Equivalents of the Axiom of Choice

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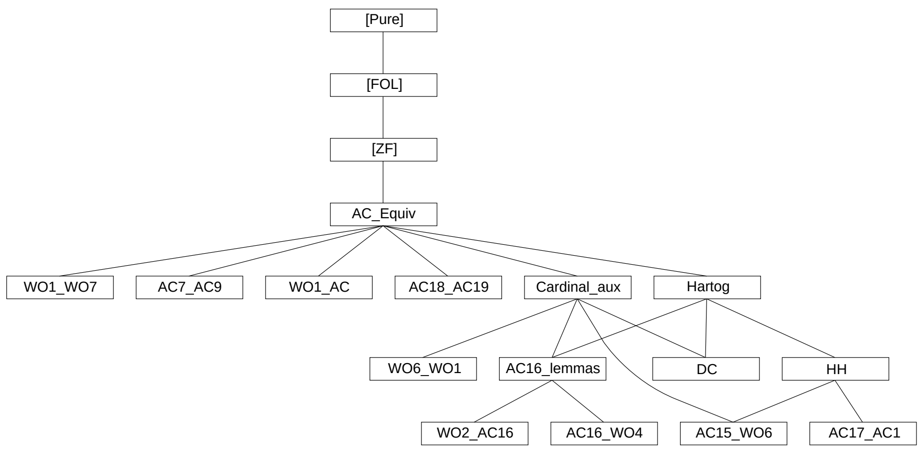
Abstract

This development [1] proves the equivalence of seven formulations of the well-ordering theorem and twenty formulations of the axiom of choice. It formalizes the first two chapters of the monograph *Equivalents of the Axiom of Choice* by Rubin and Rubin [2]. Some of this mmaterial involves extremely complex techniques.

Contents



theory *AC_Equiv* **imports** *Main* **begin**

definition
```
"WO1 == ∀A. ∃R. well_ord(A,R)"
```

definition
```
"WO2 == ∀A. ∃a. Ord(a) & A≈a"
```

definition
```
"WO3 == ∀A. ∃a. Ord(a) & (∃b. b ⊆ a & A≈b)"
```

definition
```
"WO4(m) == ∀A. ∃a f. Ord(a) & domain(f)=a &
                     (⋃b<a. f`b) = A & (∀b<a. f`b ≲ m)"
```

definition
```
"WO5 == ∃m ∈ nat. 1≤m & WO4(m)"
```

definition
```
"WO6 == ∀A. ∃m ∈ nat. 1≤m & (∃a f. Ord(a) & domain(f)=a
                        & (⋃b<a. f`b) = A & (∀b<a. f`b ≲ m))"
```

definition
```
"WO7 == ∀A. Finite(A) <-> (∀R. well_ord(A,R) --> well_ord(A,converse(R)))"
```

definition
```
"WO8 == ∀A. (∃f. f ∈ (Π X ∈ A. X)) --> (∃R. well_ord(A,R))"
```

definition

pairwise_disjoint :: "i => o" **where**
```
"pairwise_disjoint(A) == ∀A1 ∈ A. ∀A2 ∈ A. A1 Int A2 ≠ 0 --> A1=A2"
```

definition
sets_of_size_between :: "[i, i, i] => o" **where**
```
"sets_of_size_between(A,m,n) == ∀B ∈ A. m ≲ B & B ≲ n"
```

definition
```
"AC0 == ∀A. ∃f. f ∈ (Π X ∈ Pow(A)-{0}. X)"
```

definition
```
"AC1 == ∀A. 0∉A --> (∃f. f ∈ (Π X ∈ A. X))"
```

definition

```
"AC2 == ∀A. 0∉A & pairwise_disjoint(A)
             --> (∃C. ∀B ∈ A. ∃y. B Int C = {y})"
```

definition

```
"AC3 == ∀A B. ∀f ∈ A->B. ∃g. g ∈ (Π x ∈ {a ∈ A. f`a≠0}. f`x)"
```

definition

```
"AC4 == ∀R A B. (R ⊆ A*B --> (∃f. f ∈ (Π x ∈ domain(R). R``{x})))"
```

definition

```
"AC5 == ∀A B. ∀f ∈ A->B. ∃g ∈ range(f)->A. ∀x ∈ domain(g). f`(g`x)
= x"
```

definition

```
"AC6 == ∀A. 0∉A --> (Π B ∈ A. B)≠0"
```

definition

```
"AC7 == ∀A. 0∉A & (∀B1 ∈ A. ∀B2 ∈ A. B1≈B2) --> (Π B ∈ A. B)
≠ 0"
```

definition

```
"AC8 == ∀A. (∀B ∈ A. ∃B1 B2. B=<B1,B2> & B1≈B2)
             --> (∃f. ∀B ∈ A. f`B ∈ bij(fst(B),snd(B)))"
```

definition

```
"AC9 == ∀A. (∀B1 ∈ A. ∀B2 ∈ A. B1≈B2) -->
             (∃f. ∀B1 ∈ A. ∀B2 ∈ A. f`<B1,B2> ∈ bij(B1,B2))"
```

definition

```
"AC10(n) ==  ∀A. (∀B ∈ A. ~Finite(B)) -->
             (∃f. ∀B ∈ A. (pairwise_disjoint(f`B) &
             sets_of_size_between(f`B, 2, succ(n)) & Union(f`B)=B))"
```

definition

```
"AC11 == ∃n ∈ nat. 1≤n & AC10(n)"
```

definition

```
"AC12 == ∀A. (∀B ∈ A. ~Finite(B)) -->
           (∃n ∈ nat. 1≤n & (∃f. ∀B ∈ A. (pairwise_disjoint(f`B)
&
                sets_of_size_between(f`B, 2, succ(n)) & Union(f`B)=B)))"
```

definition

```
"AC13(m) == ∀A. 0∉A --> (∃f. ∀B ∈ A. f`B≠0 & f`B ⊆ B & f`B ≲
m)"
```

definition

```
"AC14 == ∃m ∈ nat. 1≤m & AC13(m)"
```

definition
  "AC15 == ∀A. 0∉A -->
    (∃m ∈ nat. 1≤m & (∃f. ∀B ∈ A. f‘B≠0 & f‘B ⊆ B &
f‘B ≲ m))"

definition
  "AC16(n, k) ==
    ∀A. ~Finite(A) -->
      (∃T. T ⊆ {X ∈ Pow(A). X≈succ(n)} &
      (∀X ∈ {X ∈ Pow(A). X≈succ(k)}. ∃! Y. Y ∈ T & X ⊆ Y))"

definition
  "AC17 == ∀A. ∀g ∈ (Pow(A)-{0} -> A) -> Pow(A)-{0}.
    ∃f ∈ Pow(A)-{0} -> A. f‘(g‘f) ∈ g‘f"

locale *AC18* =
  assumes *AC18*: "A≠0 & (∀a ∈ A. B(a) ≠ 0) -->
    ((⋂a ∈ A. ⋃b ∈ B(a). X(a,b)) =
      (⋃f ∈ Π a ∈ A. B(a). ⋂a ∈ A. X(a, f‘a)))"
  — AC18 cannot be expressed within the object-logic

definition
  "AC19 == ∀A. A≠0 & 0∉A --> ((⋂a ∈ A. ⋃b ∈ a. b) =
    (⋃f ∈ (Π B ∈ A. B). ⋂a ∈ A. f‘a))"

lemma *rvimage_id*: "rvimage(A,id(A),r) = r Int A*A"
⟨*proof*⟩

lemma *ordertype_Int*:
  "well_ord(A,r) ==> ordertype(A, r Int A*A) = ordertype(A,r)"
⟨*proof*⟩

lemma *lam_sing_bij*: "(λx ∈ A. {x}) ∈ bij(A, {{x}. x ∈ A})"
⟨*proof*⟩

lemma *inj_strengthen_type*:
  "[| f ∈ inj(A, B); !!a. a ∈ A ==> f‘a ∈ C |] ==> f ∈ inj(A,C)"
⟨*proof*⟩

lemma *nat_not_Finite*: "~ Finite(nat)"

⟨proof⟩

lemmas *le_imp_lepoll = le_imp_subset [THEN subset_imp_lepoll]*

lemma *ex1_two_eq: "[| ∃! x. P(x); P(x); P(y) |] ==> x=y"*
⟨proof⟩

lemma *surj_image_eq: "f ∈ surj(A, B) ==> f``A = B"*
⟨proof⟩

lemma *first_in_B:*
    "[| well_ord(Union(A),r); 0∉A; B ∈ A |] ==> (THE b. first(b,B,r)) ∈ B"
⟨proof⟩

lemma *ex_choice_fun: "[| well_ord(Union(A), R); 0∉A |] ==> ∃f. f:(Π X ∈ A. X)"*
⟨proof⟩

lemma *ex_choice_fun_Pow: "well_ord(A, R) ==> ∃f. f:(Π X ∈ Pow(A)-{0}. X)"*
⟨proof⟩

lemma *lepoll_m_imp_domain_lepoll_m:*
    "[| m ∈ nat; u ≲ m |] ==> domain(u) ≲ m"
⟨proof⟩

lemma *rel_domain_ex1:*

"[| succ(m) $\lesssim$ domain(r); r $\lesssim$ succ(m); m $\in$ nat |] ==> function(r)"
⟨*proof*⟩

lemma *rel_is_fun:*
"[| succ(m) $\lesssim$ domain(r); r $\lesssim$ succ(m); m $\in$ nat;
r $\subseteq$ A*B; A=domain(r) |] ==> r $\in$ A->B"
⟨*proof*⟩

end

theory *Cardinal_aux* **imports** *AC_Equiv* **begin**

lemma *Diff_lepoll:* "[| A $\lesssim$ succ(m); B $\subseteq$ A; B$\neq$0 |] ==> A-B $\lesssim$ m"
⟨*proof*⟩

lemma *lepoll_imp_ex_le_eqpoll:*
"[| A $\lesssim$ i; Ord(i) |] ==> $\exists$j. j le i & A $\approx$ j"
⟨*proof*⟩

lemma *lesspoll_imp_ex_lt_eqpoll:*
"[| A $\prec$ i; Ord(i) |] ==> $\exists$j. j<i & A $\approx$ j"
⟨*proof*⟩

lemma *Inf_Ord_imp_InfCard_cardinal:* "[| ~Finite(i); Ord(i) |] ==> InfCard(|i|)"
⟨*proof*⟩

An alternative and more general proof goes like this: A and B are both well-ordered (because they are injected into an ordinal), either A lepoll B or B lepoll A. Also both are equipollent to their cardinalities, so (if A and B are infinite) then A Un B lepoll —A—+—B— = max(—A—,—B—) lepoll i. In fact, the correctly strengthened version of this theorem appears below.

lemma *Un_lepoll_Inf_Ord_weak:*
"[|A $\approx$ i; B $\approx$ i; $\neg$ Finite(i); Ord(i)|] ==> A $\cup$ B $\lesssim$ i"
⟨*proof*⟩

lemma *Un_eqpoll_Inf_Ord:*
"[| A $\approx$ i; B $\approx$ i; ~Finite(i); Ord(i) |] ==> A Un B $\approx$ i"
⟨*proof*⟩

lemma paired_bij: "?f ∈ bij({{y,z}. y ∈ x}, x)"
⟨*proof*⟩

lemma paired_eqpoll: "{{y,z}. y ∈ x} ≈ x"
⟨*proof*⟩

lemma ex_eqpoll_disjoint: "∃B. B ≈ A & B Int C = 0"
⟨*proof*⟩

lemma Un_lepoll_Inf_Ord:
    "[| A ≲ i; B ≲ i; ~Finite(i); Ord(i) |] ==> A Un B ≲ i"
⟨*proof*⟩

lemma Least_in_Ord: "[| P(i); i ∈ j; Ord(j) |] ==> (LEAST i. P(i)) ∈ j"
⟨*proof*⟩

lemma Diff_first_lepoll:
    "[| well_ord(x,r); y ⊆ x; y ≲ succ(n); n ∈ nat |]
     ==> y - {THE b. first(b,y,r)} ≲ n"
⟨*proof*⟩

lemma UN_subset_split:
    "(⋃x ∈ X. P(x)) ⊆ (⋃x ∈ X. P(x)-Q(x)) Un (⋃x ∈ X. Q(x))"
⟨*proof*⟩

lemma UN_sing_lepoll: "Ord(a) ==> (⋃x ∈ a. {P(x)}) ≲ a"
⟨*proof*⟩

lemma UN_fun_lepoll_lemma [rule_format]:
    "[| well_ord(T, R); ~Finite(a); Ord(a); n ∈ nat |]
     ==> ∀f. (∀b ∈ a. f‘b ≲ n & f‘b ⊆ T) --> (⋃b ∈ a. f‘b) ≲ a"
⟨*proof*⟩

lemma UN_fun_lepoll:
    "[| ∀b ∈ a. f‘b ≲ n & f‘b ⊆ T; well_ord(T, R);
        ~Finite(a); Ord(a); n ∈ nat |] ==> (⋃b ∈ a. f‘b) ≲ a"
⟨*proof*⟩

lemma UN_lepoll:
    "[| ∀b ∈ a. F(b) ≲ n & F(b) ⊆ T; well_ord(T, R);
        ~Finite(a); Ord(a); n ∈ nat |]
     ==> (⋃b ∈ a. F(b)) ≲ a"
⟨*proof*⟩

lemma UN_eq_UN_Diffs:
    "Ord(a) ==> (⋃b ∈ a. F(b)) = (⋃b ∈ a. F(b) - (⋃c ∈ b. F(c)))"

