall j.\ Suc\ j{<}length\ x \longrightarrow i{\neq}j \longrightarrow x!j\ -e\rightarrow\ x!Suc\ j)$
⟨*proof*⟩

lemma *exists-ctran-Basic-None* [*rule-format*]:
$\forall s\ i.\ x \in cptn \longrightarrow x\ !\ 0 = (Some\ (Basic\ f),\ s)$
$\longrightarrow i{<}length\ x \longrightarrow fst(x!i){=}None \longrightarrow (\exists j{<}i.\ x!j\ -c\rightarrow\ x!Suc\ j)$
⟨*proof*⟩

lemma *Basic-sound*:
$[\![pre \subseteq \{s.\ f\ s \in post\};\ \{(s,\ t).\ s \in pre \wedge t = f\ s\} \subseteq guar;$
$stable\ pre\ rely;\ stable\ post\ rely]\!]$
$\Longrightarrow \models Basic\ f\ sat\ [pre,\ rely,\ guar,\ post]$
⟨*proof*⟩

Soundness of the Await rule

lemma *unique-ctran-Await* [*rule-format*]:
$\forall s\ i.\ x \in cptn \longrightarrow x\ !\ 0 = (Some\ (Await\ b\ c),\ s) \longrightarrow$
$Suc\ i{<}length\ x \longrightarrow x!i\ -c\rightarrow\ x!Suc\ i \longrightarrow$
$(\forall j.\ Suc\ j{<}length\ x \longrightarrow i{\neq}j \longrightarrow x!j\ -e\rightarrow\ x!Suc\ j)$
⟨*proof*⟩

lemma *exists-ctran-Await-None* [*rule-format*]:
$\forall s\ i.\ x \in cptn \longrightarrow x\ !\ 0 = (Some\ (Await\ b\ c),\ s)$
$\longrightarrow i{<}length\ x \longrightarrow fst(x!i){=}None \longrightarrow (\exists j{<}i.\ x!j\ -c\rightarrow\ x!Suc\ j)$
⟨*proof*⟩

lemma *Star-imp-cptn*:
$(P,\ s)\ -c{*}\rightarrow\ (R,\ t) \Longrightarrow \exists l \in cp\ P\ s.\ (last\ l){=}(R,\ t)$
$\wedge (\forall i.\ Suc\ i{<}length\ l \longrightarrow l!i\ -c\rightarrow\ l!Suc\ i)$
⟨*proof*⟩

lemma *Await-sound*:
$[\![stable\ pre\ rely;\ stable\ post\ rely;$
$\forall V.\ \vdash P\ sat\ [pre \cap b \cap \{s.\ s = V\},\ \{(s,\ t).\ s = t\},$
$UNIV,\ \{s.\ (V,\ s) \in guar\} \cap post] \wedge$
$\models P\ sat\ [pre \cap b \cap \{s.\ s = V\},\ \{(s,\ t).\ s = t\},$
$UNIV,\ \{s.\ (V,\ s) \in guar\} \cap post]\ ]\!]$
$\Longrightarrow \models Await\ b\ P\ sat\ [pre,\ rely,\ guar,\ post]$
⟨*proof*⟩

Soundness of the Conditional rule

lemma *Cond-sound*:
$[\![$ *stable pre rely*; $\models$ *P1 sat* [*pre* $\cap$ *b*, *rely*, *guar*, *post*];
$\models$ *P2 sat* [*pre* $\cap -$ *b*, *rely*, *guar*, *post*]; $\forall$ *s*. (*s*,*s*)$\in$*guar*$]\!]$
$\Longrightarrow$ $\models$ (*Cond b P1 P2*) *sat* [*pre*, *rely*, *guar*, *post*]
$\langle$*proof*$\rangle$

Soundness of the Sequential rule

inductive-cases *Seq-cases* [*elim*!]: (*Some* (*Seq P Q*), *s*) $-c\rightarrow$ *t*

lemma *last-lift-not-None*: *fst* ((*lift Q*) ((*x*#*xs*)!(*length xs*))) $\neq$ *None*
$\langle$*proof*$\rangle$

