Isabelle/HOL-Complex — Higher-Order Logic with Complex Numbers

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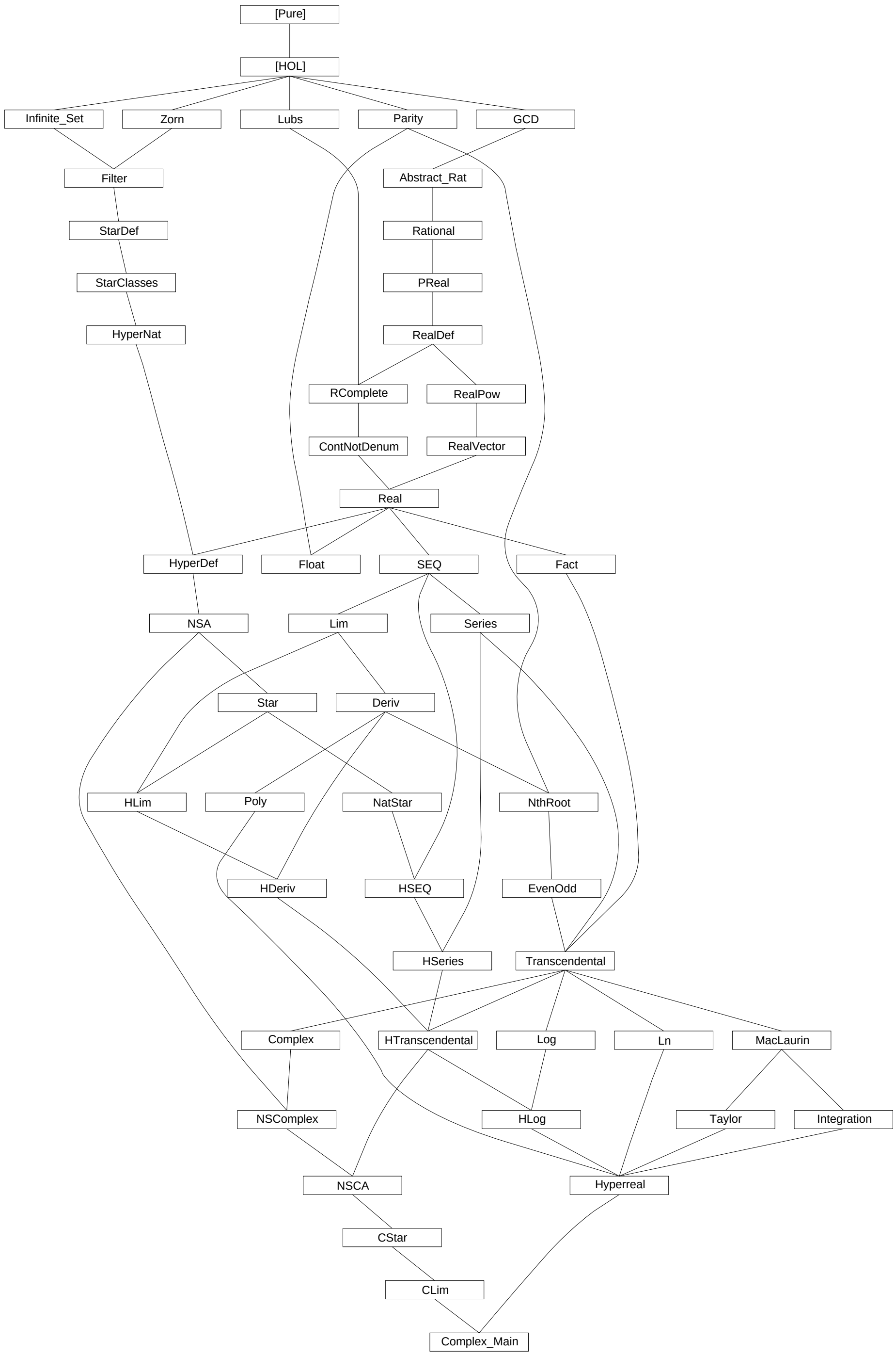
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1 Lubs: Definitions of Upper Bounds and Least Upper Bounds

theory *Lubs*
imports *Main*
begin

Thanks to suggestions by James Margetson

definition
setle :: ['a set, 'a::ord] => bool (**infixl** *<= 70) **where**
S *<= x = (ALL y: S. y <= x)

definition
setge :: ['a::ord, 'a set] => bool (**infixl** <=* 70) **where**
x <=* S = (ALL y: S. x <= y)

definition
leastP :: ['a =>bool,'a::ord] => bool **where**
leastP P x = (P x & x <=* Collect P)

definition
isUb :: ['a set, 'a set, 'a::ord] => bool **where**
isUb R S x = (S *<= x & x: R)

definition
isLub :: ['a set, 'a set, 'a::ord] => bool **where**
isLub R S x = leastP (isUb R S) x

definition
ubs :: ['a set, 'a::ord set] => 'a set **where**
ubs R S = Collect (isUb R S)

1.1 Rules for the Relations *<= and <=*

lemma *setleI*: ALL y: S. y <= x ==> S *<= x
by (simp add: setle-def)

lemma *setleD*: [| S *<= x; y: S |] ==> y <= x
by (simp add: setle-def)

lemma *setgeI*: ALL y: S. x<= y ==> x <=* S
by (simp add: setge-def)

lemma *setgeD*: [| x <=* S; y: S |] ==> x <= y
by (simp add: setge-def)

1.2 Rules about the Operators *leastP*, *ub* and *lub*

lemma *leastPD1*: leastP P x ==> P x

by (simp add: leastP-def)

lemma leastPD2: leastP P x ==> x <=∗ Collect P
by (simp add: leastP-def)

lemma leastPD3: [| leastP P x; y: Collect P |] ==> x <= y
by (blast dest!: leastPD2 setgeD)

lemma isLubD1: isLub R S x ==> S ∗<= x
by (simp add: isLub-def isUb-def leastP-def)

lemma isLubD1a: isLub R S x ==> x: R
by (simp add: isLub-def isUb-def leastP-def)

lemma isLub-isUb: isLub R S x ==> isUb R S x
apply (simp add: isUb-def)
apply (blast dest: isLubD1 isLubD1a)
done

lemma isLubD2: [| isLub R S x; y : S |] ==> y <= x
by (blast dest!: isLubD1 setleD)

lemma isLubD3: isLub R S x ==> leastP(isUb R S) x
by (simp add: isLub-def)

lemma isLubI1: leastP(isUb R S) x ==> isLub R S x
by (simp add: isLub-def)

lemma isLubI2: [| isUb R S x; x <=∗ Collect (isUb R S) |] ==> isLub R S x
by (simp add: isLub-def leastP-def)

lemma isUbD: [| isUb R S x; y : S |] ==> y <= x
by (simp add: isUb-def setle-def)

lemma isUbD2: isUb R S x ==> S ∗<= x
by (simp add: isUb-def)

lemma isUbD2a: isUb R S x ==> x: R
by (simp add: isUb-def)

lemma isUbI: [| S ∗<= x; x: R |] ==> isUb R S x
by (simp add: isUb-def)

lemma isLub-le-isUb: [| isLub R S x; isUb R S y |] ==> x <= y
apply (simp add: isLub-def)
apply (blast intro!: leastPD3)
done

lemma isLub-ubs: isLub R S x ==> x <=∗ ubs R S

apply (*simp add*: *ubs-def isLub-def*)
apply (*erule leastPD2*)
done

end

2 GCD: The Greatest Common Divisor

theory *GCD*
imports *Main*
begin

See [?].

2.1 Specification of GCD on nats

definition
  is-gcd :: *nat* ⇒ *nat* ⇒ *nat* ⇒ *bool* **where** — *gcd* as a relation
  is-gcd p m n ⟷ *p dvd m* ∧ *p dvd n* ∧
    (∀ *d*. *d dvd m* ⟶ *d dvd n* ⟶ *d dvd p*)

Uniqueness

lemma *is-gcd-unique*: *is-gcd m a b* ⟹ *is-gcd n a b* ⟹ *m* = *n*
  by (*simp add*: *is-gcd-def*) (*blast intro*: *dvd-anti-sym*)

Connection to divides relation

lemma *is-gcd-dvd*: *is-gcd m a b* ⟹ *k dvd a* ⟹ *k dvd b* ⟹ *k dvd m*
  by (*auto simp add*: *is-gcd-def*)

Commutativity

lemma *is-gcd-commute*: *is-gcd k m n* = *is-gcd k n m*
  by (*auto simp add*: *is-gcd-def*)

2.2 GCD on nat by Euclid's algorithm

fun
  gcd :: *nat* × *nat* => *nat*
where
  gcd (*m*, *n*) = (*if n* = *0 then m else gcd* (*n*, *m mod n*))

lemma *gcd-induct*:
  fixes *m n* :: *nat*
  assumes ⋀*m*. *P m 0*
    and ⋀*m n*. *0* < *n* ⟹ *P n* (*m mod n*) ⟹ *P m n*
  shows *P m n*
apply (*rule gcd.induct* [*of split P* (*m*, *n*), *unfolded Product-Type.split*])

```
apply (case-tac n = 0)
apply simp-all
using assms apply simp-all
done

lemma gcd-0 [simp]: gcd (m, 0) = m
  by simp

lemma gcd-0-left [simp]: gcd (0, m) = m
  by simp

lemma gcd-non-0: n > 0 ⟹ gcd (m, n) = gcd (n, m mod n)
  by simp

lemma gcd-1 [simp]: gcd (m, Suc 0) = 1
  by simp

declare gcd.simps [simp del]
```

gcd (m, n) divides m and n. The conjunctions don't seem provable separately.

```
lemma gcd-dvd1 [iff]: gcd (m, n) dvd m
  and gcd-dvd2 [iff]: gcd (m, n) dvd n
  apply (induct m n rule: gcd-induct)
    apply (simp-all add: gcd-non-0)
  apply (blast dest: dvd-mod-imp-dvd)
  done
```

Maximality: for all m, n, k naturals, if k divides m and k divides n then k divides *gcd* (m, n).

```
lemma gcd-greatest: k dvd m ⟹ k dvd n ⟹ k dvd gcd (m, n)
  by (induct m n rule: gcd-induct) (simp-all add: gcd-non-0 dvd-mod)
```

Function gcd yields the Greatest Common Divisor.

```
lemma is-gcd: is-gcd (gcd (m, n)) m n
  by (simp add: is-gcd-def gcd-greatest)
```

2.3 Derived laws for GCD

```
lemma gcd-greatest-iff [iff]: k dvd gcd (m, n) ⟷ k dvd m ∧ k dvd n
  by (blast intro!: gcd-greatest intro: dvd-trans)

lemma gcd-zero: gcd (m, n) = 0 ⟷ m = 0 ∧ n = 0
  by (simp only: dvd-0-left-iff [symmetric] gcd-greatest-iff)

lemma gcd-commute: gcd (m, n) = gcd (n, m)
  apply (rule is-gcd-unique)
```

apply (*rule is-gcd*)
apply (*subst is-gcd-commute*)
apply (*simp add*: *is-gcd*)
done

lemma *gcd-assoc*: *gcd* (*gcd* (*k*, *m*), *n*) = *gcd* (*k*, *gcd* (*m*, *n*))
apply (*rule is-gcd-unique*)
apply (*rule is-gcd*)
apply (*simp add*: *is-gcd-def*)
apply (*blast intro*: *dvd-trans*)
done

lemma *gcd-1-left* [*simp*]: *gcd* (*Suc 0*, *m*) = *1*
by (*simp add*: *gcd-commute*)

Multiplication laws

lemma *gcd-mult-distrib2*: *k* ∗ *gcd* (*m*, *n*) = *gcd* (*k* ∗ *m*, *k* ∗ *n*)
— [**?**, page 27]
apply (*induct m n rule*: *gcd-induct*)
apply *simp*
apply (*case-tac k* = *0*)
apply (*simp-all add*: *mod-geq gcd-non-0 mod-mult-distrib2*)
done

lemma *gcd-mult* [*simp*]: *gcd* (*k*, *k* ∗ *n*) = *k*
apply (*rule gcd-mult-distrib2* [*of k 1 n*, *simplified*, *symmetric*])
done

lemma *gcd-self* [*simp*]: *gcd* (*k*, *k*) = *k*
apply (*rule gcd-mult* [*of k 1*, *simplified*])
done

lemma *relprime-dvd-mult*: *gcd* (*k*, *n*) = *1* ==> *k dvd m* ∗ *n* ==> *k dvd m*
apply (*insert gcd-mult-distrib2* [*of m k n*])
apply *simp*
apply (*erule-tac t* = *m* **in** *ssubst*)
apply *simp*
done

lemma *relprime-dvd-mult-iff*: *gcd* (*k*, *n*) = *1* ==> (*k dvd m* ∗ *n*) = (*k dvd m*)
apply (*blast intro*: *relprime-dvd-mult dvd-trans*)
done

lemma *gcd-mult-cancel*: *gcd* (*k*, *n*) = *1* ==> *gcd* (*k* ∗ *m*, *n*) = *gcd* (*m*, *n*)
apply (*rule dvd-anti-sym*)
apply (*rule gcd-greatest*)
apply (*rule-tac n* = *k* **in** *relprime-dvd-mult*)
apply (*simp add*: *gcd-assoc*)
apply (*simp add*: *gcd-commute*)

```
  apply (simp-all add: mult-commute)
 apply (blast intro: dvd-trans)
 done
```

Addition laws

```
lemma gcd-add1 [simp]: gcd (m + n, n) = gcd (m, n)
 apply (case-tac n = 0)
  apply (simp-all add: gcd-non-0)
 done

lemma gcd-add2 [simp]: gcd (m, m + n) = gcd (m, n)
proof −
 have gcd (m, m + n) = gcd (m + n, m) by (rule gcd-commute)
 also have ... = gcd (n + m, m) by (simp add: add-commute)
 also have ... = gcd (n, m) by simp
 also have ... = gcd (m, n) by (rule gcd-commute)
 finally show ?thesis .
qed

lemma gcd-add2' [simp]: gcd (m, n + m) = gcd (m, n)
 apply (subst add-commute)
 apply (rule gcd-add2)
 done

lemma gcd-add-mult: gcd (m, k ∗ m + n) = gcd (m, n)
 by (induct k) (simp-all add: add-assoc)

lemma gcd-dvd-prod: gcd (m, n) dvd m ∗ n
 using mult-dvd-mono [of 1] by auto
```

Division by gcd yields rrelatively primes.

```
lemma div-gcd-relprime:
 assumes nz: a ≠ 0 ∨ b ≠ 0
 shows gcd (a div gcd(a,b), b div gcd(a,b)) = 1
proof −
 let ?g = gcd (a, b)
 let ?a' = a div ?g
 let ?b' = b div ?g
 let ?g' = gcd (?a', ?b')
 have dvdg: ?g dvd a ?g dvd b by simp-all
 have dvdg': ?g' dvd ?a' ?g' dvd ?b' by simp-all
 from dvdg dvdg' obtain ka kb ka' kb' where
    kab: a = ?g ∗ ka b = ?g ∗ kb ?a' = ?g' ∗ ka' ?b' = ?g' ∗ kb'
   unfolding dvd-def by blast
 then have ?g ∗ ?a' = (?g ∗ ?g') ∗ ka' ?g ∗ ?b' = (?g ∗ ?g') ∗ kb' by simp-all
 then have dvdgg':?g ∗ ?g' dvd a ?g∗ ?g' dvd b
   by (auto simp add: dvd-mult-div-cancel [OF dvdg(1)]
     dvd-mult-div-cancel [OF dvdg(2)] dvd-def)
```

have *?g* $\neq$ *0* **using** *nz* **by** (*simp add*: *gcd-zero*)
then have *gp*: *?g* > *0* **by** *simp*
from *gcd-greatest* [*OF dvdgg'*] **have** *?g* $*$ *?g'* *dvd* *?g* **.**
with *dvd-mult-cancel1* [*OF gp*] **show** *?g'* = *1* **by** *simp*
qed

2.4 LCM defined by GCD

definition
lcm :: *nat* × *nat* ⇒ *nat*
where
lcm = (λ(*m*, *n*). *m* $*$ *n* *div* *gcd* (*m*, *n*))

lemma *lcm-def*:
lcm (*m*, *n*) = *m* $*$ *n* *div* *gcd* (*m*, *n*)
unfolding *lcm-def* **by** *simp*

lemma *prod-gcd-lcm*:
m $*$ *n* = *gcd* (*m*, *n*) $*$ *lcm* (*m*, *n*)
unfolding *lcm-def* **by** (*simp add*: *dvd-mult-div-cancel* [*OF gcd-dvd-prod*])

lemma *lcm-0* [*simp*]: *lcm* (*m*, *0*) = *0*
unfolding *lcm-def* **by** *simp*

lemma *lcm-1* [*simp*]: *lcm* (*m*, *1*) = *m*
unfolding *lcm-def* **by** *simp*

lemma *lcm-0-left* [*simp*]: *lcm* (*0*, *n*) = *0*
unfolding *lcm-def* **by** *simp*

lemma *lcm-1-left* [*simp*]: *lcm* (*1*, *m*) = *m*
unfolding *lcm-def* **by** *simp*

lemma *dvd-pos*:
fixes *n m* :: *nat*
assumes *n* > *0* **and** *m dvd n*
shows *m* > *0*
using *assms* **by** (*cases m*) *auto*

lemma *lcm-least*:
assumes *m dvd k* **and** *n dvd k*
shows *lcm* (*m*, *n*) *dvd k*
proof (*cases k*)
case *0* **then show** *?thesis* **by** *auto*
next
case (*Suc -*) **then have** *pos-k*: *k* > *0* **by** *auto*
from *assms dvd-pos* [*OF this*] **have** *pos-mn*: *m* > *0* *n* > *0* **by** *auto*
with *gcd-zero* [*of m n*] **have** *pos-gcd*: *gcd* (*m*, *n*) > *0* **by** *simp*
from *assms* **obtain** *p* **where** *k-m*: *k* = *m* $*$ *p* **using** *dvd-def* **by** *blast*

```
  from assms obtain q where k-n: k = n * q using dvd-def by blast
  from pos-k k-m have pos-p: p > 0 by auto
  from pos-k k-n have pos-q: q > 0 by auto
  have k * k * gcd (q, p) = k * gcd (k * q, k * p)
    by (simp add: mult-ac gcd-mult-distrib2)
  also have ... = k * gcd (m * p * q, n * q * p)
    by (simp add: k-m [symmetric] k-n [symmetric])
  also have ... = k * p * q * gcd (m, n)
    by (simp add: mult-ac gcd-mult-distrib2)
  finally have (m * p) * (n * q) * gcd (q, p) = k * p * q * gcd (m, n)
    by (simp only: k-m [symmetric] k-n [symmetric])
  then have p * q * m * n * gcd (q, p) = p * q * k * gcd (m, n)
    by (simp add: mult-ac)
  with pos-p pos-q have m * n * gcd (q, p) = k * gcd (m, n)
    by simp
  with prod-gcd-lcm [of m n]
  have lcm (m, n) * gcd (q, p) * gcd (m, n) = k * gcd (m, n)
    by (simp add: mult-ac)
  with pos-gcd have lcm (m, n) * gcd (q, p) = k by simp
  then show ?thesis using dvd-def by auto
qed

lemma lcm-dvd1 [iff]:
  m dvd lcm (m, n)
proof (cases m)
  case 0 then show ?thesis by simp
next
  case (Suc -)
  then have mpos: m > 0 by simp
  show ?thesis
  proof (cases n)
    case 0 then show ?thesis by simp
  next
    case (Suc -)
    then have npos: n > 0 by simp
    have gcd (m, n) dvd n by simp
    then obtain k where n = gcd (m, n) * k using dvd-def by auto
    then have m * n div gcd (m, n) = m * (gcd (m, n) * k) div gcd (m, n) by
(simp add: mult-ac)
    also have ... = m * k using mpos npos gcd-zero by simp
    finally show ?thesis by (simp add: lcm-def)
  qed
qed

lemma lcm-dvd2 [iff]:
  n dvd lcm (m, n)
proof (cases n)
  case 0 then show ?thesis by simp
next
```

case (*Suc* -)
then have *npos*: $n > 0$ **by** *simp*
show *?thesis*
proof (*cases m*)
case *0* **then show** *?thesis* **by** *simp*
next
case (*Suc* -)
then have *mpos*: $m > 0$ **by** *simp*
have *gcd* (*m*, *n*) *dvd m* **by** *simp*
then obtain *k* **where** $m = gcd\ (m,\ n) * k$ **using** *dvd-def* **by** *auto*
then have $m * n\ div\ gcd\ (m,\ n) = (gcd\ (m,\ n) * k) * n\ div\ gcd\ (m,\ n)$ **by** (*simp add*: *mult-ac*)
also have $\ldots = n * k$ **using** *mpos npos gcd-zero* **by** *simp*
finally show *?thesis* **by** (*simp add*: *lcm-def*)
qed
qed