⟨*proof*⟩

lemma *lepoll_imp_eqpoll_subset:*

```
"a ≲ X ==> ∃Y. Y ⊆ X & a ≈ Y"
```

⟨*proof*⟩

lemma *Diff_lesspoll_eqpoll_Card_lemma:*

```
"[| A≈a; ~Finite(a); Card(a); B ≺ a; A-B ≺ a |] ==> P"
```

⟨*proof*⟩

lemma *Diff_lesspoll_eqpoll_Card:*

```
"[| A ≈ a; ~Finite(a); Card(a); B ≺ a |] ==> A - B ≈ a"
```

⟨*proof*⟩

end

theory *WO6_WO1* **imports** *Cardinal_aux* **begin**

definition

```
NN  :: "i => i"  where
  "NN(y) == {m ∈ nat. ∃a. ∃f. Ord(a) & domain(f)=a  &
                      (⋃b<a. f`b) = y & (∀b<a. f`b ≲ m)}"
```

definition

```
uu  :: "[i, i, i, i] => i"  where
  "uu(f, beta, gamma, delta) == (f`beta * f`gamma) Int f`delta"
```

definition

```
vv1 :: "[i, i, i] => i"  where
   "vv1(f,m,b) ==
         let g = LEAST g. (∃d. Ord(d) & (domain(uu(f,b,g,d)) ≠ 0 &

                               domain(uu(f,b,g,d)) ≲ m));
             d = LEAST d. domain(uu(f,b,g,d)) ≠ 0 &
                          domain(uu(f,b,g,d)) ≲ m
         in  if f`b ≠ 0 then domain(uu(f,b,g,d)) else 0"
```

definition

```
ww1 :: "[i, i, i] => i"  where
   "ww1(f,m,b) == f`b - vv1(f,m,b)"
```

definition
```
  gg1 :: "[i, i, i] => i"  where
     "gg1(f,a,m) == λb ∈ a++a. if b<a then vv1(f,m,b) else ww1(f,m,b--a)"
```

definition
```
  vv2 :: "[i, i, i, i] => i"  where
     "vv2(f,b,g,s) ==
           if f`g ≠ 0 then {uu(f, b, g, LEAST d. uu(f,b,g,d) ≠ 0)`s}
else 0"
```

definition
```
  ww2 :: "[i, i, i, i] => i"  where
     "ww2(f,b,g,s) == f`g - vv2(f,b,g,s)"
```

definition
```
  gg2 :: "[i, i, i, i] => i"  where
     "gg2(f,a,b,s) ==
              λg ∈ a++a. if g<a then vv2(f,b,g,s) else ww2(f,b,g--a,s)"
```

lemma `WO2_WO3: "WO2 ==> WO3"`
⟨*proof*⟩

lemma `WO3_WO1: "WO3 ==> WO1"`
⟨*proof*⟩

lemma `WO1_WO2: "WO1 ==> WO2"`
⟨*proof*⟩

lemma `lam_sets: "f ∈ A->B ==> (λx ∈ A. {f`x}): A -> {{b}. b ∈ B}"`
⟨*proof*⟩

lemma `surj_imp_eq': "f ∈ surj(A,B) ==> (⋃a ∈ A. {f`a}) = B"`
⟨*proof*⟩

lemma `surj_imp_eq: "[| f ∈ surj(A,B); Ord(A) |] ==> (⋃a<A. {f`a}) = B"`
⟨*proof*⟩

lemma `WO1_WO4: "WO1 ==> WO4(1)"`

⟨*proof*⟩

lemma *WO4_mono: "[| m≤n; WO4(m) |] ==> WO4(n)"*
⟨*proof*⟩

lemma *WO4_WO5: "[| m ∈ nat; 1≤m; WO4(m) |] ==> WO5"*
⟨*proof*⟩

lemma *WO5_WO6: "WO5 ==> WO6"*
⟨*proof*⟩

lemma *lt_oadd_odiff_disj:*

```
      "[| k < i++j;  Ord(i);  Ord(j) |]
       ==> k < i  |  (~ k<i & k = i ++ (k--i) & (k--i)<j)"
```

⟨*proof*⟩

lemma *domain_uu_subset: "domain(uu(f,b,g,d)) ⊆ f‘b"*
⟨*proof*⟩

lemma *quant_domain_uu_lepoll_m:*

```
    "∀b<a. f‘b ≲ m ==> ∀b<a. ∀g<a. ∀d<a. domain(uu(f,b,g,d)) ≲ m"
```

⟨*proof*⟩

lemma *uu_subset1: "uu(f,b,g,d) ⊆ f‘b * f‘g"*
⟨*proof*⟩

lemma *uu_subset2: "uu(f,b,g,d) ⊆ f‘d"*
⟨*proof*⟩

lemma *uu_lepoll_m: "[| ∀b<a. f‘b ≲ m; d<a |] ==> uu(f,b,g,d) ≲ m"*
⟨*proof*⟩

lemma *cases:*
```
"∀b<a. ∀g<a. ∀d<a. u(f,b,g,d) ≲ m
 ==> (∀b<a. f‘b ≠ 0 -->
              (∃g<a. ∃d<a. u(f,b,g,d) ≠ 0 & u(f,b,g,d) ≺ m))
   | (∃b<a. f‘b ≠ 0 & (∀g<a. ∀d<a. u(f,b,g,d) ≠ 0 -->
                                     u(f,b,g,d) ≈ m))"
```
⟨*proof*⟩

lemma *UN_oadd:* `"Ord(a) ==> (⋃b<a++a. C(b)) = (⋃b<a. C(b) Un C(a++b))"`
⟨*proof*⟩

lemma *vv1_subset:* `"vv1(f,m,b) ⊆ f‘b"`
⟨*proof*⟩

lemma *UN_gg1_eq:*
```
"[| Ord(a);  m ∈ nat |] ==> (⋃b<a++a. gg1(f,a,m)‘b) = (⋃b<a. f‘b)"
```
⟨*proof*⟩

lemma *domain_gg1:* `"domain(gg1(f,a,m)) = a++a"`
⟨*proof*⟩

lemma *nested_LeastI:*
```
"[| P(a, b);  Ord(a);  Ord(b);
    Least_a = (LEAST a. ∃x. Ord(x) & P(a, x)) |]
 ==> P(Least_a, LEAST b. P(Least_a, b))"
```
⟨*proof*⟩

lemmas *nested_Least_instance =*
```
  nested_LeastI [of "%g d. domain(uu(f,b,g,d)) ≠ 0 &
                           domain(uu(f,b,g,d)) ≲ m",
                 standard]
```

lemma *gg1_lepoll_m:*
```
"[| Ord(a);  m ∈ nat;
    ∀b<a. f‘b ≠0 -->
        (∃g<a. ∃d<a. domain(uu(f,b,g,d)) ≠ 0  &
                     domain(uu(f,b,g,d)) ≲ m);
    ∀b<a. f‘b ≲ succ(m);  b<a++a |]
 ==> gg1(f,a,m)‘b ≲ m"
```
⟨*proof*⟩

lemma *ex_d_uu_not_empty:*
```
"[| b<a;  g<a;  f‘b≠0;  f‘g≠0;
    y*y ⊆ y;  (⋃b<a. f‘b)=y |]
 ==> ∃d<a. uu(f,b,g,d) ≠ 0"
```
⟨*proof*⟩

lemma *uu_not_empty:*
```
"[| b<a; g<a; f‘b≠0; f‘g≠0;  y*y ⊆ y; (⋃b<a. f‘b)=y |]
 ==> uu(f,b,g,LEAST d. (uu(f,b,g,d) ≠ 0)) ≠ 0"
```
⟨*proof*⟩

lemma *not_empty_rel_imp_domain:* `"[| r ⊆ A*B; r≠0 |] ==> domain(r)≠0"`
⟨*proof*⟩

lemma *Least_uu_not_empty_lt_a:*
```
"[| b<a; g<a; f‘b≠0; f‘g≠0; y*y ⊆ y; (⋃b<a. f‘b)=y |]
 ==> (LEAST d. uu(f,b,g,d) ≠ 0) < a"
```
⟨*proof*⟩

lemma *subset_Diff_sing:* `"[| B ⊆ A; a∉B |] ==> B ⊆ A-{a}"`
⟨*proof*⟩

lemma *supset_lepoll_imp_eq:*
```
"[| A ≲ m; m ≲ B; B ⊆ A; m ∈ nat |] ==> A=B"
```
⟨*proof*⟩

lemma *uu_Least_is_fun:*
```
"[| ∀g<a. ∀d<a. domain(uu(f, b, g, d))≠0 -->
     domain(uu(f, b, g, d)) ≈ succ(m);
     ∀b<a. f‘b ≲ succ(m);  y*y ⊆ y;
```

```
      (⋃b<a. f‘b)=y;  b<a;  g<a;  d<a;
      f‘b≠0;  f‘g≠0;  m ∈ nat;  s ∈ f‘b |]
  ==> uu(f, b, g, LEAST d. uu(f,b,g,d)≠0) ∈ f‘b -> f‘g"
```
⟨proof⟩

lemma `vv2_subset:`
```
   "[| ∀g<a. ∀d<a. domain(uu(f, b, g, d))≠0 -->
                    domain(uu(f, b, g, d)) ≈ succ(m);
       ∀b<a. f‘b ≲ succ(m); y*y ⊆ y;
       (⋃b<a. f‘b)=y;  b<a;  g<a;  m ∈ nat;  s ∈ f‘b |]
    ==> vv2(f,b,g,s) ⊆ f‘g"
```
⟨proof⟩

lemma `UN_gg2_eq:`
```
   "[| ∀g<a. ∀d<a. domain(uu(f,b,g,d)) ≠ 0 -->
             domain(uu(f,b,g,d)) ≈ succ(m);
       ∀b<a. f‘b ≲ succ(m); y*y ⊆ y;
       (⋃b<a. f‘b)=y;  Ord(a);  m ∈ nat;  s ∈ f‘b;  b<a |]
    ==> (⋃g<a++a. gg2(f,a,b,s) ‘ g) = y"
```
⟨proof⟩

lemma `domain_gg2: "domain(gg2(f,a,b,s)) = a++a"`
⟨proof⟩

lemma `vv2_lepoll: "[| m ∈ nat; m≠0 |] ==> vv2(f,b,g,s) ≲ m"`
⟨proof⟩

lemma `ww2_lepoll:`
```
  "[| ∀b<a. f‘b ≲ succ(m);  g<a;  m ∈ nat;  vv2(f,b,g,d) ⊆ f‘g |]

   ==> ww2(f,b,g,d) ≲ m"
```
⟨proof⟩

lemma `gg2_lepoll_m:`
```
   "[| ∀g<a. ∀d<a. domain(uu(f,b,g,d)) ≠ 0 -->
                   domain(uu(f,b,g,d)) ≈ succ(m);
       ∀b<a. f‘b ≲ succ(m);  y*y ⊆ y;
       (⋃b<a. f‘b)=y;  b<a;  s ∈ f‘b;  m ∈ nat;  m≠ 0;  g<a++a |]

    ==> gg2(f,a,b,s) ‘ g ≲ m"
```
⟨proof⟩

lemma *lemma_ii*: "[| succ(m) ∈ NN(y); y*y ⊆ y; m ∈ nat; m≠0 |] ==> m ∈ NN(y)"
⟨*proof*⟩

lemma *z_n_subset_z_succ_n*:
"∀n ∈ nat. rec(n, x, %k r. r Un r*r) ⊆ rec(succ(n), x, %k r. r Un r*r)"
⟨*proof*⟩

lemma *le_subsets*:
"[| ∀n ∈ nat. f(n)<=f(succ(n)); n≤m; n ∈ nat; m ∈ nat |]
==> f(n)<=f(m)"
⟨*proof*⟩

lemma *le_imp_rec_subset*:
"[| n≤m; m ∈ nat |]
==> rec(n, x, %k r. r Un r*r) ⊆ rec(m, x, %k r. r Un r*r)"
⟨*proof*⟩

lemma *lemma_iv*: "∃y. x Un y*y ⊆ y"
⟨*proof*⟩

lemma *WO6_imp_NN_not_empty: "WO6 ==> NN(y) ≠ 0"*
⟨*proof*⟩

lemma *lemma1:*
"[| (⋃b<a. f`b)=y; x ∈ y; ∀b<a. f`b ≲ 1; Ord(a) |] ==> ∃c<a. f`c = {x}"
⟨*proof*⟩

