Fundamental Properties of Lambda-calculus

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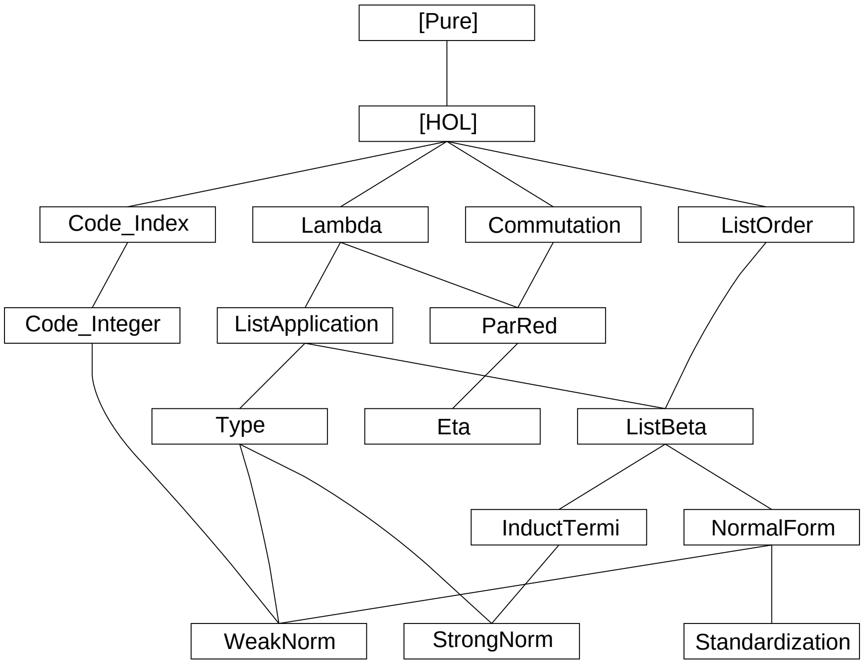
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1 Basic definitions of Lambda-calculus

theory *Lambda* **imports** *Main* **begin**

1.1 Lambda-terms in de Bruijn notation and substitution

```
datatype dB =
    Var nat
  | App dB dB (infixl ° 200)
  | Abs dB

consts
  subst :: [dB, dB, nat] => dB  (-[-'/-] [300, 0, 0] 300)
  lift :: [dB, nat] => dB

primrec
  lift (Var i) k = (if i < k then Var i else Var (i + 1))
  lift (s ° t) k = lift s k ° lift t k
  lift (Abs s) k = Abs (lift s (k + 1))

primrec
  subst-Var: (Var i)[s/k] =
    (if k < i then Var (i − 1) else if i = k then s else Var i)
  subst-App: (t ° u)[s/k] = t[s/k] ° u[s/k]
  subst-Abs: (Abs t)[s/k] = Abs (t[lift s 0 / k+1])

declare subst-Var [simp del]
```

Optimized versions of *subst* and *lift*.

```
consts
  substn :: [dB, dB, nat] => dB
  liftn :: [nat, dB, nat] => dB

primrec
  liftn n (Var i) k = (if i < k then Var i else Var (i + n))
  liftn n (s ° t) k = liftn n s k ° liftn n t k
  liftn n (Abs s) k = Abs (liftn n s (k + 1))

primrec
  substn (Var i) s k =
    (if k < i then Var (i − 1) else if i = k then liftn k s 0 else Var i)
  substn (t ° u) s k = substn t s k ° substn u s k
  substn (Abs t) s k = Abs (substn t s (k + 1))
```

1.2 Beta-reduction

```
inductive beta :: [dB, dB] => bool  (infixl →β 50)
  where
    beta [simp, intro!]: Abs s ° t →β s[t/0]
  | appL [simp, intro!]: s →β t ==> s ° u →β t ° u
```

| *appR* [*simp*, *intro*!]: $s \to_\beta t \implies u \circ s \to_\beta u \circ t$
| *abs* [*simp*, *intro*!]: $s \to_\beta t \implies Abs\ s \to_\beta Abs\ t$

abbreviation
beta-reds :: [*dB*, *dB*] => *bool* (**infixl** −>> *50*) **where**
$s \text{ −>> } t == beta\hat{}** \ s\ t$

notation (*latex*)
beta-reds (**infixl** $\to_\beta{}^*$ *50*)

inductive-cases *beta-cases* [*elim*!]:
$Var\ i \to_\beta t$
$Abs\ r \to_\beta s$
$s \circ t \to_\beta u$

declare *if-not-P* [*simp*] *not-less-eq* [*simp*]
— don't add *r-into-rtrancl*[*intro*!]

1.3 Congruence rules

lemma *rtrancl-beta-Abs* [*intro*!]:
$s \to_\beta{}^* s' \implies Abs\ s \to_\beta{}^* Abs\ s'$
⟨*proof*⟩

lemma *rtrancl-beta-AppL*:
$s \to_\beta{}^* s' \implies s \circ t \to_\beta{}^* s' \circ t$
⟨*proof*⟩

lemma *rtrancl-beta-AppR*:
$t \to_\beta{}^* t' \implies s \circ t \to_\beta{}^* s \circ t'$
⟨*proof*⟩

lemma *rtrancl-beta-App* [*intro*]:
$[\![s \to_\beta{}^* s';\ t \to_\beta{}^* t']\!] \implies s \circ t \to_\beta{}^* s' \circ t'$
⟨*proof*⟩

1.4 Substitution-lemmas

lemma *subst-eq* [*simp*]: $(Var\ k)[u/k] = u$
⟨*proof*⟩

lemma *subst-gt* [*simp*]: $i < j \implies (Var\ j)[u/i] = Var\ (j - 1)$
⟨*proof*⟩

lemma *subst-lt* [*simp*]: $j < i \implies (Var\ j)[u/i] = Var\ j$
⟨*proof*⟩

lemma *lift-lift*:
$i < k + 1 \Longrightarrow lift\ (lift\ t\ i)\ (Suc\ k) = lift\ (lift\ t\ k)\ i$
⟨*proof*⟩

lemma *lift-subst* [*simp*]:
$j < i + 1 \Longrightarrow lift\ (t[s/j])\ i = (lift\ t\ (i + 1))\ [lift\ s\ i\ /\ j]$
⟨*proof*⟩

lemma *lift-subst-lt*:
$i < j + 1 \Longrightarrow lift\ (t[s/j])\ i = (lift\ t\ i)\ [lift\ s\ i\ /\ j + 1]$
⟨*proof*⟩

lemma *subst-lift* [*simp*]:
$(lift\ t\ k)[s/k] = t$
⟨*proof*⟩

lemma *subst-subst*:
$i < j + 1 \Longrightarrow t[lift\ v\ i\ /\ Suc\ j][u[v/j]/i] = t[u/i][v/j]$
⟨*proof*⟩

1.5 Equivalence proof for optimized substitution

lemma *liftn-0* [*simp*]: $liftn\ 0\ t\ k = t$
⟨*proof*⟩

lemma *liftn-lift* [*simp*]: $liftn\ (Suc\ n)\ t\ k = lift\ (liftn\ n\ t\ k)\ k$
⟨*proof*⟩

lemma *substn-subst-n* [*simp*]: $substn\ t\ s\ n = t[liftn\ n\ s\ 0\ /\ n]$
⟨*proof*⟩

theorem *substn-subst-0*: $substn\ t\ s\ 0 = t[s/0]$
⟨*proof*⟩

1.6 Preservation theorems

Not used in Church-Rosser proof, but in Strong Normalization.

theorem *subst-preserves-beta* [*simp*]:
$r \rightarrow_\beta s ==> r[t/i] \rightarrow_\beta s[t/i]$
⟨*proof*⟩

theorem *subst-preserves-beta'*: $r \rightarrow_\beta{}^* s ==> r[t/i] \rightarrow_\beta{}^* s[t/i]$
⟨*proof*⟩

theorem *lift-preserves-beta* [*simp*]:
$r \rightarrow_\beta s ==> lift\ r\ i \rightarrow_\beta lift\ s\ i$
⟨*proof*⟩

theorem *lift-preserves-beta'*: $r \rightarrow_\beta{}^* s ==> lift\ r\ i \rightarrow_\beta{}^* lift\ s\ i$
⟨*proof*⟩

theorem *subst-preserves-beta2* [*simp*]: $r \rightarrow_\beta s$ ==> $t[r/i] \rightarrow_\beta{}^* t[s/i]$
⟨*proof*⟩

theorem *subst-preserves-beta2'*: $r \rightarrow_\beta{}^* s$ ==> $t[r/i] \rightarrow_\beta{}^* t[s/i]$
⟨*proof*⟩

end

2 Abstract commutation and confluence notions

theory *Commutation* **imports** *Main* **begin**

2.1 Basic definitions

definition
square :: ['a => 'a => bool, 'a => 'a => bool, 'a => 'a => bool, 'a => 'a => bool] => bool **where**
square R S T U =
($\forall x\ y.\ R\ x\ y$ --> ($\forall z.\ S\ x\ z$ --> ($\exists u.\ T\ y\ u \wedge U\ z\ u$)))

definition
commute :: ['a => 'a => bool, 'a => 'a => bool] => bool **where**
commute R S = *square R S S R*

definition
diamond :: ('a => 'a => bool) => bool **where**
diamond R = *commute R R*

definition
Church-Rosser :: ('a => 'a => bool) => bool **where**
Church-Rosser R =
($\forall x\ y.$ (*sup R* (R^--1))^** x y --> ($\exists z.$ R^** x z $\wedge$ R^** y z))

abbreviation
confluent :: ('a => 'a => bool) => bool **where**
confluent R == *diamond* (R^**)