2.5 GCD and LCM on integers

definition
igcd :: *int* ⇒ *int* ⇒ *int* **where**
igcd i j = *int* (*gcd* (*nat* (*abs i*), *nat* (*abs j*)))

lemma *igcd-dvd1* [*simp*]: *igcd i j dvd i*
by (*simp add*: *igcd-def int-dvd-iff*)

lemma *igcd-dvd2* [*simp*]: *igcd i j dvd j*
by (*simp add*: *igcd-def int-dvd-iff*)

lemma *igcd-pos*: $igcd\ i\ j \geq 0$
by (*simp add*: *igcd-def*)

lemma *igcd0* [*simp*]: $(igcd\ i\ j = 0) = (i = 0 \wedge j = 0)$
by (*simp add*: *igcd-def gcd-zero*) *arith*

lemma *igcd-commute*: *igcd i j* = *igcd j i*
unfolding *igcd-def* **by** (*simp add*: *gcd-commute*)

lemma *igcd-neg1* [*simp*]: $igcd\ (-\ i)\ j = igcd\ i\ j$
unfolding *igcd-def* **by** *simp*

lemma *igcd-neg2* [*simp*]: $igcd\ i\ (-\ j) = igcd\ i\ j$
unfolding *igcd-def* **by** *simp*

lemma *zrelprime-dvd-mult*: $igcd\ i\ j = 1 \Longrightarrow i\ dvd\ k * j \Longrightarrow i\ dvd\ k$
unfolding *igcd-def*
proof −
assume $int\ (gcd\ (nat\ |i|,\ nat\ |j|)) = 1\ i\ dvd\ k * j$
then have *g*: $gcd\ (nat\ |i|,\ nat\ |j|) = 1$ **by** *simp*

```
  from ‹i dvd k * j› obtain h where h: k*j = i*h unfolding dvd-def by blast
  have th: nat |i| dvd nat |k| * nat |j|
    unfolding dvd-def
    by (rule-tac x= nat |h| in exI, simp add: h nat-abs-mult-distrib [symmetric])
  from relprime-dvd-mult [OF g th] obtain h' where h': nat |k| = nat |i| * h'
    unfolding dvd-def by blast
  from h' have int (nat |k|) = int (nat |i| * h') by simp
  then have |k| = |i| * int h' by (simp add: int-mult)
  then show ?thesis
    apply (subst zdvd-abs1 [symmetric])
    apply (subst zdvd-abs2 [symmetric])
    apply (unfold dvd-def)
    apply (rule-tac x = int h' in exI, simp)
    done
qed

lemma int-nat-abs: int (nat (abs x)) = abs x  by arith

lemma igcd-greatest:
  assumes k dvd m and k dvd n
  shows k dvd igcd m n
proof -
  let ?k' = nat |k|
  let ?m' = nat |m|
  let ?n' = nat |n|
  from ‹k dvd m› and ‹k dvd n› have dvd': ?k' dvd ?m' ?k' dvd ?n'
    unfolding zdvd-int by (simp-all only: int-nat-abs zdvd-abs1 zdvd-abs2)
  from gcd-greatest [OF dvd'] have int (nat |k|) dvd igcd m n
    unfolding igcd-def by (simp only: zdvd-int)
  then have |k| dvd igcd m n by (simp only: int-nat-abs)
  then show k dvd igcd m n by (simp add: zdvd-abs1)
qed

lemma div-igcd-relprime:
  assumes nz: a ≠ 0 ∨ b ≠ 0
  shows igcd (a div (igcd a b)) (b div (igcd a b)) = 1
proof -
  from nz have nz': nat |a| ≠ 0 ∨ nat |b| ≠ 0 by arith
  let ?g = igcd a b
  let ?a' = a div ?g
  let ?b' = b div ?g
  let ?g' = igcd ?a' ?b'
  have dvdg: ?g dvd a ?g dvd b by (simp-all add: igcd-dvd1 igcd-dvd2)
  have dvdg': ?g' dvd ?a' ?g' dvd ?b' by (simp-all add: igcd-dvd1 igcd-dvd2)
  from dvdg dvdg' obtain ka kb ka' kb' where
   kab: a = ?g*ka b = ?g*kb ?a' = ?g'*ka' ?b' = ?g' * kb'
    unfolding dvd-def by blast
  then have ?g* ?a' = (?g * ?g') * ka' ?g* ?b' = (?g * ?g') * kb' by simp-all
  then have dvdgg':?g * ?g' dvd a ?g* ?g' dvd b
```

```
    by (auto simp add: zdvd-mult-div-cancel [OF dvdg(1)]
      zdvd-mult-div-cancel [OF dvdg(2)] dvd-def)
  have ?g ≠ 0 using nz by simp
  then have gp: ?g ≠ 0 using igcd-pos[where i=a and j=b] by arith
  from igcd-greatest [OF dvdgg'] have ?g * ?g' dvd ?g .
  with zdvd-mult-cancel1 [OF gp] have |?g'| = 1 by simp
  with igcd-pos show ?g' = 1 by simp
qed

definition ilcm = (λi j. int (lcm(nat(abs i),nat(abs j))))

lemma dvd-ilcm-self1[simp]: i dvd ilcm i j
by(simp add:ilcm-def dvd-int-iff)

lemma dvd-ilcm-self2[simp]: j dvd ilcm i j
by(simp add:ilcm-def dvd-int-iff)


lemma dvd-imp-dvd-ilcm1:
  assumes k dvd i shows k dvd (ilcm i j)
proof -
  have nat(abs k) dvd nat(abs i) using ‹k dvd i›
    by(simp add:int-dvd-iff[symmetric] dvd-int-iff[symmetric] zdvd-abs1)
  thus ?thesis by(simp add:ilcm-def dvd-int-iff)(blast intro: dvd-trans)
qed

lemma dvd-imp-dvd-ilcm2:
  assumes k dvd j shows k dvd (ilcm i j)
proof -
  have nat(abs k) dvd nat(abs j) using ‹k dvd j›
    by(simp add:int-dvd-iff[symmetric] dvd-int-iff[symmetric] zdvd-abs1)
  thus ?thesis by(simp add:ilcm-def dvd-int-iff)(blast intro: dvd-trans)
qed


lemma zdvd-self-abs1: (d::int) dvd (abs d)
by (case-tac d <0, simp-all)

lemma zdvd-self-abs2: (abs (d::int)) dvd d
by (case-tac d<0, simp-all)



lemma lcm-pos:
  assumes mpos: m > 0
  and npos: n>0
  shows lcm (m,n) > 0
proof(rule ccontr, simp add: lcm-def gcd-zero)
assume h:m*n div gcd(m,n) = 0
```

from *mpos npos* **have** $gcd\ (m,n) \neq 0$ **using** *gcd-zero* **by** *simp*
hence *gcdp*: $gcd(m,n) > 0$ **by** *simp*
with *h*
have $m*n < gcd(m,n)$
 by (*cases* $m * n < gcd\ (m,\ n)$) (*auto simp add*: *div-if* [*OF gcdp*, **where** $m=m*n$])
moreover
have *gcd*(*m*,*n*) *dvd m* **by** *simp*
 with *mpos dvd-imp-le* **have** *t1*:$gcd(m,n) \leq m$ **by** *simp*
 with *npos* **have** *t1*:$gcd(m,n)*n \leq m*n$ **by** *simp*
 have $gcd(m,n) \leq gcd(m,n)*n$ **using** *npos* **by** *simp*
 with *t1* **have** $gcd(m,n) \leq m*n$ **by** *arith*
ultimately show *False* **by** *simp*
qed

lemma *ilcm-pos*:
 assumes *anz*: $a \neq 0$
 and *bnz*: $b \neq 0$
 shows $0 < ilcm\ a\ b$
proof−
 let *?na* = *nat* (*abs a*)
 let *?nb* = *nat* (*abs b*)
 have *nap*: *?na* >*0* **using** *anz* **by** *simp*
 have *nbp*: *?nb* >*0* **using** *bnz* **by** *simp*
 have *0* < *lcm* (*?na*,*?nb*) **by** (*rule lcm-pos*[*OF nap nbp*])
 thus *?thesis* **by** (*simp add*: *ilcm-def*)
qed

end

3 Abstract-Rat: Abstract rational numbers

theory *Abstract-Rat*
imports *GCD*
begin

types $Num = int \times int$

abbreviation
 Num0-syn :: *Num* (0_N)
where $0_N \equiv (0,\ 0)$

abbreviation
 Numi-syn :: $int \Rightarrow Num$ (-_N)
where $i_N \equiv (i,\ 1)$

definition
 isnormNum :: $Num \Rightarrow bool$
where

```
  isnormNum = (λ(a,b). (if a = 0 then b = 0 else b > 0 ∧ igcd a b = 1))

definition
  normNum :: Num ⇒ Num
where
  normNum = (λ(a,b). (if a=0 ∨ b = 0 then (0,0) else
  (let g = igcd a b
   in if b > 0 then (a div g, b div g) else (− (a div g), − (b div g)))))

lemma normNum-isnormNum [simp]: isnormNum (normNum x)
proof −
  have ∃ a b. x = (a,b) by auto
  then obtain a b where x[simp]: x = (a,b) by blast
 {assume a=0 ∨ b = 0 hence ?thesis by (simp add: normNum-def isnormNum-def)}

  moreover
  {assume anz: a ≠ 0 and bnz: b ≠ 0
    let ?g = igcd a b
    let ?a' = a div ?g
    let ?b' = b div ?g
    let ?g' = igcd ?a' ?b'
    from anz bnz have ?g ≠ 0 by simp  with igcd-pos[of a b]
    have gpos: ?g > 0  by arith
    have gdvd: ?g dvd a ?g dvd b by (simp-all add: igcd-dvd1 igcd-dvd2)
    from zdvd-mult-div-cancel[OF gdvd(1)] zdvd-mult-div-cancel[OF gdvd(2)]
    anz bnz
    have nz':?a' ≠ 0 ?b' ≠ 0
      by − (rule notI,simp add:igcd-def)+
    from anz bnz have stupid: a ≠ 0 ∨ b ≠ 0 by blast
    from div-igcd-relprime[OF stupid] have gp1: ?g' = 1 .
    from bnz have b < 0 ∨ b > 0 by arith
    moreover
    {assume b: b > 0
      from pos-imp-zdiv-nonneg-iff[OF gpos] b
      have ?b' ≥ 0 by simp
      with nz' have b': ?b' > 0 by simp
      from b b' anz bnz nz' gp1 have ?thesis
           by (simp add: isnormNum-def normNum-def Let-def split-def fst-conv
snd-conv)}
    moreover {assume b: b < 0
      {assume b': ?b' ≥ 0
        from gpos have th: ?g ≥ 0 by arith
        from mult-nonneg-nonneg[OF th b'] zdvd-mult-div-cancel[OF gdvd(2)]
        have False using b by simp }
      hence b': ?b' < 0 by (presburger add: linorder-not-le[symmetric])
      from anz bnz nz' b b' gp1 have ?thesis
           by (simp add: isnormNum-def normNum-def Let-def split-def fst-conv
snd-conv)}
    ultimately have ?thesis by blast
```

}
ultimately show *?thesis* **by** *blast*
qed

Arithmetic over Num

definition
$Nadd :: Num \Rightarrow Num \Rightarrow Num$ (**infixl** $+_N$ *60*)
where
$Nadd = (\lambda(a,b)\ (a',b').\ \mathit{if}\ a = 0 \lor b = 0\ \mathit{then}\ normNum(a',b')$
$\mathit{else\ if}\ a'=0 \lor b' = 0\ \mathit{then}\ normNum(a,b)$
$\mathit{else}\ normNum(a*b' + b*a',\ b*b'))$

definition
$Nmul :: Num \Rightarrow Num \Rightarrow Num$ (**infixl** $*_N$ *60*)
where
$Nmul = (\lambda(a,b)\ (a',b').\ \mathit{let}\ g = igcd\ (a*a')\ (b*b')$
$\mathit{in}\ (a*a'\ \mathit{div}\ g,\ b*b'\ \mathit{div}\ g))$

definition
$Nneg :: Num \Rightarrow Num$ ($\sim_N$)
where
$Nneg \equiv (\lambda(a,b).\ (-a,b))$

definition
$Nsub :: Num \Rightarrow Num \Rightarrow Num$ (**infixl** $-_N$ *60*)
where
$Nsub = (\lambda a\ b.\ a +_N \sim_N b)$

definition
$Ninv :: Num \Rightarrow Num$
where
$Ninv \equiv \lambda(a,b).\ \mathit{if}\ a < 0\ \mathit{then}\ (-b,\ |a|)\ \mathit{else}\ (b,a)$

definition
$Ndiv :: Num \Rightarrow Num \Rightarrow Num$ (**infixl** $\div_N$ *60*)
where
$Ndiv \equiv \lambda a\ b.\ a *_N Ninv\ b$

lemma *Nneg-normN*[*simp*]: $isnormNum\ x \Longrightarrow isnormNum\ (\sim_N x)$
by(*simp add*: *isnormNum-def Nneg-def split-def*)
lemma *Nadd-normN*[*simp*]: $isnormNum\ (x +_N y)$
by (*simp add*: *Nadd-def split-def*)
lemma *Nsub-normN*[*simp*]: $[\![\ isnormNum\ y]\!] \Longrightarrow isnormNum\ (x -_N y)$
by (*simp add*: *Nsub-def split-def*)
lemma *Nmul-normN*[*simp*]: **assumes** *xn*:*isnormNum x* **and** *yn*: *isnormNum y*
shows $isnormNum\ (x *_N y)$
proof−
have $\exists\ a\ b.\ x = (a,b)$ **and** $\exists\ a'\ b'.\ y = (a',b')$ **by** *auto*
then obtain $a\ b\ a'\ b'$ **where** *ab*: $x = (a,b)$ **and** *ab'*: $y = (a',b')$ **by** *blast*

{**assume** $a = 0$
hence *?thesis* **using** *xn ab ab'*
by (*simp add*: *igcd-def isnormNum-def Let-def Nmul-def split-def*)}
moreover
{**assume** $a' = 0$
hence *?thesis* **using** *yn ab ab'*
by (*simp add*: *igcd-def isnormNum-def Let-def Nmul-def split-def*)}
moreover
{**assume** *a*: $a \neq 0$ **and** *a'*: $a' \neq 0$
hence *bp*: $b > 0$ $b' > 0$ **using** *xn yn ab ab'* **by** (*simp-all add*: *isnormNum-def*)
from *mult-pos-pos*[*OF bp*] **have** $x *_N y = normNum\ (a*a',\ b*b')$
using *ab ab' a a' bp* **by** (*simp add*: *Nmul-def Let-def split-def normNum-def*)
hence *?thesis* **by** *simp*}
ultimately show *?thesis* **by** *blast*
qed

lemma *Ninv-normN*[*simp*]: $isnormNum\ x \Longrightarrow isnormNum\ (Ninv\ x)$
by (*simp add*: *Ninv-def isnormNum-def split-def*)
(*cases fst* $x = 0$, *auto simp add*: *igcd-commute*)

lemma *isnormNum-int*[*simp*]:
$isnormNum\ 0_N\ \ isnormNum\ (1::int)_N\ \ i \neq 0 \Longrightarrow isnormNum\ i_N$
by (*simp-all add*: *isnormNum-def igcd-def*)