lemma *lemma2:*
"[| (⋃b<a. f`b)=y; x ∈ y; ∀b<a. f`b ≲ 1; Ord(a) |]
==> f` (LEAST i. f`i = {x}) = {x}"
⟨*proof*⟩

lemma *NN_imp_ex_inj: "1 ∈ NN(y) ==> ∃a f. Ord(a) & f ∈ inj(y, a)"*
⟨*proof*⟩

lemma *y_well_ord: "[| y*y ⊆ y; 1 ∈ NN(y) |] ==> ∃r. well_ord(y, r)"*
⟨*proof*⟩

lemma *rev_induct_lemma [rule_format]:*
"[| n ∈ nat; !!m. [| m ∈ nat; m≠0; P(succ(m)) |] ==> P(m) |]
==> n≠0 --> P(n) --> P(1)"
⟨*proof*⟩

lemma *rev_induct:*
"[| n ∈ nat; P(n); n≠0;
!!m. [| m ∈ nat; m≠0; P(succ(m)) |] ==> P(m) |]
==> P(1)"
⟨*proof*⟩

lemma *NN_into_nat: "n ∈ NN(y) ==> n ∈ nat"*
⟨*proof*⟩

lemma *lemma3: "[| n ∈ NN(y); y*y ⊆ y; n≠0 |] ==> 1 ∈ NN(y)"*
⟨*proof*⟩

lemma *NN_y_0: "0 ∈ NN(y) ==> y=0"*
⟨*proof*⟩

lemma *WO6_imp_WO1: "WO6 ==> WO1"*
⟨*proof*⟩

end

theory *WO1_WO7* **imports** *AC_Equiv* **begin**

definition

```
"LEMMA ==
 ∀X. ~Finite(X) --> (∃R. well_ord(X,R) & ~well_ord(X,converse(R)))"
```

lemma *WO7_iff_LEMMA: "WO7 <-> LEMMA"*
⟨*proof*⟩

lemma *LEMMA_imp_WO1: "LEMMA ==> WO1"*
⟨*proof*⟩

lemma *converse_Memrel_not_wf_on:*

```
"[| Ord(a); ~Finite(a) |] ==> ~wf[a](converse(Memrel(a)))"
```

⟨*proof*⟩

lemma *converse_Memrel_not_well_ord:*

"[| Ord(a); ~Finite(a) |] ==> ~well_ord(a,converse(Memrel(a)))"
⟨*proof*⟩

lemma *well_ord_rvimage_ordertype:*
"well_ord(A,r) $\Longrightarrow$
rvimage (ordertype(A,r), converse(ordermap(A,r)),r) =
Memrel(ordertype(A,r))"
⟨*proof*⟩

lemma *well_ord_converse_Memrel:*
"[| well_ord(A,r); well_ord(A,converse(r)) |]
==> well_ord(ordertype(A,r), converse(Memrel(ordertype(A,r))))"

⟨*proof*⟩

lemma *WO1_imp_LEMMA: "WO1 ==> LEMMA"*
⟨*proof*⟩

lemma *WO1_iff_WO7: "WO1 <-> WO7"*
⟨*proof*⟩

lemma *WO1_WO8: "WO1 ==> WO8"*
⟨*proof*⟩

lemma *WO8_WO1: "WO8 ==> WO1"*
⟨*proof*⟩

end

theory *AC7_AC9* **imports** *AC_Equiv* **begin**

lemma *Sigma_fun_space_not0: "[| 0$\notin$A; B $\in$ A |] ==> (nat->Union(A)) * B $\neq$ 0"*

⟨*proof*⟩

lemma *inj_lemma:*

```
"C ∈ A ==> (λg ∈ (nat->Union(A))*C.
        (λn ∈ nat. if(n=0, snd(g), fst(g)`(n #- 1))))
        ∈ inj((nat->Union(A))*C, (nat->Union(A)) ) "
```

⟨*proof*⟩

lemma *Sigma_fun_space_eqpoll:*

```
"[| C ∈ A; 0∉A |] ==> (nat->Union(A)) * C ≈ (nat->Union(A))"
```

⟨*proof*⟩

lemma *AC6_AC7:* `"AC6 ==> AC7"`

⟨*proof*⟩

lemma *lemma1_1:* `"y ∈ (Π B ∈ A. Y*B) ==> (λB ∈ A. snd(y`B)) ∈ (Π B ∈ A. B)"`

⟨*proof*⟩

lemma *lemma1_2:*

```
"y ∈ (Π B ∈ {Y*C. C ∈ A}. B) ==> (λB ∈ A. y`(Y*B)) ∈ (Π B ∈ A. Y*B)"
```

⟨*proof*⟩

lemma *AC7_AC6_lemma1:*

```
"(Π B ∈ {(nat->Union(A))*C. C ∈ A}. B) ≠ 0 ==> (Π B ∈ A. B) ≠ 0"
```

⟨*proof*⟩

lemma *AC7_AC6_lemma2:* `"0 ∉ A ==> 0 ∉ {(nat -> Union(A)) * C. C ∈ A}"`

⟨*proof*⟩

lemma *AC7_AC6:* `"AC7 ==> AC6"`

⟨*proof*⟩

lemma *AC1_AC8_lemma1:*

```
        "∀B ∈ A. ∃B1 B2. B=<B1,B2> & B1 ≈ B2
        ==> 0 ∉ { bij(fst(B),snd(B)). B ∈ A }"
```

⟨*proof*⟩

lemma *AC1_AC8_lemma2:*

```
     "[| f ∈ (Π X ∈ RepFun(A,p). X); D ∈ A |] ==> (λx ∈ A. f‘p(x))‘D
∈ p(D)"
```

⟨*proof*⟩

lemma *AC1_AC8: "AC1 ==> AC8"*

⟨*proof*⟩

lemma *AC8_AC9_lemma:*

```
     "∀B1 ∈ A. ∀B2 ∈ A. B1 ≈ B2
      ==> ∀B ∈ A*A. ∃B1 B2. B=<B1,B2> & B1 ≈ B2"
```

⟨*proof*⟩

lemma *AC8_AC9: "AC8 ==> AC9"*

⟨*proof*⟩

lemma *snd_lepoll_SigmaI: "b ∈ B ⟹ X ≲ B × X"*

⟨*proof*⟩

lemma *nat_lepoll_lemma:*

```
     "[|0 ∉ A; B ∈ A|] ==> nat ≲ ((nat → Union(A)) × B) × nat"
```

⟨*proof*⟩

lemma *AC9_AC1_lemma1:*

```
     "[| 0∉A;  A≠0;
         C = {((nat->Union(A))*B)*nat. B ∈ A}  Un
```

```
          {cons(0,((nat->Union(A))*B)*nat). B ∈ A};
       B1 ∈ C;  B2 ∈ C |]
    ==> B1 ≈ B2"
```
⟨*proof*⟩

lemma *AC9_AC1_lemma2:*
```
    "∀B1 ∈ {(F*B)*N. B ∈ A} Un {cons(0,(F*B)*N). B ∈ A}.
     ∀B2 ∈ {(F*B)*N. B ∈ A} Un {cons(0,(F*B)*N). B ∈ A}.
       f`<B1,B2> ∈ bij(B1, B2)
     ==> (λB ∈ A. snd(fst((f`<cons(0,(F*B)*N),(F*B)*N>)`0))) ∈ (Π X
∈ A. X)"
```
⟨*proof*⟩

lemma *AC9_AC1: "AC9 ==> AC1"*
⟨*proof*⟩

end

theory *WO1_AC* **imports** *AC_Equiv* **begin**

theorem *WO1_AC1: "WO1 ==> AC1"*
⟨*proof*⟩

lemma *lemma1: "[| WO1; ∀B ∈ A. ∃C ∈ D(B). P(C,B) |] ==> ∃f. ∀B ∈ A. P(f`B,B)"*
⟨*proof*⟩

lemma *lemma2_1: "[| ~Finite(B); WO1 |] ==> |B| + |B| ≈ B"*
⟨*proof*⟩

lemma *lemma2_2:*
```
    "f ∈ bij(D+D, B) ==> {{f`Inl(i), f`Inr(i)}. i ∈ D} ∈ Pow(Pow(B))"
```
⟨*proof*⟩

lemma *lemma2_3:*
```
       "f ∈ bij(D+D, B) ==> pairwise_disjoint({{f`Inl(i), f`Inr(i)}.
i ∈ D})"
```
⟨*proof*⟩

lemma *lemma2_4:*

```
"[| f ∈ bij(D+D, B); 1≤n |]
 ==> sets_of_size_between({{f`Inl(i), f`Inr(i)}. i ∈ D}, 2, succ(n))"
```

⟨*proof*⟩

lemma *lemma2_5:*

```
"f ∈ bij(D+D, B) ==> Union({{f`Inl(i), f`Inr(i)}. i ∈ D})=B"
```

⟨*proof*⟩

lemma *lemma2:*

```
"[| WO1; ~Finite(B); 1≤n  |]
 ==> ∃C ∈ Pow(Pow(B)). pairwise_disjoint(C) &
            sets_of_size_between(C, 2, succ(n)) &
            Union(C)=B"
```

⟨*proof*⟩

theorem *WO1_AC10:* *"[| WO1; 1≤n |] ==> AC10(n)"*

⟨*proof*⟩

end

theory *Hartog* **imports** *AC_Equiv* **begin**

definition

Hartog :: "i => i" **where**

```
"Hartog(X) == LEAST i. ~ i ≲ X"
```

lemma *Ords_in_set:* *"∀a. Ord(a) --> a ∈ X ==> P"*

⟨*proof*⟩

lemma *Ord_lepoll_imp_ex_well_ord:*

```
"[| Ord(a); a ≲ X |]
 ==> ∃Y. Y ⊆ X & (∃R. well_ord(Y,R) & ordertype(Y,R)=a)"
```

⟨*proof*⟩

lemma *Ord_lepoll_imp_eq_ordertype:*

```
"[| Ord(a); a ≲ X |] ==> ∃Y. Y ⊆ X & (∃R. R ⊆ X*X & ordertype(Y,R)=a)"
```

⟨*proof*⟩

lemma *Ords_lepoll_set_lemma:*

```
"(∀a. Ord(a) --> a ≲ X) ==>
   ∀a. Ord(a) -->
     a ∈ {b. Z ∈ Pow(X)*Pow(X*X), ∃Y R. Z=<Y,R> & ordertype(Y,R)=b}"
```

⟨*proof*⟩

lemma *Ords_lepoll_set:* "$\forall$a. Ord(a) --> a $\lesssim$ X ==> P"
⟨*proof*⟩

lemma *ex_Ord_not_lepoll:* "$\exists$a. Ord(a) & ~a $\lesssim$ X"
⟨*proof*⟩

lemma *not_Hartog_lepoll_self:* "~ Hartog(A) $\lesssim$ A"
⟨*proof*⟩

lemmas *Hartog_lepoll_selfE = not_Hartog_lepoll_self [THEN notE, standard]*

lemma *Ord_Hartog:* "Ord(Hartog(A))"
⟨*proof*⟩

lemma *less_HartogE1:* "[| i < Hartog(A); ~ i $\lesssim$ A |] ==> P"
⟨*proof*⟩

lemma *less_HartogE:* "[| i < Hartog(A); i $\approx$ Hartog(A) |] ==> P"
⟨*proof*⟩

lemma *Card_Hartog:* "Card(Hartog(A))"
⟨*proof*⟩

end

theory *HH* **imports** *AC_Equiv Hartog* **begin**

definition
  HH :: "[i, i, i] => i" **where**
    "HH(f,x,a) == transrec(a, %b r. let z = x - ($\bigcup$c $\in$ b. r`c)
                                    in if f`z $\in$ Pow(z)-{0} then f`z else
{x})"