2.2 Basic lemmas

square

lemma *square-sym*: *square R S T U* ==> *square S R U T*
⟨*proof*⟩

lemma *square-subset*:
[| *square R S T U*; $T \le T'$ |] ==> *square R S T' U*
⟨*proof*⟩

lemma *square-reflcl*:
  [| *square R S T* (*R*ˆ==); *S* ≤ *T* |] ==> *square* (*R*ˆ==) *S T* (*R*ˆ==)
⟨*proof*⟩

lemma *square-rtrancl*:
  square R S S T ==> *square* (*R*ˆ∗∗) *S S* (*T*ˆ∗∗)
⟨*proof*⟩

lemma *square-rtrancl-reflcl-commute*:
  square R S (*S*ˆ∗∗) (*R*ˆ==) ==> *commute* (*R*ˆ∗∗) (*S*ˆ∗∗)
⟨*proof*⟩

commute

lemma *commute-sym*: *commute R S* ==> *commute S R*
⟨*proof*⟩

lemma *commute-rtrancl*: *commute R S* ==> *commute* (*R*ˆ∗∗) (*S*ˆ∗∗)
⟨*proof*⟩

lemma *commute-Un*:
  [| *commute R T*; *commute S T* |] ==> *commute* (*sup R S*) *T*
⟨*proof*⟩

diamond, confluence, and union

lemma *diamond-Un*:
  [| *diamond R*; *diamond S*; *commute R S* |] ==> *diamond* (*sup R S*)
⟨*proof*⟩

lemma *diamond-confluent*: *diamond R* ==> *confluent R*
⟨*proof*⟩

lemma *square-reflcl-confluent*:
  square R R (*R*ˆ==) (*R*ˆ==) ==> *confluent R*
⟨*proof*⟩

lemma *confluent-Un*:
  [| *confluent R*; *confluent S*; *commute* (*R*ˆ∗∗) (*S*ˆ∗∗) |] ==> *confluent* (*sup R S*)
⟨*proof*⟩

lemma *diamond-to-confluence*:
  [| *diamond R*; *T* ≤ *R*; *R* ≤ *T*ˆ∗∗ |] ==> *confluent T*
⟨*proof*⟩

2.3 Church-Rosser

lemma *Church-Rosser-confluent*: *Church-Rosser R* = *confluent R*
⟨*proof*⟩

2.4 Newman's lemma

Proof by Stefan Berghofer

theorem *newman*:
assumes *wf*: *wfP* (R^{-1-1})
and *lc*: $\bigwedge a\ b\ c.\ R\ a\ b \Longrightarrow R\ a\ c \Longrightarrow$
$\exists\, d.\ R^{**}\ b\ d \wedge R^{**}\ c\ d$
shows $\bigwedge b\ c.\ R^{**}\ a\ b \Longrightarrow R^{**}\ a\ c \Longrightarrow$
$\exists\, d.\ R^{**}\ b\ d \wedge R^{**}\ c\ d$
⟨*proof*⟩

Alternative version. Partly automated by Tobias Nipkow. Takes 2 minutes (2002).

This is the maximal amount of automation possible at the moment.

theorem *newman′*:
assumes *wf*: *wfP* (R^{-1-1})
and *lc*: $\bigwedge a\ b\ c.\ R\ a\ b \Longrightarrow R\ a\ c \Longrightarrow$
$\exists\, d.\ R^{**}\ b\ d \wedge R^{**}\ c\ d$
shows $\bigwedge b\ c.\ R^{**}\ a\ b \Longrightarrow R^{**}\ a\ c \Longrightarrow$
$\exists\, d.\ R^{**}\ b\ d \wedge R^{**}\ c\ d$
⟨*proof*⟩

end

3 Parallel reduction and a complete developments

theory *ParRed* **imports** *Lambda Commutation* **begin**

3.1 Parallel reduction

inductive *par-beta* :: [*dB*, *dB*] => *bool* (**infixl** => *50*)
where
var [*simp*, *intro*!]: *Var n* => *Var n*
| *abs* [*simp*, *intro*!]: *s* => *t* ==> *Abs s* => *Abs t*
| *app* [*simp*, *intro*!]: [| *s* => *s′*; *t* => *t′* |] ==> *s* ° *t* => *s′* ° *t′*
| *beta* [*simp*, *intro*!]: [| *s* => *s′*; *t* => *t′* |] ==> (*Abs s*) ° *t* => *s′*[*t′*/*0*]

inductive-cases *par-beta-cases* [*elim*!]:
Var n => *t*
Abs s => *Abs t*
(*Abs s*) ° *t* => *u*
s ° *t* => *u*
Abs s => *t*

3.2 Inclusions

beta ⊆ *par-beta* ⊆ *beta*ˆ∗

lemma *par-beta-varL* [*simp*]:
$(Var\ n => t) = (t = Var\ n)$
⟨*proof*⟩

lemma *par-beta-refl* [*simp*]: $t => t$
⟨*proof*⟩

lemma *beta-subset-par-beta*: *beta* <= *par-beta*
⟨*proof*⟩

lemma *par-beta-subset-beta*: *par-beta* <= *beta*^**
⟨*proof*⟩

3.3 Misc properties of par-beta

lemma *par-beta-lift* [*simp*]:
$t => t' \Longrightarrow lift\ t\ n => lift\ t'\ n$
⟨*proof*⟩

lemma *par-beta-subst*:
$s => s' \Longrightarrow t => t' \Longrightarrow t[s/n] => t'[s'/n]$
⟨*proof*⟩

3.4 Confluence (directly)

lemma *diamond-par-beta*: *diamond par-beta*
⟨*proof*⟩

3.5 Complete developments

consts
cd :: *dB* => *dB*
recdef *cd measure size*
cd (*Var n*) = *Var n*
cd (*Var n* ° *t*) = *Var n* ° *cd t*
cd ((*s1* ° *s2*) ° *t*) = *cd* (*s1* ° *s2*) ° *cd t*
cd (*Abs u* ° *t*) = (*cd u*)[*cd t*/*0*]
cd (*Abs s*) = *Abs* (*cd s*)

lemma *par-beta-cd*: $s => t \Longrightarrow t => cd\ s$
⟨*proof*⟩

3.6 Confluence (via complete developments)

lemma *diamond-par-beta2*: *diamond par-beta*
⟨*proof*⟩

theorem *beta-confluent*: *confluent beta*
⟨*proof*⟩

end

4 Eta-reduction

theory *Eta* **imports** *ParRed* **begin**

4.1 Definition of eta-reduction and relatives

consts
  free :: *dB* => *nat* => *bool*
primrec
  $free\ (Var\ j)\ i = (j = i)$
  $free\ (s \circ t)\ i = (free\ s\ i \lor free\ t\ i)$
  $free\ (Abs\ s)\ i = free\ s\ (i + 1)$

inductive *eta* :: [*dB*, *dB*] => *bool* (**infixl** $\to_\eta$ *50*)
  where
    eta [*simp*, *intro*]: $\neg\ free\ s\ 0 ==> Abs\ (s \circ Var\ 0) \to_\eta s[dummy/0]$
  | *appL* [*simp*, *intro*]: $s \to_\eta t ==> s \circ u \to_\eta t \circ u$
  | *appR* [*simp*, *intro*]: $s \to_\eta t ==> u \circ s \to_\eta u \circ t$
  | *abs* [*simp*, *intro*]: $s \to_\eta t ==> Abs\ s \to_\eta Abs\ t$

abbreviation
  eta-reds :: [*dB*, *dB*] => *bool* (**infixl** −*e*>> *50*) **where**
  s −*e*>> *t* == *eta*ˆ** *s t*

abbreviation
  eta-red0 :: [*dB*, *dB*] => *bool* (**infixl** −*e*>= *50*) **where**
  s −*e*>= *t* == *eta*ˆ== *s t*

notation (*xsymbols*)
  eta-reds (**infixl** ${\to_\eta}^*$ *50*) **and**
  eta-red0 (**infixl** ${\to_\eta}^=$ *50*)

inductive-cases *eta-cases* [*elim*!]:
  $Abs\ s \to_\eta z$
  $s \circ t \to_\eta u$
  $Var\ i \to_\eta t$