Relations over Num

definition
$Nlt0 :: Num \Rightarrow bool\ (0{>_N})$
where
$Nlt0 = (\lambda(a,b).\ a < 0)$

definition
$Nle0 :: Num \Rightarrow bool\ (0{\geq_N})$
where
$Nle0 = (\lambda(a,b).\ a \leq 0)$

definition
$Ngt0 :: Num \Rightarrow bool\ (0{<_N})$
where
$Ngt0 = (\lambda(a,b).\ a > 0)$

definition
$Nge0 :: Num \Rightarrow bool\ (0{\leq_N})$
where
$Nge0 = (\lambda(a,b).\ a \geq 0)$

definition
$Nlt :: Num \Rightarrow Num \Rightarrow bool$ (**infix** $<_N$ *55*)
where
$Nlt = (\lambda a\ b.\ 0{>_N}\ (a -_N b))$

definition
  Nle :: *Num* ⇒ *Num* ⇒ *bool* (**infix** $\leq_N$ *55*)
where
  Nle = (λ*a b*. *0*$\geq_N$ (*a* $-_N$ *b*))

definition
  INum = (λ(*a*,*b*). *of-int a* / *of-int b*)

lemma *INum-int* [*simp*]: *INum* i_N = ((*of-int i*) ::′*a*::*field*) *INum* 0_N = (*0*::′*a*::*field*)
  by (*simp-all add*: *INum-def*)

lemma *isnormNum-unique*[*simp*]:
  assumes *na*: *isnormNum x* **and** *nb*: *isnormNum y*
  shows ((*INum x* ::′*a*::{*ring-char-0*,*field*, *division-by-zero*}) = *INum y*) = (*x* = *y*) (**is** *?lhs* = *?rhs*)
proof
  have ∃ *a b a*′ *b*′. *x* = (*a*,*b*) ∧ *y* = (*a*′,*b*′) **by** *auto*
  then obtain *a b a*′ *b*′ **where** *xy*[*simp*]: *x* = (*a*,*b*) *y*=(*a*′,*b*′) **by** *blast*
  assume *H*: *?lhs*
  {**assume** *a* = *0* ∨ *b* = *0* ∨ *a*′ = *0* ∨ *b*′ = *0* **hence** *?rhs*
    using *na nb H*
    apply (*simp add*: *INum-def split-def isnormNum-def*)
    apply (*cases a* = *0*, *simp-all*)
    apply (*cases b* = *0*, *simp-all*)
    apply (*cases a*′ = *0*, *simp-all*)
    apply (*cases a*′ = *0*, *simp-all add*: *of-int-eq-0-iff*)
    done}
  moreover
  { **assume** *az*: *a* ≠ *0* **and** *bz*: *b* ≠ *0* **and** *a*′*z*: *a*′≠*0* **and** *b*′*z*: *b*′≠*0*
   from *az bz a*′*z b*′*z na nb* **have** *pos*: *b* > *0 b*′ > *0* **by** (*simp-all add*: *isnormNum-def*)
   from *prems* **have** *eq*:*a* ∗ *b*′ = *a*′∗*b*
    by (*simp add*: *INum-def eq-divide-eq divide-eq-eq of-int-mult*[*symmetric*] *del*: *of-int-mult*)
   from *prems* **have** *gcd1*: *igcd a b* = *1 igcd b a* = *1 igcd a*′ *b*′ = *1 igcd b*′ *a*′ = *1*
    by (*simp-all add*: *isnormNum-def add*: *igcd-commute*)
   from *eq* **have** *raw-dvd*: *a dvd a*′∗*b b dvd b*′∗*a a*′ *dvd a*∗*b*′ *b*′ *dvd b*∗*a*′
    apply(*unfold dvd-def*)
    apply (*rule-tac x*=*b*′ **in** *exI*, *simp add*: *mult-ac*)
    apply (*rule-tac x*=*a*′ **in** *exI*, *simp add*: *mult-ac*)
    apply (*rule-tac x*=*b* **in** *exI*, *simp add*: *mult-ac*)
    apply (*rule-tac x*=*a* **in** *exI*, *simp add*: *mult-ac*)
    done
   from *zdvd-dvd-eq*[*OF bz zrelprime-dvd-mult*[*OF gcd1*(*2*) *raw-dvd*(*2*)]
    zrelprime-dvd-mult[*OF gcd1*(*4*) *raw-dvd*(*4*)]]
    have *eq1*: *b* = *b*′ **using** *pos* **by** *simp-all*
    with *eq* **have** *a* = *a*′ **using** *pos* **by** *simp*
    with *eq1* **have** *?rhs* **by** *simp*}

```
  ultimately show ?rhs by blast
next
  assume ?rhs thus ?lhs by simp
qed


lemma isnormNum0[simp]: isnormNum x ⟹ (INum x = (0::'a::{ring-char-0,
field,division-by-zero})) = (x = 0_N)
  unfolding INum-int(2)[symmetric]
  by (rule isnormNum-unique, simp-all)

lemma of-int-div-aux: d ~= 0 ==> ((of-int x)::'a::{field, ring-char-0}) / (of-int
d) =
    of-int (x div d) + (of-int (x mod d)) / ((of-int d)::'a)
proof -
  assume d ~= 0
  hence dz: of-int d ≠ (0::'a) by (simp add: of-int-eq-0-iff)
  let ?t = of-int (x div d) * ((of-int d)::'a) + of-int(x mod d)
  let ?f = λx. x / of-int d
  have x = (x div d) * d + x mod d
    by auto
  then have eq: of-int x = ?t
    by (simp only: of-int-mult[symmetric] of-int-add [symmetric])
  then have of-int x / of-int d = ?t / of-int d
    using cong[OF refl[of ?f] eq] by simp
  then show ?thesis by (simp add: add-divide-distrib ring-simps prems)
qed

lemma of-int-div: (d::int) ~= 0 ==> d dvd n ==>
    (of-int(n div d)::'a::{field, ring-char-0}) = of-int n / of-int d
  apply (frule of-int-div-aux [of d n, where ?'a = 'a])
  apply simp
  apply (simp add: zdvd-iff-zmod-eq-0)
done


lemma normNum[simp]: INum (normNum x) = (INum x :: 'a::{ring-char-0,field,
division-by-zero})
proof−
  have ∃ a b. x = (a,b) by auto
  then obtain a b where x[simp]: x = (a,b) by blast
  {assume a=0 ∨ b = 0 hence ?thesis
     by (simp add: INum-def normNum-def split-def Let-def)}
  moreover
  {assume a: a≠0 and b: b≠0
    let ?g = igcd a b
    from a b have g: ?g ≠ 0by simp
    from of-int-div[OF g, where ?'a = 'a]
    have ?thesis by (auto simp add: INum-def normNum-def split-def Let-def)}
```

ultimately show *?thesis* **by** *blast*
qed

lemma *INum-normNum-iff* [*code*]: (*INum x* ::'*a*::{*field*, *division-by-zero*, *ring-char-0*}) = *INum y* ⟷ *normNum x* = *normNum y* (**is** *?lhs* = *?rhs*)
proof −
have *normNum x* = *normNum y* ⟷ (*INum* (*normNum x*) :: '*a*) = *INum* (*normNum y*)
by (*simp del*: *normNum*)
also have ... = *?lhs* **by** *simp*
finally show *?thesis* **by** *simp*
qed

lemma *Nadd*[*simp*]: *INum* ($x +_N y$) = *INum x* + (*INum y* :: '*a* :: {*ring-char-0*,*division-by-zero*,*field*})
proof−
let *?z* = *0*:: '*a*
have ∃ *a b*. *x* = (*a*,*b*) ∃ *a*' *b*'. *y* = (*a*',*b*') **by** *auto*
then obtain *a b a*' *b*' **where** *x*[*simp*]: *x* = (*a*,*b*)
and *y*[*simp*]: *y* = (*a*',*b*') **by** *blast*
{**assume** *a*=*0* ∨ *a*'= *0* ∨ *b* =*0* ∨ *b*' = *0* **hence** *?thesis*
apply (*cases a*=*0*,*simp-all add*: *Nadd-def*)
apply (*cases b*= *0*,*simp-all add*: *INum-def*)
apply (*cases a*'= *0*,*simp-all*)
apply (*cases b*'= *0*,*simp-all*)
done }
moreover
{**assume** *aa*':*a* ≠ *0 a*'≠ *0* **and** *bb*': *b* ≠ *0 b*' ≠ *0*
{**assume** *z*: *a* ∗ *b*' + *b* ∗ *a*' = *0*
hence *of-int* (*a*∗*b*' + *b*∗*a*') / (*of-int b*∗ *of-int b*') = *?z* **by** *simp*
hence *of-int b*' ∗ *of-int a* / (*of-int b* ∗ *of-int b*') + *of-int b* ∗ *of-int a*' / (*of-int b* ∗ *of-int b*') = *?z* **by** (*simp add*:*add-divide-distrib*)
hence *th*: *of-int a* / *of-int b* + *of-int a*' / *of-int b*' = *?z* **using** *bb*' *aa*' **by** *simp*
from *z aa*' *bb*' **have** *?thesis*
by (*simp add*: *th Nadd-def normNum-def INum-def split-def*)}
moreover {**assume** *z*: *a* ∗ *b*' + *b* ∗ *a*' ≠ *0*
let *?g* = *igcd* (*a* ∗ *b*' + *b* ∗ *a*') (*b*∗*b*')
have *gz*: *?g* ≠ *0* **using** *z* **by** *simp*
have *?thesis* **using** *aa*' *bb*' *z gz*
of-int-div[**where** *?*'*a* = '*a*,
OF gz igcd-dvd1[**where** *i*=*a* ∗ *b*' + *b* ∗ *a*' **and** *j*=*b*∗*b*']]
of-int-div[**where** *?*'*a* = '*a*,
OF gz igcd-dvd2[**where** *i*=*a* ∗ *b*' + *b* ∗ *a*' **and** *j*=*b*∗*b*']]
by (*simp add*: *x y Nadd-def INum-def normNum-def Let-def add-divide-distrib*)}
ultimately have *?thesis* **using** *aa*' *bb*'
by (*simp add*: *Nadd-def INum-def normNum-def x y Let-def*) }
ultimately show *?thesis* **by** *blast*
qed

lemma *Nmul*[*simp*]: *INum* $(x *_N y)$ = *INum x* $*$ (*INum y*:: *'a* :: {*ring-char-0*,*division-by-zero*,*field*})

proof−
  let *?z* = *0*::*'a*
  have $\exists$ *a b*. *x* = (*a*,*b*) $\exists$ *a' b'*. *y* = (*a'*,*b'*) **by** *auto*
  then obtain *a b a' b'* **where** *x*: *x* = (*a*,*b*) **and** *y*: *y* = (*a'*,*b'*) **by** *blast*
  {**assume** *a*=*0* $\vee$ *a'*= *0* $\vee$ *b* = *0* $\vee$ *b'* = *0* **hence** *?thesis*
    apply (*cases a*=*0*,*simp-all add*: *x y Nmul-def INum-def Let-def*)
    apply (*cases b*=*0*,*simp-all*)
    apply (*cases a'*=*0*,*simp-all*)
    done }
  moreover
  {**assume** *z*: *a* $\neq$ *0* *a'* $\neq$ *0* *b* $\neq$ *0* *b'* $\neq$ *0*
   let *?g*=*igcd* (*a*$*$*a'*) (*b*$*$*b'*)
   have *gz*: *?g* $\neq$ *0* **using** *z* **by** *simp*
   from *z of-int-div*[**where** *?'a* = *'a*, *OF gz igcd-dvd1*[**where** *i*=*a*$*$*a'* **and** *j*=*b*$*$*b'*]]

    of-int-div[**where** *?'a* = *'a* , *OF gz igcd-dvd2*[**where** *i*=*a*$*$*a'* **and** *j*=*b*$*$*b'*]]
   have *?thesis* **by** (*simp add*: *Nmul-def x y Let-def INum-def*)}
  ultimately show *?thesis* **by** *blast*
qed

lemma *Nneg*[*simp*]: *INum* ($\sim_N$ *x*) = − (*INum x* ::*'a*:: *field*)
  by (*simp add*: *Nneg-def split-def INum-def*)

lemma *Nsub*[*simp*]: **shows** *INum* $(x -_N y)$ = *INum x* − (*INum y*:: *'a* :: {*ring-char-0*,*division-by-zero*,*field*})
by (*simp add*: *Nsub-def split-def*)

lemma *Ninv*[*simp*]: *INum* (*Ninv x*) = (*1*::*'a* :: {*division-by-zero*,*field*}) / (*INum x*)
  by (*simp add*: *Ninv-def INum-def split-def*)

lemma *Ndiv*[*simp*]: *INum* $(x \div_N y)$ = *INum x* / (*INum y* ::*'a* :: {*ring-char-0*, *division-by-zero*,*field*}) **by** (*simp add*: *Ndiv-def*)

lemma *Nlt0-iff*[*simp*]: **assumes** *nx*: *isnormNum x*
 shows ((*INum x* :: *'a* :: {*ring-char-0*,*division-by-zero*,*ordered-field*})< *0*) = $0 >_N x$
proof−
  have $\exists$ *a b*. *x* = (*a*,*b*) **by** *simp*
  then obtain *a b* **where** *x*[*simp*]:*x* = (*a*,*b*) **by** *blast*
  {**assume** *a* = *0* **hence** *?thesis* **by** (*simp add*: *Nlt0-def INum-def*) }
  moreover
  {**assume** *a*: *a*$\neq$*0* **hence** *b*: (*of-int b*::*'a*) > *0* **using** *nx* **by** (*simp add*: *isnormNum-def*)
   from *pos-divide-less-eq*[*OF b*, **where** *b*=*of-int a* **and** *a*=*0*::*'a*]
   have *?thesis* **by** (*simp add*: *Nlt0-def INum-def*)}
  ultimately show *?thesis* **by** *blast*
qed

lemma *Nle0-iff* [*simp*]:**assumes** *nx*: *isnormNum x*
shows $((INum\ x :: 'a :: \{ring\text{-}char\text{-}0, division\text{-}by\text{-}zero, ordered\text{-}field\}) \le 0) = 0 \ge_N x$
proof−
have $\exists\ a\ b.\ x = (a,b)$ **by** *simp*
then obtain *a b* **where** *x*[*simp*]:$x = (a,b)$ **by** *blast*
{**assume** $a = 0$ **hence** *?thesis* **by** (*simp add*: *Nle0-def INum-def*) }
moreover
{**assume** *a*: $a \ne 0$ **hence** *b*: $(of\text{-}int\ b :: 'a) > 0$ **using** *nx* **by** (*simp add*: *isnormNum-def*)
from *pos-divide-le-eq*[*OF b*, **where** *b=of-int a* **and** $a=0::'a$]
have *?thesis* **by** (*simp add*: *Nle0-def INum-def*)}
ultimately show *?thesis* **by** *blast*
qed

lemma *Ngt0-iff* [*simp*]:**assumes** *nx*: *isnormNum x* **shows** $((INum\ x :: 'a :: \{ring\text{-}char\text{-}0, division\text{-}by\text{-}zero, ordere$
$0) = 0 <_N x$
proof−
have $\exists\ a\ b.\ x = (a,b)$ **by** *simp*
then obtain *a b* **where** *x*[*simp*]:$x = (a,b)$ **by** *blast*
{**assume** $a = 0$ **hence** *?thesis* **by** (*simp add*: *Ngt0-def INum-def*) }
moreover
{**assume** *a*: $a \ne 0$ **hence** *b*: $(of\text{-}int\ b::'a) > 0$ **using** *nx* **by** (*simp add*: *isnormNum-def*)
from *pos-less-divide-eq*[*OF b*, **where** *b=of-int a* **and** $a=0::'a$]
have *?thesis* **by** (*simp add*: *Ngt0-def INum-def*)}
ultimately show *?thesis* **by** *blast*
qed
lemma *Nge0-iff* [*simp*]:**assumes** *nx*: *isnormNum x*
shows $((INum\ x :: 'a :: \{ring\text{-}char\text{-}0, division\text{-}by\text{-}zero, ordered\text{-}field\}) \ge 0) = 0 \le_N x$
proof−
have $\exists\ a\ b.\ x = (a,b)$ **by** *simp*
then obtain *a b* **where** *x*[*simp*]:$x = (a,b)$ **by** *blast*
{**assume** $a = 0$ **hence** *?thesis* **by** (*simp add*: *Nge0-def INum-def*) }
moreover
{**assume** *a*: $a \ne 0$ **hence** *b*: $(of\text{-}int\ b::'a) > 0$ **using** *nx* **by** (*simp add*: *isnormNum-def*)
from *pos-le-divide-eq*[*OF b*, **where** *b=of-int a* **and** $a=0::'a$]
have *?thesis* **by** (*simp add*: *Nge0-def INum-def*)}
ultimately show *?thesis* **by** *blast*
qed

lemma *Nlt-iff* [*simp*]: **assumes** *nx*: *isnormNum x* **and** *ny*: *isnormNum y*
shows $((INum\ x :: 'a :: \{ring\text{-}char\text{-}0, division\text{-}by\text{-}zero, ordered\text{-}field\}) < INum\ y) = (x <_N y)$
proof−
let $?z = 0::'a$
have $((INum\ x ::'a) < INum\ y) = (INum\ (x -_N y) < ?z)$ **using** *nx ny* **by** *simp*
also have $\ldots = (0 >_N (x -_N y))$ **using** *Nlt0-iff* [*OF Nsub-normN*[*OF ny*]] **by** *simp*
finally show *?thesis* **by** (*simp add*: *Nlt-def*)

qed

lemma *Nle-iff* [*simp*]: **assumes** *nx*: *isnormNum x* **and** *ny*: *isnormNum y*
shows $((INum\ x :: 'a :: \{ring\text{-}char\text{-}0, division\text{-}by\text{-}zero, ordered\text{-}field\}) \leq INum\ y) = (x \leq_N y)$
proof−
have $((INum\ x ::'a) \leq INum\ y) = (INum\ (x -_N y) \leq (0::'a))$ **using** *nx ny* **by** *simp*
also have $\ldots = (0 \geq_N (x -_N y))$ **using** *Nle0-iff* [*OF Nsub-normN*[*OF ny*]] **by** *simp*
finally show *?thesis* **by** (*simp add*: *Nle-def*)
qed

lemma *Nadd-commute*: $x +_N y = y +_N x$
proof−
have *n*: *isnormNum* $(x +_N y)$ *isnormNum* $(y +_N x)$ **by** *simp-all*
have $(INum\ (x +_N y)::'a :: \{ring\text{-}char\text{-}0, division\text{-}by\text{-}zero, field\}) = INum\ (y +_N x)$ **by** *simp*
with *isnormNum-unique*[*OF n*] **show** *?thesis* **by** *simp*
qed

lemma[*simp*]: $(0,\ b) +_N y = normNum\ y$ $(a,\ 0) +_N y = normNum\ y$
$x +_N (0,\ b) = normNum\ x$ $x +_N (a,\ 0) = normNum\ x$
apply (*simp add*: *Nadd-def split-def*, *simp add*: *Nadd-def split-def*)
apply (*subst Nadd-commute*,*simp add*: *Nadd-def split-def*)
apply (*subst Nadd-commute*,*simp add*: *Nadd-def split-def*)
done

lemma *normNum-nilpotent-aux*[*simp*]: **assumes** *nx*: *isnormNum x*
shows *normNum x = x*
proof−
let *?a = normNum x*
have *n*: *isnormNum ?a* **by** *simp*
have *th*:$INum\ ?a = (INum\ x ::'a :: \{ring\text{-}char\text{-}0, division\text{-}by\text{-}zero, field\})$ **by** *simp*
with *isnormNum-unique*[*OF n nx*]
show *?thesis* **by** *simp*
qed

lemma *normNum-nilpotent*[*simp*]: *normNum* (*normNum x*) = *normNum x*
by *simp*
lemma *normNum0*[*simp*]: $normNum\ (0,b) = 0_N$ $normNum\ (a,0) = 0_N$
by (*simp-all add*: *normNum-def*)
lemma *normNum-Nadd*: $normNum\ (x +_N y) = x +_N y$ **by** *simp*
lemma *Nadd-normNum1*[*simp*]: $normNum\ x +_N y = x +_N y$
proof−
have *n*: *isnormNum* $(normNum\ x +_N y)$ *isnormNum* $(x +_N y)$ **by** *simp-all*
have $INum\ (normNum\ x +_N y) = INum\ x + (INum\ y :: 'a :: \{ring\text{-}char\text{-}0, division\text{-}by\text{-}zero, field\})$ **by** *simp*
also have $\ldots = INum\ (x +_N y)$ **by** *simp*