0.1 Lemmas useful in each of the three proofs

lemma *HH_def_satisfies_eq:*
     "HH(f,x,a) = (let z = x - ($\bigcup$b $\in$ a. HH(f,x,b))
                  in if f`z $\in$ Pow(z)-{0} then f`z else {x})"
⟨*proof*⟩

lemma *HH_values:* "HH(f,x,a) $\in$ Pow(x)-{0} | HH(f,x,a)={x}"
⟨*proof*⟩

lemma *subset_imp_Diff_eq:*
     "B $\subseteq$ A ==> X-($\bigcup$a $\in$ A. P(a)) = X-($\bigcup$a $\in$ A-B. P(a))-($\bigcup$b $\in$ B. P(b))"
⟨*proof*⟩

lemma `Ord_DiffE: "[| c ∈ a-b; b<a |] ==> c=b | b<c & c<a"`
⟨*proof*⟩

lemma `Diff_UN_eq_self: "(!!y. y∈A ==> P(y) = {x}) ==> x - (⋃y ∈ A. P(y)) = x"`
⟨*proof*⟩

lemma `HH_eq: "x - (⋃b ∈ a. HH(f,x,b)) = x - (⋃b ∈ a1. HH(f,x,b))`
`==> HH(f,x,a) = HH(f,x,a1)"`
⟨*proof*⟩

lemma `HH_is_x_gt_too: "[| HH(f,x,b)={x}; b<a |] ==> HH(f,x,a)={x}"`
⟨*proof*⟩

lemma `HH_subset_x_lt_too:`
`"[| HH(f,x,a) ∈ Pow(x)-{0}; b<a |] ==> HH(f,x,b) ∈ Pow(x)-{0}"`
⟨*proof*⟩

lemma `HH_subset_x_imp_subset_Diff_UN:`
`"HH(f,x,a) ∈ Pow(x)-{0} ==> HH(f,x,a) ∈ Pow(x - (⋃b ∈ a. HH(f,x,b)))-{0}"`
⟨*proof*⟩

lemma `HH_eq_arg_lt:`
`"[| HH(f,x,v)=HH(f,x,w); HH(f,x,v) ∈ Pow(x)-{0}; v ∈ w |] ==> P"`
⟨*proof*⟩

lemma `HH_eq_imp_arg_eq:`
`"[| HH(f,x,v)=HH(f,x,w); HH(f,x,w) ∈ Pow(x)-{0}; Ord(v); Ord(w) |] ==> v=w"`
⟨*proof*⟩

lemma `HH_subset_x_imp_lepoll:`
`"[| HH(f, x, i) ∈ Pow(x)-{0}; Ord(i) |] ==> i lepoll Pow(x)-{0}"`
⟨*proof*⟩

lemma `HH_Hartog_is_x: "HH(f, x, Hartog(Pow(x)-{0})) = {x}"`
⟨*proof*⟩

lemma `HH_Least_eq_x: "HH(f, x, LEAST i. HH(f, x, i) = {x}) = {x}"`
⟨*proof*⟩

lemma `less_Least_subset_x:`
`"a ∈ (LEAST i. HH(f,x,i)={x}) ==> HH(f,x,a) ∈ Pow(x)-{0}"`
⟨*proof*⟩

0.2 Lemmas used in the proofs of AC1 ==¿ WO2 and AC17 ==¿ AC1

lemma `lam_Least_HH_inj_Pow:`

```
"(λa ∈ (LEAST i. HH(f,x,i)={x}). HH(f,x,a))
 ∈ inj(LEAST i. HH(f,x,i)={x}, Pow(x)-{0})"
```
⟨*proof*⟩

lemma *lam_Least_HH_inj:*
```
"∀a ∈ (LEAST i. HH(f,x,i)={x}). ∃z ∈ x. HH(f,x,a) = {z}
 ==> (λa ∈ (LEAST i. HH(f,x,i)={x}). HH(f,x,a))
     ∈ inj(LEAST i. HH(f,x,i)={x}, {{y}. y ∈ x})"
```
⟨*proof*⟩

lemma *lam_surj_sing:*
```
"[| x - (⋃a ∈ A. F(a)) = 0;  ∀a ∈ A. ∃z ∈ x. F(a) = {z} |]

 ==> (λa ∈ A. F(a)) ∈ surj(A, {{y}. y ∈ x})"
```
⟨*proof*⟩

lemma *not_emptyI2:* `"y ∈ Pow(x)-{0} ==> x ≠ 0"`
⟨*proof*⟩

lemma *f_subset_imp_HH_subset:*
```
"f`(x - (⋃j ∈ i. HH(f,x,j))) ∈ Pow(x - (⋃j ∈ i. HH(f,x,j)))-{0}

 ==> HH(f, x, i) ∈ Pow(x) - {0}"
```
⟨*proof*⟩

lemma *f_subsets_imp_UN_HH_eq_x:*
```
"∀z ∈ Pow(x)-{0}. f`z ∈ Pow(z)-{0}
 ==> x - (⋃j ∈ (LEAST i. HH(f,x,i)={x}). HH(f,x,j)) = 0"
```
⟨*proof*⟩

lemma *HH_values2:* `"HH(f,x,i) = f`(x - (⋃j ∈ i. HH(f,x,j))) | HH(f,x,i)={x}"`
⟨*proof*⟩

lemma *HH_subset_imp_eq:*
```
"HH(f,x,i): Pow(x)-{0} ==> HH(f,x,i)=f`(x - (⋃j ∈ i. HH(f,x,j)))"
```
⟨*proof*⟩

lemma *f_sing_imp_HH_sing:*
```
"[| f ∈ (Pow(x)-{0}) -> {{z}. z ∈ x};
    a ∈ (LEAST i. HH(f,x,i)={x}) |] ==> ∃z ∈ x. HH(f,x,a) = {z}"
```
⟨*proof*⟩

lemma *f_sing_lam_bij:*
```
"[| x - (⋃j ∈ (LEAST i. HH(f,x,i)={x}). HH(f,x,j)) = 0;
    f ∈ (Pow(x)-{0}) -> {{z}. z ∈ x} |]
 ==> (λa ∈ (LEAST i. HH(f,x,i)={x}). HH(f,x,a))
     ∈ bij(LEAST i. HH(f,x,i)={x}, {{y}. y ∈ x})"
```
⟨*proof*⟩

lemma *lam_singI:*
  "f ∈ (Π X ∈ Pow(x)-{0}. F(X))
   ==> (λX ∈ Pow(x)-{0}. {f'X}) ∈ (Π X ∈ Pow(x)-{0}. {{z}. z ∈ F(X)})"
⟨*proof*⟩

lemmas *bij_Least_HH_x =*
  comp_bij [OF f_sing_lam_bij [OF _ lam_singI]
    lam_sing_bij [THEN bij_converse_bij], standard]

0.3 The proof of AC1 ==¿ WO2

lemma *bijection:*
  "f ∈ (Π X ∈ Pow(x) - {0}. X)
   ==> ∃g. g ∈ bij(x, LEAST i. HH(λX ∈ Pow(x)-{0}. {f'X}, x, i) =
{x})"
⟨*proof*⟩

lemma *AC1_WO2: "AC1 ==> WO2"*
⟨*proof*⟩

end

theory *AC15_WO6* **imports** *HH Cardinal_aux* **begin**

lemma *lepoll_Sigma: "A≠0 ==> B ≲ A*B"*
⟨*proof*⟩

lemma *cons_times_nat_not_Finite:*
  *"0∉A ==> ∀B ∈ {cons(0,x*nat). x ∈ A}. ~Finite(B)"*
⟨*proof*⟩

lemma *lemma1: "[| Union(C)=A; a ∈ A |] ==> ∃B ∈ C. a ∈ B & B ⊆ A"*
⟨*proof*⟩

lemma *lemma2:*
  "[| pairwise_disjoint(A); B ∈ A; C ∈ A; a ∈ B; a ∈ C |] ==>
B=C"

⟨*proof*⟩

lemma *lemma3:*
```
"∀B ∈ {cons(0, x*nat). x ∈ A}. pairwise_disjoint(f‘B) &
        sets_of_size_between(f‘B, 2, n) & Union(f‘B)=B
==> ∀B ∈ A. ∃! u. u ∈ f‘cons(0, B*nat) & u ⊆ cons(0, B*nat) &

        0 ∈ u & 2 ≲ u & u ≲ n"
```
⟨*proof*⟩

lemma *lemma4:* `"[| A ≲ i; Ord(i) |] ==> {P(a). a ∈ A} ≲ i"`
⟨*proof*⟩

lemma *lemma5_1:*
```
"[| B ∈ A; 2 ≲ u(B) |] ==> (λx ∈ A. {fst(x). x ∈ u(x)-{0}})‘B ≠
0"
```
⟨*proof*⟩

lemma *lemma5_2:*
```
"[|  B ∈ A; u(B) ⊆ cons(0, B*nat) |]
 ==> (λx ∈ A. {fst(x). x ∈ u(x)-{0}})‘B ⊆ B"
```
⟨*proof*⟩

lemma *lemma5_3:*
```
"[| n ∈ nat; B ∈ A; 0 ∈ u(B); u(B) ≲ succ(n) |]
 ==> (λx ∈ A. {fst(x). x ∈ u(x)-{0}})‘B ≲ n"
```
⟨*proof*⟩

lemma *ex_fun_AC13_AC15:*
```
"[| ∀B ∈ {cons(0, x*nat). x ∈ A}.
          pairwise_disjoint(f‘B) &
          sets_of_size_between(f‘B, 2, succ(n)) & Union(f‘B)=B;

    n ∈ nat |]
 ==> ∃f. ∀B ∈ A. f‘B ≠ 0 & f‘B ⊆ B & f‘B ≲ n"
```
⟨*proof*⟩

theorem *AC10_AC11:* `"[| n ∈ nat; 1≤n; AC10(n) |] ==> AC11"`
⟨*proof*⟩

theorem *AC11_AC12: "AC11 ==> AC12"*
⟨*proof*⟩

theorem *AC12_AC15: "AC12 ==> AC15"*
⟨*proof*⟩

lemma *OUN_eq_UN: "Ord(x) ==> (⋃a<x. F(a)) = (⋃a ∈ x. F(a))"*
⟨*proof*⟩

lemma *AC15_WO6_aux1:*
"∀x ∈ Pow(A)-{0}. f‘x≠0 & f‘x ⊆ x & f‘x ≲ m
==> (⋃i<LEAST x. HH(f,A,x)={A}. HH(f,A,i)) = A"
⟨*proof*⟩

lemma *AC15_WO6_aux2:*
"∀x ∈ Pow(A)-{0}. f‘x≠0 & f‘x ⊆ x & f‘x ≲ m
==> ∀x < (LEAST x. HH(f,A,x)={A}). HH(f,A,x) ≲ m"
⟨*proof*⟩

theorem *AC15_WO6: "AC15 ==> WO6"*
⟨*proof*⟩

theorem *AC10_AC13: "[| n ∈ nat; 1≤n; AC10(n) |] ==> AC13(n)"*
⟨*proof*⟩

lemma *AC1_AC13: "AC1 ==> AC13(1)"*
⟨*proof*⟩

lemma *AC13_mono: "[| m≤n; AC13(m) |] ==> AC13(n)"*
⟨*proof*⟩

theorem *AC13_AC14: "[| n ∈ nat; 1≤n; AC13(n) |] ==> AC14"*
⟨*proof*⟩

theorem *AC14_AC15: "AC14 ==> AC15"*
⟨*proof*⟩

lemma *lemma_aux: "[| A≠0; A ≲ 1 |] ==> ∃a. A={a}"*
⟨*proof*⟩

lemma *AC13_AC1_lemma:*

```
  "∀B ∈ A. f(B)≠0 & f(B)<=B & f(B) ≲ 1
   ==> (λx ∈ A. THE y. f(x)={y}) ∈ (Π X ∈ A. X)"
```
⟨proof⟩

theorem *AC13_AC1: "AC13(1) ==> AC1"*
⟨proof⟩

theorem *AC11_AC14: "AC11 ==> AC14"*
⟨proof⟩

end

theory *AC16_lemmas* **imports** *AC_Equiv Hartog Cardinal_aux* **begin**

lemma *cons_Diff_eq: "a∉A ==> cons(a,A)-{a}=A"*
⟨proof⟩

lemma *nat_1_lepoll_iff: "1≲X <-> (∃x. x ∈ X)"*
⟨proof⟩

lemma *eqpoll_1_iff_singleton: "X≈1 <-> (∃x. X={x})"*
⟨proof⟩

lemma *cons_eqpoll_succ: "[| x≈n; y∉x |] ==> cons(y,x)≈succ(n)"*
⟨proof⟩

lemma *subsets_eqpoll_1_eq: "{Y ∈ Pow(X). Y≈1} = {{x}. x ∈ X}"*
⟨proof⟩

lemma *eqpoll_RepFun_sing: "X≈{{x}. x ∈ X}"*
⟨proof⟩

lemma *subsets_eqpoll_1_eqpoll: "{Y ∈ Pow(X). Y≈1}≈X"*
⟨proof⟩

lemma *InfCard_Least_in:*
```
  "[| InfCard(x); y ⊆ x; y ≈ succ(z) |] ==> (LEAST i. i ∈ y) ∈ y"
```
⟨proof⟩

lemma *subsets_lepoll_lemma1:*
```
  "[| InfCard(x); n ∈ nat |]
   ==> {y ∈ Pow(x). y≈succ(succ(n))} ≲ x*{y ∈ Pow(x). y≈succ(n)}"
```