declare *map-eq-Cons-conv* [*simp del*] *Cons-eq-map-conv* [*simp del*]
lemma *Seq-sound1* [*rule-format*]:
x$\in$ *cptn-mod* $\Longrightarrow$ $\forall$ *s P*. *x* !*0*=(*Some* (*Seq P Q*), *s*) $\longrightarrow$
($\forall$ *i*<*length x*. *fst*(*x*!*i*)$\neq$*Some Q*) $\longrightarrow$
($\exists$ *xs*$\in$ *cp* (*Some P*) *s*. *x*=*map* (*lift Q*) *xs*)
$\langle$*proof*$\rangle$
declare *map-eq-Cons-conv* [*simp del*] *Cons-eq-map-conv* [*simp del*]

lemma *Seq-sound2* [*rule-format*]:
x $\in$ *cptn* $\Longrightarrow$ $\forall$ *s P i*. *x*!*0*=(*Some* (*Seq P Q*), *s*) $\longrightarrow$ *i*<*length x*
$\longrightarrow$ *fst*(*x*!*i*)=*Some Q* $\longrightarrow$
($\forall$ *j*<*i*. *fst*(*x*!*j*)$\neq$(*Some Q*)) $\longrightarrow$
($\exists$ *xs ys*. *xs* $\in$ *cp* (*Some P*) *s* $\wedge$ *length xs*=*Suc i*
$\wedge$ *ys* $\in$ *cp* (*Some Q*) (*snd*(*xs* !*i*)) $\wedge$ *x*=(*map* (*lift Q*) *xs*)@*tl ys*)
$\langle$*proof*$\rangle$

lemma *last-lift-not-None2*: *fst* ((*lift Q*) (*last* (*x*#*xs*))) $\neq$ *None*
$\langle$*proof*$\rangle$

lemma *Seq-sound*:
$[\![\models$ *P sat* [*pre*, *rely*, *guar*, *mid*]; $\models$ *Q sat* [*mid*, *rely*, *guar*, *post*]$]\!]$
$\Longrightarrow$ $\models$ *Seq P Q sat* [*pre*, *rely*, *guar*, *post*]
$\langle$*proof*$\rangle$

Soundness of the While rule

lemma *last-append*[*rule-format*]:
$\forall$ *xs*. *ys*$\neq$[] $\longrightarrow$ ((*xs*@*ys*)!(*length* (*xs*@*ys*) $-$ (*Suc 0*)))=(*ys*!(*length ys* $-$ (*Suc 0*)))
$\langle$*proof*$\rangle$

lemma *assum-after-body*:
$[\![$ $\models$ *P sat* [*pre* $\cap$ *b*, *rely*, *guar*, *pre*];
(*Some P*, *s*) # *xs* $\in$ *cptn-mod*; *fst* (*last* ((*Some P*, *s*) # *xs*)) = *None*; *s* $\in$ *b*;
(*Some* (*While b P*), *s*) # (*Some* (*Seq P* (*While b P*)), *s*) #

$map\ (lift\ (While\ b\ P))\ xs\ @\ ys \in assum\ (pre,\ rely)]\!]$
$\Longrightarrow (Some\ (While\ b\ P),\ snd\ (last\ ((Some\ P,\ s)\ \#\ xs)))\ \#\ ys \in assum\ (pre,\ rely)$
$\langle proof \rangle$

lemma *While-sound-aux* [*rule-format*]:
$[\![\ pre \cap -\ b \subseteq post;\ \models P\ sat\ [pre \cap b,\ rely,\ guar,\ pre];\ \forall s.\ (s,\ s) \in guar;$
$stable\ pre\ rely;\ \ stable\ post\ rely;\ x \in cptn\text{-}mod\ ]\!]$
$\Longrightarrow\ \forall s\ xs.\ x{=}(Some(While\ b\ P),s)\#xs \longrightarrow x{\in}assum(pre,\ rely) \longrightarrow x \in comm$
$(guar,\ post)$
$\langle proof \rangle$

lemma *While-sound*:
$[\![stable\ pre\ rely;\ pre \cap -\ b \subseteq post;\ stable\ post\ rely;$
$\models P\ sat\ [pre \cap b,\ rely,\ guar,\ pre];\ \forall s.\ (s,s){\in}guar]\!]$
$\Longrightarrow \models While\ b\ P\ sat\ [pre,\ rely,\ guar,\ post]$
$\langle proof \rangle$

Soundness of the Rule of Consequence

lemma *Conseq-sound*:
$[\![pre \subseteq pre';\ rely \subseteq rely';\ guar' \subseteq guar;\ post' \subseteq post;$
$\models P\ sat\ [pre',\ rely',\ guar',\ post']]\!]$
$\Longrightarrow \models P\ sat\ [pre,\ rely,\ guar,\ post]$
$\langle proof \rangle$