finally show *?thesis* **using** *isnormNum-unique*[*OF n*] **by** *simp*
qed
lemma *Nadd-normNum2*[*simp*]: $x +_N$ *normNum* $y = x +_N y$
proof−
have *n*: *isnormNum* $(x +_N$ *normNum* $y)$ *isnormNum* $(x +_N y)$ **by** *simp-all*
have *INum* $(x +_N$ *normNum* $y)$ = *INum* x + (*INum* y :: *'a* :: {*ring-char-0*, *division-by-zero*,*field*}) **by** *simp*
also have ... = *INum* $(x +_N y)$ **by** *simp*
finally show *?thesis* **using** *isnormNum-unique*[*OF n*] **by** *simp*
qed

lemma *Nadd-assoc*: $x +_N y +_N z = x +_N (y +_N z)$
proof−
have *n*: *isnormNum* $(x +_N y +_N z)$ *isnormNum* $(x +_N (y +_N z))$ **by** *simp-all*
have *INum* $(x +_N y +_N z)$ = (*INum* $(x +_N (y +_N z))$ :: *'a* :: {*ring-char-0*, *division-by-zero*,*field*}) **by** *simp*
with *isnormNum-unique*[*OF n*] **show** *?thesis* **by** *simp*
qed

lemma *Nmul-commute*: *isnormNum* $x \Longrightarrow$ *isnormNum* $y \Longrightarrow x *_N y = y *_N x$
by (*simp add*: *Nmul-def split-def Let-def igcd-commute mult-commute*)

lemma *Nmul-assoc*: **assumes** *nx*: *isnormNum x* **and** *ny*:*isnormNum y* **and** *nz*:*isnormNum z*
shows $x *_N y *_N z = x *_N (y *_N z)$
proof−
from *nx ny nz* **have** *n*: *isnormNum* $(x *_N y *_N z)$ *isnormNum* $(x *_N (y *_N z))$

by *simp-all*
have *INum* $(x +_N y +_N z)$ = (*INum* $(x +_N (y +_N z))$ :: *'a* :: {*ring-char-0*, *division-by-zero*,*field*}) **by** *simp*
with *isnormNum-unique*[*OF n*] **show** *?thesis* **by** *simp*
qed

lemma *Nsub0*: **assumes** *x*: *isnormNum x* **and** *y*:*isnormNum y* **shows** $(x -_N y = 0_N) = (x = y)$
proof−
{**fix** *h* :: *'a* :: {*ring-char-0*,*division-by-zero*,*ordered-field*}
from *isnormNum-unique*[**where** *?'a* = *'a*, *OF Nsub-normN*[*OF y*], **where** $y{=}0_N$]
have $(x -_N y = 0_N)$ = (*INum* $(x -_N y)$ = (*INum* 0_N :: *'a*)) **by** *simp*
also have ... = (*INum* x = (*INum* y:: *'a*)) **by** *simp*
also have ... = $(x = y)$ **using** *x y* **by** *simp*
finally show *?thesis* .}
qed

lemma *Nmul0*[*simp*]: $c *_N 0_N = 0_N$ $\;0_N *_N c = 0_N$
by (*simp-all add*: *Nmul-def Let-def split-def*)

lemma *Nmul-eq0*[*simp*]: **assumes** *nx*:*isnormNum x* **and** *ny*: *isnormNum y*
shows $(x *_N y = 0_N) = (x = 0_N \lor y = 0_N)$
proof−
{**fix** *h* :: $'a$ :: {*ring-char-0*,*division-by-zero*,*ordered-field*}
have $\exists\ a\ b\ a'\ b'.\ x = (a,b) \land y= (a',b')$ **by** *auto*
then obtain $a\ b\ a'\ b'$ **where** *xy*[*simp*]: $x = (a,b)$ $y = (a',b')$ **by** *blast*
have *n0*: *isnormNum* 0_N **by** *simp*
show *?thesis* **using** *nx ny*
apply (*simp only*: *isnormNum-unique*[**where** $?'a = 'a$, *OF Nmul-normN*[*OF nx ny*] *n0*, *symmetric*] *Nmul*[**where** $?'a = 'a$])
apply (*simp add*: *INum-def split-def isnormNum-def fst-conv snd-conv*)
apply (*cases* $a=0$,*simp-all*)
apply (*cases* $a'=0$,*simp-all*)
done }
qed
lemma *Nneg-Nneg*[*simp*]: $\sim_N (\sim_N c) = c$
by (*simp add*: *Nneg-def split-def*)

lemma *Nmul1*[*simp*]:
isnormNum $c \Longrightarrow 1_N *_N c = c$
isnormNum $c \Longrightarrow c *_N 1_N = c$
apply (*simp-all add*: *Nmul-def Let-def split-def isnormNum-def*)
by (*cases fst* $c = 0$, *simp-all*,*cases c*, *simp-all*)+

end

4 Rational: Rational numbers

theory *Rational*
imports *Abstract-Rat*
uses (*rat-arith.ML*)
begin

4.1 Rational numbers

4.1.1 Equivalence of fractions

definition
fraction :: (*int* × *int*) *set* **where**
fraction = $\{x.\ snd\ x \neq 0\}$

definition
ratrel :: ((*int* × *int*) × (*int* × *int*)) *set* **where**
ratrel = $\{(x,y).\ snd\ x \neq 0 \land snd\ y \neq 0 \land fst\ x * snd\ y = fst\ y * snd\ x\}$

lemma *fraction-iff* [*simp*]: $(x \in fraction) = (snd\ x \neq 0)$
by (*simp add*: *fraction-def*)

lemma *ratrel-iff* [*simp*]:

$((x,y) \in ratrel) =$
$(snd\ x \neq 0 \wedge snd\ y \neq 0 \wedge fst\ x * snd\ y = fst\ y * snd\ x)$
by (*simp add*: *ratrel-def*)

lemma *refl-ratrel*: *refl fraction ratrel*
by (*auto simp add*: *refl-def fraction-def ratrel-def*)

lemma *sym-ratrel*: *sym ratrel*
by (*simp add*: *ratrel-def sym-def*)

lemma *trans-ratrel-lemma*:
assumes *1*: $a * b' = a' * b$
assumes *2*: $a' * b'' = a'' * b'$
assumes *3*: $b' \neq (0::int)$
shows $a * b'' = a'' * b$
proof −
have $b' * (a * b'') = b'' * (a * b')$ **by** *simp*
also note *1*
also have $b'' * (a' * b) = b * (a' * b'')$ **by** *simp*
also note *2*
also have $b * (a'' * b') = b' * (a'' * b)$ **by** *simp*
finally have $b' * (a * b'') = b' * (a'' * b)$ **.**
with *3* **show** $a * b'' = a'' * b$ **by** *simp*
qed

lemma *trans-ratrel*: *trans ratrel*
by (*auto simp add*: *trans-def elim*: *trans-ratrel-lemma*)

lemma *equiv-ratrel*: *equiv fraction ratrel*
by (*rule equiv.intro* [*OF refl-ratrel sym-ratrel trans-ratrel*])

lemmas *equiv-ratrel-iff* [*iff*] = *eq-equiv-class-iff* [*OF equiv-ratrel*]

lemma *equiv-ratrel-iff2*:
$[\![snd\ x \neq 0;\ snd\ y \neq 0]\!]$
$\Longrightarrow (ratrel\ ``\ \{x\} = ratrel\ ``\ \{y\}) = ((x,y) \in ratrel)$
by (*rule eq-equiv-class-iff* [*OF equiv-ratrel*], *simp-all*)

4.1.2 The type of rational numbers

typedef (*Rat*) *rat* = *fraction*//*ratrel*
proof
have $(0,1) \in fraction$ **by** (*simp add*: *fraction-def*)
thus $ratrel``\{(0,1)\} \in fraction//ratrel$ **by** (*rule quotientI*)
qed

lemma *ratrel-in-Rat* [*simp*]: $snd\ x \neq 0 \Longrightarrow ratrel``\{x\} \in Rat$
by (*simp add*: *Rat-def quotientI*)

declare *Abs-Rat-inject* [*simp*] *Abs-Rat-inverse* [*simp*]

definition
Fract :: *int* ⇒ *int* ⇒ *rat* **where**
[*code func del*]: *Fract a b* = *Abs-Rat* (*ratrel*‘‘{(*a*,*b*)})

lemma *Fract-zero*:
Fract k 0 = *Fract l 0*
by (*simp add*: *Fract-def ratrel-def*)

theorem *Rat-cases* [*case-names Fract*, *cases type*: *rat*]:
(!!*a b*. *q* = *Fract a b* ==> *b* ≠ *0* ==> *C*) ==> *C*
by (*cases q*) (*clarsimp simp add*: *Fract-def Rat-def fraction-def quotient-def*)

theorem *Rat-induct* [*case-names Fract*, *induct type*: *rat*]:
(!!*a b*. *b* ≠ *0* ==> *P* (*Fract a b*)) ==> *P q*
by (*cases q*) *simp*

4.1.3 Congruence lemmas

lemma *add-congruent2*:
(λ*x y*. *ratrel*‘‘{(*fst x* ∗ *snd y* + *fst y* ∗ *snd x*, *snd x* ∗ *snd y*)})
respects2 ratrel
apply (*rule equiv-ratrel* [*THEN congruent2-commuteI*])
apply (*simp-all add*: *left-distrib*)
done

lemma *minus-congruent*:
(λ*x*. *ratrel*‘‘{(− *fst x*, *snd x*)}) *respects ratrel*
by (*simp add*: *congruent-def*)

lemma *mult-congruent2*:
(λ*x y*. *ratrel*‘‘{(*fst x* ∗ *fst y*, *snd x* ∗ *snd y*)}) *respects2 ratrel*
by (*rule equiv-ratrel* [*THEN congruent2-commuteI*], *simp-all*)

lemma *inverse-congruent*:
(λ*x*. *ratrel*‘‘{*if fst x*=*0 then* (*0*,*1*) *else* (*snd x*, *fst x*)}) *respects ratrel*
by (*auto simp add*: *congruent-def mult-commute*)

lemma *le-congruent2*:
(λ*x y*. {(*fst x* ∗ *snd y*)∗(*snd x* ∗ *snd y*) ≤ (*fst y* ∗ *snd x*)∗(*snd x* ∗ *snd y*)})
respects2 ratrel
proof (*clarsimp simp add*: *congruent2-def*)
fix *a b a′ b′ c d c′ d′*::*int*
assume *neq*: *b* ≠ *0* *b′* ≠ *0* *d* ≠ *0* *d′* ≠ *0*
assume *eq1*: *a* ∗ *b′* = *a′* ∗ *b*
assume *eq2*: *c* ∗ *d′* = *c′* ∗ *d*

```
  let ?le = λa b c d. ((a ∗ d) ∗ (b ∗ d) ≤ (c ∗ b) ∗ (b ∗ d))
  {
    fix a b c d x :: int assume x: x ≠ 0
    have ?le a b c d = ?le (a ∗ x) (b ∗ x) c d
    proof −
      from x have 0 < x ∗ x by (auto simp add: zero-less-mult-iff)
      hence ?le a b c d =
          ((a ∗ d) ∗ (b ∗ d) ∗ (x ∗ x) ≤ (c ∗ b) ∗ (b ∗ d) ∗ (x ∗ x))
        by (simp add: mult-le-cancel-right)
      also have ... = ?le (a ∗ x) (b ∗ x) c d
        by (simp add: mult-ac)
      finally show ?thesis .
    qed
  } note le-factor = this

  let ?D = b ∗ d and ?D′ = b′ ∗ d′
  from neq have D: ?D ≠ 0 by simp
  from neq have ?D′ ≠ 0 by simp
  hence ?le a b c d = ?le (a ∗ ?D′) (b ∗ ?D′) c d
    by (rule le-factor)
  also have ... = ((a ∗ b′) ∗ ?D ∗ ?D′ ∗ d ∗ d′ ≤ (c ∗ d′) ∗ ?D ∗ ?D′ ∗ b ∗ b′)
    by (simp add: mult-ac)
  also have ... = ((a′ ∗ b) ∗ ?D ∗ ?D′ ∗ d ∗ d′ ≤ (c′ ∗ d) ∗ ?D ∗ ?D′ ∗ b ∗ b′)
    by (simp only: eq1 eq2)
  also have ... = ?le (a′ ∗ ?D) (b′ ∗ ?D) c′ d′
    by (simp add: mult-ac)
  also from D have ... = ?le a′ b′ c′ d′
    by (rule le-factor [symmetric])
  finally show ?le a b c d = ?le a′ b′ c′ d′ .
qed

lemmas UN-ratrel = UN-equiv-class [OF equiv-ratrel]
lemmas UN-ratrel2 = UN-equiv-class2 [OF equiv-ratrel equiv-ratrel]
```

4.1.4 Standard operations on rational numbers

```
instance rat :: zero
  Zero-rat-def: 0 == Fract 0 1 ..
lemmas [code func del] = Zero-rat-def

instance rat :: one
  One-rat-def: 1 == Fract 1 1 ..
lemmas [code func del] = One-rat-def

instance rat :: plus
  add-rat-def:
   q + r ==
       Abs-Rat (⋃x ∈ Rep-Rat q. ⋃y ∈ Rep-Rat r.
           ratrel``{(fst x ∗ snd y + fst y ∗ snd x, snd x ∗ snd y)}) ..
```

lemmas [*code func del*] = *add-rat-def*

instance *rat* :: *minus*
minus-rat-def:
$- \; q == Abs\text{-}Rat\ (\bigcup x \in Rep\text{-}Rat\ q.\ ratrel``\{(- \; fst\ x,\ snd\ x)\})$
diff-rat-def: $q - r == q + - (r::rat)$ **..**
lemmas [*code func del*] = *minus-rat-def diff-rat-def*

instance *rat* :: *times*
mult-rat-def:
$q * r ==$
$Abs\text{-}Rat\ (\bigcup x \in Rep\text{-}Rat\ q.\ \bigcup y \in Rep\text{-}Rat\ r.$
$ratrel``\{(fst\ x * fst\ y,\ snd\ x * snd\ y)\})$ **..**
lemmas [*code func del*] = *mult-rat-def*

instance *rat* :: *inverse*
inverse-rat-def:
$inverse\ q ==$
$Abs\text{-}Rat\ (\bigcup x \in Rep\text{-}Rat\ q.$
$ratrel``\{if\ fst\ x{=}0\ then\ (0,1)\ else\ (snd\ x,\ fst\ x)\})$
divide-rat-def: $q\ /\ r == q * inverse\ (r::rat)$ **..**
lemmas [*code func del*] = *inverse-rat-def divide-rat-def*

instance *rat* :: *ord*
le-rat-def:
$q \le r == contents\ (\bigcup x \in Rep\text{-}Rat\ q.\ \bigcup y \in Rep\text{-}Rat\ r.$
$\{(fst\ x * snd\ y)*(snd\ x * snd\ y) \le (fst\ y * snd\ x)*(snd\ x * snd\ y)\})$
less-rat-def: $(z < (w::rat)) == (z \le w\ \&\ z \ne w)$ **..**
lemmas [*code func del*] = *le-rat-def less-rat-def*

instance *rat* :: *abs*
abs-rat-def: $|q| == if\ q < 0\ then\ -q\ else\ (q::rat)$ **..**

instance *rat* :: *sgn*
sgn-rat-def: $sgn(q::rat) == (if\ q{=}0\ then\ 0\ else\ if\ 0{<}q\ then\ 1\ else\ -\ 1)$ **..**

instance *rat* :: *power* **..**

primrec (*rat*)
rat-power-0: $q\ \hat{}\ 0 \quad = 1$
rat-power-Suc: $q\ \hat{}\ (Suc\ n) = (q::rat) * (q\ \hat{}\ n)$

theorem *eq-rat*: $b \ne 0 ==> d \ne 0 ==>$
$(Fract\ a\ b = Fract\ c\ d) = (a * d = c * b)$
by (*simp add*: *Fract-def*)

theorem *add-rat*: $b \ne 0 ==> d \ne 0 ==>$
$Fract\ a\ b + Fract\ c\ d = Fract\ (a * d + c * b)\ (b * d)$
by (*simp add*: *Fract-def add-rat-def add-congruent2 UN-ratrel2*)

theorem *minus-rat*: $b \neq 0 ==> -(Fract\ a\ b) = Fract\ (-a)\ b$
by (*simp add*: *Fract-def minus-rat-def minus-congruent UN-ratrel*)

theorem *diff-rat*: $b \neq 0 ==> d \neq 0 ==>$
  $Fract\ a\ b - Fract\ c\ d = Fract\ (a * d - c * b)\ (b * d)$
by (*simp add*: *diff-rat-def add-rat minus-rat*)

theorem *mult-rat*: $b \neq 0 ==> d \neq 0 ==>$
  $Fract\ a\ b * Fract\ c\ d = Fract\ (a * c)\ (b * d)$
by (*simp add*: *Fract-def mult-rat-def mult-congruent2 UN-ratrel2*)

theorem *inverse-rat*: $a \neq 0 ==> b \neq 0 ==>$
  $inverse\ (Fract\ a\ b) = Fract\ b\ a$
by (*simp add*: *Fract-def inverse-rat-def inverse-congruent UN-ratrel*)

theorem *divide-rat*: $c \neq 0 ==> b \neq 0 ==> d \neq 0 ==>$
  $Fract\ a\ b\ /\ Fract\ c\ d = Fract\ (a * d)\ (b * c)$
by (*simp add*: *divide-rat-def inverse-rat mult-rat*)

theorem *le-rat*: $b \neq 0 ==> d \neq 0 ==>$
  $(Fract\ a\ b \leq Fract\ c\ d) = ((a * d) * (b * d) \leq (c * b) * (b * d))$
by (*simp add*: *Fract-def le-rat-def le-congruent2 UN-ratrel2*)

theorem *less-rat*: $b \neq 0 ==> d \neq 0 ==>$
  $(Fract\ a\ b < Fract\ c\ d) = ((a * d) * (b * d) < (c * b) * (b * d))$
by (*simp add*: *less-rat-def le-rat eq-rat order-less-le*)

theorem *abs-rat*: $b \neq 0 ==> |Fract\ a\ b| = Fract\ |a|\ |b|$
  by (*simp add*: *abs-rat-def minus-rat Zero-rat-def less-rat eq-rat*)
    (*auto simp add*: *mult-less-0-iff zero-less-mult-iff order-le-less*
        split: *abs-split*)