⟨*proof*⟩

lemma *set_of_Ord_succ_Union: "(∀y ∈ z. Ord(y)) ==> z ⊆ succ(Union(z))"*
⟨*proof*⟩

lemma *subset_not_mem: "j ⊆ i ==> i ∉ j"*
⟨*proof*⟩

lemma *succ_Union_not_mem:*
"(!!y. y ∈ z ==> Ord(y)) ==> succ(Union(z)) ∉ z"
⟨*proof*⟩

lemma *Union_cons_eq_succ_Union:*
"Union(cons(succ(Union(z)),z)) = succ(Union(z))"
⟨*proof*⟩

lemma *Un_Ord_disj: "[| Ord(i); Ord(j) |] ==> i Un j = i | i Un j = j"*
⟨*proof*⟩

lemma *Union_eq_Un: "x ∈ X ==> Union(X) = x Un Union(X-{x})"*
⟨*proof*⟩

lemma *Union_in_lemma [rule_format]:*
"n ∈ nat ==> ∀z. (∀y ∈ z. Ord(y)) & z≈n & z≠0 --> Union(z) ∈ z"
⟨*proof*⟩

lemma *Union_in: "[| ∀x ∈ z. Ord(x); z≈n; z≠0; n ∈ nat |] ==> Union(z) ∈ z"*
⟨*proof*⟩

lemma *succ_Union_in_x:*
"[| InfCard(x); z ∈ Pow(x); z≈n; n ∈ nat |] ==> succ(Union(z)) ∈ x"
⟨*proof*⟩

lemma *succ_lepoll_succ_succ:*
"[| InfCard(x); n ∈ nat |]
==> {y ∈ Pow(x). y≈succ(n)} ≲ {y ∈ Pow(x). y≈succ(succ(n))}"
⟨*proof*⟩

lemma *subsets_eqpoll_X:*
"[| InfCard(X); n ∈ nat |] ==> {Y ∈ Pow(X). Y≈succ(n)} ≈ X"
⟨*proof*⟩

lemma *image_vimage_eq:*
"[| f ∈ surj(A,B); y ⊆ B |] ==> f``(converse(f)``y) = y"
⟨*proof*⟩

lemma *vimage_image_eq: "[| f ∈ inj(A,B); y ⊆ A |] ==> converse(f)``(f``y) = y"*
⟨*proof*⟩

lemma *subsets_eqpoll:*
"A≈B ==> {Y ∈ Pow(A). Y≈n}≈{Y ∈ Pow(B). Y≈n}"
⟨*proof*⟩

lemma *WO2_imp_ex_Card: "WO2 ==> ∃a. Card(a) & X≈a"*
⟨*proof*⟩

lemma *lepoll_infinite: "[| X≲Y; ~Finite(X) |] ==> ~Finite(Y)"*
⟨*proof*⟩

lemma *infinite_Card_is_InfCard: "[| ~Finite(X); Card(X) |] ==> InfCard(X)"*
⟨*proof*⟩

lemma *WO2_infinite_subsets_eqpoll_X: "[| WO2; n ∈ nat; ~Finite(X) |]*

==> {Y ∈ Pow(X). Y≈succ(n)}≈X"
⟨*proof*⟩

lemma *well_ord_imp_ex_Card: "well_ord(X,R) ==> ∃a. Card(a) & X≈a"*
⟨*proof*⟩

lemma *well_ord_infinite_subsets_eqpoll_X:*
"[| well_ord(X,R); n ∈ nat; ~Finite(X) |] ==> {Y ∈ Pow(X). Y≈succ(n)}≈X"
⟨*proof*⟩

end

theory *WO2_AC16* **imports** *AC_Equiv AC16_lemmas Cardinal_aux* **begin**

definition
recfunAC16 :: "[i,i,i,i] => i" **where**
"recfunAC16(f,h,i,a) ==
transrec2(i, 0,
%g r. if (∃y ∈ r. h`g ⊆ y) then r
else r Un {f`(LEAST i. h`g ⊆ f`i &
(∀b<a. (h`b ⊆ f`i --> (∀t ∈ r. ~ h`b ⊆ t))))})"

lemma *recfunAC16_0: "recfunAC16(f,h,0,a) = 0"*
⟨*proof*⟩

lemma *recfunAC16_succ:*
```
    "recfunAC16(f,h,succ(i),a) =
     (if (∃y ∈ recfunAC16(f,h,i,a). h ‘ i ⊆ y) then recfunAC16(f,h,i,a)

      else recfunAC16(f,h,i,a) Un
           {f ‘ (LEAST j. h ‘ i ⊆ f ‘ j &
            (∀b<a. (h‘b ⊆ f‘j
             --> (∀t ∈ recfunAC16(f,h,i,a). ~ h‘b ⊆ t))))})"
```
⟨*proof*⟩

lemma *recfunAC16_Limit:*
```
"Limit(i)
        ==> recfunAC16(f,h,i,a) = (⋃j<i. recfunAC16(f,h,j,a))"
```
⟨*proof*⟩

lemma *transrec2_mono_lemma [rule_format]:*
```
    "[| !!g r. r ⊆ B(g,r);  Ord(i) |]
     ==> j<i --> transrec2(j, 0, B) ⊆ transrec2(i, 0, B)"
```
⟨*proof*⟩

lemma *transrec2_mono:*
```
    "[| !!g r. r ⊆ B(g,r); j≤i |]
     ==> transrec2(j, 0, B) ⊆ transrec2(i, 0, B)"
```
⟨*proof*⟩

lemma *recfunAC16_mono:*
```
       "i≤j ==> recfunAC16(f, g, i, a) ⊆ recfunAC16(f, g, j, a)"
```
⟨*proof*⟩

lemma *lemma3_1:*
```
   "[| ∀y<x. ∀z<a. z<y | (∃Y ∈ F(y). f(z)<=Y) --> (∃! Y. Y ∈ F(y)
& f(z)<=Y);
       ∀i j. i≤j --> F(i) ⊆ F(j); j≤i; i<x; z<a;
       V ∈ F(i); f(z)<=V; W ∈ F(j); f(z)<=W |]
```

```
      ==> V = W"
```
⟨*proof*⟩

lemma *lemma3:*
```
    "[| ∀y<x. ∀z<a. z<y | (∃Y ∈ F(y). f(z)<=Y) --> (∃! Y. Y ∈ F(y)
& f(z)<=Y);
        ∀i j. i≤j --> F(i) ⊆ F(j); i<x; j<x; z<a;
        V ∈ F(i); f(z)<=V; W ∈ F(j); f(z)<=W |]
      ==> V = W"
```
⟨*proof*⟩

lemma *lemma4:*
```
     "[| ∀y<x. F(y) ⊆ X &
                (∀x<a. x < y | (∃Y ∈ F(y). h(x) ⊆ Y) -->
                 (∃! Y. Y ∈ F(y) & h(x) ⊆ Y));
         x < a |]
      ==> ∀y<x. ∀z<a. z < y | (∃Y ∈ F(y). h(z) ⊆ Y) -->
                       (∃! Y. Y ∈ F(y) & h(z) ⊆ Y)"
```
⟨*proof*⟩

lemma *lemma5:*
```
     "[| ∀y<x. F(y) ⊆ X &
               (∀x<a. x < y | (∃Y ∈ F(y). h(x) ⊆ Y) -->
                (∃! Y. Y ∈ F(y) & h(x) ⊆ Y));
         x < a; Limit(x); ∀i j. i≤j --> F(i) ⊆ F(j) |]
      ==> (⋃x<x. F(x)) ⊆ X &
          (∀xa<a. xa < x | (∃x ∈ ⋃x<x. F(x). h(xa) ⊆ x)
                --> (∃! Y. Y ∈ (⋃x<x. F(x)) & h(xa) ⊆ Y))"
```
⟨*proof*⟩

lemma *dbl_Diff_eqpoll_Card:*
```
      "[| A≈a; Card(a); ~Finite(a); B≺a; C≺a |] ==> A - B - C≈a"
```
⟨*proof*⟩

lemma *Finite_lesspoll_infinite_Ord:*
"[| Finite(X); ~Finite(a); Ord(a) |] ==> X≺a"
⟨*proof*⟩

lemma *Union_lesspoll:*
"[| ∀x ∈ X. x lepoll n & x ⊆ T; well_ord(T, R); X lepoll b;
b<a; ~Finite(a); Card(a); n ∈ nat |]
==> Union(X)≺a"
⟨*proof*⟩

lemma *Un_sing_eq_cons: "A Un {a} = cons(a, A)"*
⟨*proof*⟩

lemma *Un_lepoll_succ: "A lepoll B ==> A Un {a} lepoll succ(B)"*
⟨*proof*⟩

lemma *Diff_UN_succ_empty: "Ord(a) ==> F(a) - (⋃b<succ(a). F(b)) = 0"*
⟨*proof*⟩

lemma *Diff_UN_succ_subset: "Ord(a) ==> F(a) Un X - (⋃b<succ(a). F(b)) ⊆ X"*
⟨*proof*⟩

lemma *recfunAC16_Diff_lepoll_1:*
"Ord(x)
==> recfunAC16(f, g, x, a) - (⋃i<x. recfunAC16(f, g, i, a)) lepoll 1"
⟨*proof*⟩

lemma *in_Least_Diff:*
"[| z ∈ F(x); Ord(x) |]
==> z ∈ F(LEAST i. z ∈ F(i)) - (⋃j<(LEAST i. z ∈ F(i)). F(j))"
⟨*proof*⟩

lemma *Least_eq_imp_ex:*
"[| (LEAST i. w ∈ F(i)) = (LEAST i. z ∈ F(i));
w ∈ (⋃i<a. F(i)); z ∈ (⋃i<a. F(i)) |]
==> ∃b<a. w ∈ (F(b) - (⋃c<b. F(c))) & z ∈ (F(b) - (⋃c<b. F(c)))"
⟨*proof*⟩

lemma *two_in_lepoll_1: "[| A lepoll 1; a ∈ A; b ∈ A |] ==> a=b"*
⟨*proof*⟩

lemma *UN_lepoll_index:*

```
"[| ∀i<a. F(i)-(⋃j<i. F(j)) lepoll 1; Limit(a) |]
 ==> (⋃x<a. F(x)) lepoll a"
```

⟨*proof*⟩

lemma *recfunAC16_lepoll_index:* `"Ord(y) ==> recfunAC16(f, h, y, a) lepoll y"`

⟨*proof*⟩

lemma *Union_recfunAC16_lesspoll:*

```
"[| recfunAC16(f,g,y,a) ⊆ {X ∈ Pow(A). X≈n};
    A≈a;  y<a;  ~Finite(a);  Card(a);  n ∈ nat |]
 ==> Union(recfunAC16(f,g,y,a))≺a"
```

⟨*proof*⟩

lemma *dbl_Diff_eqpoll:*

```
"[| recfunAC16(f, h, y, a) ⊆ {X ∈ Pow(A) . X≈succ(k #+ m)};
    Card(a); ~ Finite(a); A≈a;
    k ∈ nat;  y<a;
    h ∈ bij(a, {Y ∈ Pow(A). Y≈succ(k)}) |]
 ==> A - Union(recfunAC16(f, h, y, a)) - h`y≈a"
```

⟨*proof*⟩

lemmas *disj_Un_eqpoll_nat_sum =*

```
eqpoll_trans [THEN eqpoll_trans,
              OF disj_Un_eqpoll_sum sum_eqpoll_cong nat_sum_eqpoll_sum,
              standard]
```

lemma *Un_in_Collect:*

```
"[| x ∈ Pow(A - B - h`i); x≈m;
    h ∈ bij(a, {x ∈ Pow(A) . x≈k}); i<a; k ∈ nat; m ∈ nat |]
 ==> h ` i Un x ∈ {x ∈ Pow(A) . x≈k #+ m}"
```

⟨*proof*⟩

lemma *lemma6:*

```
"[| ∀y<succ(j). F(y)<=X & (∀x<a. x<y | P(x,y) --> Q(x,y)); succ(j)<a |]
 ==> F(j)<=X & (∀x<a. x<j | P(x,j) --> Q(x,j))"
```

⟨proof⟩

lemma *lemma7:*

```
"[| ∀x<a. x<j | P(x,j) --> Q(x,j);  succ(j)<a |]
 ==> P(j,j) --> (∀x<a. x≤j | P(x,j) --> Q(x,j))"
```

⟨proof⟩

lemma *ex_subset_eqpoll:*

```
"[| A≈a; ~ Finite(a); Ord(a); m ∈ nat |] ==> ∃X ∈ Pow(A). X≈m"
```

⟨proof⟩

lemma *subset_Un_disjoint:* `"[| A ⊆ B Un C; A Int C = 0 |] ==> A ⊆ B"`

⟨proof⟩

lemma *Int_empty:*

```
"[| X ∈ Pow(A - Union(B) -C); T ∈ B; F ⊆ T |] ==> F Int X = 0"
```

⟨proof⟩

lemma *subset_imp_eq_lemma:*

```
"m ∈ nat ==> ∀A B. A ⊆ B & m lepoll A & B lepoll m --> A=B"
```

⟨proof⟩

lemma *subset_imp_eq:* `"[| A ⊆ B; m lepoll A; B lepoll m; m ∈ nat |] ==> A=B"`

⟨proof⟩

lemma *bij_imp_arg_eq:*

```
"[| f ∈ bij(a, {Y ∈ X. Y≈succ(k)}); k ∈ nat; f`b ⊆ f`y; b<a; y<a |]
 ==> b=y"
```