4.2 Properties of eta, subst and free

lemma *subst-not-free* [*simp*]: $\neg\ free\ s\ i \Longrightarrow s[t/i] = s[u/i]$
  ⟨*proof*⟩

lemma *free-lift* [*simp*]:
    $free\ (lift\ t\ k)\ i = (i < k \land free\ t\ i \lor k < i \land free\ t\ (i - 1))$
  ⟨*proof*⟩

lemma *free-subst* [*simp*]:
  $free\ (s[t/k])\ i =$
   $(free\ s\ k \wedge free\ t\ i \vee free\ s\ (if\ i < k\ then\ i\ else\ i + 1))$
 ⟨*proof*⟩

lemma *free-eta*: $s \rightarrow_\eta t \Longrightarrow free\ t\ i = free\ s\ i$
 ⟨*proof*⟩

lemma *not-free-eta*:
  $[|\ s \rightarrow_\eta t;\ \neg\ free\ s\ i\ |] \Longrightarrow \neg\ free\ t\ i$
 ⟨*proof*⟩

lemma *eta-subst* [*simp*]:
  $s \rightarrow_\eta t \Longrightarrow s[u/i] \rightarrow_\eta t[u/i]$
 ⟨*proof*⟩

theorem *lift-subst-dummy*: $\neg\ free\ s\ i \Longrightarrow lift\ (s[dummy/i])\ i = s$
 ⟨*proof*⟩

4.3 Confluence of eta

lemma *square-eta*: *square eta eta* (*eta*^==) (*eta*^==)
 ⟨*proof*⟩

theorem *eta-confluent*: *confluent eta*
 ⟨*proof*⟩

4.4 Congruence rules for eta*

lemma *rtrancl-eta-Abs*: $s \rightarrow_\eta{}^* s' \Longrightarrow Abs\ s \rightarrow_\eta{}^* Abs\ s'$
 ⟨*proof*⟩

lemma *rtrancl-eta-AppL*: $s \rightarrow_\eta{}^* s' \Longrightarrow s \circ t \rightarrow_\eta{}^* s' \circ t$
 ⟨*proof*⟩

lemma *rtrancl-eta-AppR*: $t \rightarrow_\eta{}^* t' \Longrightarrow s \circ t \rightarrow_\eta{}^* s \circ t'$
 ⟨*proof*⟩

lemma *rtrancl-eta-App*:
  $[|\ s \rightarrow_\eta{}^* s';\ t \rightarrow_\eta{}^* t'\ |] \Longrightarrow s \circ t \rightarrow_\eta{}^* s' \circ t'$
 ⟨*proof*⟩

4.5 Commutation of beta and eta

lemma *free-beta*:
  $s \rightarrow_\beta t \Longrightarrow free\ t\ i \Longrightarrow free\ s\ i$
 ⟨*proof*⟩

lemma *beta-subst* [*intro*]: $s \rightarrow_\beta t \Longrightarrow s[u/i] \rightarrow_\beta t[u/i]$
 ⟨*proof*⟩

lemma *subst-Var-Suc* [*simp*]: $t[Var\ i/i] = t[Var(i)/i + 1]$
  ⟨*proof*⟩

lemma *eta-lift* [*simp*]: $s \to_\eta t ==> lift\ s\ i \to_\eta lift\ t\ i$
  ⟨*proof*⟩

lemma *rtrancl-eta-subst*: $s \to_\eta t \Longrightarrow u[s/i] \to_\eta{}^* u[t/i]$
  ⟨*proof*⟩

lemma *rtrancl-eta-subst′*:
  fixes $s\ t :: dB$
  assumes *eta*: $s \to_\eta{}^* t$
  shows $s[u/i] \to_\eta{}^* t[u/i]$ ⟨*proof*⟩

lemma *rtrancl-eta-subst′′*:
  fixes $s\ t :: dB$
  assumes *eta*: $s \to_\eta{}^* t$
  shows $u[s/i] \to_\eta{}^* u[t/i]$ ⟨*proof*⟩

lemma *square-beta-eta*: *square beta eta* (*eta*ˆ∗∗) (*beta*ˆ==)
  ⟨*proof*⟩

lemma *confluent-beta-eta*: *confluent* (*sup beta eta*)
  ⟨*proof*⟩

4.6 Implicit definition of eta

$Abs\ (lift\ s\ 0 \circ Var\ 0) \to_\eta s$

lemma *not-free-iff-lifted*:
    $(\neg\ free\ s\ i) = (\exists\, t.\ s = lift\ t\ i)$
  ⟨*proof*⟩

theorem *explicit-is-implicit*:
  $(\forall\, s\ u.\ (\neg\ free\ s\ 0) \dashrightarrow R\ (Abs\ (s \circ Var\ 0))\ (s[u/0])) =$
    $(\forall\, s.\ R\ (Abs\ (lift\ s\ 0 \circ Var\ 0))\ s)$
  ⟨*proof*⟩

4.7 Eta-postponement theorem

Based on a paper proof due to Andreas Abel. Unlike the proof by Masako Takahashi [4], it does not use parallel eta reduction, which only seems to complicate matters unnecessarily.

theorem *eta-case*:
  fixes $s :: dB$
  assumes *free*: $\neg\ free\ s\ 0$
  and *s*: $s[dummy/0] => u$
  shows $\exists\, t'.\ Abs\ (s \circ Var\ 0) => t' \wedge t' \to_\eta{}^* u$

$\langle proof \rangle$

theorem *eta-par-beta*:
  assumes *st*: $s \to_\eta t$
  and *tu*: $t => u$
  shows $\exists t'.\ s => t' \wedge t' \to_\eta{}^* u$ $\langle proof \rangle$

theorem *eta-postponement'*:
  assumes *eta*: $s \to_\eta{}^* t$ **and** *beta*: $t => u$
  shows $\exists t'.\ s => t' \wedge t' \to_\eta{}^* u$ $\langle proof \rangle$

theorem *eta-postponement*:
  assumes *st*: $(sup\ beta\ eta)^{**}\ s\ t$
  shows $(eta^{**}\ OO\ beta^{**})\ s\ t$ $\langle proof \rangle$

end

5 Application of a term to a list of terms

theory *ListApplication* **imports** *Lambda* **begin**

abbreviation
  $list\text{-}application :: dB => dB\ list => dB$ (**infixl** $^{\circ\circ}$ *150*) **where**
  $t\ ^{\circ\circ}\ ts == foldl\ (op\ ^{\circ})\ t\ ts$

lemma *apps-eq-tail-conv* [*iff*]: $(r\ ^{\circ\circ}\ ts = s\ ^{\circ\circ}\ ts) = (r = s)$
  $\langle proof \rangle$

lemma *Var-eq-apps-conv* [*iff*]: $(Var\ m = s\ ^{\circ\circ}\ ss) = (Var\ m = s \wedge ss = [])$
  $\langle proof \rangle$

lemma *Var-apps-eq-Var-apps-conv* [*iff*]:
    $(Var\ m\ ^{\circ\circ}\ rs = Var\ n\ ^{\circ\circ}\ ss) = (m = n \wedge rs = ss)$
  $\langle proof \rangle$

lemma *App-eq-foldl-conv*:
  $(r\ ^{\circ}\ s = t\ ^{\circ\circ}\ ts) =$
    $(if\ ts = []\ then\ r\ ^{\circ}\ s = t$
    $else\ (\exists ss.\ ts = ss\ @\ [s] \wedge r = t\ ^{\circ\circ}\ ss))$
  $\langle proof \rangle$

lemma *Abs-eq-apps-conv* [*iff*]:
    $(Abs\ r = s\ ^{\circ\circ}\ ss) = (Abs\ r = s \wedge ss = [])$
  $\langle proof \rangle$

lemma *apps-eq-Abs-conv* [*iff*]: $(s\ ^{\circ\circ}\ ss = Abs\ r) = (s = Abs\ r \wedge ss = [])$
  $\langle proof \rangle$

lemma *Abs-apps-eq-Abs-apps-conv* [*iff*]:
$(Abs\ r\ ^{\circ\circ}\ rs = Abs\ s\ ^{\circ\circ}\ ss) = (r = s \wedge rs = ss)$
⟨*proof*⟩

lemma *Abs-App-neq-Var-apps* [*iff*]:
$Abs\ s\ ^{\circ}\ t \neq Var\ n\ ^{\circ\circ}\ ss$
⟨*proof*⟩

lemma *Var-apps-neq-Abs-apps* [*iff*]:
$Var\ n\ ^{\circ\circ}\ ts \neq Abs\ r\ ^{\circ\circ}\ ss$
⟨*proof*⟩

lemma *ex-head-tail*:
$\exists\, ts\ h.\ t = h\ ^{\circ\circ}\ ts \wedge ((\exists\, n.\ h = Var\ n) \vee (\exists\, u.\ h = Abs\ u))$
⟨*proof*⟩

lemma *size-apps* [*simp*]:
$size\ (r\ ^{\circ\circ}\ rs) = size\ r + foldl\ (op\ +)\ 0\ (map\ size\ rs) + length\ rs$
⟨*proof*⟩

lemma *lem0*: $[|\ (0::nat) < k;\ m <= n\ |] ==> m < n + k$
⟨*proof*⟩

lemma *lift-map* [*simp*]:
$lift\ (t\ ^{\circ\circ}\ ts)\ i = lift\ t\ i\ ^{\circ\circ}\ map\ (\lambda t.\ lift\ t\ i)\ ts$
⟨*proof*⟩

lemma *subst-map* [*simp*]:
$subst\ (t\ ^{\circ\circ}\ ts)\ u\ i = subst\ t\ u\ i\ ^{\circ\circ}\ map\ (\lambda t.\ subst\ t\ u\ i)\ ts$
⟨*proof*⟩

lemma *app-last*: $(t\ ^{\circ\circ}\ ts)\ ^{\circ}\ u = t\ ^{\circ\circ}\ (ts\ @\ [u])$
⟨*proof*⟩