4.1.5 The ordered field of rational numbers

instance *rat* :: *field*
proof
  fix *q r s* :: *rat*
  show $(q + r) + s = q + (r + s)$
    by (*induct q*, *induct r*, *induct s*)
      (*simp add*: *add-rat add-ac mult-ac int-distrib*)
  show $q + r = r + q$
    by (*induct q*, *induct r*) (*simp add*: *add-rat add-ac mult-ac*)
  show $0 + q = q$
    by (*induct q*) (*simp add*: *Zero-rat-def add-rat*)
  show $(-q) + q = 0$
    by (*induct q*) (*simp add*: *Zero-rat-def minus-rat add-rat eq-rat*)
  show $q - r = q + (-r)$
    by (*induct q*, *induct r*) (*simp add*: *add-rat minus-rat diff-rat*)

```
  show (q * r) * s = q * (r * s)
    by (induct q, induct r, induct s) (simp add: mult-rat mult-ac)
  show q * r = r * q
    by (induct q, induct r) (simp add: mult-rat mult-ac)
  show 1 * q = q
    by (induct q) (simp add: One-rat-def mult-rat)
  show (q + r) * s = q * s + r * s
    by (induct q, induct r, induct s)
       (simp add: add-rat mult-rat eq-rat int-distrib)
  show q ≠ 0 ==> inverse q * q = 1
    by (induct q) (simp add: inverse-rat mult-rat One-rat-def Zero-rat-def eq-rat)
  show q / r = q * inverse r
    by (simp add: divide-rat-def)
  show 0 ≠ (1::rat)
    by (simp add: Zero-rat-def One-rat-def eq-rat)
qed

instance rat :: linorder
proof
  fix q r s :: rat
  {
    assume q ≤ r and r ≤ s
    show q ≤ s
    proof (insert prems, induct q, induct r, induct s)
      fix a b c d e f :: int
      assume neq: b ≠ 0  d ≠ 0  f ≠ 0
      assume 1: Fract a b ≤ Fract c d and 2: Fract c d ≤ Fract e f
      show Fract a b ≤ Fract e f
      proof -
        from neq obtain bb: 0 < b * b and dd: 0 < d * d and ff: 0 < f * f
          by (auto simp add: zero-less-mult-iff linorder-neq-iff)
        have (a * d) * (b * d) * (f * f) ≤ (c * b) * (b * d) * (f * f)
        proof -
          from neq 1 have (a * d) * (b * d) ≤ (c * b) * (b * d)
            by (simp add: le-rat)
          with ff show ?thesis by (simp add: mult-le-cancel-right)
        qed
        also have ... = (c * f) * (d * f) * (b * b)
          by (simp only: mult-ac)
        also have ... ≤ (e * d) * (d * f) * (b * b)
        proof -
          from neq 2 have (c * f) * (d * f) ≤ (e * d) * (d * f)
            by (simp add: le-rat)
          with bb show ?thesis by (simp add: mult-le-cancel-right)
        qed
        finally have (a * f) * (b * f) * (d * d) ≤ e * b * (b * f) * (d * d)
          by (simp only: mult-ac)
        with dd have (a * f) * (b * f) ≤ (e * b) * (b * f)
          by (simp add: mult-le-cancel-right)
```

```
      with neq show ?thesis by (simp add: le-rat)
    qed
  qed
next
  assume q ≤ r and r ≤ q
  show q = r
  proof (insert prems, induct q, induct r)
    fix a b c d :: int
    assume neq: b ≠ 0  d ≠ 0
    assume 1: Fract a b ≤ Fract c d and 2: Fract c d ≤ Fract a b
    show Fract a b = Fract c d
    proof −
      from neq 1 have (a ∗ d) ∗ (b ∗ d) ≤ (c ∗ b) ∗ (b ∗ d)
        by (simp add: le-rat)
      also have ... ≤ (a ∗ d) ∗ (b ∗ d)
      proof −
        from neq 2 have (c ∗ b) ∗ (d ∗ b) ≤ (a ∗ d) ∗ (d ∗ b)
          by (simp add: le-rat)
        thus ?thesis by (simp only: mult-ac)
      qed
      finally have (a ∗ d) ∗ (b ∗ d) = (c ∗ b) ∗ (b ∗ d) .
      moreover from neq have b ∗ d ≠ 0 by simp
      ultimately have a ∗ d = c ∗ b by simp
      with neq show ?thesis by (simp add: eq-rat)
    qed
  qed
next
  show q ≤ q
    by (induct q) (simp add: le-rat)
  show (q < r) = (q ≤ r ∧ q ≠ r)
    by (simp only: less-rat-def)
  show q ≤ r ∨ r ≤ q
    by (induct q, induct r)
       (simp add: le-rat mult-commute, rule linorder-linear)
}
qed

instance rat :: distrib-lattice
  inf r s ≡ min r s
  sup r s ≡ max r s
  by default (auto simp add: min-max.sup-inf-distrib1 inf-rat-def sup-rat-def)

instance rat :: ordered-field
proof
  fix q r s :: rat
  show q ≤ r ==> s + q ≤ s + r
  proof (induct q, induct r, induct s)
    fix a b c d e f :: int
    assume neq: b ≠ 0  d ≠ 0  f ≠ 0
```

```
    assume le: Fract a b ≤ Fract c d
    show Fract e f + Fract a b ≤ Fract e f + Fract c d
    proof −
      let ?F = f ∗ f from neq have F: 0 < ?F
        by (auto simp add: zero-less-mult-iff)
      from neq le have (a ∗ d) ∗ (b ∗ d) ≤ (c ∗ b) ∗ (b ∗ d)
        by (simp add: le-rat)
      with F have (a ∗ d) ∗ (b ∗ d) ∗ ?F ∗ ?F ≤ (c ∗ b) ∗ (b ∗ d) ∗ ?F ∗ ?F
        by (simp add: mult-le-cancel-right)
      with neq show ?thesis by (simp add: add-rat le-rat mult-ac int-distrib)
    qed
  qed
  show q < r ==> 0 < s ==> s ∗ q < s ∗ r
  proof (induct q, induct r, induct s)
    fix a b c d e f :: int
    assume neq: b ≠ 0  d ≠ 0  f ≠ 0
    assume le: Fract a b < Fract c d
    assume gt: 0 < Fract e f
    show Fract e f ∗ Fract a b < Fract e f ∗ Fract c d
    proof −
      let ?E = e ∗ f and ?F = f ∗ f
      from neq gt have 0 < ?E
        by (auto simp add: Zero-rat-def less-rat le-rat order-less-le eq-rat)
      moreover from neq have 0 < ?F
        by (auto simp add: zero-less-mult-iff)
      moreover from neq le have (a ∗ d) ∗ (b ∗ d) < (c ∗ b) ∗ (b ∗ d)
        by (simp add: less-rat)
      ultimately have (a ∗ d) ∗ (b ∗ d) ∗ ?E ∗ ?F < (c ∗ b) ∗ (b ∗ d) ∗ ?E ∗ ?F
        by (simp add: mult-less-cancel-right)
      with neq show ?thesis
        by (simp add: less-rat mult-rat mult-ac)
    qed
  qed
  show |q| = (if q < 0 then −q else q)
    by (simp only: abs-rat-def)
qed (auto simp: sgn-rat-def)

instance rat :: division-by-zero
proof
  show inverse 0 = (0::rat)
    by (simp add: Zero-rat-def Fract-def inverse-rat-def
                inverse-congruent UN-ratrel)
qed

instance rat :: recpower
proof
  fix q :: rat
  fix n :: nat
  show q ^ 0 = 1 by simp
```

```
  show q ^ (Suc n) = q * (q ^ n) by simp
qed
```

4.2 Various Other Results

```
lemma minus-rat-cancel [simp]: b ≠ 0 ==> Fract (−a) (−b) = Fract a b
by (simp add: eq-rat)

theorem Rat-induct-pos [case-names Fract, induct type: rat]:
  assumes step: !!a b. 0 < b ==> P (Fract a b)
    shows P q
proof (cases q)
  have step': !!a b. b < 0 ==> P (Fract a b)
  proof −
    fix a::int and b::int
    assume b: b < 0
    hence 0 < −b by simp
    hence P (Fract (−a) (−b)) by (rule step)
    thus P (Fract a b) by (simp add: order-less-imp-not-eq [OF b])
  qed
  case (Fract a b)
  thus P q by (force simp add: linorder-neq-iff step step')
qed

lemma zero-less-Fract-iff:
    0 < b ==> (0 < Fract a b) = (0 < a)
by (simp add: Zero-rat-def less-rat order-less-imp-not-eq2 zero-less-mult-iff)

lemma Fract-add-one: n ≠ 0 ==> Fract (m + n) n = Fract m n + 1
apply (insert add-rat [of concl: m n 1 1])
apply (simp add: One-rat-def [symmetric])
done

lemma of-nat-rat: of-nat k = Fract (of-nat k) 1
by (induct k) (simp-all add: Zero-rat-def One-rat-def add-rat)

lemma of-int-rat: of-int k = Fract k 1
by (cases k rule: int-diff-cases, simp add: of-nat-rat diff-rat)

lemma Fract-of-nat-eq: Fract (of-nat k) 1 = of-nat k
by (rule of-nat-rat [symmetric])

lemma Fract-of-int-eq: Fract k 1 = of-int k
by (rule of-int-rat [symmetric])

lemma Fract-of-int-quotient: Fract k l = (if l = 0 then Fract 1 0 else of-int k /
of-int l)
by (auto simp add: Fract-zero Fract-of-int-eq [symmetric] divide-rat)
```

4.3 Numerals and Arithmetic

instance *rat* :: *number*
rat-number-of-def: (*number-of w* :: *rat*) ≡ *of-int w* **..**

instance *rat* :: *number-ring*
by *default* (*simp add*: *rat-number-of-def*)

use *rat-arith.ML*
declaration ⟪ *K rat-arith-setup* ⟫

4.4 Embedding from Rationals to other Fields

class *field-char-0* = *field* + *ring-char-0*

instance *ordered-field* < *field-char-0* **..**

definition
of-rat :: *rat* ⇒ *'a*::*field-char-0*
where
[*code func del*]: *of-rat q* = *contents* ($\bigcup$(*a*,*b*) ∈ *Rep-Rat q*. {*of-int a* / *of-int b*})

lemma *of-rat-congruent*:
(λ(*a*, *b*). {*of-int a* / *of-int b*::*'a*::*field-char-0*}) *respects ratrel*
apply (*rule congruent.intro*)
apply (*clarsimp simp add*: *nonzero-divide-eq-eq nonzero-eq-divide-eq*)
apply (*simp only*: *of-int-mult* [*symmetric*])
done

lemma *of-rat-rat*:
b ≠ *0* ⟹ *of-rat* (*Fract a b*) = *of-int a* / *of-int b*
unfolding *Fract-def of-rat-def*
by (*simp add*: *UN-ratrel of-rat-congruent*)

lemma *of-rat-0* [*simp*]: *of-rat 0* = *0*
by (*simp add*: *Zero-rat-def of-rat-rat*)

lemma *of-rat-1* [*simp*]: *of-rat 1* = *1*
by (*simp add*: *One-rat-def of-rat-rat*)

lemma *of-rat-add*: *of-rat* (*a* + *b*) = *of-rat a* + *of-rat b*
by (*induct a*, *induct b*, *simp add*: *add-rat of-rat-rat add-frac-eq*)

lemma *of-rat-minus*: *of-rat* (− *a*) = − *of-rat a*
by (*induct a*, *simp add*: *minus-rat of-rat-rat*)

lemma *of-rat-diff*: *of-rat* (*a* − *b*) = *of-rat a* − *of-rat b*
by (*simp only*: *diff-minus of-rat-add of-rat-minus*)

lemma *of-rat-mult*: *of-rat* (*a* ∗ *b*) = *of-rat a* ∗ *of-rat b*

apply (*induct a*, *induct b*, *simp add*: *mult-rat of-rat-rat*)
apply (*simp add*: *divide-inverse nonzero-inverse-mult-distrib mult-ac*)
done

lemma *nonzero-of-rat-inverse*:
  $a \neq 0 \Longrightarrow$ *of-rat* (*inverse a*) = *inverse* (*of-rat a*)
apply (*rule inverse-unique* [*symmetric*])
apply (*simp add*: *of-rat-mult* [*symmetric*])
done

lemma *of-rat-inverse*:
  (*of-rat* (*inverse a*)::'*a*::{*field-char-0*,*division-by-zero*}) =
   inverse (*of-rat a*)
by (*cases a* = *0*, *simp-all add*: *nonzero-of-rat-inverse*)

lemma *nonzero-of-rat-divide*:
  $b \neq 0 \Longrightarrow$ *of-rat* (*a* / *b*) = *of-rat a* / *of-rat b*
by (*simp add*: *divide-inverse of-rat-mult nonzero-of-rat-inverse*)

lemma *of-rat-divide*:
  (*of-rat* (*a* / *b*)::'*a*::{*field-char-0*,*division-by-zero*})
   = *of-rat a* / *of-rat b*
by (*cases b* = *0*, *simp-all add*: *nonzero-of-rat-divide*)

lemma *of-rat-power*:
  (*of-rat* (*a* ˆ *n*)::'*a*::{*field-char-0*,*recpower*}) = *of-rat a* ˆ *n*
by (*induct n*) (*simp-all add*: *of-rat-mult power-Suc*)

lemma *of-rat-eq-iff* [*simp*]: (*of-rat a* = *of-rat b*) = (*a* = *b*)
apply (*induct a*, *induct b*)
apply (*simp add*: *of-rat-rat eq-rat*)
apply (*simp add*: *nonzero-divide-eq-eq nonzero-eq-divide-eq*)
apply (*simp only*: *of-int-mult* [*symmetric*] *of-int-eq-iff*)
done

lemmas *of-rat-eq-0-iff* [*simp*] = *of-rat-eq-iff* [*of - 0*, *simplified*]

lemma *of-rat-eq-id* [*simp*]: *of-rat* = (*id* :: *rat* ⇒ *rat*)
proof
  fix *a*
  show *of-rat a* = *id a*
  by (*induct a*)
    (*simp add*: *of-rat-rat divide-rat Fract-of-int-eq* [*symmetric*])
qed

Collapse nested embeddings

lemma *of-rat-of-nat-eq* [*simp*]: *of-rat* (*of-nat n*) = *of-nat n*
by (*induct n*) (*simp-all add*: *of-rat-add*)

lemma *of-rat-of-int-eq* [*simp*]: *of-rat* (*of-int z*) = *of-int z*
by (*cases z rule*: *int-diff-cases*, *simp add*: *of-rat-diff*)

lemma *of-rat-number-of-eq* [*simp*]:
of-rat (*number-of w*) = (*number-of w* :: *'a*::{*number-ring*,*field-char-0*})
by (*simp add*: *number-of-eq*)

lemmas *zero-rat* = *Zero-rat-def*
lemmas *one-rat* = *One-rat-def*

abbreviation
rat-of-nat :: *nat* ⇒ *rat*
where
rat-of-nat ≡ *of-nat*

abbreviation
rat-of-int :: *int* ⇒ *rat*
where
rat-of-int ≡ *of-int*

4.5 Implementation of rational numbers as pairs of integers

definition
Rational :: *int* × *int* ⇒ *rat*
where
Rational = *INum*

code-datatype *Rational*

lemma *Rational-simp*:
Rational (*k*, *l*) = *rat-of-int k* / *rat-of-int l*
unfolding *Rational-def INum-def* **by** *simp*

lemma *Rational-zero* [*simp*]: *Rational* 0_N = *0*
by (*simp add*: *Rational-simp*)

lemma *Rational-lit* [*simp*]: *Rational* i_N = *rat-of-int i*
by (*simp add*: *Rational-simp*)

lemma *zero-rat-code* [*code*, *code unfold*]:
0 = *Rational* 0_N **by** *simp*

lemma *zero-rat-code* [*code*, *code unfold*]:
1 = *Rational* 1_N **by** *simp*

lemma [*code*, *code unfold*]:
number-of k = *rat-of-int* (*number-of k*)
by (*simp add*: *number-of-is-id rat-number-of-def*)

definition
[*code func del*]: *Fract'* (*b*::*bool*) *k l* = *Fract k l*

lemma [*code*]:
Fract k l = *Fract'* (*l* $\neq$ *0*) *k l*
unfolding *Fract'-def* **..**

lemma [*code*]:
Fract' True k l = (*if l* $\neq$ *0 then Rational* (*k*, *l*) *else Fract 1 0*)
by (*simp add*: *Fract'-def Rational-simp Fract-of-int-quotient* [*of k l*])

lemma [*code*]:
of-rat (*Rational* (*k*, *l*)) = (*if l* $\neq$ *0 then of-int k* / *of-int l else 0*)
by (*cases l* = *0*)
(*auto simp add*: *Rational-simp of-rat-rat* [*simplified Fract-of-int-quotient* [*of k l*], *symmetric*])

instance *rat* :: *eq* **..**

lemma *rat-eq-code* [*code*]: *Rational x* = *Rational y* $\longleftrightarrow$ *normNum x* = *normNum y*
unfolding *Rational-def INum-normNum-iff* **..**

lemma *rat-less-eq-code* [*code*]: *Rational x* $\leq$ *Rational y* $\longleftrightarrow$ *normNum x* $\leq_N$ *normNum y*
proof −
have *normNum x* $\leq_N$ *normNum y* $\longleftrightarrow$ *Rational* (*normNum x*) $\leq$ *Rational* (*normNum y*)
by (*simp add*: *Rational-def del*: *normNum*)
also have . . . = (*Rational x* $\leq$ *Rational y*) **by** (*simp add*: *Rational-def*)
finally show *?thesis* **by** *simp*
qed

lemma *rat-less-code* [*code*]: *Rational x* < *Rational y* $\longleftrightarrow$ *normNum x* $<_N$ *normNum y*
proof −
have *normNum x* $<_N$ *normNum y* $\longleftrightarrow$ *Rational* (*normNum x*) < *Rational* (*normNum y*)
by (*simp add*: *Rational-def del*: *normNum*)
also have . . . = (*Rational x* < *Rational y*) **by** (*simp add*: *Rational-def*)
finally show *?thesis* **by** *simp*
qed

lemma *rat-add-code* [*code*]: *Rational x* + *Rational y* = *Rational* ($x +_N y$)
unfolding *Rational-def* **by** *simp*

lemma *rat-mul-code* [*code*]: *Rational x* ∗ *Rational y* = *Rational* ($x *_N y$)
unfolding *Rational-def* **by** *simp*

lemma *rat-neg-code* [*code*]: $-$ *Rational* x = *Rational* ($\sim_N$ x)
 unfolding *Rational-def* **by** *simp*

lemma *rat-sub-code* [*code*]: *Rational* x $-$ *Rational* y = *Rational* (x $-_N$ y)
 unfolding *Rational-def* **by** *simp*

lemma *rat-inv-code* [*code*]: *inverse* (*Rational* x) = *Rational* (*Ninv* x)
 unfolding *Rational-def Ninv divide-rat-def* **by** *simp*

lemma *rat-div-code* [*code*]: *Rational* x / *Rational* y = *Rational* (x $\div_N$ y)
 unfolding *Rational-def* **by** *simp*