Soundness of the system for sequential component programs

theorem *rgsound*:
$\vdash P\ sat\ [pre,\ rely,\ guar,\ post] \Longrightarrow \models P\ sat\ [pre,\ rely,\ guar,\ post]$
$\langle proof \rangle$

3.5.2 Soundness of the System for Parallel Programs

constdefs
$ParallelCom :: ('a\ rgformula)\ list \Rightarrow 'a\ par\text{-}com$
$ParallelCom\ Ps \equiv map\ (Some \circ fst)\ Ps$

lemma *two*:
$[\![\ \forall i{<}length\ xs.\ rely \cup (\bigcup j{\in}\{j.\ j < length\ xs \wedge j \neq i\}.\ Guar\ (xs\ !\ j))$
$\subseteq Rely\ (xs\ !\ i);$
$pre \subseteq (\bigcap i{\in}\{i.\ i < length\ xs\}.\ Pre\ (xs\ !\ i));$
$\forall i{<}length\ xs.$
$\models Com\ (xs\ !\ i)\ sat\ [Pre\ (xs\ !\ i),\ Rely\ (xs\ !\ i),\ Guar\ (xs\ !\ i),\ Post\ (xs\ !\ i)];$
$length\ xs{=}length\ clist;\ x \in par\text{-}cp\ (ParallelCom\ xs)\ s;\ x{\in}par\text{-}assum(pre,\ rely);$
$\forall i{<}length\ clist.\ clist!i{\in}cp\ (Some(Com(xs!i)))\ s;\ x \propto clist\ ]\!]$
$\Longrightarrow \forall j\ i.\ i{<}length\ clist \wedge Suc\ j{<}length\ x \longrightarrow (clist!i!j)\ {-c\rightarrow}\ (clist!i!Suc\ j)$
$\longrightarrow (snd(clist!i!j),\ snd(clist!i!Suc\ j)) \in Guar(xs!i)$
$\langle proof \rangle$

lemma *three* [*rule-format*]:
$[\![xs{\neq}[]; \forall i{<}length\ xs.\ rely \cup (\bigcup j{\in}\{j.\ j < length\ xs \wedge j \neq i\}.\ Guar\ (xs\ !\ j))$
$\subseteq Rely\ (xs\ !\ i);$
$pre \subseteq (\bigcap i{\in}\{i.\ i < length\ xs\}.\ Pre\ (xs\ !\ i));$
$\forall i{<}length\ xs.$
$\models Com\ (xs\ !\ i)\ sat\ [Pre\ (xs\ !\ i),\ Rely\ (xs\ !\ i),\ Guar\ (xs\ !\ i),\ Post\ (xs\ !\ i)];$
$length\ xs{=}length\ clist;\ x \in par\text{-}cp\ (ParallelCom\ xs)\ s;\ x \in par\text{-}assum(pre,\ rely);$
$\forall i{<}length\ clist.\ clist!i{\in}cp\ (Some(Com(xs!i)))\ s;\ x \propto clist\ ]\!]$
$\Longrightarrow \forall j\ i.\ i{<}length\ clist \wedge Suc\ j{<}length\ x \longrightarrow (clist!i!j)\ -e\rightarrow\ (clist!i!Suc\ j)$
$\longrightarrow (snd(clist!i!j),\ snd(clist!i!Suc\ j)) \in rely \cup (\bigcup j{\in}\{j.\ j < length\ xs \wedge j \neq i\}.$
$Guar\ (xs\ !\ j))$
$\langle proof \rangle$

lemma *four*:
$[\![xs{\neq}[]; \forall i < length\ xs.\ rely \cup (\bigcup j{\in}\{j.\ j < length\ xs \wedge j \neq i\}.\ Guar\ (xs\ !\ j))$
$\subseteq Rely\ (xs\ !\ i);$
$(\bigcup j{\in}\{j.\ j < length\ xs\}.\ Guar\ (xs\ !\ j)) \subseteq guar;$
$pre \subseteq (\bigcap i{\in}\{i.\ i < length\ xs\}.\ Pre\ (xs\ !\ i));$
$\forall i < length\ xs.$
$\models Com\ (xs\ !\ i)\ sat\ [Pre\ (xs\ !\ i),\ Rely\ (xs\ !\ i),\ Guar\ (xs\ !\ i),\ Post\ (xs\ !\ i)];$
$x \in par\text{-}cp\ (ParallelCom\ xs)\ s;\ x \in par\text{-}assum\ (pre,\ rely);\ Suc\ i < length\ x;$
$x\ !\ i\ -pc\rightarrow\ x\ !\ Suc\ i]\!]$
$\Longrightarrow (snd\ (x\ !\ i),\ snd\ (x\ !\ Suc\ i)) \in guar$
$\langle proof \rangle$