⟨proof⟩

lemma *ex_next_set:*

```
"[| recfunAC16(f, h, y, a) ⊆ {X ∈ Pow(A) . X≈succ(k #+ m)};
    Card(a); ~ Finite(a); A≈a;
    k ∈ nat; m ∈ nat; y<a;
    h ∈ bij(a, {Y ∈ Pow(A). Y≈succ(k)});
    ~ (∃Y ∈ recfunAC16(f, h, y, a). h`y ⊆ Y) |]
 ==> ∃X ∈ {Y ∈ Pow(A). Y≈succ(k #+ m)}. h`y ⊆ X &
          (∀b<a. h`b ⊆ X -->
          (∀T ∈ recfunAC16(f, h, y, a). ~ h`b ⊆ T))"
```
⟨*proof*⟩

lemma *ex_next_Ord:*
```
"[| recfunAC16(f, h, y, a) ⊆ {X ∈ Pow(A) . X≈succ(k #+ m)};
    Card(a); ~ Finite(a); A≈a;
    k ∈ nat; m ∈ nat; y<a;
    h ∈ bij(a, {Y ∈ Pow(A). Y≈succ(k)});
    f ∈ bij(a, {Y ∈ Pow(A). Y≈succ(k #+ m)});
    ~ (∃Y ∈ recfunAC16(f, h, y, a). h`y ⊆ Y) |]
 ==> ∃c<a. h`y ⊆ f`c &
          (∀b<a. h`b ⊆ f`c -->
          (∀T ∈ recfunAC16(f, h, y, a). ~ h`b ⊆ T))"
```
⟨*proof*⟩

lemma *lemma8:*
```
"[| ∀x<a. x<j | (∃xa ∈ F(j). P(x, xa))
    --> (∃! Y. Y ∈ F(j) & P(x, Y)); F(j) ⊆ X;
    L ∈ X; P(j, L) & (∀x<a. P(x, L) --> (∀xa ∈ F(j). ~P(x, xa)))
|]
 ==> F(j) Un {L} ⊆ X &
     (∀x<a. x≤j | (∃xa ∈ (F(j) Un {L}). P(x, xa)) -->
            (∃! Y. Y ∈ (F(j) Un {L}) & P(x, Y)))"
```
⟨*proof*⟩

lemma *main_induct:*
```
"[| b < a; f ∈ bij(a, {Y ∈ Pow(A) . Y≈succ(k #+ m)});
```

```
        h ∈ bij(a, {Y ∈ Pow(A) . Y≈succ(k)});
        ~Finite(a); Card(a); A≈a; k ∈ nat; m ∈ nat |]
      ==> recfunAC16(f, h, b, a) ⊆ {X ∈ Pow(A) . X≈succ(k #+ m)} &
          (∀x<a. x < b | (∃Y ∈ recfunAC16(f, h, b, a). h ` x ⊆ Y) -->

          (∃! Y. Y ∈ recfunAC16(f, h, b, a) & h ` x ⊆ Y))"
```
⟨*proof*⟩

lemma `lemma_simp_induct:`

```
     "[| ∀b. b<a --> F(b) ⊆ S & (∀x<a. (x<b | (∃Y ∈ F(b). f`x ⊆ Y))

                                  --> (∃! Y. Y ∈ F(b) & f`x ⊆ Y));
        f ∈ a->f``(a); Limit(a);
        ∀i j. i≤j --> F(i) ⊆ F(j) |]
       ==> (⋃j<a. F(j)) ⊆ S &
           (∀x ∈ f``a. ∃! Y. Y ∈ (⋃j<a. F(j)) & x ⊆ Y)"
```
⟨*proof*⟩

theorem `WO2_AC16: "[| WO2; 0<m; k ∈ nat; m ∈ nat |] ==> AC16(k #+ m,k)"`
⟨*proof*⟩

end

theory `AC16_WO4` **imports** `AC16_lemmas` **begin**

lemma `lemma1:`

```
     "[| Finite(A); 0<m; m ∈ nat |]
      ==> ∃a f. Ord(a) & domain(f) = a &
                (⋃b<a. f`b) = A & (∀b<a. f`b ≲ m)"
```
⟨*proof*⟩

lemmas *well_ord_paired = paired_bij [THEN bij_is_inj, THEN well_ord_rvimage]*

lemma *lepoll_trans1: "[| A $\lesssim$ B; ~ A $\lesssim$ C |] ==> ~ B $\lesssim$ C"*
⟨*proof*⟩

lemmas *lepoll_paired = paired_eqpoll [THEN eqpoll_sym, THEN eqpoll_imp_lepoll]*

lemma *lemma2: "$\exists$y R. well_ord(y,R) & x Int y = 0 & ~y $\lesssim$ z & ~Finite(y)"*
⟨*proof*⟩

lemma *infinite_Un: "~Finite(B) ==> ~Finite(A Un B)"*
⟨*proof*⟩

lemma *succ_not_lepoll_lemma:*
"[| ~($\exists$x $\in$ A. f`x=y); f $\in$ inj(A, B); y $\in$ B |]
==> (λa $\in$ succ(A). if(a=A, y, f`a)) $\in$ inj(succ(A), B)"
⟨*proof*⟩

lemma *succ_not_lepoll_imp_eqpoll: "[| ~A $\approx$ B; A $\lesssim$ B |] ==> succ(A) $\lesssim$ B"*
⟨*proof*⟩

lemmas *ordertype_eqpoll =*
ordermap_bij [THEN exI [THEN eqpoll_def [THEN def_imp_iff, THEN iffD2]]]

lemma *cons_cons_subset:*
"[| a ⊆ y; b ∈ y-a; u ∈ x |] ==> cons(b, cons(u, a)) ∈ Pow(x Un y)"
⟨*proof*⟩

lemma *cons_cons_eqpoll:*
"[| a ≈ k; a ⊆ y; b ∈ y-a; u ∈ x; x Int y = 0 |]
==> cons(b, cons(u, a)) ≈ succ(succ(k))"
⟨*proof*⟩

lemma *set_eq_cons:*
"[| succ(k) ≈ A; k ≈ B; B ⊆ A; a ∈ A-B; k ∈ nat |] ==> A = cons(a, B)"
⟨*proof*⟩

lemma *cons_eqE:* "[| cons(x,a) = cons(y,a); x ∉ a |] ==> x = y "
⟨*proof*⟩

lemma *eq_imp_Int_eq:* "A = B ==> A Int C = B Int C"
⟨*proof*⟩

lemma *eqpoll_sum_imp_Diff_lepoll_lemma [rule_format]:*
"[| k ∈ nat; m ∈ nat |]
==> ∀A B. A ≈ k #+ m & k ≲ B & B ⊆ A --> A-B ≲ m"
⟨*proof*⟩

lemma *eqpoll_sum_imp_Diff_lepoll:*
"[| A ≈ succ(k #+ m); B ⊆ A; succ(k) ≲ B; k ∈ nat; m ∈ nat |]

==> A-B ≲ m"
⟨*proof*⟩

lemma *eqpoll_sum_imp_Diff_eqpoll_lemma [rule_format]:*
"[| k ∈ nat; m ∈ nat |]
==> ∀A B. A ≈ k #+ m & k ≈ B & B ⊆ A --> A-B ≈ m"
⟨*proof*⟩

lemma *eqpoll_sum_imp_Diff_eqpoll:*
"[| A ≈ succ(k #+ m); B ⊆ A; succ(k) ≈ B; k ∈ nat; m ∈ nat |]

```
    ==> A-B ≈ m"
⟨proof⟩
```

lemma *subsets_lepoll_0_eq_unit: "{x ∈ Pow(X). x ≲ 0} = {0}"*
⟨*proof*⟩

lemma *subsets_lepoll_succ:*

```
    "n ∈ nat ==> {z ∈ Pow(y). z ≲ succ(n)} =
                 {z ∈ Pow(y). z ≲ n} Un {z ∈ Pow(y). z ≈ succ(n)}"
```

⟨*proof*⟩

lemma *Int_empty:*

```
    "n ∈ nat ==> {z ∈ Pow(y). z ≲ n} Int {z ∈ Pow(y). z ≈ succ(n)}
= 0"
```

⟨*proof*⟩

locale *(***open***) AC16 =*
fixes *x* **and** *y* **and** *k* **and** *l* **and** *m* **and** *t_n* **and** *R* **and** *MM* **and** *LL* **and** *GG* **and** *s*

```
  defines k_def:     "k   == succ(l)"
      and MM_def:    "MM  == {v ∈ t_n. succ(k) ≲ v Int y}"
      and LL_def:    "LL  == {v Int y. v ∈ MM}"
      and GG_def:    "GG  == λv ∈ LL. (THE w. w ∈ MM & v ⊆ w) - v"
      and s_def:     "s(u) == {v ∈ t_n. u ∈ v & k ≲ v Int y}"
  assumes all_ex:    "∀z ∈ {z ∈ Pow(x Un y) . z ≈ succ(k)}.
                       ∃! w. w ∈ t_n & z ⊆ w "
    and disjoint[iff]:  "x Int y = 0"
    and "includes":  "t_n ⊆ {v ∈ Pow(x Un y). v ≈ succ(k #+ m)}"
    and WO_R[iff]:      "well_ord(y,R)"
    and lnat[iff]:      "l ∈ nat"
    and mnat[iff]:      "m ∈ nat"
    and mpos[iff]:      "0<m"
    and Infinite[iff]:  "~ Finite(y)"
    and noLepoll:  "~ y ≲ {v ∈ Pow(x). v ≈ m}"
```

lemma *(***in*** AC16) knat [iff]: "k ∈ nat"*
⟨*proof*⟩

lemma *(in AC16) Diff_Finite_eqpoll:* "[| l ≈ a; a ⊆ y |] ==> y - a ≈ y"
⟨*proof*⟩

lemma *(in AC16) s_subset:* "s(u) ⊆ t_n"
⟨*proof*⟩

lemma *(in AC16) sI:*

```
"[| w ∈ t_n; cons(b,cons(u,a)) ⊆ w; a ⊆ y; b ∈ y-a; l ≈ a |]

 ==> w ∈ s(u)"
```

⟨*proof*⟩

lemma *(in AC16) in_s_imp_u_in:* "v ∈ s(u) ==> u ∈ v"
⟨*proof*⟩

lemma *(in AC16) ex1_superset_a:*

```
"[| l ≈ a;  a ⊆ y;  b ∈ y - a;  u ∈ x |]
 ==> ∃! c. c ∈ s(u) & a ⊆ c & b ∈ c"
```

⟨*proof*⟩

lemma *(in AC16) the_eq_cons:*

```
"[| ∀v ∈ s(u). succ(l) ≈ v Int y;
    l ≈ a;  a ⊆ y;  b ∈ y - a;  u ∈ x |]
 ==> (THE c. c ∈ s(u) & a ⊆ c & b ∈ c) Int y = cons(b, a)"
```

⟨*proof*⟩

lemma *(in AC16) y_lepoll_subset_s:*

```
"[| ∀v ∈ s(u). succ(l) ≈ v Int y;
    l ≈ a;  a ⊆ y;  u ∈ x |]
 ==> y ≲ {v ∈ s(u). a ⊆ v}"
```

⟨*proof*⟩

lemma *(in AC16) x_imp_not_y [dest]:* "a ∈ x ==> a ∉ y"
⟨*proof*⟩

lemma *(in AC16) w_Int_eq_w_Diff:*

```
"w ⊆ x Un y ==> w Int (x - {u}) = w - cons(u, w Int y)"
```

⟨*proof*⟩

lemma *(in AC16) w_Int_eqpoll_m:*

```
"[| w ∈ {v ∈ s(u). a ⊆ v};
    l ≈ a;  u ∈ x;
    ∀v ∈ s(u). succ(l) ≈ v Int y |]
 ==> w Int (x - {u}) ≈ m"
```

⟨*proof*⟩

lemma *(in AC16) eqpoll_m_not_empty:* `"a ≈ m ==> a ≠ 0"`

⟨*proof*⟩

lemma *(in AC16) cons_cons_in:*

```
"[| z ∈ xa Int (x - {u}); l ≈ a; a ⊆ y; u ∈ x |]
 ==> ∃! w. w ∈ t_n & cons(z, cons(u, a)) ⊆ w"
```

⟨*proof*⟩

lemma *(in AC16) subset_s_lepoll_w:*

```
"[| ∀v ∈ s(u). succ(l) ≈ v Int y; a ⊆ y; l ≈ a; u ∈ x |]
 ==> {v ∈ s(u). a ⊆ v} ≲ {v ∈ Pow(x). v ≈ m}"
```

⟨*proof*⟩

lemma *(in AC16) well_ord_subsets_eqpoll_n:*

```
"n ∈ nat ==> ∃S. well_ord({z ∈ Pow(y) . z ≈ succ(n)}, S)"
```