Setup for SML code generator

```
types-code
  rat ((int */ int))
attach (term-of) ⟪
fun term-of-rat (p, q) =
  let
    val rT = Type (Rational.rat, [])
  in
    if q = 1 orelse p = 0 then HOLogic.mk-number rT p
    else Const (HOL.inverse-class.divide, rT --> rT --> rT) $
      HOLogic.mk-number rT p $ HOLogic.mk-number rT q
  end;
⟫
attach (test) ⟪
fun gen-rat i =
  let
    val p = random-range 0 i;
    val q = random-range 1 (i + 1);
    val g = Integer.gcd p q;
    val p' = p div g;
    val q' = q div g;
  in
    (if one-of [true, false] then p' else ~ p',
     if p' = 0 then 0 else q')
  end;
⟫

consts-code
  Rational ((-))

consts-code
  of-int :: int ⇒ rat (⟨module⟩rat'-of'-int)
attach ⟪
fun rat-of-int 0 = (0, 0)
  | rat-of-int i = (i, 1);
⟫
```

end

5 PReal: Positive real numbers

theory *PReal*
imports *Rational*
begin

Could be generalized and moved to *Ring-and-Field*

```
lemma add-eq-exists: ∃ x. a+x = (b::rat)
by (rule-tac x=b−a in exI, simp)

definition
  cut :: rat set => bool where
  cut A = ({} ⊂ A &
          A < {r. 0 < r} &
          (∀ y ∈ A. ((∀ z. 0<z & z < y −−> z ∈ A) & (∃ u ∈ A. y < u))))

lemma cut-of-rat:
  assumes q: 0 < q shows cut {r::rat. 0 < r & r < q} (is cut ?A)
proof −
  from q have pos: ?A < {r. 0 < r} by force
  have nonempty: {} ⊂ ?A
  proof
    show {} ⊆ ?A by simp
    show {} ≠ ?A
      by (force simp only: q eq-commute [of {}] interval-empty-iff)
  qed
  show ?thesis
    by (simp add: cut-def pos nonempty,
        blast dest: dense intro: order-less-trans)
qed


typedef preal = {A. cut A}
  by (blast intro: cut-of-rat [OF zero-less-one])

instance preal :: {ord, plus, minus, times, inverse, one} ..

definition
  preal-of-rat :: rat => preal where
  preal-of-rat q = Abs-preal {x::rat. 0 < x & x < q}

definition
  psup :: preal set => preal where
  psup P = Abs-preal (⋃ X ∈ P. Rep-preal X)

definition
```

add-set :: [*rat set*,*rat set*] => *rat set* **where**
add-set A B = $\{w.\ \exists x \in A.\ \exists y \in B.\ w = x + y\}$

definition
diff-set :: [*rat set*,*rat set*] => *rat set* **where**
diff-set A B = $\{w.\ \exists x.\ 0 < w\ \&\ 0 < x\ \&\ x \notin B\ \&\ x + w \in A\}$

definition
mult-set :: [*rat set*,*rat set*] => *rat set* **where**
mult-set A B = $\{w.\ \exists x \in A.\ \exists y \in B.\ w = x * y\}$

definition
inverse-set :: *rat set* => *rat set* **where**
inverse-set A = $\{x.\ \exists y.\ 0 < x\ \&\ x < y\ \&\ inverse\ y \notin A\}$

defs (**overloaded**)

preal-less-def:
$R < S$ == *Rep-preal R* < *Rep-preal S*

preal-le-def:
$R \leq S$ == *Rep-preal R* $\subseteq$ *Rep-preal S*

preal-add-def:
$R + S$ == *Abs-preal* (*add-set* (*Rep-preal R*) (*Rep-preal S*))

preal-diff-def:
$R - S$ == *Abs-preal* (*diff-set* (*Rep-preal R*) (*Rep-preal S*))

preal-mult-def:
$R * S$ == *Abs-preal* (*mult-set* (*Rep-preal R*) (*Rep-preal S*))

preal-inverse-def:
inverse R == *Abs-preal* (*inverse-set* (*Rep-preal R*))

preal-one-def:
1 == *preal-of-rat 1*

Reduces equality on abstractions to equality on representatives

declare *Abs-preal-inject* [*simp*]
declare *Abs-preal-inverse* [*simp*]

lemma *rat-mem-preal*: $0 < q$ ==> $\{r{::}rat.\ 0 < r\ \&\ r < q\} \in preal$
by (*simp add*: *preal-def cut-of-rat*)

lemma *preal-nonempty*: $A \in preal$ ==> $\exists x{\in}A.\ 0 < x$
by (*unfold preal-def cut-def*, *blast*)

lemma *preal-Ex-mem*: $A \in preal \Longrightarrow \exists x.\ x \in A$
by (*drule preal-nonempty*, *fast*)

lemma *preal-imp-psubset-positives*: $A \in preal ==> A < \{r.\ 0 < r\}$
by (*force simp add*: *preal-def cut-def*)

lemma *preal-exists-bound*: $A \in preal ==> \exists x.\ 0 < x\ \&\ x \notin A$
by (*drule preal-imp-psubset-positives*, *auto*)

lemma *preal-exists-greater*: $[|\ A \in preal;\ y \in A\ |] ==> \exists u \in A.\ y < u$
by (*unfold preal-def cut-def*, *blast*)

lemma *preal-downwards-closed*: $[|\ A \in preal;\ y \in A;\ 0 < z;\ z < y\ |] ==> z \in A$
by (*unfold preal-def cut-def*, *blast*)

Relaxing the final premise

lemma *preal-downwards-closed'*:
  $[|\ A \in preal;\ y \in A;\ 0 < z;\ z \le y\ |] ==> z \in A$
apply (*simp add*: *order-le-less*)
apply (*blast intro*: *preal-downwards-closed*)
done

A positive fraction not in a positive real is an upper bound. Gleason p. 122 - Remark (1)

lemma *not-in-preal-ub*:
 assumes *A*: $A \in preal$
  and *notx*: $x \notin A$
  and *y*: $y \in A$
  and *pos*: $0 < x$
 shows $y < x$
proof (*cases rule*: *linorder-cases*)
 assume $x<y$
 with *notx* **show** *?thesis*
  by (*simp add*: *preal-downwards-closed* [*OF A y*] *pos*)
next
 assume $x=y$
 with *notx* **and** *y* **show** *?thesis* **by** *simp*
next
 assume $y<x$
 thus *?thesis* **.**
qed

preal lemmas instantiated to *Rep-preal X*

lemma *mem-Rep-preal-Ex*: $\exists x.\ x \in Rep\text{-}preal\ X$
by (*rule preal-Ex-mem* [*OF Rep-preal*])

lemma *Rep-preal-exists-bound*: $\exists x>0.\ x \notin Rep\text{-}preal\ X$
by (*rule preal-exists-bound* [*OF Rep-preal*])

lemmas *not-in-Rep-preal-ub* = *not-in-preal-ub* [*OF Rep-preal*]

5.1 *preal-of-prat*: the Injection from prat to preal

lemma *rat-less-set-mem-preal*: *0* < *y* ==> {*u*::*rat*. *0* < *u* & *u* < *y*} ∈ *preal*
by (*simp add*: *preal-def cut-of-rat*)

lemma *rat-subset-imp-le*:
  [|{*u*::*rat*. *0* < *u* & *u* < *x*} ⊆ {*u*. *0* < *u* & *u* < *y*}; *0*<*x*|] ==> *x* ≤ *y*
apply (*simp add*: *linorder-not-less* [*symmetric*])
apply (*blast dest*: *dense intro*: *order-less-trans*)
done

lemma *rat-set-eq-imp-eq*:
  [|{*u*::*rat*. *0* < *u* & *u* < *x*} = {*u*. *0* < *u* & *u* < *y*};
    0 < *x*; *0* < *y*|] ==> *x* = *y*
by (*blast intro*: *rat-subset-imp-le order-antisym*)

5.2 Properties of Ordering

lemma *preal-le-refl*: *w* ≤ (*w*::*preal*)
by (*simp add*: *preal-le-def*)

lemma *preal-le-trans*: [| *i* ≤ *j*; *j* ≤ *k* |] ==> *i* ≤ (*k*::*preal*)
by (*force simp add*: *preal-le-def*)

lemma *preal-le-anti-sym*: [| *z* ≤ *w*; *w* ≤ *z* |] ==> *z* = (*w*::*preal*)
apply (*simp add*: *preal-le-def*)
apply (*rule Rep-preal-inject* [*THEN iffD1*], *blast*)
done

lemma *preal-less-le*: ((*w*::*preal*) < *z*) = (*w* ≤ *z* & *w* ≠ *z*)
by (*simp add*: *preal-le-def preal-less-def Rep-preal-inject psubset-def*)

instance *preal* :: *order*
 by *intro-classes*
  (*assumption* |
   rule preal-le-refl preal-le-trans preal-le-anti-sym preal-less-le)+

lemma *preal-imp-pos*: [|*A* ∈ *preal*; *r* ∈ *A*|] ==> *0* < *r*
by (*insert preal-imp-psubset-positives*, *blast*)

lemma *preal-le-linear*: *x* <= *y* | *y* <= (*x*::*preal*)
apply (*auto simp add*: *preal-le-def*)
apply (*rule ccontr*)
apply (*blast dest*: *not-in-Rep-preal-ub intro*: *preal-imp-pos* [*OF Rep-preal*]
    elim: *order-less-asym*)
done

instance *preal* :: *linorder*
  by *intro-classes* (*rule preal-le-linear*)

instance *preal* :: *distrib-lattice*
  inf ≡ *min*
  sup ≡ *max*
  by *intro-classes*
    (*auto simp add*: *inf-preal-def sup-preal-def min-max.sup-inf-distrib1*)

5.3 Properties of Addition

lemma *preal-add-commute*: (*x*::*preal*) + *y* = *y* + *x*
apply (*unfold preal-add-def add-set-def*)
apply (*rule-tac f* = *Abs-preal* **in** *arg-cong*)
apply (*force simp add*: *add-commute*)
done

Lemmas for proving that addition of two positive reals gives a positive real

lemma *empty-psubset-nonempty*: *a* ∈ *A* ==> {} ⊂ *A*
by *blast*

Part 1 of Dedekind sections definition

lemma *add-set-not-empty*:
    [|*A* ∈ *preal*; *B* ∈ *preal*|] ==> {} ⊂ *add-set A B*
apply (*drule preal-nonempty*)+
apply (*auto simp add*: *add-set-def*)
done

Part 2 of Dedekind sections definition. A structured version of this proof is *preal-not-mem-mult-set-Ex* below.

lemma *preal-not-mem-add-set-Ex*:
    [|*A* ∈ *preal*; *B* ∈ *preal*|] ==> ∃ *q*>*0*. *q* ∉ *add-set A B*
apply (*insert preal-exists-bound* [*of A*] *preal-exists-bound* [*of B*], *auto*)
apply (*rule-tac x* = *x*+*xa* **in** *exI*)
apply (*simp add*: *add-set-def*, *clarify*)
apply (*drule* (*3*) *not-in-preal-ub*)+
apply (*force dest*: *add-strict-mono*)
done

lemma *add-set-not-rat-set*:
   assumes *A*: *A* ∈ *preal*
       and *B*: *B* ∈ *preal*
     shows *add-set A B* < {*r*. *0* < *r*}
proof
  from *preal-imp-pos* [*OF A*] *preal-imp-pos* [*OF B*]
  show *add-set A B* ⊆ {*r*. *0* < *r*} **by** (*force simp add*: *add-set-def*)
next
  show *add-set A B* ≠ {*r*. *0* < *r*}
    by (*insert preal-not-mem-add-set-Ex* [*OF A B*], *blast*)

qed

Part 3 of Dedekind sections definition

lemma *add-set-lemma3*:
  $[|A \in preal;\ B \in preal;\ u \in add\text{-}set\ A\ B;\ 0 < z;\ z < u|]$
  $==> z \in add\text{-}set\ A\ B$
proof (*unfold add-set-def*, *clarify*)
 fix x::*rat* **and** y::*rat*
 assume A: $A \in preal$
  and B: $B \in preal$
  and [*simp*]: $0 < z$
  and *zless*: $z < x + y$
  and x: $x \in A$
  and y: $y \in B$
 have *xpos* [*simp*]: $0<x$ **by** (*rule preal-imp-pos* [*OF A x*])
 have *ypos* [*simp*]: $0<y$ **by** (*rule preal-imp-pos* [*OF B y*])
 have *xypos* [*simp*]: $0 < x+y$ **by** (*simp add*: *pos-add-strict*)
 let $?f = z/(x+y)$
 have *fless*: $?f < 1$ **by** (*simp add*: *zless pos-divide-less-eq*)
 show $\exists x' \in A.\ \exists y'{\in}B.\ z = x' + y'$
 proof (*intro bexI*)
  show $z = x*?f + y*?f$
   by (*simp add*: *left-distrib* [*symmetric*] *divide-inverse mult-ac*
    order-less-imp-not-eq2)
 next
  show $y * ?f \in B$
  proof (*rule preal-downwards-closed* [*OF B y*])
   show $0 < y * ?f$
    by (*simp add*: *divide-inverse zero-less-mult-iff*)
  next
   show $y * ?f < y$
    by (*insert mult-strict-left-mono* [*OF fless ypos*], *simp*)
  qed
 next
  show $x * ?f \in A$
  proof (*rule preal-downwards-closed* [*OF A x*])
   show $0 < x * ?f$
    by (*simp add*: *divide-inverse zero-less-mult-iff*)
  next
   show $x * ?f < x$
    by (*insert mult-strict-left-mono* [*OF fless xpos*], *simp*)
  qed
 qed
qed

Part 4 of Dedekind sections definition

lemma *add-set-lemma4*:
  $[|A \in preal;\ B \in preal;\ y \in add\text{-}set\ A\ B|] ==> \exists u \in add\text{-}set\ A\ B.\ y < u$
apply (*auto simp add*: *add-set-def*)

apply (*frule preal-exists-greater* [*of A*], *auto*)
apply (*rule-tac x=u + y* **in** *exI*)
apply (*auto intro*: *add-strict-left-mono*)
done

lemma *mem-add-set*:
[|$A \in preal$; $B \in preal$|] ==> *add-set A B* $\in$ *preal*
apply (*simp* (*no-asm-simp*) *add*: *preal-def cut-def*)
apply (*blast intro*!: *add-set-not-empty add-set-not-rat-set*
add-set-lemma3 add-set-lemma4)
done

lemma *preal-add-assoc*: $((x::preal) + y) + z = x + (y + z)$
apply (*simp add*: *preal-add-def mem-add-set Rep-preal*)
apply (*force simp add*: *add-set-def add-ac*)
done

instance *preal* :: *ab-semigroup-add*
proof
fix *a b c* :: *preal*
show $(a + b) + c = a + (b + c)$ **by** (*rule preal-add-assoc*)
show $a + b = b + a$ **by** (*rule preal-add-commute*)
qed

lemma *preal-add-left-commute*: $x + (y + z) = y + ((x + z)::preal)$
by (*rule add-left-commute*)

Positive Real addition is an AC operator

lemmas *preal-add-ac* = *preal-add-assoc preal-add-commute preal-add-left-commute*

5.4 Properties of Multiplication

Proofs essentially same as for addition

lemma *preal-mult-commute*: $(x::preal) * y = y * x$
apply (*unfold preal-mult-def mult-set-def*)
apply (*rule-tac f = Abs-preal* **in** *arg-cong*)
apply (*force simp add*: *mult-commute*)
done

Multiplication of two positive reals gives a positive real.

Lemmas for proving positive reals multiplication set in *preal*

Part 1 of Dedekind sections definition

lemma *mult-set-not-empty*:
[|$A \in preal$; $B \in preal$|] ==> $\{\} \subset$ *mult-set A B*
apply (*insert preal-nonempty* [*of A*] *preal-nonempty* [*of B*])
apply (*auto simp add*: *mult-set-def*)
done

Part 2 of Dedekind sections definition

```
lemma preal-not-mem-mult-set-Ex:
  assumes A: A ∈ preal
    and B: B ∈ preal
   shows ∃ q. 0 < q & q ∉ mult-set A B
proof −
  from preal-exists-bound [OF A]
  obtain x where [simp]: 0 < x x ∉ A by blast
  from preal-exists-bound [OF B]
  obtain y where [simp]: 0 < y y ∉ B by blast
  show ?thesis
  proof (intro exI conjI)
    show 0 < x*y by (simp add: mult-pos-pos)
    show x * y ∉ mult-set A B
    proof −
      { fix u::rat and v::rat
             assume u ∈ A and v ∈ B and x*y = u*v
             moreover
             with prems have u<x and v<y by (blast dest: not-in-preal-ub)+
             moreover
             with prems have 0≤v
               by (blast intro: preal-imp-pos [OF B]  order-less-imp-le prems)
             moreover
        from calculation
             have u*v < x*y by (blast intro: mult-strict-mono prems)
             ultimately have False by force }
      thus ?thesis by (auto simp add: mult-set-def)
    qed
  qed
qed
```

```
lemma mult-set-not-rat-set:
  assumes A: A ∈ preal
    and B: B ∈ preal
  shows mult-set A B < {r. 0 < r}
proof
  show mult-set A B ⊆ {r. 0 < r}
    by (force simp add: mult-set-def
      intro: preal-imp-pos [OF A] preal-imp-pos [OF B] mult-pos-pos)
  show mult-set A B ≠ {r. 0 < r}
    using preal-not-mem-mult-set-Ex [OF A B] by blast
qed
```

Part 3 of Dedekind sections definition

```
lemma mult-set-lemma3:
     [|A ∈ preal; B ∈ preal; u ∈ mult-set A B; 0 < z; z < u|]
      ==> z ∈ mult-set A B
proof (unfold mult-set-def, clarify)
  fix x::rat and y::rat
```

```
  assume A: A ∈ preal
    and B: B ∈ preal
    and [simp]: 0 < z
    and zless: z < x * y
    and x:  x ∈ A
    and y:  y ∈ B
  have [simp]: 0<y by (rule preal-imp-pos [OF B y])
  show ∃ x' ∈ A. ∃ y' ∈ B. z = x' * y'
  proof
    show ∃ y'∈B. z = (z/y) * y'
    proof
      show z = (z/y)*y
        by (simp add: divide-inverse mult-commute [of y] mult-assoc
                      order-less-imp-not-eq2)
      show y ∈ B by fact
    qed
  next
    show z/y ∈ A
    proof (rule preal-downwards-closed [OF A x])
      show 0 < z/y
        by (simp add: zero-less-divide-iff)
      show z/y < x by (simp add: pos-divide-less-eq zless)
    qed
  qed
qed
```

Part 4 of Dedekind sections definition

```
lemma mult-set-lemma4:
     [|A ∈ preal; B ∈ preal; y ∈ mult-set A B|] ==> ∃ u ∈ mult-set A B. y < u
apply (auto simp add: mult-set-def)
apply (frule preal-exists-greater [of A], auto)
apply (rule-tac x=u * y in exI)
apply (auto intro: preal-imp-pos [of A] preal-imp-pos [of B]
                   mult-strict-right-mono)
done
```

```
lemma mem-mult-set:
     [|A ∈ preal; B ∈ preal|] ==> mult-set A B ∈ preal
apply (simp (no-asm-simp) add: preal-def cut-def)
apply (blast intro!: mult-set-not-empty mult-set-not-rat-set
                     mult-set-lemma3 mult-set-lemma4)
done
```

```
lemma preal-mult-assoc: ((x::preal) * y) * z = x * (y * z)
apply (simp add: preal-mult-def mem-mult-set Rep-preal)
apply (force simp add: mult-set-def mult-ac)
done
```

```
instance preal :: ab-semigroup-mult
proof
  fix a b c :: preal
  show (a * b) * c = a * (b * c) by (rule preal-mult-assoc)
  show a * b = b * a by (rule preal-mult-commute)
qed

lemma preal-mult-left-commute: x * (y * z) = y * ((x * z)::preal)
by (rule mult-left-commute)
```