lemma *parcptn-not-empty* [*simp*]:$[] \notin par\text{-}cptn$
$\langle proof \rangle$

lemma *five*:
$[\![xs{\neq}[]; \forall i{<}length\ xs.\ rely \cup (\bigcup j{\in}\{j.\ j < length\ xs \wedge j \neq i\}.\ Guar\ (xs\ !\ j))$
$\subseteq Rely\ (xs\ !\ i);$
$pre \subseteq (\bigcap i{\in}\{i.\ i < length\ xs\}.\ Pre\ (xs\ !\ i));$
$(\bigcap i{\in}\{i.\ i < length\ xs\}.\ Post\ (xs\ !\ i)) \subseteq post;$
$\forall i < length\ xs.$
$\models Com\ (xs\ !\ i)\ sat\ [Pre\ (xs\ !\ i),\ Rely\ (xs\ !\ i),\ Guar\ (xs\ !\ i),\ Post\ (xs\ !\ i)];$
$x \in par\text{-}cp\ (ParallelCom\ xs)\ s;\ x \in par\text{-}assum\ (pre,\ rely);$
$All\text{-}None\ (fst\ (last\ x))\ ]\!] \Longrightarrow snd\ (last\ x) \in post$
$\langle proof \rangle$

lemma *ParallelEmpty* [*rule-format*]:
$\forall i\ s.\ x \in par\text{-}cp\ (ParallelCom\ [])\ s \longrightarrow$
$Suc\ i < length\ x \longrightarrow (x\ !\ i,\ x\ !\ Suc\ i) \notin par\text{-}ctran$
$\langle proof \rangle$

theorem *par-rgsound*:
$\vdash c\ SAT\ [pre,\ rely,\ guar,\ post] \Longrightarrow$
$\models (ParallelCom\ c)\ SAT\ [pre,\ rely,\ guar,\ post]$
$\langle proof \rangle$

end

3.6 Concrete Syntax

theory *RG-Syntax*
imports *RG-Hoare Quote-Antiquote*
begin

syntax
-Assign :: *idt* ⇒ *'b* ⇒ *'a com* ((´- :=/ -) [70, 65] 61)
-skip :: *'a com* (SKIP)
-Seq :: *'a com* ⇒ *'a com* ⇒ *'a com* ((-;;/ -) [60,61] 60)
-Cond :: *'a bexp* ⇒ *'a com* ⇒ *'a com* ⇒ *'a com* ((0IF -/ THEN -/ ELSE -/FI) [0, 0, 0] 61)
-Cond2 :: *'a bexp* ⇒ *'a com* ⇒ *'a com* ((0IF - THEN - FI) [0,0] 56)
-While :: *'a bexp* ⇒ *'a com* ⇒ *'a com* ((0WHILE - /DO - /OD) [0, 0] 61)
-Await :: *'a bexp* ⇒ *'a com* ⇒ *'a com* ((0AWAIT - /THEN /- /END) [0,0] 61)
-Atom :: *'a com* ⇒ *'a com* ((⟨-⟩) 61)
-Wait :: *'a bexp* ⇒ *'a com* ((0WAIT - END) 61)

translations
´ x := a ⇀ Basic ≪´ (-update-name x (K-record a))≫
SKIP ⇌ Basic id
c1;; c2 ⇌ Seq c1 c2
IF b THEN c1 ELSE c2 FI ⇀ Cond .{b}. c1 c2
IF b THEN c FI ⇌ IF b THEN c ELSE SKIP FI
WHILE b DO c OD ⇀ While .{b}. c
AWAIT b THEN c END ⇌ Await .{b}. c
⟨c⟩ ⇌ AWAIT True THEN c END
WAIT b END ⇌ AWAIT b THEN SKIP END

nonterminals
prgs

syntax
-PAR :: *prgs* ⇒ *'a* (COBEGIN//-//COEND 60)
-prg :: *'a* ⇒ *prgs* (- 57)
-prgs :: [*'a*, *prgs*] ⇒ *prgs* (-//‖//- [60,57] 57)

translations
-prg a ⇀ [a]
-prgs a ps ⇀ a # ps
-PAR ps ⇀ ps

syntax
-prg-scheme :: [*'a*, *'a*, *'a*, *'a*] ⇒ *prgs* (SCHEME [- ≤ - < -] - [0,0,0,60] 57)

translations
  -prg-scheme j i k c $\rightleftharpoons$ (*map* ($\lambda i.$ *c*) [*j*..<*k*])