A customized induction schema for $^{\circ\circ}$.

lemma *lem*:
assumes $!!n\ ts.\ \forall\, t \in set\ ts.\ P\ t ==> P\ (Var\ n\ ^{\circ\circ}\ ts)$
and $!!u\ ts.\ [|\ P\ u;\ \forall\, t \in set\ ts.\ P\ t\ |] ==> P\ (Abs\ u\ ^{\circ\circ}\ ts)$
shows $size\ t = n \Longrightarrow P\ t$
⟨*proof*⟩

theorem *Apps-dB-induct*:
assumes $!!n\ ts.\ \forall\, t \in set\ ts.\ P\ t ==> P\ (Var\ n\ ^{\circ\circ}\ ts)$
and $!!u\ ts.\ [|\ P\ u;\ \forall\, t \in set\ ts.\ P\ t\ |] ==> P\ (Abs\ u\ ^{\circ\circ}\ ts)$
shows $P\ t$
⟨*proof*⟩

end

6 Simply-typed lambda terms

theory *Type* **imports** *ListApplication* **begin**

6.1 Environments

definition
shift :: $(nat \Rightarrow 'a) \Rightarrow nat \Rightarrow 'a \Rightarrow nat \Rightarrow 'a$ (-<-:-> [90, 0, 0] 91) **where**
$e\text{<}i\text{:}a\text{>} = (\lambda j.\ if\ j < i\ then\ e\ j\ else\ if\ j = i\ then\ a\ else\ e\ (j - 1))$

notation (*xsymbols*)
shift $(\text{-}\langle\text{-:-}\rangle\ [90,\ 0,\ 0]\ 91)$

notation (*HTML* **output**)
shift $(\text{-}\langle\text{-:-}\rangle\ [90,\ 0,\ 0]\ 91)$

lemma *shift-eq* [*simp*]: $i = j \Longrightarrow (e\langle i{:}T\rangle)\ j = T$
⟨*proof*⟩

lemma *shift-gt* [*simp*]: $j < i \Longrightarrow (e\langle i{:}T\rangle)\ j = e\ j$
⟨*proof*⟩

lemma *shift-lt* [*simp*]: $i < j \Longrightarrow (e\langle i{:}T\rangle)\ j = e\ (j - 1)$
⟨*proof*⟩

lemma *shift-commute* [*simp*]: $e\langle i{:}U\rangle\langle 0{:}T\rangle = e\langle 0{:}T\rangle\langle Suc\ i{:}U\rangle$
⟨*proof*⟩

6.2 Types and typing rules

datatype *type* =
Atom nat
| *Fun type type* (**infixr** ⇒ *200*)

inductive *typing* :: $(nat \Rightarrow type) \Rightarrow dB \Rightarrow type \Rightarrow bool$ $(\text{-} \vdash \text{-} : \text{-}\ [50,\ 50,\ 50]\ 50)$
where
Var [*intro*!]: $env\ x = T \Longrightarrow env \vdash Var\ x : T$
| *Abs* [*intro*!]: $env\langle 0{:}T\rangle \vdash t : U \Longrightarrow env \vdash Abs\ t : (T \Rightarrow U)$
| *App* [*intro*!]: $env \vdash s : T \Rightarrow U \Longrightarrow env \vdash t : T \Longrightarrow env \vdash (s \circ t) : U$

inductive-cases *typing-elims* [*elim*!]:
$e \vdash Var\ i : T$
$e \vdash t \circ u : T$
$e \vdash Abs\ t : T$

consts
typings :: $(nat \Rightarrow type) \Rightarrow dB\ list \Rightarrow type\ list \Rightarrow bool$

abbreviation
funs :: $type\ list \Rightarrow type \Rightarrow type$ (**infixr** =>> *200*) **where**

$Ts =\!>\!> T == foldr\ Fun\ Ts\ T$

abbreviation
$typings\text{-}rel :: (nat \Rightarrow type) \Rightarrow dB\ list \Rightarrow type\ list \Rightarrow bool$
(- ||− - : - [50, 50, 50] 50) **where**
$env\ ||\!-\ ts : Ts == typings\ env\ ts\ Ts$

notation (*latex*)
funs (**infixr** $\Rrightarrow$ *200*) **and**
typings-rel (- $\Vdash$ - : - [*50*, *50*, *50*] *50*)

primrec
$(e \Vdash [] : Ts) = (Ts = [])$
$(e \Vdash (t \# ts) : Ts) =$
$(case\ Ts\ of$
$[] \Rightarrow False$
$|\ T \# Ts \Rightarrow e \vdash t : T \wedge e \Vdash ts : Ts)$

6.3 Some examples

lemma $e \vdash Abs\ (Abs\ (Abs\ (Var\ 1 \circ (Var\ 2 \circ Var\ 1 \circ Var\ 0)))) : ?T$
⟨*proof*⟩

lemma $e \vdash Abs\ (Abs\ (Abs\ (Var\ 2 \circ Var\ 0 \circ (Var\ 1 \circ Var\ 0)))) : ?T$
⟨*proof*⟩

6.4 Lists of types

lemma *lists-typings*:
$e \Vdash ts : Ts \Longrightarrow listsp\ (\lambda t.\ \exists T.\ e \vdash t : T)\ ts$
⟨*proof*⟩

lemma *types-snoc*: $e \Vdash ts : Ts \Longrightarrow e \vdash t : T \Longrightarrow e \Vdash ts\ @\ [t] : Ts\ @\ [T]$
⟨*proof*⟩

lemma *types-snoc-eq*: $e \Vdash ts\ @\ [t] : Ts\ @\ [T] =$
$(e \Vdash ts : Ts \wedge e \vdash t : T)$
⟨*proof*⟩

lemma *rev-exhaust2* [*case-names Nil snoc*, *extraction-expand*]:
$(xs = [] \Longrightarrow P) \Longrightarrow (\bigwedge ys\ y.\ xs = ys\ @\ [y] \Longrightarrow P) \Longrightarrow P$
— Cannot use *rev-exhaust* from the *List* theory, since it is not constructive
⟨*proof*⟩

lemma *types-snocE*: $e \Vdash ts\ @\ [t] : Ts \Longrightarrow$
$(\bigwedge Us\ U.\ Ts = Us\ @\ [U] \Longrightarrow e \Vdash ts : Us \Longrightarrow e \vdash t : U \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

6.5 n-ary function types

lemma *list-app-typeD*:
$e \vdash t \circ\circ ts : T \Longrightarrow \exists Ts.\ e \vdash t : Ts \Rrightarrow T \wedge e \Vdash ts : Ts$
⟨*proof*⟩

lemma *list-app-typeE*:
$e \vdash t \circ\circ ts : T \Longrightarrow (\bigwedge Ts.\ e \vdash t : Ts \Rrightarrow T \Longrightarrow e \Vdash ts : Ts \Longrightarrow C) \Longrightarrow C$
⟨*proof*⟩

lemma *list-app-typeI*:
$e \vdash t : Ts \Rrightarrow T \Longrightarrow e \Vdash ts : Ts \Longrightarrow e \vdash t \circ\circ ts : T$
⟨*proof*⟩

For the specific case where the head of the term is a variable, the following theorems allow to infer the types of the arguments without analyzing the typing derivation. This is crucial for program extraction.

theorem *var-app-type-eq*:
$e \vdash Var\ i \circ\circ ts : T \Longrightarrow e \vdash Var\ i \circ\circ ts : U \Longrightarrow T = U$
⟨*proof*⟩

lemma *var-app-types*: $e \vdash Var\ i \circ\circ ts \circ\circ us : T \Longrightarrow e \Vdash ts : Ts \Longrightarrow$
$e \vdash Var\ i \circ\circ ts : U \Longrightarrow \exists Us.\ U = Us \Rrightarrow T \wedge e \Vdash us : Us$
⟨*proof*⟩

lemma *var-app-typesE*: $e \vdash Var\ i \circ\circ ts : T \Longrightarrow$
$(\bigwedge Ts.\ e \vdash Var\ i : Ts \Rrightarrow T \Longrightarrow e \Vdash ts : Ts \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

lemma *abs-typeE*: $e \vdash Abs\ t : T \Longrightarrow (\bigwedge U\ V.\ e\langle 0{:}U\rangle \vdash t : V \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