Positive Real multiplication is an AC operator

```
lemmas preal-mult-ac =
    preal-mult-assoc preal-mult-commute preal-mult-left-commute
```

Positive real 1 is the multiplicative identity element

```
lemma preal-mult-1: (1::preal) * z = z
unfolding preal-one-def
proof (induct z)
  fix A :: rat set
  assume A: A ∈ preal
  have {w. ∃ u. 0 < u ∧ u < 1 & (∃ v ∈ A. w = u * v)} = A (is ?lhs = A)
  proof
    show ?lhs ⊆ A
    proof clarify
      fix x::rat and u::rat and v::rat
      assume upos: 0<u and u<1 and v: v ∈ A
      have vpos: 0<v by (rule preal-imp-pos [OF A v])
      hence u*v < 1*v by (simp only: mult-strict-right-mono prems)
      thus u * v ∈ A
        by (force intro: preal-downwards-closed [OF A v] mult-pos-pos
          upos vpos)
    qed
  next
    show A ⊆ ?lhs
    proof clarify
      fix x::rat
      assume x: x ∈ A
      have xpos: 0<x by (rule preal-imp-pos [OF A x])
      from preal-exists-greater [OF A x]
      obtain v where v: v ∈ A and xlessv: x < v ..
      have vpos: 0<v by (rule preal-imp-pos [OF A v])
      show ∃ u. 0 < u ∧ u < 1 ∧ (∃ v∈A. x = u * v)
      proof (intro exI conjI)
        show 0 < x/v
          by (simp add: zero-less-divide-iff xpos vpos)
        show x / v < 1
          by (simp add: pos-divide-less-eq vpos xlessv)
        show ∃ v'∈A. x = (x / v) * v'
        proof
```

```
        show x = (x/v)*v
          by (simp add: divide-inverse mult-assoc vpos
                  order-less-imp-not-eq2)
        show v ∈ A by fact
      qed
    qed
  qed
 qed
 thus preal-of-rat 1 * Abs-preal A = Abs-preal A
  by (simp add: preal-of-rat-def preal-mult-def mult-set-def
          rat-mem-preal A)
qed

instance preal :: comm-monoid-mult
by intro-classes (rule preal-mult-1)

lemma preal-mult-1-right: z * (1::preal) = z
by (rule mult-1-right)
```

5.5 Distribution of Multiplication across Addition

```
lemma mem-Rep-preal-add-iff:
   (z ∈ Rep-preal(R+S)) = (∃ x ∈ Rep-preal R. ∃ y ∈ Rep-preal S. z = x + y)
apply (simp add: preal-add-def mem-add-set Rep-preal)
apply (simp add: add-set-def)
done

lemma mem-Rep-preal-mult-iff:
   (z ∈ Rep-preal(R*S)) = (∃ x ∈ Rep-preal R. ∃ y ∈ Rep-preal S. z = x * y)
apply (simp add: preal-mult-def mem-mult-set Rep-preal)
apply (simp add: mult-set-def)
done

lemma distrib-subset1:
  Rep-preal (w * (x + y)) ⊆ Rep-preal (w * x + w * y)
apply (auto simp add: Bex-def mem-Rep-preal-add-iff mem-Rep-preal-mult-iff)
apply (force simp add: right-distrib)
done

lemma preal-add-mult-distrib-mean:
 assumes a: a ∈ Rep-preal w
  and b: b ∈ Rep-preal w
  and d: d ∈ Rep-preal x
  and e: e ∈ Rep-preal y
 shows ∃ c ∈ Rep-preal w. a * d + b * e = c * (d + e)
proof
 let ?c = (a*d + b*e)/(d+e)
 have [simp]: 0<a 0<b 0<d 0<e 0<d+e
  by (blast intro: preal-imp-pos [OF Rep-preal] a b d e pos-add-strict)+
```

```
  have cpos: 0 < ?c
    by (simp add: zero-less-divide-iff zero-less-mult-iff pos-add-strict)
  show a ∗ d + b ∗ e = ?c ∗ (d + e)
    by (simp add: divide-inverse mult-assoc order-less-imp-not-eq2)
  show ?c ∈ Rep-preal w
  proof (cases rule: linorder-le-cases)
    assume a ≤ b
    hence ?c ≤ b
      by (simp add: pos-divide-le-eq right-distrib mult-right-mono
                    order-less-imp-le)
    thus ?thesis by (rule preal-downwards-closed' [OF Rep-preal b cpos])
  next
    assume b ≤ a
    hence ?c ≤ a
      by (simp add: pos-divide-le-eq right-distrib mult-right-mono
                    order-less-imp-le)
    thus ?thesis by (rule preal-downwards-closed' [OF Rep-preal a cpos])
  qed
qed

lemma distrib-subset2:
     Rep-preal (w ∗ x + w ∗ y) ⊆ Rep-preal (w ∗ (x + y))
apply (auto simp add: Bex-def mem-Rep-preal-add-iff mem-Rep-preal-mult-iff)
apply (drule-tac w=w and x=x and y=y in preal-add-mult-distrib-mean, auto)
done

lemma preal-add-mult-distrib2: (w ∗ ((x::preal) + y)) = (w ∗ x) + (w ∗ y)
apply (rule Rep-preal-inject [THEN iffD1])
apply (rule equalityI [OF distrib-subset1 distrib-subset2])
done

lemma preal-add-mult-distrib: (((x::preal) + y) ∗ w) = (x ∗ w) + (y ∗ w)
by (simp add: preal-mult-commute preal-add-mult-distrib2)

instance preal :: comm-semiring
by intro-classes (rule preal-add-mult-distrib)
```

5.6 Existence of Inverse, a Positive Real

```
lemma mem-inv-set-ex:
  assumes A: A ∈ preal shows ∃ x y. 0 < x & x < y & inverse y ∉ A
proof −
  from preal-exists-bound [OF A]
  obtain x where [simp]: 0<x x ∉ A by blast
  show ?thesis
  proof (intro exI conjI)
    show 0 < inverse (x+1)
      by (simp add: order-less-trans [OF - less-add-one])
    show inverse(x+1) < inverse x
```

```
    by (simp add: less-imp-inverse-less less-add-one)
  show inverse (inverse x) ∉ A
    by (simp add: order-less-imp-not-eq2)
 qed
qed
```

Part 1 of Dedekind sections definition

```
lemma inverse-set-not-empty:
    A ∈ preal ==> {} ⊂ inverse-set A
apply (insert mem-inv-set-ex [of A])
apply (auto simp add: inverse-set-def)
done
```

Part 2 of Dedekind sections definition

```
lemma preal-not-mem-inverse-set-Ex:
  assumes A: A ∈ preal  shows ∃ q. 0 < q & q ∉ inverse-set A
proof −
 from preal-nonempty [OF A]
 obtain x where x: x ∈ A and  xpos [simp]: 0<x ..
 show ?thesis
 proof (intro exI conjI)
  show 0 < inverse x by simp
  show inverse x ∉ inverse-set A
  proof −
   { fix y::rat
     assume ygt: inverse x < y
     have [simp]: 0 < y by (simp add: order-less-trans [OF - ygt])
     have iyless: inverse y < x
       by (simp add: inverse-less-imp-less [of x] ygt)
     have inverse y ∈ A
       by (simp add: preal-downwards-closed [OF A x] iyless)}
   thus ?thesis by (auto simp add: inverse-set-def)
  qed
 qed
qed

lemma inverse-set-not-rat-set:
  assumes A: A ∈ preal  shows inverse-set A < {r. 0 < r}
proof
 show inverse-set A ⊆ {r. 0 < r}  by (force simp add: inverse-set-def)
next
 show inverse-set A ≠ {r. 0 < r}
  by (insert preal-not-mem-inverse-set-Ex [OF A], blast)
qed
```

Part 3 of Dedekind sections definition

```
lemma inverse-set-lemma3:
    [|A ∈ preal; u ∈ inverse-set A; 0 < z; z < u|]
     ==> z ∈ inverse-set A
```

```
apply (auto simp add: inverse-set-def)
apply (auto intro: order-less-trans)
done
```

Part 4 of Dedekind sections definition

```
lemma inverse-set-lemma4:
    [|A ∈ preal; y ∈ inverse-set A|] ==> ∃ u ∈ inverse-set A. y < u
apply (auto simp add: inverse-set-def)
apply (drule dense [of y])
apply (blast intro: order-less-trans)
done
```

```
lemma mem-inverse-set:
    A ∈ preal ==> inverse-set A ∈ preal
apply (simp (no-asm-simp) add: preal-def cut-def)
apply (blast intro!: inverse-set-not-empty inverse-set-not-rat-set
                inverse-set-lemma3 inverse-set-lemma4)
done
```

5.7 Gleason’s Lemma 9-3.4, page 122

```
lemma Gleason9-34-exists:
  assumes A: A ∈ preal
    and ∀ x∈A. x + u ∈ A
    and 0 ≤ z
  shows ∃ b∈A. b + (of-int z) ∗ u ∈ A
proof (cases z rule: int-cases)
  case (nonneg n)
  show ?thesis
  proof (simp add: prems, induct n)
    case 0
      from preal-nonempty [OF A]
      show ?case by force
    case (Suc k)
      from this obtain b where b ∈ A b + of-nat k ∗ u ∈ A ..
      hence b + of-int (int k)∗u + u ∈ A by (simp add: prems)
      thus ?case by (force simp add: left-distrib add-ac prems)
  qed
next
  case (neg n)
  with prems show ?thesis by simp
qed
```

```
lemma Gleason9-34-contra:
  assumes A: A ∈ preal
    shows [|∀ x∈A. x + u ∈ A; 0 < u; 0 < y; y ∉ A|] ==> False
proof (induct u, induct y)
  fix a::int and b::int
```

```
  fix c::int and d::int
  assume bpos [simp]: 0 < b
    and dpos [simp]: 0 < d
    and closed: ∀x∈A. x + (Fract c d) ∈ A
    and upos: 0 < Fract c d
    and ypos: 0 < Fract a b
    and notin: Fract a b ∉ A
  have cpos [simp]: 0 < c
    by (simp add: zero-less-Fract-iff [OF dpos, symmetric] upos)
  have apos [simp]: 0 < a
    by (simp add: zero-less-Fract-iff [OF bpos, symmetric] ypos)
  let ?k = a*d
  have frle: Fract a b ≤ Fract ?k 1 * (Fract c d)
  proof −
    have ?thesis = ((a * d * b * d) ≤ c * b * (a * d * b * d))
      by (simp add: mult-rat le-rat order-less-imp-not-eq2 mult-ac)
    moreover
    have (1 * (a * d * b * d)) ≤ c * b * (a * d * b * d)
      by (rule mult-mono,
          simp-all add: int-one-le-iff-zero-less zero-less-mult-iff
                        order-less-imp-le)
    ultimately
    show ?thesis by simp
  qed
  have k: 0 ≤ ?k by (simp add: order-less-imp-le zero-less-mult-iff)
  from Gleason9-34-exists [OF A closed k]
  obtain z where z: z ∈ A
           and mem: z + of-int ?k * Fract c d ∈ A ..
  have less: z + of-int ?k * Fract c d < Fract a b
    by (rule not-in-preal-ub [OF A notin mem ypos])
  have 0<z by (rule preal-imp-pos [OF A z])
  with frle and less show False by (simp add: Fract-of-int-eq)
qed


lemma Gleason9-34:
  assumes A: A ∈ preal
    and upos: 0 < u
  shows ∃r ∈ A. r + u ∉ A
proof (rule ccontr, simp)
  assume closed: ∀r∈A. r + u ∈ A
  from preal-exists-bound [OF A]
  obtain y where y: y ∉ A and ypos: 0 < y by blast
  show False
    by (rule Gleason9-34-contra [OF A closed upos ypos y])
qed
```

5.8 Gleason's Lemma 9-3.6

```
lemma lemma-gleason9-36:
  assumes A: A ∈ preal
    and x: 1 < x
  shows ∃ r ∈ A. r*x ∉ A
proof −
  from preal-nonempty [OF A]
  obtain y where y: y ∈ A and  ypos: 0<y ..
  show ?thesis
  proof (rule classical)
    assume ~(∃ r∈A. r * x ∉ A)
    with y have ymem: y * x ∈ A by blast
    from ypos mult-strict-left-mono [OF x]
    have yless: y < y*x by simp
    let ?d = y*x − y
    from yless have dpos: 0 < ?d and eq: y + ?d = y*x by auto
    from Gleason9-34 [OF A dpos]
    obtain r where r: r∈A and notin: r + ?d ∉ A ..
    have rpos: 0<r by (rule preal-imp-pos [OF A r])
    with dpos have rdpos: 0 < r + ?d by arith
    have ~ (r + ?d ≤ y + ?d)
    proof
      assume le: r + ?d ≤ y + ?d
      from ymem have yd: y + ?d ∈ A by (simp add: eq)
      have r + ?d ∈ A by (rule preal-downwards-closed' [OF A yd rdpos le])
      with notin show False by simp
    qed
    hence y < r by simp
    with ypos have  dless: ?d < (r * ?d)/y
      by (simp add: pos-less-divide-eq mult-commute [of ?d]
                mult-strict-right-mono dpos)
    have r + ?d < r*x
    proof −
      have r + ?d < r + (r * ?d)/y by (simp add: dless)
      also with ypos have ... = (r/y) * (y + ?d)
        by (simp only: right-distrib divide-inverse mult-ac, simp)
      also have ... = r*x using ypos
        by (simp add: times-divide-eq-left)
      finally show r + ?d < r*x .
    qed
    with r notin rdpos
    show ∃ r∈A. r * x ∉ A by (blast dest:  preal-downwards-closed [OF A])
  qed
qed
```

5.9 Existence of Inverse: Part 2

```
lemma mem-Rep-preal-inverse-iff:
    (z ∈ Rep-preal(inverse R)) =
```

```
    (0 < z ∧ (∃ y. z < y ∧ inverse y ∉ Rep-preal R))
apply (simp add: preal-inverse-def mem-inverse-set Rep-preal)
apply (simp add: inverse-set-def)
done

lemma Rep-preal-of-rat:
    0 < q ==> Rep-preal (preal-of-rat q) = {x. 0 < x ∧ x < q}
by (simp add: preal-of-rat-def rat-mem-preal)

lemma subset-inverse-mult-lemma:
  assumes xpos: 0 < x and xless: x < 1
  shows ∃ r u y. 0 < r & r < y & inverse y ∉ Rep-preal R &
    u ∈ Rep-preal R & x = r * u
proof −
  from xpos and xless have 1 < inverse x by (simp add: one-less-inverse-iff)
  from lemma-gleason9-36 [OF Rep-preal this]
  obtain r where r: r ∈ Rep-preal R
           and notin: r * (inverse x) ∉ Rep-preal R ..
  have rpos: 0<r by (rule preal-imp-pos [OF Rep-preal r])
  from preal-exists-greater [OF Rep-preal r]
  obtain u where u: u ∈ Rep-preal R and rless: r < u ..
  have upos: 0<u by (rule preal-imp-pos [OF Rep-preal u])
  show ?thesis
  proof (intro exI conjI)
    show 0 < x/u using xpos upos
      by (simp add: zero-less-divide-iff)
    show x/u < x/r using xpos upos rpos
      by (simp add: divide-inverse mult-less-cancel-left rless)
    show inverse (x / r) ∉ Rep-preal R using notin
      by (simp add: divide-inverse mult-commute)
    show u ∈ Rep-preal R by (rule u)
    show x = x / u * u using upos
      by (simp add: divide-inverse mult-commute)
  qed
qed

lemma subset-inverse-mult:
    Rep-preal(preal-of-rat 1) ⊆ Rep-preal(inverse R * R)
apply (auto simp add: Bex-def Rep-preal-of-rat mem-Rep-preal-inverse-iff
                    mem-Rep-preal-mult-iff)
apply (blast dest: subset-inverse-mult-lemma)
done

lemma inverse-mult-subset-lemma:
  assumes rpos: 0 < r
    and rless: r < y
    and notin: inverse y ∉ Rep-preal R
    and q: q ∈ Rep-preal R
  shows r*q < 1
```

```
proof -
  have q < inverse y using rpos rless
    by (simp add: not-in-preal-ub [OF Rep-preal notin] q)
  hence r * q < r/y using rpos
    by (simp add: divide-inverse mult-less-cancel-left)
  also have ... ≤ 1 using rpos rless
    by (simp add: pos-divide-le-eq)
  finally show ?thesis .
qed

lemma inverse-mult-subset:
     Rep-preal(inverse R * R) ⊆ Rep-preal(preal-of-rat 1)
apply (auto simp add: Bex-def Rep-preal-of-rat mem-Rep-preal-inverse-iff
                      mem-Rep-preal-mult-iff)
apply (simp add: zero-less-mult-iff preal-imp-pos [OF Rep-preal])
apply (blast intro: inverse-mult-subset-lemma)
done

lemma preal-mult-inverse: inverse R * R = (1::preal)
unfolding preal-one-def
apply (rule Rep-preal-inject [THEN iffD1])
apply (rule equalityI [OF inverse-mult-subset subset-inverse-mult])
done

lemma preal-mult-inverse-right: R * inverse R = (1::preal)
apply (rule preal-mult-commute [THEN subst])
apply (rule preal-mult-inverse)
done
```

Theorems needing *Gleason9-34*

```
lemma Rep-preal-self-subset: Rep-preal (R) ⊆ Rep-preal(R + S)
proof
  fix r
  assume r: r ∈ Rep-preal R
  have rpos: 0<r by (rule preal-imp-pos [OF Rep-preal r])
  from mem-Rep-preal-Ex
  obtain y where y: y ∈ Rep-preal S ..
  have ypos: 0<y by (rule preal-imp-pos [OF Rep-preal y])
  have ry: r+y ∈ Rep-preal(R + S) using r y
    by (auto simp add: mem-Rep-preal-add-iff)
  show r ∈ Rep-preal(R + S) using r ypos rpos
    by (simp add:  preal-downwards-closed [OF Rep-preal ry])
qed

lemma Rep-preal-sum-not-subset: ~ Rep-preal (R + S) ⊆ Rep-preal(R)
proof -
  from mem-Rep-preal-Ex
  obtain y where y: y ∈ Rep-preal S ..
  have ypos: 0<y by (rule preal-imp-pos [OF Rep-preal y])
```

from *Gleason9-34* [*OF Rep-preal ypos*]
obtain *r* **where** *r*: $r \in$ *Rep-preal R* **and** *notin*: $r + y \notin$ *Rep-preal R* **..**
have $r + y \in$ *Rep-preal* $(R + S)$ **using** *r y*
by (*auto simp add*: *mem-Rep-preal-add-iff*)
thus *?thesis* **using** *notin* **by** *blast*
qed

lemma *Rep-preal-sum-not-eq*: *Rep-preal* $(R + S) \neq$ *Rep-preal*(R)
by (*insert Rep-preal-sum-not-subset*, *blast*)

at last, Gleason prop. 9-3.5(iii) page 123

lemma *preal-self-less-add-left*: $(R$::*preal*$) < R + S$
apply (*unfold preal-less-def psubset-def*)
apply (*simp add*: *Rep-preal-self-subset Rep-preal-sum-not-eq* [*THEN not-sym*])
done

lemma *preal-self-less-add-right*: $(R$::*preal*$) < S + R$
by (*simp add*: *preal-add-commute preal-self-less-add-left*)

lemma *preal-not-eq-self*: $x \neq x + (y$::*preal*$)$
by (*insert preal-self-less-add-left* [*of x y*], *auto*)