Translations for variables before and after a transition:

syntax
  -before :: *id* $\Rightarrow$ *'a* (^{o}-)
  -after :: *id* $\Rightarrow$ *'a* (^{a}-)

translations
  $^{o}x \rightleftharpoons x$ *´fst*
  $^{a}x \rightleftharpoons x$ *´snd*

⟨*ML*⟩

end

3.7 Examples

theory *RG-Examples* **imports** *RG-Syntax* **begin**

lemmas *definitions* [*simp*]= *stable-def Pre-def Rely-def Guar-def Post-def Com-def*

3.7.1 Set Elements of an Array to Zero

lemma *le-less-trans2*: $[\![(j::nat)<k;\ i\leq j]\!] \Longrightarrow i<k$
⟨*proof*⟩

lemma *add-le-less-mono*: $[\![\ (a::nat) < c;\ b\leq d\ ]\!] \Longrightarrow a + b < c + d$
⟨*proof*⟩

record *Example1* =
  A :: *nat list*

lemma *Example1*:
$\vdash$ *COBEGIN*
      SCHEME [*0* $\leq i < n$]
    (´*A* := ´*A* [*i* := *0*],
    $\{\!|$ $n <$ *length* ´*A* $|\!\}$,
    $\{\!|$ *length* ^{o}A = *length* $^{a}A \wedge {}^{o}A$! i = ^{a}A ! i $|\!\}$,
    $\{\!|$ *length* ^{o}A = *length* $^{a}A \wedge (\forall j<n.\ i \neq j \longrightarrow {}^{o}A$! j = ^{a}A ! $j)$ $|\!\}$,
    $\{\!|$ ´*A* ! i = *0* $|\!\}$)
    COEND
 SAT [$\{\!|$ $n <$ *length* ´*A* $|\!\}$, $\{\!|$ $^{o}A = {}^{a}A$ $|\!\}$, $\{\!|$ *True* $|\!\}$, $\{\!|$ $\forall i < n.$ ´*A* ! i = *0* $|\!\}$]
⟨*proof*⟩

lemma *Example1-parameterized*:
$k < t \Longrightarrow$
 $\vdash$ *COBEGIN*

$SCHEME\ [k*n \le i<(Suc\ k)*n]\ (\acute{}A{:=}\acute{}A[i{:=}0],$
$\{\!|t*n < length\ \acute{}A|\!\},$
$\{\!|t*n < length\ {}^{o}A \wedge length\ {}^{o}A{=}length\ {}^{a}A \wedge {}^{o}A!i = {}^{a}A!i|\!\},$
$\{\!|t*n < length\ {}^{o}A \wedge length\ {}^{o}A{=}length\ {}^{a}A \wedge (\forall j<length\ {}^{o}A\ .\ i\neq j \longrightarrow {}^{o}A!j = {}^{a}A!j)|\!\},$
$\{\!|\acute{}A!i{=}0|\!\})$
$COEND$
$SAT\ [\{\!|t*n < length\ \acute{}A|\!\},$
$\{\!|t*n < length\ {}^{o}A \wedge length\ {}^{o}A{=}length\ {}^{a}A \wedge (\forall i<n.\ {}^{o}A!(k*n+i){=}{}^{a}A!(k*n+i))|\!\},$

$\{\!|t*n < length\ {}^{o}A \wedge length\ {}^{o}A{=}length\ {}^{a}A \wedge$
$(\forall i<length\ {}^{o}A\ .\ (i<k*n \longrightarrow {}^{o}A!i = {}^{a}A!i) \wedge ((Suc\ k)*n \le i \longrightarrow {}^{o}A!i = {}^{a}A!i))|\!\},$
$\{\!|\forall i<n.\ \acute{}A!(k*n+i) = 0|\!\}]$
$\langle proof \rangle$