6.6 Lifting preserves well-typedness

lemma *lift-type* [*intro*!]: $e \vdash t : T \Longrightarrow e\langle i{:}U\rangle \vdash lift\ t\ i : T$
⟨*proof*⟩

lemma *lift-types*:
$e \Vdash ts : Ts \Longrightarrow e\langle i{:}U\rangle \Vdash (map\ (\lambda t.\ lift\ t\ i)\ ts) : Ts$
⟨*proof*⟩

6.7 Substitution lemmas

lemma *subst-lemma*:
$e \vdash t : T \Longrightarrow e' \vdash u : U \Longrightarrow e = e'\langle i{:}U\rangle \Longrightarrow e' \vdash t[u/i] : T$
⟨*proof*⟩

lemma *substs-lemma*:
$e \vdash u : T \Longrightarrow e\langle i{:}T\rangle \Vdash ts : Ts \Longrightarrow$
$e \Vdash (map\ (\lambda t.\ t[u/i])\ ts) : Ts$

⟨*proof*⟩

6.8 Subject reduction

lemma *subject-reduction*: $e \vdash t : T \Longrightarrow t \to_\beta t' \Longrightarrow e \vdash t' : T$
⟨*proof*⟩

theorem *subject-reduction'*: $t \to_\beta{}^* t' \Longrightarrow e \vdash t : T \Longrightarrow e \vdash t' : T$
⟨*proof*⟩

6.9 Alternative induction rule for types

lemma *type-induct* [*induct type*]:
assumes
$(\bigwedge T.\ (\bigwedge T1\ T2.\ T = T1 \Rightarrow T2 \Longrightarrow P\ T1) \Longrightarrow$
$(\bigwedge T1\ T2.\ T = T1 \Rightarrow T2 \Longrightarrow P\ T2) \Longrightarrow P\ T)$
shows $P\ T$
⟨*proof*⟩

end

7 Lifting an order to lists of elements

theory *ListOrder* **imports** *Main* **begin**

Lifting an order to lists of elements, relating exactly one element.

definition
step1 :: (*'a* => *'a* => *bool*) => *'a list* => *'a list* => *bool* **where**
step1 r =
(λ*ys xs*. $\exists$ *us z z' vs*. *xs* = *us* @ *z* # *vs* $\wedge$ *r z' z* $\wedge$ *ys* =
us @ *z'* # *vs*)

lemma *step1-converse* [*simp*]: *step1* (*r*^−−*1*) = (*step1 r*)^−−*1*
⟨*proof*⟩

lemma *in-step1-converse* [*iff*]: (*step1* (*r*^−−*1*) *x y*) = ((*step1 r*)^−−*1 x y*)
⟨*proof*⟩

lemma *not-Nil-step1* [*iff*]: ¬ *step1 r* [] *xs*
⟨*proof*⟩

lemma *not-step1-Nil* [*iff*]: ¬ *step1 r xs* []
⟨*proof*⟩

lemma *Cons-step1-Cons* [*iff*]:
(*step1 r* (*y* # *ys*) (*x* # *xs*)) =
(*r y x* $\wedge$ *xs* = *ys* $\vee$ *x* = *y* $\wedge$ *step1 r ys xs*)

⟨*proof*⟩

lemma *append-step1I*:
$\mathit{step1}\ r\ ys\ xs \wedge vs = us \vee ys = xs \wedge \mathit{step1}\ r\ vs\ us$
$\quad ==> \mathit{step1}\ r\ (ys\ @\ vs)\ (xs\ @\ us)$
⟨*proof*⟩

lemma *Cons-step1E* [*elim*!]:
assumes $\mathit{step1}\ r\ ys\ (x\ \#\ xs)$
and $!!y.\ ys = y\ \#\ xs \Longrightarrow r\ y\ x \Longrightarrow R$
and $!!zs.\ ys = x\ \#\ zs \Longrightarrow \mathit{step1}\ r\ zs\ xs \Longrightarrow R$
shows R
⟨*proof*⟩

lemma *Snoc-step1-SnocD*:
$\mathit{step1}\ r\ (ys\ @\ [y])\ (xs\ @\ [x])$
$\quad ==> (\mathit{step1}\ r\ ys\ xs \wedge y = x \vee ys = xs \wedge r\ y\ x)$
⟨*proof*⟩

lemma *Cons-acc-step1I* [*intro*!]:
$\mathit{accp}\ r\ x ==> \mathit{accp}\ (\mathit{step1}\ r)\ xs \Longrightarrow \mathit{accp}\ (\mathit{step1}\ r)\ (x\ \#\ xs)$
⟨*proof*⟩

lemma *lists-accD*: $\mathit{listsp}\ (\mathit{accp}\ r)\ xs ==> \mathit{accp}\ (\mathit{step1}\ r)\ xs$
⟨*proof*⟩

lemma *ex-step1I*:
$[|\ x \in \mathit{set}\ xs;\ r\ y\ x\ |]$
$\quad ==> \exists ys.\ \mathit{step1}\ r\ ys\ xs \wedge y \in \mathit{set}\ ys$
⟨*proof*⟩

lemma *lists-accI*: $\mathit{accp}\ (\mathit{step1}\ r)\ xs ==> \mathit{listsp}\ (\mathit{accp}\ r)\ xs$
⟨*proof*⟩

end

8 Lifting beta-reduction to lists

theory *ListBeta* **imports** *ListApplication ListOrder* **begin**

Lifting beta-reduction to lists of terms, reducing exactly one element.

abbreviation
list-beta :: *dB list* => *dB list* => *bool* (**infixl** => *50*) **where**
rs => *ss* == *step1 beta rs ss*

lemma *head-Var-reduction*:
$\mathit{Var}\ n\ {}^{\circ\circ}\ rs \rightarrow_\beta v \Longrightarrow \exists ss.\ rs => ss \wedge v = \mathit{Var}\ n\ {}^{\circ\circ}\ ss$
⟨*proof*⟩

lemma *apps-betasE* [*elim*!]:
  assumes *major*: $r \circ\circ\ rs \rightarrow_\beta s$
    and *cases*: $!!r'.\ [|\ r \rightarrow_\beta r';\ s = r' \circ\circ\ rs\ |] ==> R$
      $!!rs'.\ [|\ rs => rs';\ s = r \circ\circ\ rs'\ |] ==> R$
      $!!t\ u\ us.\ [|\ r = Abs\ t;\ rs = u\ \#\ us;\ s = t[u/0] \circ\circ\ us\ |] ==> R$
  shows *R*
⟨*proof*⟩

lemma *apps-preserves-beta* [*simp*]:
  $r \rightarrow_\beta s ==> r \circ\circ\ ss \rightarrow_\beta s \circ\circ\ ss$
  ⟨*proof*⟩

lemma *apps-preserves-beta2* [*simp*]:
  $r ->> s ==> r \circ\circ\ ss ->> s \circ\circ\ ss$
  ⟨*proof*⟩

lemma *apps-preserves-betas* [*simp*]:
  $rs => ss \Longrightarrow r \circ\circ\ rs \rightarrow_\beta r \circ\circ\ ss$
  ⟨*proof*⟩

end

9 Inductive characterization of terminating lambda terms

theory *InductTermi* **imports** *ListBeta* **begin**

9.1 Terminating lambda terms

inductive *IT* :: *dB* => *bool*
  where
    Var [*intro*]: *listsp IT rs* ==> *IT* (*Var n* ∘∘ *rs*)
  | *Lambda* [*intro*]: *IT r* ==> *IT* (*Abs r*)
  | *Beta* [*intro*]: *IT* ((*r*[*s*/*0*]) ∘∘ *ss*) ==> *IT s* ==> *IT* ((*Abs r* ∘ *s*) ∘∘ *ss*)

9.2 Every term in IT terminates

lemma *double-induction-lemma* [*rule-format*]:
  termip beta s ==> ∀ *t*. *termip beta t* −−>
    (∀ *r ss*. *t* = *r*[*s*/*0*] ∘∘ *ss* −−> *termip beta* (*Abs r* ∘ *s* ∘∘ *ss*))
  ⟨*proof*⟩

lemma *IT-implies-termi*: *IT t* ==> *termip beta t*
  ⟨*proof*⟩

9.3 Every terminating term is in IT

declare *Var-apps-neq-Abs-apps* [*symmetric*, *simp*]

lemma [*simp*, *THEN not-sym*, *simp*]: $Var\ n \circ\circ ss \neq Abs\ r \circ s \circ\circ ts$
⟨*proof*⟩

lemma [*simp*]:
$(Abs\ r \circ s \circ\circ ss = Abs\ r' \circ s' \circ\circ ss') = (r = r' \wedge s = s' \wedge ss = ss')$
⟨*proof*⟩

inductive-cases [*elim*!]:
$IT\ (Var\ n \circ\circ ss)$
$IT\ (Abs\ t)$
$IT\ (Abs\ r \circ s \circ\circ ts)$

theorem *termi-implies-IT*: *termip beta r ==> IT r*
⟨*proof*⟩

end

10 Strong normalization for simply-typed lambda calculus

theory *StrongNorm* **imports** *Type InductTermi* **begin**

Formalization by Stefan Berghofer. Partly based on a paper proof by Felix Joachimski and Ralph Matthes [1].