⟨*proof*⟩

lemma *(in AC16) well_ord_subsets_lepoll_n:*

```
"n ∈ nat ==> ∃R. well_ord({z ∈ Pow(y). z ≲ n}, R)"
```

⟨*proof*⟩

lemma *(in AC16) LL_subset:* `"LL ⊆ {z ∈ Pow(y). z ≲ succ(k #+ m)}"`

⟨*proof*⟩

lemma *(in AC16) well_ord_LL:* `"∃S. well_ord(LL,S)"`

⟨*proof*⟩

lemma *(in AC16) unique_superset_in_MM:*
"v ∈ LL ==> ∃! w. w ∈ MM & v ⊆ w"
⟨*proof*⟩

lemma *(in AC16) Int_in_LL: "w ∈ MM ==> w Int y ∈ LL"*
⟨*proof*⟩

lemma *(in AC16) in_LL_eq_Int:*
"v ∈ LL ==> v = (THE x. x ∈ MM & v ⊆ x) Int y"
⟨*proof*⟩

lemma *(in AC16) unique_superset1: "a ∈ LL ⟹ (THE x. x ∈ MM ∧ a ⊆ x) ∈ MM"*
⟨*proof*⟩

lemma *(in AC16) the_in_MM_subset:*
"v ∈ LL ==> (THE x. x ∈ MM & v ⊆ x) ⊆ x Un y"
⟨*proof*⟩

lemma *(in AC16) GG_subset: "v ∈ LL ==> GG ‘ v ⊆ x"*
⟨*proof*⟩

lemma *(in AC16) nat_lepoll_ordertype: "nat ≲ ordertype(y, R)"*
⟨*proof*⟩

lemma *(in AC16) ex_subset_eqpoll_n: "n ∈ nat ==> ∃z. z ⊆ y & n ≈ z"*
⟨*proof*⟩

lemma *(in AC16) exists_proper_in_s: "u ∈ x ==> ∃v ∈ s(u). succ(k) ≲ v Int y"*
⟨*proof*⟩

lemma *(in AC16) exists_in_MM: "u ∈ x ==> ∃w ∈ MM. u ∈ w"*
⟨*proof*⟩

lemma *(in AC16) exists_in_LL: "u ∈ x ==> ∃w ∈ LL. u ∈ GG‘w"*
⟨*proof*⟩

lemma *(in AC16) OUN_eq_x:* "well_ord(LL,S) ==>
($\bigcup$b<ordertype(LL,S). GG ` (converse(ordermap(LL,S)) ` b)) = x"
⟨*proof*⟩

lemma *(in AC16) in_MM_eqpoll_n:* "w ∈ MM ==> w ≈ succ(k #+ m)"
⟨*proof*⟩

lemma *(in AC16) in_LL_eqpoll_n:* "w ∈ LL ==> succ(k) $\lesssim$ w"
⟨*proof*⟩

lemma *(in AC16) in_LL:* "w ∈ LL ==> w ⊆ (THE x. x ∈ MM ∧ w ⊆ x)"
⟨*proof*⟩

lemma *(in AC16) all_in_lepoll_m:*
"well_ord(LL,S) ==>
∀b<ordertype(LL,S). GG ` (converse(ordermap(LL,S)) ` b) $\lesssim$ m"
⟨*proof*⟩

lemma *(in AC16) conclusion:*
"∃a f. Ord(a) & domain(f) = a & ($\bigcup$b<a. f ` b) = x & (∀b<a. f ` b $\lesssim$ m)"
⟨*proof*⟩

theorem *AC16_WO4:*
"[| AC16(k #+ m, k); 0 < k; 0 < m; k ∈ nat; m ∈ nat |] ==> WO4(m)"
⟨*proof*⟩

end

theory *AC17_AC1* **imports** *HH* **begin**

lemma *AC0_AC1_lemma:* "[| f:(Π X ∈ A. X); D ⊆ A |] ==> ∃g. g:(Π X ∈ D. X)"
⟨*proof*⟩

lemma *AC0_AC1: "AC0 ==> AC1"*
⟨*proof*⟩

lemma *AC1_AC0: "AC1 ==> AC0"*
⟨*proof*⟩

lemma *AC1_AC17_lemma: "f ∈ (Π X ∈ Pow(A) - {0}. X) ==> f ∈ (Pow(A) - {0} -> A)"*
⟨*proof*⟩

lemma *AC1_AC17: "AC1 ==> AC17"*
⟨*proof*⟩

lemma *UN_eq_imp_well_ord:*
```
"[| x - (⋃j ∈ LEAST i. HH(λX ∈ Pow(x)-{0}. {f‘X}, x, i) = {x}.

      HH(λX ∈ Pow(x)-{0}. {f‘X}, x, j)) = 0;
      f ∈ Pow(x)-{0} -> x |]
      ==> ∃r. well_ord(x,r)"
```
⟨*proof*⟩

lemma *not_AC1_imp_ex:*
```
"~AC1 ==> ∃A. ∀f ∈ Pow(A)-{0} -> A. ∃u ∈ Pow(A)-{0}. f‘u ∉ u"
```
⟨*proof*⟩

lemma *AC17_AC1_aux1:*
```
"[| ∀f ∈ Pow(x) - {0} -> x. ∃u ∈ Pow(x) - {0}. f‘u∉u;
    ∃f ∈ Pow(x)-{0}->x.
       x - (⋃a ∈ (LEAST i. HH(λX ∈ Pow(x)-{0}. {f‘X},x,i)={x}).

       HH(λX ∈ Pow(x)-{0}. {f‘X},x,a)) = 0 |]
    ==> P"
```
⟨*proof*⟩

lemma *AC17_AC1_aux2:*

```
"~ (∃f ∈ Pow(x)-{0}->x. x - F(f) = 0)
 ==> (λf ∈ Pow(x)-{0}->x . x - F(f))
      ∈ (Pow(x) -{0} -> x) -> Pow(x) - {0}"
```

⟨*proof*⟩

lemma *AC17_AC1_aux3:*

```
"[| f`Z ∈ Z; Z ∈ Pow(x)-{0} |]
 ==> (λX ∈ Pow(x)-{0}. {f`X})`Z ∈ Pow(Z)-{0}"
```

⟨*proof*⟩

lemma *AC17_AC1_aux4:*

```
"∃f ∈ F. f`((λf ∈ F. Q(f))`f) ∈ (λf ∈ F. Q(f))`f
 ==> ∃f ∈ F. f`Q(f) ∈ Q(f)"
```

⟨*proof*⟩

lemma *AC17_AC1: "AC17 ==> AC1"*

⟨*proof*⟩

lemma *AC1_AC2_aux1:*

```
"[| f:(Π X ∈ A. X);  B ∈ A;  0∉A |] ==> {f`B} ⊆ B Int {f`C. C
∈ A}"
```

⟨*proof*⟩

lemma *AC1_AC2_aux2:*

```
"[| pairwise_disjoint(A); B ∈ A; C ∈ A; D ∈ B; D ∈ C |] ==>
f`B = f`C"
```

⟨*proof*⟩

lemma *AC1_AC2: "AC1 ==> AC2"*

⟨*proof*⟩

lemma *AC2_AC1_aux1: "0∉A ==> 0 ∉ {B*{B}. B ∈ A}"*

⟨*proof*⟩

lemma *AC2_AC1_aux2:* *"[| X*{X} Int C = {y}; X ∈ A |]*

```
==> (THE y. X*{X} Int C = {y}): X*A"
```

⟨*proof*⟩

lemma *AC2_AC1_aux3:*
  "$\forall$D $\in$ {E{E}. E $\in$ A}. $\exists$y. D Int C = {y}*
  ==> (λx $\in$ A. fst(THE z. (x{x} Int C = {z}))) $\in$ (Π X $\in$ A. X)"*

⟨*proof*⟩

lemma *AC2_AC1: "AC2 ==> AC1"*

⟨*proof*⟩

lemma *empty_notin_images: "0 $\notin$ {R``{x}. x $\in$ domain(R)}"*

⟨*proof*⟩

lemma *AC1_AC4: "AC1 ==> AC4"*

⟨*proof*⟩

lemma *AC4_AC3_aux1: "f $\in$ A->B ==> ($\bigcup$z $\in$ A. {z}*f`z) $\subseteq$ A*Union(B)"*

⟨*proof*⟩

lemma *AC4_AC3_aux2: "domain($\bigcup$z $\in$ A. {z}*f(z)) = {a $\in$ A. f(a)$\neq$0}"*

⟨*proof*⟩

lemma *AC4_AC3_aux3: "x $\in$ A ==> ($\bigcup$z $\in$ A. {z}*f(z))``{x} = f(x)"*

⟨*proof*⟩

lemma *AC4_AC3: "AC4 ==> AC3"*

⟨*proof*⟩

lemma *AC3_AC1_lemma:*
  "b$\notin$A ==> (Π x $\in$ {a $\in$ A. id(A)`a$\neq$b}. id(A)`x) = (Π x $\in$ A. x)"

⟨*proof*⟩

lemma *AC3_AC1: "AC3 ==> AC1"*

⟨*proof*⟩

lemma `AC4_AC5: "AC4 ==> AC5"`
⟨*proof*⟩

lemma `AC5_AC4_aux1: "R ⊆ A*B ==> (λx ∈ R. fst(x)) ∈ R -> A"`
⟨*proof*⟩

lemma `AC5_AC4_aux2: "R ⊆ A*B ==> range(λx ∈ R. fst(x)) = domain(R)"`
⟨*proof*⟩

lemma `AC5_AC4_aux3: "[| ∃f ∈ A->C. P(f,domain(f)); A=B |] ==> ∃f ∈ B->C. P(f,B)"`
⟨*proof*⟩

lemma `AC5_AC4_aux4: "[| R ⊆ A*B; g ∈ C->R; ∀x ∈ C. (λz ∈ R. fst(z))`‘`(g‘x) = x |]`
`==> (λx ∈ C. snd(g‘x)): (Π x ∈ C. R‘‘{x})"`
⟨*proof*⟩

lemma `AC5_AC4: "AC5 ==> AC4"`
⟨*proof*⟩

lemma `AC1_iff_AC6: "AC1 <-> AC6"`
⟨*proof*⟩

end

theory `AC18_AC19` **imports** `AC_Equiv` **begin**

definition
  `uu    :: "i => i"` **where**
    `"uu(a) == {c Un {0}. c ∈ a}"`

lemma *PROD_subsets:*
    "[| $f \in (\Pi\ b \in \{P(a).\ a \in A\}.\ b)$; $\forall a \in A.\ P(a) <= Q(a)$ |]
    ==> $(\lambda a \in A.\ f`P(a)) \in (\Pi\ a \in A.\ Q(a))$"
⟨*proof*⟩

lemma *lemma_AC18:*
    "[| $\forall A.\ 0 \notin A$ --> $(\exists f.\ f \in (\Pi\ X \in A.\ X))$; $A \neq 0$ |]
    ==> $(\bigcap a \in A.\ \bigcup b \in B(a).\ X(a, b)) \subseteq$
        $(\bigcup f \in \Pi\ a \in A.\ B(a).\ \bigcap a \in A.\ X(a, f`a))$"
⟨*proof*⟩

lemma *AC1_AC18:* "AC1 ==> PROP AC18"
⟨*proof*⟩

theorem *(***in** *AC18) AC19*
⟨*proof*⟩

lemma *RepRep_conj:*
    "[| $A \neq 0$; $0 \notin A$ |] ==> $\{uu(a).\ a \in A\} \neq 0$ & $0 \notin \{uu(a).\ a \in A\}$"
⟨*proof*⟩

lemma *lemma1_1:* "[|$c \in a$; $x = c$ Un $\{0\}$; $x \notin a$ |] ==> $x - \{0\} \in a$"
⟨*proof*⟩

lemma *lemma1_2:*
    "[| $f`(uu(a)) \notin a$; $f \in (\Pi\ B \in \{uu(a).\ a \in A\}.\ B)$; $a \in A$ |]