3.7.2 Increment a Variable in Parallel

Two components

record *Example2* =
x :: nat
$c\text{-}0$ :: nat
$c\text{-}1$ :: nat

lemma *Example2*:
$\vdash\ COBEGIN$
$(\langle\ \acute{}x{:=}\acute{}x+1;;\ \acute{}c\text{-}0{:=}\acute{}c\text{-}0 + 1\ \rangle,$
$\{\!|\acute{}x{=}\acute{}c\text{-}0 + \acute{}c\text{-}1\ \wedge \acute{}c\text{-}0{=}0|\!\},$
$\{\!|{}^{o}c\text{-}0 = {}^{a}c\text{-}0 \wedge$
$({}^{o}x{=}{}^{o}c\text{-}0 + {}^{o}c\text{-}1$
$\longrightarrow {}^{a}x = {}^{a}c\text{-}0 + {}^{a}c\text{-}1)|\!\},$
$\{\!|{}^{o}c\text{-}1 = {}^{a}c\text{-}1 \wedge$
$({}^{o}x{=}{}^{o}c\text{-}0 + {}^{o}c\text{-}1$
$\longrightarrow {}^{a}x = {}^{a}c\text{-}0 + {}^{a}c\text{-}1)|\!\},$
$\{\!|\acute{}x{=}\acute{}c\text{-}0 + \acute{}c\text{-}1 \wedge \acute{}c\text{-}0{=}1\ |\!\})$
$\|$
$(\langle\ \acute{}x{:=}\acute{}x+1;;\ \acute{}c\text{-}1{:=}\acute{}c\text{-}1+1\ \rangle,$
$\{\!|\acute{}x{=}\acute{}c\text{-}0 + \acute{}c\text{-}1 \wedge \acute{}c\text{-}1{=}0\ |\!\},$
$\{\!|{}^{o}c\text{-}1 = {}^{a}c\text{-}1 \wedge$
$({}^{o}x{=}{}^{o}c\text{-}0 + {}^{o}c\text{-}1$
$\longrightarrow {}^{a}x = {}^{a}c\text{-}0 + {}^{a}c\text{-}1)|\!\},$
$\{\!|{}^{o}c\text{-}0 = {}^{a}c\text{-}0 \wedge$
$({}^{o}x{=}{}^{o}c\text{-}0 + {}^{o}c\text{-}1$
$\longrightarrow {}^{a}x = {}^{a}c\text{-}0 + {}^{a}c\text{-}1)|\!\},$
$\{\!|\acute{}x{=}\acute{}c\text{-}0 + \acute{}c\text{-}1 \wedge \acute{}c\text{-}1{=}1|\!\})$
$COEND$
$SAT\ [\{\!|\acute{}x{=}0 \wedge \acute{}c\text{-}0{=}0 \wedge \acute{}c\text{-}1{=}0|\!\},$
$\{\!|{}^{o}x{=}{}^{a}x \wedge {}^{o}c\text{-}0{=}\ {}^{a}c\text{-}0 \wedge {}^{o}c\text{-}1{=}{}^{a}c\text{-}1|\!\},$
$\{\!|True|\!\},$

$\{\!\!\{ \acute{}x{=}2 \}\!\!\}]$
⟨*proof*⟩

Parameterized

lemma *Example2-lemma2-aux*: $j<n \Longrightarrow$
$(\sum i{=}0..<n.\ (b\ i{::}nat)) =$
$(\sum i{=}0..<j.\ b\ i) + b\ j + (\sum i{=}0..<n-(Suc\ j)\ .\ b\ (Suc\ j + i))$
⟨*proof*⟩

lemma *Example2-lemma2-aux2*:
$j\leq s \Longrightarrow (\sum i{::}nat{=}0..<j.\ (b\ (s{:=}t))\ i) = (\sum i{=}0..<j.\ b\ i)$
⟨*proof*⟩

lemma *Example2-lemma2*:
$[\![j<n;\ b\ j{=}0]\!] \Longrightarrow Suc\ (\sum i{::}nat{=}0..<n.\ b\ i){=}(\sum i{=}0..<n.\ (b\ (j := Suc\ 0))\ i)$
⟨*proof*⟩

lemma *Example2-lemma2-Suc0*: $[\![j<n;\ b\ j{=}0]\!] \Longrightarrow$
$Suc\ (\sum i{::}nat{=}0..<\ n.\ b\ i){=}(\sum i{=}0..<\ n.\ (b\ (j{:=}Suc\ 0))\ i)$
⟨*proof*⟩