10.1 Properties of *IT*

lemma *lift-IT* [*intro*!]: $IT\ t \Longrightarrow IT\ (lift\ t\ i)$
⟨*proof*⟩

lemma *lifts-IT*: $listsp\ IT\ ts \Longrightarrow listsp\ IT\ (map\ (\lambda t.\ lift\ t\ 0)\ ts)$
⟨*proof*⟩

lemma *subst-Var-IT*: $IT\ r \Longrightarrow IT\ (r[Var\ i/j])$
⟨*proof*⟩

lemma *Var-IT*: $IT\ (Var\ n)$
⟨*proof*⟩

lemma *app-Var-IT*: $IT\ t \Longrightarrow IT\ (t \circ Var\ i)$
⟨*proof*⟩

10.2 Well-typed substitution preserves termination

lemma *subst-type-IT*:
  $\bigwedge t\ e\ T\ u\ i.\ IT\ t \Longrightarrow e\langle i{:}U\rangle \vdash t : T \Longrightarrow$
    $IT\ u \Longrightarrow e \vdash u : U \Longrightarrow IT\ (t[u/i])$
  (**is** $PROP\ ?P\ U$ **is** $\bigwedge t\ e\ T\ u\ i.\ \text{-} \Longrightarrow PROP\ ?Q\ t\ e\ T\ u\ i\ U$)
⟨*proof*⟩

10.3 Well-typed terms are strongly normalizing

lemma *type-implies-IT*:
  assumes $e \vdash t : T$
  shows $IT\ t$
  ⟨*proof*⟩

theorem *type-implies-termi*: $e \vdash t : T \Longrightarrow termip\ beta\ t$
⟨*proof*⟩

end

11 Inductive characterization of lambda terms in normal form

theory *NormalForm*
imports *ListBeta*
begin

11.1 Terms in normal form

definition
  $listall :: ('a \Rightarrow bool) \Rightarrow 'a\ list \Rightarrow bool$ **where**
  $listall\ P\ xs \equiv (\forall i.\ i < length\ xs \longrightarrow P\ (xs\ !\ i))$

declare *listall-def* [*extraction-expand*]

theorem *listall-nil*: $listall\ P\ []$
  ⟨*proof*⟩

theorem *listall-nil-eq* [*simp*]: $listall\ P\ [] = True$
  ⟨*proof*⟩

theorem *listall-cons*: $P\ x \Longrightarrow listall\ P\ xs \Longrightarrow listall\ P\ (x\ \#\ xs)$
  ⟨*proof*⟩

theorem *listall-cons-eq* [*simp*]: $listall\ P\ (x\ \#\ xs) = (P\ x \wedge listall\ P\ xs)$
  ⟨*proof*⟩

lemma *listall-conj1*: $listall\ (\lambda x.\ P\ x \wedge Q\ x)\ xs \Longrightarrow listall\ P\ xs$

⟨*proof*⟩

lemma *listall-conj2*: $listall\ (\lambda x.\ P\ x \wedge Q\ x)\ xs \Longrightarrow listall\ Q\ xs$
⟨*proof*⟩

lemma *listall-app*: $listall\ P\ (xs\ @\ ys) = (listall\ P\ xs \wedge listall\ P\ ys)$
⟨*proof*⟩

lemma *listall-snoc* [*simp*]: $listall\ P\ (xs\ @\ [x]) = (listall\ P\ xs \wedge P\ x)$
⟨*proof*⟩

lemma *listall-cong* [*cong*, *extraction-expand*]:
$xs = ys \Longrightarrow listall\ P\ xs = listall\ P\ ys$
— Currently needed for strange technical reasons
⟨*proof*⟩

listsp is equivalent to *listall*, but cannot be used for program extraction.

lemma *listall-listsp-eq*: $listall\ P\ xs = listsp\ P\ xs$
⟨*proof*⟩

inductive $NF :: dB \Rightarrow bool$
where
App: $listall\ NF\ ts \Longrightarrow NF\ (Var\ x\ {}^{\circ\circ}\ ts)$
| *Abs*: $NF\ t \Longrightarrow NF\ (Abs\ t)$
monos *listall-def*

lemma *nat-eq-dec*: $\bigwedge n::nat.\ m = n \vee m \neq n$
⟨*proof*⟩

lemma *nat-le-dec*: $\bigwedge n::nat.\ m < n \vee \neg\ (m < n)$
⟨*proof*⟩

lemma *App-NF-D*: **assumes** *NF*: $NF\ (Var\ n\ {}^{\circ\circ}\ ts)$
shows $listall\ NF\ ts$ ⟨*proof*⟩

11.2 Properties of *NF*

lemma *Var-NF*: $NF\ (Var\ n)$
⟨*proof*⟩

lemma *Abs-NF*:
assumes *NF*: $NF\ (Abs\ t\ {}^{\circ\circ}\ ts)$
shows $ts = []$ ⟨*proof*⟩

lemma *subst-terms-NF*: $listall\ NF\ ts \Longrightarrow$
$listall\ (\lambda t.\ \forall\, i\ j.\ NF\ (t[Var\ i/j]))\ ts \Longrightarrow$
$listall\ NF\ (map\ (\lambda t.\ t[Var\ i/j])\ ts)$
⟨*proof*⟩

lemma *subst-Var-NF*: $NF\ t \Longrightarrow NF\ (t[Var\ i/j])$
⟨*proof*⟩

lemma *app-Var-NF*: $NF\ t \Longrightarrow \exists\, t'.\ t \circ\ Var\ i \rightarrow_\beta{}^*\ t' \wedge NF\ t'$
⟨*proof*⟩

lemma *lift-terms-NF*: $listall\ NF\ ts \Longrightarrow$
$listall\ (\lambda t.\ \forall\, i.\ NF\ (lift\ t\ i))\ ts \Longrightarrow$
$listall\ NF\ (map\ (\lambda t.\ lift\ t\ i)\ ts)$
⟨*proof*⟩

lemma *lift-NF*: $NF\ t \Longrightarrow NF\ (lift\ t\ i)$
⟨*proof*⟩

NF characterizes exactly the terms that are in normal form.

lemma *NF-eq*: $NF\ t = (\forall\, t'.\ \neg\ t \rightarrow_\beta t')$
⟨*proof*⟩

end

12 Standardization

theory *Standardization*
imports *NormalForm*
begin

Based on lecture notes by Ralph Matthes [3], original proof idea due to Ralph Loader [2].

12.1 Standard reduction relation

declare *listrel-mono* [*mono-set*]

inductive
sred :: $dB \Rightarrow dB \Rightarrow bool$ (**infixl** $\rightarrow_s$ *50*)
and *sredlist* :: $dB\ list \Rightarrow dB\ list \Rightarrow bool$ (**infixl** $[\rightarrow_s]$ *50*)
where
$s\ [\rightarrow_s]\ t \equiv listrelp\ op \rightarrow_s\ s\ t$
| *Var*: $rs\ [\rightarrow_s]\ rs' \Longrightarrow Var\ x \circ\circ\ rs \rightarrow_s Var\ x \circ\circ\ rs'$
| *Abs*: $r \rightarrow_s r' \Longrightarrow ss\ [\rightarrow_s]\ ss' \Longrightarrow Abs\ r \circ\circ\ ss \rightarrow_s Abs\ r' \circ\circ\ ss'$
| *Beta*: $r[s/0] \circ\circ\ ss \rightarrow_s t \Longrightarrow Abs\ r \circ\ s \circ\circ\ ss \rightarrow_s t$

lemma *refl-listrelp*: $\forall\, x \in set\ xs.\ R\ x\ x \Longrightarrow listrelp\ R\ xs\ xs$
⟨*proof*⟩

lemma *refl-sred*: $t \rightarrow_s t$
⟨*proof*⟩

lemma *refl-sreds*: $ts\ [\rightarrow_s]\ ts$
  $\langle proof \rangle$

lemma *listrelp-conj1*: $listrelp\ (\lambda x\ y.\ R\ x\ y \wedge S\ x\ y)\ x\ y \Longrightarrow listrelp\ R\ x\ y$
  $\langle proof \rangle$

lemma *listrelp-conj2*: $listrelp\ (\lambda x\ y.\ R\ x\ y \wedge S\ x\ y)\ x\ y \Longrightarrow listrelp\ S\ x\ y$
  $\langle proof \rangle$

lemma *listrelp-app*:
  assumes *xsys*: $listrelp\ R\ xs\ ys$
  shows $listrelp\ R\ xs'\ ys' \Longrightarrow listrelp\ R\ (xs\ @\ xs')\ (ys\ @\ ys')$ $\langle proof \rangle$

lemma *lemma1*:
  assumes r: $r \rightarrow_s r'$ **and** s: $s \rightarrow_s s'$
  shows $r\ ^\circ\ s \rightarrow_s r'\ ^\circ\ s'$ $\langle proof \rangle$

lemma *lemma1'*:
  assumes *ts*: $ts\ [\rightarrow_s]\ ts'$
  shows $r \rightarrow_s r' \Longrightarrow r\ ^{\circ\circ}\ ts \rightarrow_s r'\ ^{\circ\circ}\ ts'$ $\langle proof \rangle$