    ==> $f`(uu(a))-\{0\} \in a$"
⟨*proof*⟩

lemma *lemma1:* "$\exists f.\ f \in (\Pi\ B \in \{uu(a).\ a \in A\}.\ B)$ ==> $\exists f.\ f \in (\Pi\ B \in A.\ B)$"
⟨*proof*⟩

lemma *lemma2_1:* "$a \neq 0$ ==> $0 \in (\bigcup b \in uu(a).\ b)$"
⟨*proof*⟩

lemma *lemma2:* "[| A≠0; 0∉A |] ==> (⋂x ∈ {uu(a). a ∈ A}. ⋃b ∈ x. b) ≠ 0"
⟨*proof*⟩

lemma *AC19_AC1:* "AC19 ==> AC1"
⟨*proof*⟩

end

theory *DC* **imports** *AC_Equiv Hartog Cardinal_aux* **begin**

lemma *RepFun_lepoll:* "Ord(a) ==> {P(b). b ∈ a} ≲ a"
⟨*proof*⟩

Trivial in the presence of AC, but here we need a wellordering of X

lemma *image_Ord_lepoll:* "[| f ∈ X->Y; Ord(X) |] ==> f``X ≲ X"
⟨*proof*⟩

lemma *range_subset_domain:*
```
      "[| R ⊆ X*X;   !!g. g ∈ X ==> ∃u. <g,u> ∈ R |]
       ==> range(R) ⊆ domain(R)"
```
⟨*proof*⟩

lemma *cons_fun_type:* "g ∈ n->X ==> cons(<n,x>, g) ∈ succ(n) -> cons(x, X)"
⟨*proof*⟩

lemma *cons_fun_type2:*
```
     "[| g ∈ n->X; x ∈ X |] ==> cons(<n,x>, g) ∈ succ(n) -> X"
```
⟨*proof*⟩

lemma *cons_image_n:* "n ∈ nat ==> cons(<n,x>, g)``n = g``n"
⟨*proof*⟩

lemma *cons_val_n:* "g ∈ n->X ==> cons(<n,x>, g)`n = x"
⟨*proof*⟩

lemma *cons_image_k:* "k ∈ n ==> cons(<n,x>, g)``k = g``k"
⟨*proof*⟩

lemma *cons_val_k:* "[| k ∈ n; g ∈ n->X |] ==> cons(<n,x>, g)`k = g`k"
⟨*proof*⟩

lemma *domain_cons_eq_succ:* "domain(f)=x ==> domain(cons(<x,y>, f)) = succ(x)"
⟨*proof*⟩

```
lemma restrict_cons_eq: "g ∈ n->X ==> restrict(cons(<n,x>, g), n) =
g"
⟨proof⟩

lemma succ_in_succ: "[| Ord(k); i ∈ k |] ==> succ(i) ∈ succ(k)"
⟨proof⟩

lemma restrict_eq_imp_val_eq:
      "[|restrict(f, domain(g)) = g; x ∈ domain(g)|]
       ==> f`x = g`x"
⟨proof⟩

lemma domain_eq_imp_fun_type: "[| domain(f)=A; f ∈ B->C |] ==> f ∈ A->C"
⟨proof⟩

lemma ex_in_domain: "[| R ⊆ A * B; R ≠ 0 |] ==> ∃x. x ∈ domain(R)"
⟨proof⟩


definition
  DC  :: "i => o"  where
    "DC(a) == ∀X R. R ⊆ Pow(X)*X  &
                    (∀Y ∈ Pow(X). Y ≺ a --> (∃x ∈ X. <Y,x> ∈ R))
                    --> (∃f ∈ a->X. ∀b<a. <f``b,f`b> ∈ R)"

definition
  DC0 :: o  where
    "DC0 == ∀A B R. R ⊆ A*B & R≠0 & range(R) ⊆ domain(R)
                    --> (∃f ∈ nat->domain(R). ∀n ∈ nat. <f`n,f`succ(n)>:R)"

definition
  ff  :: "[i, i, i, i] => i"  where
    "ff(b, X, Q, R) ==
           transrec(b, %c r. THE x. first(x, {x ∈ X. <r``c, x> ∈ R},
Q))"


locale (open) DC0_imp =
  fixes XX and RR and X and R

  assumes all_ex: "∀Y ∈ Pow(X). Y ≺ nat --> (∃x ∈ X. <Y, x> ∈ R)"

  defines XX_def: "XX == (⋃n ∈ nat. {f ∈ n->X. ∀k ∈ n. <f``k, f`k>
∈ R})"
     and RR_def:  "RR == {<z1,z2>:XX*XX. domain(z2)=succ(domain(z1))
                                  & restrict(z2, domain(z1)) = z1}"
```

```
lemma (in DC0_imp) lemma1_1: "RR ⊆ XX*XX"
⟨proof⟩
```

```
lemma (in DC0_imp) lemma1_2: "RR ≠ 0"
⟨proof⟩
```

```
lemma (in DC0_imp) lemma1_3: "range(RR) ⊆ domain(RR)"
⟨proof⟩
```

```
lemma (in DC0_imp) lemma2:
     "[| ∀n ∈ nat. <f‘n, f‘succ(n)> ∈ RR;  f ∈ nat -> XX;  n ∈ nat |]

      ==> ∃k ∈ nat. f‘succ(n) ∈ k -> X & n ∈ k
                  & <f‘succ(n)‘‘n, f‘succ(n)‘n> ∈ R"
⟨proof⟩
```

```
lemma (in DC0_imp) lemma3_1:
     "[| ∀n ∈ nat. <f‘n, f‘succ(n)> ∈ RR;  f ∈ nat -> XX;  m ∈ nat |]

      ==>  {f‘succ(x)‘x. x ∈ m} = {f‘succ(m)‘x. x ∈ m}"
⟨proof⟩
```

```
lemma (in DC0_imp) lemma3:
```

"[| $\forall$n $\in$ nat. <f`n, f`succ(n)> $\in$ RR; f $\in$ nat -> XX; m $\in$ nat |]

==> (λx $\in$ nat. f`succ(x)`x) `` m = f`succ(m)``m"
$\langle$*proof*$\rangle$

theorem *DC0_imp_DC_nat: "DC0 ==> DC(nat)"*
$\langle$*proof*$\rangle$

lemma *singleton_in_funs:*
"x $\in$ X ==> {<0,x>} $\in$
($\bigcup$n $\in$ nat. {f $\in$ succ(n)->X. $\forall$k $\in$ n. <f`k, f`succ(k)> $\in$
R})"
$\langle$*proof*$\rangle$

locale **(open)** *imp_DC0* =
fixes *XX* **and** *RR* **and** *x* **and** *R* **and** *f* **and** *allRR*
defines *XX_def: "XX == ($\bigcup$n $\in$ nat.
{f $\in$ succ(n)->domain(R). $\forall$k $\in$ n. <f`k, f`succ(k)>
$\in$ R})"*
and *RR_def:*
"RR == {<z1,z2>:Fin(XX)*XX.
(domain(z2)=succ($\bigcup$f $\in$ z1. domain(f))
& ($\forall$f $\in$ z1. restrict(z2, domain(f)) = f))
| (~ ($\exists$g $\in$ XX. domain(g)=succ($\bigcup$f $\in$ z1. domain(f))

& ($\forall$f $\in$ z1. restrict(g, domain(f)) = f)) & z2={<0,x>})}"
and *allRR_def:*
"allRR == $\forall$b<nat.
<f``b, f`b> $\in$
{<z1,z2>$\in$Fin(XX)*XX. (domain(z2)=succ($\bigcup$f $\in$ z1. domain(f))
& ($\bigcup$f $\in$ z1. domain(f)) = b
& ($\forall$f $\in$ z1. restrict(z2,domain(f))
= f))}"

lemma **(in** *imp_DC0***)** *lemma4:*
"[| range(R) $\subseteq$ domain(R); x $\in$ domain(R) |]
==> RR $\subseteq$ Pow(XX)*XX &
($\forall$Y $\in$ Pow(XX). Y $\prec$ nat --> ($\exists$x $\in$ XX. <Y,x>:RR))"
$\langle$*proof*$\rangle$

lemma **(in** *imp_DC0***)** *UN_image_succ_eq:*
"[| f $\in$ nat->X; n $\in$ nat |]
==> ($\bigcup$x $\in$ f``succ(n). P(x)) =  P(f`n) Un ($\bigcup$x $\in$ f``n. P(x))"
$\langle$*proof*$\rangle$

```
lemma (in imp_DC0) UN_image_succ_eq_succ:
     "[| (⋃x ∈ f``n. P(x)) = y; P(f`n) = succ(y);
         f ∈ nat -> X; n ∈ nat |] ==> (⋃x ∈ f``succ(n). P(x)) = succ(y)"
```
⟨*proof*⟩

```
lemma (in imp_DC0) apply_domain_type:
     "[| h ∈ succ(n) -> D;  n ∈ nat; domain(h)=succ(y) |] ==> h`y ∈ D"
```
⟨*proof*⟩

```
lemma (in imp_DC0) image_fun_succ:
     "[| h ∈ nat -> X; n ∈ nat |] ==> h``succ(n) = cons(h`n, h``n)"
```
⟨*proof*⟩

```
lemma (in imp_DC0) f_n_type:
     "[| domain(f`n) = succ(k); f ∈ nat -> XX;  n ∈ nat |]
      ==> f`n ∈ succ(k) -> domain(R)"
```
⟨*proof*⟩

```
lemma (in imp_DC0) f_n_pairs_in_R [rule_format]:
     "[| h ∈ nat -> XX;  domain(h`n) = succ(k);  n ∈ nat |]
      ==> ∀i ∈ k. <h`n`i, h`n`succ(i)> ∈ R"
```
⟨*proof*⟩

```
lemma (in imp_DC0) restrict_cons_eq_restrict:
     "[| restrict(h, domain(u))=u;  h ∈ n->X;  domain(u) ⊆ n |]
      ==> restrict(cons(<n, y>, h), domain(u)) = u"
```
⟨*proof*⟩

```
lemma (in imp_DC0) all_in_image_restrict_eq:
     "[| ∀x ∈ f``n. restrict(f`n, domain(x))=x;
         f ∈ nat -> XX;
         n ∈ nat;  domain(f`n) = succ(n);
         (⋃x ∈ f``n. domain(x)) ⊆ n |]
      ==> ∀x ∈ f``succ(n). restrict(cons(<succ(n),y>, f`n), domain(x))
= x"
```
⟨*proof*⟩

```
lemma (in imp_DC0) simplify_recursion:
     "[| ∀b<nat. <f``b, f`b> ∈ RR;
         f ∈ nat -> XX; range(R) ⊆ domain(R); x ∈ domain(R)|]
      ==> allRR"
```
⟨*proof*⟩

```
lemma (in imp_DC0) lemma2:
     "[| allRR; f ∈ nat->XX; range(R) ⊆ domain(R); x ∈ domain(R); n
∈ nat |]
      ==> f`n ∈ succ(n) -> domain(R) & (∀i ∈ n. <f`n`i, f`n`succ(i)>:R)"
```

⟨proof⟩

lemma *(in imp_DC0) lemma3:*

```
"[| allRR; f ∈ nat->XX; n∈nat; range(R) ⊆ domain(R);  x ∈ domain(R)
|]
    ==> f‘n‘n = f‘succ(n)‘n"
```

⟨proof⟩

theorem *DC_nat_imp_DC0: "DC(nat) ==> DC0"*

⟨proof⟩

lemma *fun_Ord_inj:*

```
"[| f ∈ a->X;  Ord(a);
    !!b c. [| b<c; c ∈ a |] ==> f‘b≠f‘c |]
  ==> f ∈ inj(a, X)"
```

⟨proof⟩

lemma *value_in_image: "[| f ∈ X->Y; A ⊆ X; a ∈ A |] ==> f‘a ∈ f‘‘A"*

⟨proof⟩

theorem *DC_WO3: "(∀K. Card(K) --> DC(K)) ==> WO3"*

⟨proof⟩

lemma *images_eq:*

```
"[| ∀x ∈ A. f‘x=g‘x; f ∈ Df->Cf; g ∈ Dg->Cg; A ⊆ Df; A ⊆ Dg |]

  ==> f‘‘A = g‘‘A"
```

⟨proof⟩

lemma *lam_images_eq:*

```
"[| Ord(a); b ∈ a |] ==> (λx ∈ a. h(x))‘‘b = (λx ∈ b. h(x))‘‘b"
```

⟨proof⟩

lemma *lam_type_RepFun: "(λb ∈ a. h(b)) ∈ a -> {h(b). b ∈ a}"*

⟨proof⟩

lemma *lemmaX:*

```
"[| ∀Y ∈ Pow(X). Y ≺ K --> (∃x ∈ X. <Y, x> ∈ R);
    b ∈ K; Z ∈ Pow(X); Z ≺ K |]
  ==> {x ∈ X. <Z,x> ∈ R} ≠ 0"
```

⟨*proof*⟩

lemma `WO1_DC_lemma:`
```
     "[| Card(K); well_ord(X,Q);
         ∀Y ∈ Pow(X). Y ≺ K --> (∃x ∈ X. &lt;Y, x&gt; ∈ R); b ∈ K |]
      ==> ff(b, X, Q, R) ∈ {x ∈ X. &lt;(λc ∈ b. ff(c, X, Q, R))``b, x&gt;
∈ R}"
```
⟨*proof*⟩

theorem `WO1_DC_Card: "WO1 ==> ∀K. Card(K) --> DC(K)"`
⟨*proof*⟩

end

References

[1] Lawrence C. Paulson and Krzysztof Grąbczewski. Mechanizing set theory: Cardinal arithmetic and the axiom of choice. *Journal of Automated Reasoning*, 17(3):291–323, December 1996.

[2] Herman Rubin and Jean E. Rubin. *Equivalents of the Axiom of Choice, II*. North-Holland, 1985.