record *Example2-parameterized* =
  $C :: nat \Rightarrow nat$
  $y\ :: nat$

lemma *Example2-parameterized*: $0<n \Longrightarrow$
$\vdash COBEGIN\ SCHEME\ \ [0{\leq}i{<}n]$
  $(\langle\ \acute{}y{:=}\acute{}y{+}1;;\ \acute{}C{:=}\acute{}C\ (i{:=}1)\ \rangle,$
  $\{\!\!\{\acute{}y{=}(\sum i{=}0..<n.\ \acute{}C\ i) \wedge \acute{}C\ i{=}0\}\!\!\},$
  $\{\!\!\{{}^{o}C\ i = {}^{a}C\ i\ \wedge$
  $({}^{o}y{=}(\sum i{=}0..<n.\ {}^{o}C\ i) \longrightarrow {}^{a}y = (\sum i{=}0..<n.\ {}^{a}C\ i))\}\!\!\},$
  $\{\!\!\{(\forall j{<}n.\ i{\neq}j \longrightarrow {}^{o}C\ j = {}^{a}C\ j)\ \wedge$
  $({}^{o}y{=}(\sum i{=}0..<n.\ {}^{o}C\ i) \longrightarrow {}^{a}y = (\sum i{=}0..<n.\ {}^{a}C\ i))\}\!\!\},$
  $\{\!\!\{\acute{}y{=}(\sum i{=}0..<n.\ \acute{}C\ i) \wedge \acute{}C\ i{=}1\}\!\!\})$
  $COEND$
$SAT\ [\{\!\!\{\acute{}y{=}0 \wedge (\sum i{=}0..<n.\ \acute{}C\ i){=}0\ \}\!\!\},\ \{\!\!\{{}^{o}C{=}{}^{a}C \wedge {}^{o}y{=}{}^{a}y\}\!\!\},\ \{\!\!\{True\}\!\!\},\ \{\!\!\{\acute{}y{=}n\}\!\!\}]$
⟨*proof*⟩

3.7.3 Find Least Element

A previous lemma:

lemma *mod-aux* :$[\![i < (n{::}nat);\ a\ mod\ n = i;\ \ j < a + n;\ j\ mod\ n = i;\ a < j]\!]$
$\Longrightarrow False$
⟨*proof*⟩

record *Example3* =
  $X :: nat \Rightarrow nat$
  $Y :: nat \Rightarrow nat$

lemma *Example3*: $m\ mod\ n{=}0 \Longrightarrow$
$\vdash$ *COBEGIN*
SCHEME $[0{\le}i{<}n]$
$(WHILE\ (\forall j{<}n.\ {}^\prime X\ i < {}^\prime Y\ j)\ \ DO$
$IF\ P(B!({}^\prime X\ i))\ THEN\ {}^\prime Y{:=}{}^\prime Y\ (i{:=}{}^\prime X\ i)$
$ELSE\ {}^\prime X{:=}\ {}^\prime X\ (i{:=}({}^\prime X\ i){+}\ n)\ FI$
$OD,$
$\{\!|({}^\prime X\ i)\ mod\ n{=}i \wedge (\forall j{<}{}^\prime X\ i.\ j\ mod\ n{=}i \longrightarrow \neg P(B!j)) \wedge ({}^\prime Y\ i{<}m \longrightarrow P(B!({}^\prime Y\ i)) \wedge {}^\prime Y\ i{\le}\ m{+}i)|\!\},$
$\{\!|(\forall j{<}n.\ i{\ne}j \longrightarrow {}^a Y\ j \le {}^o Y\ j) \wedge {}^o X\ i = {}^a X\ i \wedge$
${}^o Y\ i = {}^a Y\ i|\!\},$
$\{\!|(\forall j{<}n.\ i{\ne}j \longrightarrow {}^o X\ j = {}^a X\ j \wedge {}^o Y\ j = {}^a Y\ j) \wedge$
${}^a Y\ i \le {}^o Y\ i|\!\},$
$\{\!|({}^\prime X\ i)\ mod\ n{=}i \wedge (\forall j{<}{}^\prime X\ i.\ j\ mod\ n{=}i \longrightarrow \neg P(B!j)) \wedge ({}^\prime Y\ i{<}m \longrightarrow P(B!({}^\prime Y\ i)) \wedge {}^\prime Y\ i{\le}\ m{+}i) \wedge (\exists j{<}n.\ {}^\prime Y\ j \le {}^\prime X\ i)\ |\!\})$
COEND
$SAT\ [\{\!|\ \forall i{<}n.\ {}^\prime X\ i{=}i \wedge {}^\prime Y\ i{=}m{+}i\ |\!\},\{\!|{}^o X{=}{}^a X \wedge {}^o Y{=}{}^a Y|\!\},\{\!|True|\!\},$
$\{\!|\forall i{<}n.\ ({}^\prime X\ i)\ mod\ n{=}i \wedge (\forall j{<}{}^\prime X\ i.\ j\ mod\ n{=}i \longrightarrow \neg P(B!j)) \wedge$
$({}^\prime Y\ i{<}m \longrightarrow P(B!({}^\prime Y\ i)) \wedge {}^\prime Y\ i{\le}\ m{+}i) \wedge (\exists j{<}n.\ {}^\prime Y\ j \le {}^\prime X\ i)|\!\}]$
$\langle proof \rangle$