lemma *lemma2-1*:
  assumes *beta*: $t \rightarrow_\beta u$
  shows $t \rightarrow_s u$ $\langle proof \rangle$

lemma *listrelp-betas*:
  assumes *ts*: $listrelp\ op \rightarrow_\beta{}^*\ ts\ ts'$
  shows $\bigwedge t\ t'.\ t \rightarrow_\beta{}^*\ t' \Longrightarrow t\ ^{\circ\circ}\ ts \rightarrow_\beta{}^*\ t'\ ^{\circ\circ}\ ts'$ $\langle proof \rangle$

lemma *lemma2-2*:
  assumes t: $t \rightarrow_s u$
  shows $t \rightarrow_\beta{}^*\ u$ $\langle proof \rangle$

lemma *sred-lift*:
  assumes s: $s \rightarrow_s t$
  shows $lift\ s\ i \rightarrow_s lift\ t\ i$ $\langle proof \rangle$

lemma *lemma3*:
  assumes r: $r \rightarrow_s r'$
  shows $s \rightarrow_s s' \Longrightarrow r[s/x] \rightarrow_s r'[s'/x]$ $\langle proof \rangle$

lemma *lemma4-aux*:
  assumes *rs*: $listrelp\ (\lambda t\ u.\ t \rightarrow_s u \wedge (\forall r.\ u \rightarrow_\beta r \longrightarrow t \rightarrow_s r))\ rs\ rs'$
  shows $rs' => ss \Longrightarrow rs\ [\rightarrow_s]\ ss$ $\langle proof \rangle$

lemma *lemma4*:
  assumes r: $r \rightarrow_s r'$
  shows $r' \rightarrow_\beta r'' \Longrightarrow r \rightarrow_s r''$ $\langle proof \rangle$

lemma *rtrancl-beta-sred*:
assumes r: $r \rightarrow_\beta{}^* r'$
shows $r \rightarrow_s r'$ $\langle proof \rangle$

12.2 Leftmost reduction and weakly normalizing terms

inductive
lred :: $dB \Rightarrow dB \Rightarrow bool$ (**infixl** $\rightarrow_l$ *50*)
and *lredlist* :: $dB\ list \Rightarrow dB\ list \Rightarrow bool$ (**infixl** $[\rightarrow_l]$ *50*)
where
$s\ [\rightarrow_l]\ t \equiv listrelp\ op \rightarrow_l\ s\ t$
| *Var*: $rs\ [\rightarrow_l]\ rs' \Longrightarrow Var\ x\ {}^{\circ\circ}\ rs \rightarrow_l Var\ x\ {}^{\circ\circ}\ rs'$
| *Abs*: $r \rightarrow_l r' \Longrightarrow Abs\ r \rightarrow_l Abs\ r'$
| *Beta*: $r[s/0]\ {}^{\circ\circ}\ ss \rightarrow_l t \Longrightarrow Abs\ r\ {}^\circ\ s\ {}^{\circ\circ}\ ss \rightarrow_l t$

lemma *lred-imp-sred*:
assumes *lred*: $s \rightarrow_l t$
shows $s \rightarrow_s t$ $\langle proof \rangle$

inductive *WN* :: *dB => bool*
where
Var: $listsp\ WN\ rs \Longrightarrow WN\ (Var\ n\ {}^{\circ\circ}\ rs)$
| *Lambda*: $WN\ r \Longrightarrow WN\ (Abs\ r)$
| *Beta*: $WN\ ((r[s/0])\ {}^{\circ\circ}\ ss) \Longrightarrow WN\ ((Abs\ r\ {}^\circ\ s)\ {}^{\circ\circ}\ ss)$

lemma *listrelp-imp-listsp1*:
assumes H: $listrelp\ (\lambda x\ y.\ P\ x)\ xs\ ys$
shows $listsp\ P\ xs$ $\langle proof \rangle$

lemma *listrelp-imp-listsp2*:
assumes H: $listrelp\ (\lambda x\ y.\ P\ y)\ xs\ ys$
shows $listsp\ P\ ys$ $\langle proof \rangle$

lemma *lemma5*:
assumes *lred*: $r \rightarrow_l r'$
shows $WN\ r$ **and** $NF\ r'$ $\langle proof \rangle$

lemma *lemma6*:
assumes *wn*: $WN\ r$
shows $\exists\, r'.\ r \rightarrow_l r'$ $\langle proof \rangle$

lemma *lemma7*:
assumes r: $r \rightarrow_s r'$
shows $NF\ r' \Longrightarrow r \rightarrow_l r'$ $\langle proof \rangle$

lemma *WN-eq*: $WN\ t = (\exists\, t'.\ t \rightarrow_\beta{}^* t' \wedge NF\ t')$
$\langle proof \rangle$

end

13 Weak normalization for simply-typed lambda calculus

theory *WeakNorm*
imports *Type NormalForm Code-Integer*
begin

Formalization by Stefan Berghofer. Partly based on a paper proof by Felix Joachimski and Ralph Matthes [1].

13.1 Main theorems

lemma *norm-list*:
  assumes *f-compat*: $\bigwedge t\ t'.\ t \to_\beta{}^*\ t' \Longrightarrow f\ t \to_\beta{}^*\ f\ t'$
  and *f-NF*: $\bigwedge t.\ NF\ t \Longrightarrow NF\ (f\ t)$
  and *uNF*: $NF\ u$ **and** *uT*: $e \vdash u : T$
  shows $\bigwedge Us.\ e\langle i{:}T\rangle \Vdash as : Us \Longrightarrow$
    $listall\ (\lambda t.\ \forall\, e\ T'\ u\ i.\ e\langle i{:}T\rangle \vdash t : T' \longrightarrow$
      $NF\ u \longrightarrow e \vdash u : T \longrightarrow (\exists\, t'.\ t[u/i] \to_\beta{}^*\ t' \wedge NF\ t'))\ as \Longrightarrow$
    $\exists\, as'.\ \forall j.\ Var\ j\ {}^{\circ\circ}\ map\ (\lambda t.\ f\ (t[u/i]))\ as \to_\beta{}^*$
      $Var\ j\ {}^{\circ\circ}\ map\ f\ as' \wedge NF\ (Var\ j\ {}^{\circ\circ}\ map\ f\ as')$
  (**is** $\bigwedge Us.\ \text{-} \Longrightarrow listall\ ?R\ as \Longrightarrow \exists\, as'.\ ?ex\ Us\ as\ as'$)
$\langle proof\rangle$

lemma *subst-type-NF*:
  $\bigwedge t\ e\ T\ u\ i.\ NF\ t \Longrightarrow e\langle i{:}U\rangle \vdash t : T \Longrightarrow NF\ u \Longrightarrow e \vdash u : U \Longrightarrow \exists\, t'.\ t[u/i] \to_\beta{}^*\ t' \wedge NF\ t'$
  (**is** $PROP\ ?P\ U$ **is** $\bigwedge t\ e\ T\ u\ i.\ \text{-} \Longrightarrow PROP\ ?Q\ t\ e\ T\ u\ i\ U$)
$\langle proof\rangle$
inductive *rtyping* :: $(nat \Rightarrow type) \Rightarrow dB \Rightarrow type \Rightarrow bool$ $(\text{-} \vdash_R \text{-} : \text{-}\ [50,\ 50,\ 50]\ 50)$
  where
    Var: $e\ x = T \Longrightarrow e \vdash_R Var\ x : T$
  | *Abs*: $e\langle 0{:}T\rangle \vdash_R t : U \Longrightarrow e \vdash_R Abs\ t : (T \Rightarrow U)$
  | *App*: $e \vdash_R s : T \Rightarrow U \Longrightarrow e \vdash_R t : T \Longrightarrow e \vdash_R (s\ {}^\circ\ t) : U$

lemma *rtyping-imp-typing*: $e \vdash_R t : T \Longrightarrow e \vdash t : T$
  $\langle proof\rangle$

theorem *type-NF*:
  assumes $e \vdash_R t : T$
  shows $\exists\, t'.\ t \to_\beta{}^*\ t' \wedge NF\ t'$ $\langle proof\rangle$

13.2 Extracting the program

declare *NF.induct* [*ind-realizer*]
declare *rtranclp.induct* [*ind-realizer irrelevant*]
declare *rtyping.induct* [*ind-realizer*]
lemmas [*extraction-expand*] = *conj-assoc listall-cons-eq*

extract *type-NF*

lemma *rtranclR-rtrancl-eq*: $rtranclpR\ r\ a\ b = r^{**}\ a\ b$
⟨*proof*⟩

lemma *NFR-imp-NF*: $NFR\ nf\ t \Longrightarrow NF\ t$
⟨*proof*⟩

The program corresponding to the proof of the central lemma, which performs substitution and normalization, is shown in Figure 1. The correctness theorem corresponding to the program *subst-type-NF* is

$$\begin{aligned}&\bigwedge x.\ NFR\ x\ t \Longrightarrow\\ &\quad e\langle i{:}U\rangle \vdash t : T \Longrightarrow\\ &\quad (\bigwedge xa.\ NFR\ xa\ u \Longrightarrow\\ &\qquad e \vdash u : U \Longrightarrow\\ &\qquad t[u/i] \rightarrow_\beta{}^*\ fst\ (subst\text{-}type\text{-}NF\ t\ e\ i\ U\ T\ u\ x\ xa) \wedge\\ &\qquad NFR\ (snd\ (subst\text{-}type\text{-}NF\ t\ e\ i\ U\ T\ u\ x\ xa))\ (fst\ (subst\text{-}type\text{-}NF\ t\ e\ i\ U\\ &T\ u\ x\ xa)))\end{aligned}$$

where *NFR* is the realizability predicate corresponding to the datatype *NFT*, which is inductively defined by the rules

```
subst-type-NF ≡
λx xa xb xc xd xe H Ha.
  type-induct-P xc
   (λx H2 H2a xa xb xc xd xe H.
       NFT-rec arbitrary
        (λts xa xaa r xb xc xd xe H.
            var-app-typesE-P (xb⟨xe:x⟩) xa ts
             (λUs--. case nat-eq-dec xa xe of
                      Left ⇒ case ts of [] ⇒ (xd, H)
                             | a # list ⇒
                                case Us-- of [] ⇒ arbitrary
                                | T''-- # Ts-- ⇒
                                   let (x, y) =
                                          norm-list (λt. lift t 0) xd xb xe list Ts--
                                           (λt. lift-NF 0) H
                                           (listall-conj2-P-Q list (λi. (xaa (Suc i), r (Suc i))));
                                       (xa, ya) = snd (xaa 0, r 0) xb T''-- xd xe H;
                                       (xd, yb) = app-Var-NF 0 (lift-NF 0 H);
                                       (xa, ya) =
                                         H2 T''-- (Ts-- ⇛ xc) xd xb (Ts-- ⇛ xc) xa 0 yb ya;
                                       (x, y) =
                                         H2a T''-- (Ts-- ⇛ xc) (dB.Var 0 °° map (λt. lift t 0) x)
                                          xb xc xa 0 (y 0) ya
                                   in (x, y)
                      | Right ⇒
                         let (x, y) =
                               let (x, y) =
                                     norm-list (λt. t) xd xb xe ts Us-- (λx H. H) H
                                      (listall-conj2-P-Q ts (λz. (xaa z, r z)))
                               in (x, λx. y x)
                         in case nat-le-dec xe xa of
                             Left ⇒ (dB.Var (xa − Suc 0) °° x, y (xa − Suc 0))
                             | Right ⇒ (dB.Var xa °° x, y xa)))
        (λt x r xa xb xc xd H.
            abs-typeE-P xb
             (λU V. let (x, y) =
                         let (x, y) = r (λa. (xa⟨0:U⟩) a) V (lift xc 0) (Suc xd) (lift-NF 0 H)
                         in (dB.Abs x, NFT.Abs x y)
                     in (x, y)))
        H (λa. xb a) xc xd xe)
   x xa xd xe xb H Ha
```

Figure 1: Program extracted from *subst-type-NF*

```
subst-Var-NF ≡
λx xa H.
  NFT-rec arbitrary
   (λts x xa r xb xc.
       case nat-eq-dec x xc of
       Left ⇒ NFT.App (map (λt. t[dB.Var xb/xc]) ts) xb
                 (subst-terms-NF ts xb xc (listall-conj1-P-Q ts (λz. (xa z, r z)))
                   (listall-conj2-P-Q ts (λz. (xa z, r z))))
       | Right ⇒
           case nat-le-dec xc x of
           Left ⇒ NFT.App (map (λt. t[dB.Var xb/xc]) ts) (x − Suc 0)
                     (subst-terms-NF ts xb xc (listall-conj1-P-Q ts (λz. (xa z, r z)))
                       (listall-conj2-P-Q ts (λz. (xa z, r z))))
           | Right ⇒
               NFT.App (map (λt. t[dB.Var xb/xc]) ts) x
                (subst-terms-NF ts xb xc (listall-conj1-P-Q ts (λz. (xa z, r z)))
                  (listall-conj2-P-Q ts (λz. (xa z, r z)))))
   (λt x r xa xb. NFT.Abs (t[dB.Var (Suc xa)/Suc xb]) (r (Suc xa) (Suc xb))) H x xa

app-Var-NF ≡
λx. NFT-rec arbitrary
     (λts xa xaa r.
         (dB.Var xa °° (ts @ [dB.Var x]),
          NFT.App (ts @ [dB.Var x]) xa
           (snd (listall-app-P ts)
             (listall-conj1-P-Q ts (λz. (xaa z, r z)),
              listall-cons-P (Var-NF x) listall-nil-eq-P))))
     (λt xa r. (t[dB.Var x/0], subst-Var-NF x 0 xa))

lift-NF ≡
λx H. NFT-rec arbitrary
        (λts x xa r xb.
            case nat-le-dec x xb of
            Left ⇒ NFT.App (map (λt. lift t xb) ts) x
                      (lift-terms-NF ts xb (listall-conj1-P-Q ts (λz. (xa z, r z)))
                        (listall-conj2-P-Q ts (λz. (xa z, r z))))
            | Right ⇒
                NFT.App (map (λt. lift t xb) ts) (Suc x)
                 (lift-terms-NF ts xb (listall-conj1-P-Q ts (λz. (xa z, r z)))
                   (listall-conj2-P-Q ts (λz. (xa z, r z)))))
        (λt x r xa. NFT.Abs (lift t (Suc xa)) (r (Suc xa))) H x

type-NF ≡
λH. rtypingT-rec (λe x T. (dB.Var x, Var-NF x))
      (λe T t U x r. let (x, y) = r in (dB.Abs x, NFT.Abs x y))
      (λe s T U t x xa r ra.
          let (x, y) = r; (xa, ya) = ra;
              (x, y) =
                let (x, y) =
                      subst-type-NF (dB.App (dB.Var 0) (lift xa 0)) e 0 (T ⇒ U) U x
                       (NFT.App [lift xa 0] 0 (listall-cons-P (lift-NF 0 ya) listall-nil-P)) y
                in (x, y)
          in (x, y))
      H
```

Figure 2: Program extracted from lemmas and main theorem

$\forall\, i{<}length\ ts.\ NFR\ (nfs\ i)\ (ts\ !\ i) \Longrightarrow NFR\ (NFT.App\ ts\ x\ nfs)\ (dB.Var\ x\ {}^{\circ\circ}\ ts)$
$NFR\ nf\ t \Longrightarrow NFR\ (NFT.Abs\ t\ nf)\ (dB.Abs\ t)$

The programs corresponding to the main theorem *type-NF*, as well as to some lemmas, are shown in Figure 2. The correctness statement for the main function *type-NF* is

$\bigwedge x.\ rtypingR\ x\ e\ t\ T \Longrightarrow t \rightarrow_\beta{}^*\ fst\ (type\text{-}NF\ x) \wedge NFR\ (snd\ (type\text{-}NF\ x))\ (fst\ (type\text{-}NF\ x))$

where the realizability predicate *rtypingR* corresponding to the computationally relevant version of the typing judgement is inductively defined by the rules

$e\ x = T \Longrightarrow rtypingR\ (rtypingT.Var\ e\ x\ T)\ e\ (dB.Var\ x)\ T$
$rtypingR\ ty\ (e\langle 0{:}T\rangle)\ t\ U \Longrightarrow rtypingR\ (rtypingT.Abs\ e\ T\ t\ U\ ty)\ e\ (dB.Abs\ t)\ (T \Rightarrow U)$
$rtypingR\ ty\ e\ s\ (T \Rightarrow U) \Longrightarrow$
$rtypingR\ ty'\ e\ t\ T \Longrightarrow rtypingR\ (rtypingT.App\ e\ s\ T\ U\ t\ ty\ ty')\ e\ (dB.App\ s\ t)\ U$

13.3 Generating executable code

consts-code
  arbitrary :: *'a*   ((*error arbitrary*))
  arbitrary :: *'a* ⇒ *'b* ((*fn '- => error arbitrary*))

code-module *Norm*
contains
  test = *type-NF*

The following functions convert between Isabelle's built-in `term` datatype and the generated `dB` datatype. This allows to generate example terms using Isabelle's parser and inspect normalized terms using Isabelle's pretty printer.

⟨*ML*⟩

We now try out the extracted program *type-NF* on some example terms.

⟨*ML*⟩

The same story again for code next generation.

⟨*ML*⟩

definition
  int-of-nat :: *nat* ⇒ *int* **where**
  int-of-nat = *of-nat*

export-code *type-NF nat int-of-nat* **in** *SML* **module-name** *Norm*

$\langle ML \rangle$

end

References

[1] F. Joachimski and R. Matthes. Short proofs of normalization for the simply-typed λ-calculus, permutative conversions and Gödel's T. *Archive for Mathematical Logic*, 42(1):59–87, 2003.

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[4] M. Takahashi. Parallel reductions in λ-calculus. *Information and Computation*, 118(1):120–127, April 1995.