Same but with a list as auxiliary variable:

record *Example3-list* =
X :: *nat list*
Y :: *nat list*

lemma *Example3-list*: $m\ mod\ n{=}0 \Longrightarrow \vdash (COBEGIN\ SCHEME\ [0{\le}i{<}n]$
$(WHILE\ (\forall j{<}n.\ {}^\prime X!i < {}^\prime Y!j)\ \ DO$
$IF\ P(B!({}^\prime X!i))\ THEN\ {}^\prime Y{:=}{}^\prime Y[i{:=}{}^\prime X!i]\ ELSE\ {}^\prime X{:=}\ {}^\prime X[i{:=}({}^\prime X!i){+}\ n]\ FI$
$OD,$
$\{\!|n{<}length\ {}^\prime X \wedge n{<}length\ {}^\prime Y \wedge ({}^\prime X!i)\ mod\ n{=}i \wedge (\forall j{<}{}^\prime X!i.\ j\ mod\ n{=}i \longrightarrow \neg P(B!j)) \wedge ({}^\prime Y!i{<}m \longrightarrow P(B!({}^\prime Y!i)) \wedge {}^\prime Y!i{\le}\ m{+}i)|\!\},$
$\{\!|(\forall j{<}n.\ i{\ne}j \longrightarrow {}^a Y!j \le {}^o Y!j) \wedge {}^o X!i = {}^a X!i \wedge$
${}^o Y!i = {}^a Y!i \wedge length\ {}^o X = length\ {}^a X \wedge length\ {}^o Y = length\ {}^a Y|\!\},$
$\{\!|(\forall j{<}n.\ i{\ne}j \longrightarrow {}^o X!j = {}^a X!j \wedge {}^o Y!j = {}^a Y!j) \wedge$
${}^a Y!i \le {}^o Y!i \wedge length\ {}^o X = length\ {}^a X \wedge length\ {}^o Y = length\ {}^a Y|\!\},$
$\{\!|({}^\prime X!i)\ mod\ n{=}i \wedge (\forall j{<}{}^\prime X!i.\ j\ mod\ n{=}i \longrightarrow \neg P(B!j)) \wedge ({}^\prime Y!i{<}m \longrightarrow P(B!({}^\prime Y!i)) \wedge {}^\prime Y!i{\le}\ m{+}i) \wedge (\exists j{<}n.\ {}^\prime Y!j \le {}^\prime X!i)\ |\!\})\ COEND)$
$SAT\ [\{\!|n{<}length\ {}^\prime X \wedge n{<}length\ {}^\prime Y \wedge (\forall i{<}n.\ {}^\prime X!i{=}i \wedge {}^\prime Y!i{=}m{+}i)\ |\!\},$
$\{\!|{}^o X{=}{}^a X \wedge {}^o Y{=}{}^a Y|\!\},$
$\{\!|True|\!\},$
$\{\!|\forall i{<}n.\ ({}^\prime X!i)\ mod\ n{=}i \wedge (\forall j{<}{}^\prime X!i.\ j\ mod\ n{=}i \longrightarrow \neg P(B!j)) \wedge$
$({}^\prime Y!i{<}m \longrightarrow P(B!({}^\prime Y!i)) \wedge {}^\prime Y!i{\le}\ m{+}i) \wedge (\exists j{<}n.\ {}^\prime Y!j \le {}^\prime X!i)|\!\}]$
$\langle proof \rangle$

end

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