5.10 Subtraction for Positive Reals

Gleason prop. 9-3.5(iv), page 123: proving $A < B \Longrightarrow \exists D.\ A + D = B$. We define the claimed D and show that it is a positive real

Part 1 of Dedekind sections definition

lemma *diff-set-not-empty*:
$R < S$ ==> $\{\} \subset$ *diff-set* (*Rep-preal S*) (*Rep-preal R*)
apply (*auto simp add*: *preal-less-def diff-set-def elim*!: *equalityE*)
apply (*frule-tac x1* = *S* **in** *Rep-preal* [*THEN preal-exists-greater*])
apply (*drule preal-imp-pos* [*OF Rep-preal*], *clarify*)
apply (*cut-tac a*=*x* **and** *b*=*u* **in** *add-eq-exists*, *force*)
done

Part 2 of Dedekind sections definition

lemma *diff-set-nonempty*:
$\exists q.\ 0 < q$ & $q \notin$ *diff-set* (*Rep-preal S*) (*Rep-preal R*)
apply (*cut-tac X* = *S* **in** *Rep-preal-exists-bound*)
apply (*erule exE*)
apply (*rule-tac x* = *x* **in** *exI*, *auto*)
apply (*simp add*: *diff-set-def*)
apply (*auto dest*: *Rep-preal* [*THEN preal-downwards-closed*])
done

lemma *diff-set-not-rat-set*:
diff-set (*Rep-preal S*) (*Rep-preal R*) $< \{r.\ 0 < r\}$ (**is** *?lhs* < *?rhs*)

```
proof
  show ?lhs ⊆ ?rhs by (auto simp add: diff-set-def)
  show ?lhs ≠ ?rhs using diff-set-nonempty by blast
qed
```

Part 3 of Dedekind sections definition

```
lemma diff-set-lemma3:
     [|R < S; u ∈ diff-set (Rep-preal S) (Rep-preal R); 0 < z; z < u|]
      ==> z ∈ diff-set (Rep-preal S) (Rep-preal R)
apply (auto simp add: diff-set-def)
apply (rule-tac x=x in exI)
apply (drule Rep-preal [THEN preal-downwards-closed], auto)
done
```

Part 4 of Dedekind sections definition

```
lemma diff-set-lemma4:
     [|R < S; y ∈ diff-set (Rep-preal S) (Rep-preal R)|]
      ==> ∃ u ∈ diff-set (Rep-preal S) (Rep-preal R). y < u
apply (auto simp add: diff-set-def)
apply (drule Rep-preal [THEN preal-exists-greater], clarify)
apply (cut-tac a=x+y and b=u in add-eq-exists, clarify)
apply (rule-tac x=y+xa in exI)
apply (auto simp add: add-ac)
done

lemma mem-diff-set:
     R < S ==> diff-set (Rep-preal S) (Rep-preal R) ∈ preal
apply (unfold preal-def cut-def)
apply (blast intro!: diff-set-not-empty diff-set-not-rat-set
                     diff-set-lemma3 diff-set-lemma4)
done

lemma mem-Rep-preal-diff-iff:
      R < S ==>
       (z ∈ Rep-preal(S−R)) =
       (∃ x. 0 < x & 0 < z & x ∉ Rep-preal R & x + z ∈ Rep-preal S)
apply (simp add: preal-diff-def mem-diff-set Rep-preal)
apply (force simp add: diff-set-def)
done
```

proving that $R + D \leq S$

```
lemma less-add-left-lemma:
  assumes Rless: R < S
    and a: a ∈ Rep-preal R
    and cb: c + b ∈ Rep-preal S
    and c ∉ Rep-preal R
    and 0 < b
    and 0 < c
  shows a + b ∈ Rep-preal S
```

```
proof -
  have 0<a by (rule preal-imp-pos [OF Rep-preal a])
  moreover
  have a < c using prems
    by (blast intro: not-in-Rep-preal-ub )
  ultimately show ?thesis using prems
    by (simp add: preal-downwards-closed [OF Rep-preal cb])
qed

lemma less-add-left-le1:
      R < (S::preal) ==> R + (S−R) ≤ S
apply (auto simp add: Bex-def preal-le-def mem-Rep-preal-add-iff
                  mem-Rep-preal-diff-iff)
apply (blast intro: less-add-left-lemma)
done
```

5.11 proving that $S \leq R + D$ — trickier

```
lemma lemma-sum-mem-Rep-preal-ex:
     x ∈ Rep-preal S ==> ∃ e. 0 < e & x + e ∈ Rep-preal S
apply (drule Rep-preal [THEN preal-exists-greater], clarify)
apply (cut-tac a=x and b=u in add-eq-exists, auto)
done

lemma less-add-left-lemma2:
  assumes Rless: R < S
    and x:     x ∈ Rep-preal S
    and xnot: x ∉ Rep-preal R
  shows ∃ u v z. 0 < v & 0 < z & u ∈ Rep-preal R & z ∉ Rep-preal R &
                  z + v ∈ Rep-preal S & x = u + v
proof -
  have xpos: 0<x by (rule preal-imp-pos [OF Rep-preal x])
  from lemma-sum-mem-Rep-preal-ex [OF x]
  obtain e where epos: 0 < e and xe: x + e ∈ Rep-preal S by blast
  from  Gleason9-34 [OF Rep-preal epos]
  obtain r where r: r ∈ Rep-preal R and notin: r + e ∉ Rep-preal R ..
  with x xnot xpos have rless: r < x by (blast intro: not-in-Rep-preal-ub)
  from add-eq-exists [of r x]
  obtain y where eq: x = r+y by auto
  show ?thesis
  proof (intro exI conjI)
    show r ∈ Rep-preal R by (rule r)
    show r + e ∉ Rep-preal R by (rule notin)
    show r + e + y ∈ Rep-preal S using xe eq by (simp add: add-ac)
    show x = r + y by (simp add: eq)
    show 0 < r + e using epos preal-imp-pos [OF Rep-preal r]
      by simp
    show 0 < y using rless eq by arith
  qed
```

qed

lemma *less-add-left-le2*: $R < (S::preal) ==> S \le R + (S-R)$
apply (*auto simp add*: *preal-le-def*)
apply (*case-tac* $x \in$ *Rep-preal R*)
apply (*cut-tac Rep-preal-self-subset* [*of R*], *force*)
apply (*auto simp add*: *Bex-def mem-Rep-preal-add-iff mem-Rep-preal-diff-iff*)
apply (*blast dest*: *less-add-left-lemma2*)
done

lemma *less-add-left*: $R < (S::preal) ==> R + (S-R) = S$
by (*blast intro*: *preal-le-anti-sym* [*OF less-add-left-le1 less-add-left-le2*])

lemma *less-add-left-Ex*: $R < (S::preal) ==> \exists D.\ R + D = S$
by (*fast dest*: *less-add-left*)

lemma *preal-add-less2-mono1*: $R < (S::preal) ==> R + T < S + T$
apply (*auto dest*!: *less-add-left-Ex simp add*: *preal-add-assoc*)
apply (*rule-tac y1 = D* **in** *preal-add-commute* [*THEN subst*])
apply (*auto intro*: *preal-self-less-add-left simp add*: *preal-add-assoc* [*symmetric*])
done

lemma *preal-add-less2-mono2*: $R < (S::preal) ==> T + R < T + S$
by (*auto intro*: *preal-add-less2-mono1 simp add*: *preal-add-commute* [*of T*])

lemma *preal-add-right-less-cancel*: $R + T < S + T ==> R < (S::preal)$
apply (*insert linorder-less-linear* [*of R S*], *auto*)
apply (*drule-tac R = S* **and** *T = T* **in** *preal-add-less2-mono1*)
apply (*blast dest*: *order-less-trans*)
done

lemma *preal-add-left-less-cancel*: $T + R < T + S ==> R < (S::preal)$
by (*auto elim*: *preal-add-right-less-cancel simp add*: *preal-add-commute* [*of T*])

lemma *preal-add-less-cancel-right*: $((R::preal) + T < S + T) = (R < S)$
by (*blast intro*: *preal-add-less2-mono1 preal-add-right-less-cancel*)

lemma *preal-add-less-cancel-left*: $(T + (R::preal) < T + S) = (R < S)$
by (*blast intro*: *preal-add-less2-mono2 preal-add-left-less-cancel*)

lemma *preal-add-le-cancel-right*: $((R::preal) + T \le S + T) = (R \le S)$
by (*simp add*: *linorder-not-less* [*symmetric*] *preal-add-less-cancel-right*)

lemma *preal-add-le-cancel-left*: $(T + (R::preal) \le T + S) = (R \le S)$
by (*simp add*: *linorder-not-less* [*symmetric*] *preal-add-less-cancel-left*)

lemma *preal-add-less-mono*:
    $[|\ x1 < y1;\ x2 < y2\ |] ==> x1 + x2 < y1 + (y2::preal)$
apply (*auto dest*!: *less-add-left-Ex simp add*: *preal-add-ac*)

apply (*rule preal-add-assoc* [*THEN subst*])
apply (*rule preal-self-less-add-right*)
done

lemma *preal-add-right-cancel*: $(R{::}preal) + T = S + T ==> R = S$
apply (*insert linorder-less-linear* [*of R S*], *safe*)
apply (*drule-tac* [!] $T = T$ **in** *preal-add-less2-mono1*, *auto*)
done

lemma *preal-add-left-cancel*: $C + A = C + B ==> A = (B{::}preal)$
by (*auto intro*: *preal-add-right-cancel simp add*: *preal-add-commute*)

lemma *preal-add-left-cancel-iff*: $(C + A = C + B) = ((A{::}preal) = B)$
by (*fast intro*: *preal-add-left-cancel*)

lemma *preal-add-right-cancel-iff*: $(A + C = B + C) = ((A{::}preal) = B)$
by (*fast intro*: *preal-add-right-cancel*)

lemmas *preal-cancels* =
preal-add-less-cancel-right preal-add-less-cancel-left
preal-add-le-cancel-right preal-add-le-cancel-left
preal-add-left-cancel-iff preal-add-right-cancel-iff

instance *preal* :: *ordered-cancel-ab-semigroup-add*
proof
fix $a\ b\ c$ :: *preal*
show $a + b = a + c \Longrightarrow b = c$ **by** (*rule preal-add-left-cancel*)
show $a \le b \Longrightarrow c + a \le c + b$ **by** (*simp only*: *preal-add-le-cancel-left*)
qed

5.12 Completeness of type *preal*

Prove that supremum is a cut

Part 1 of Dedekind sections definition

lemma *preal-sup-set-not-empty*:
$P \neq \{\} ==> \{\} \subset (\bigcup X \in P.\ Rep\text{-}preal(X))$
apply *auto*
apply (*cut-tac* $X = x$ **in** *mem-Rep-preal-Ex*, *auto*)
done

Part 2 of Dedekind sections definition

lemma *preal-sup-not-exists*:
$\forall X \in P.\ X \le Y ==> \exists q.\ 0 < q\ \&\ q \notin (\bigcup X \in P.\ Rep\text{-}preal(X))$
apply (*cut-tac* $X = Y$ **in** *Rep-preal-exists-bound*)
apply (*auto simp add*: *preal-le-def*)
done

lemma *preal-sup-set-not-rat-set*:

$\forall X \in P.\ X \leq Y$ ==> $(\bigcup X \in P.\ \textit{Rep-preal}(X)) < \{r.\ 0 < r\}$
apply (*drule preal-sup-not-exists*)
apply (*blast intro*: *preal-imp-pos* [*OF Rep-preal*])
done

Part 3 of Dedekind sections definition

lemma *preal-sup-set-lemma3*:
[|$P \neq \{\}$; $\forall X \in P.\ X \leq Y$; $u \in (\bigcup X \in P.\ \textit{Rep-preal}(X))$; $0 < z$; $z < u$|]
==> $z \in (\bigcup X \in P.\ \textit{Rep-preal}(X))$
by (*auto elim*: *Rep-preal* [*THEN preal-downwards-closed*])

Part 4 of Dedekind sections definition

lemma *preal-sup-set-lemma4*:
[|$P \neq \{\}$; $\forall X \in P.\ X \leq Y$; $y \in (\bigcup X \in P.\ \textit{Rep-preal}(X))$ |]
==> $\exists u \in (\bigcup X \in P.\ \textit{Rep-preal}(X)).\ y < u$
by (*blast dest*: *Rep-preal* [*THEN preal-exists-greater*])

lemma *preal-sup*:
[|$P \neq \{\}$; $\forall X \in P.\ X \leq Y$|] ==> $(\bigcup X \in P.\ \textit{Rep-preal}(X)) \in \textit{preal}$
apply (*unfold preal-def cut-def*)
apply (*blast intro*!: *preal-sup-set-not-empty preal-sup-set-not-rat-set*
preal-sup-set-lemma3 preal-sup-set-lemma4)
done

lemma *preal-psup-le*:
[| $\forall X \in P.\ X \leq Y$; $x \in P$ |] ==> $x \leq \textit{psup}\ P$
apply (*simp* (*no-asm-simp*) *add*: *preal-le-def*)
apply (*subgoal-tac* $P \neq \{\}$)
apply (*auto simp add*: *psup-def preal-sup*)
done

lemma *psup-le-ub*: [| $P \neq \{\}$; $\forall X \in P.\ X \leq Y$ |] ==> $\textit{psup}\ P \leq Y$
apply (*simp* (*no-asm-simp*) *add*: *preal-le-def*)
apply (*simp add*: *psup-def preal-sup*)
apply (*auto simp add*: *preal-le-def*)
done

Supremum property

lemma *preal-complete*:
[| $P \neq \{\}$; $\forall X \in P.\ X \leq Y$ |] ==> $(\exists X \in P.\ Z < X) = (Z < \textit{psup}\ P)$
apply (*simp add*: *preal-less-def psup-def preal-sup*)
apply (*auto simp add*: *preal-le-def*)
apply (*rename-tac U*)
apply (*cut-tac* $x = U$ **and** $y = Z$ **in** *linorder-less-linear*)
apply (*auto simp add*: *preal-less-def*)
done

5.13 The Embedding from *rat* into *preal*

lemma *preal-of-rat-add-lemma1*:
  [|$x < y + z$; $0 < x$; $0 < y$|] ==> $x * y * inverse\ (y + z) < (y::rat)$
apply (*frule-tac* $c = y * inverse\ (y + z)$ **in** *mult-strict-right-mono*)
apply (*simp add*: *zero-less-mult-iff*)
apply (*simp add*: *mult-ac*)
done

lemma *preal-of-rat-add-lemma2*:
 assumes $u < x + y$
  and $0 < x$
  and $0 < y$
  and $0 < u$
 shows $\exists\, v\ w::rat.\ w < y\ \&\ 0 < v\ \&\ v < x\ \&\ 0 < w\ \&\ u = v + w$
proof (*intro exI conjI*)
 show $u * x * inverse(x+y) < x$ **using** *prems*
  by (*simp add*: *preal-of-rat-add-lemma1*)
 show $u * y * inverse(x+y) < y$ **using** *prems*
  by (*simp add*: *preal-of-rat-add-lemma1 add-commute* [*of x*])
 show $0 < u * x * inverse\ (x + y)$ **using** *prems*
  by (*simp add*: *zero-less-mult-iff*)
 show $0 < u * y * inverse\ (x + y)$ **using** *prems*
  by (*simp add*: *zero-less-mult-iff*)
 show $u = u * x * inverse\ (x + y) + u * y * inverse\ (x + y)$ **using** *prems*
  by (*simp add*: *left-distrib* [*symmetric*] *right-distrib* [*symmetric*] *mult-ac*)
qed

lemma *preal-of-rat-add*:
  [| $0 < x$; $0 < y$|]
  ==> *preal-of-rat* $((x::rat) + y)$ = *preal-of-rat* x + *preal-of-rat* y
apply (*unfold preal-of-rat-def preal-add-def*)
apply (*simp add*: *rat-mem-preal*)
apply (*rule-tac* f = *Abs-preal* **in** *arg-cong*)
apply (*auto simp add*: *add-set-def*)
apply (*blast dest*: *preal-of-rat-add-lemma2*)
done

lemma *preal-of-rat-mult-lemma1*:
  [|$x < y$; $0 < x$; $0 < z$|] ==> $x * z * inverse\ y < (z::rat)$
apply (*frule-tac* $c = z * inverse\ y$ **in** *mult-strict-right-mono*)
apply (*simp add*: *zero-less-mult-iff*)
apply (*subgoal-tac* $y * (z * inverse\ y) = z * (y * inverse\ y)$)
apply (*simp-all add*: *mult-ac*)
done

lemma *preal-of-rat-mult-lemma2*:
 assumes *xless*: $x < y * z$
  and *xpos*: $0 < x$
  and *ypos*: $0 < y$

```
  shows x * z * inverse y * inverse z < (z::rat)
proof -
  have 0 < y * z using prems by simp
  hence zpos:  0 < z using prems by (simp add: zero-less-mult-iff)
  have x * z * inverse y * inverse z = x * inverse y * (z * inverse z)
    by (simp add: mult-ac)
  also have ... = x/y using zpos
    by (simp add: divide-inverse)
  also from xless have ... < z
    by (simp add: pos-divide-less-eq [OF ypos] mult-commute)
  finally show ?thesis .
qed

lemma preal-of-rat-mult-lemma3:
  assumes uless: u < x * y
    and 0 < x
    and 0 < y
    and 0 < u
  shows ∃ v w::rat. v < x & w < y & 0 < v & 0 < w & u = v * w
proof -
  from dense [OF uless]
  obtain r where u < r r < x * y by blast
  thus ?thesis
  proof (intro exI conjI)
  show u * x * inverse r < x using prems
    by (simp add: preal-of-rat-mult-lemma1)
  show r * y * inverse x * inverse y < y using prems
    by (simp add: preal-of-rat-mult-lemma2)
  show 0 < u * x * inverse r using prems
    by (simp add: zero-less-mult-iff)
  show 0 < r * y * inverse x * inverse y using prems
    by (simp add: zero-less-mult-iff)
  have u * x * inverse r * (r * y * inverse x * inverse y) =
      u * (r * inverse r) * (x * inverse x) * (y * inverse y)
    by (simp only: mult-ac)
  thus u = u * x * inverse r * (r * y * inverse x * inverse y) using prems
    by simp
  qed
qed

lemma preal-of-rat-mult:
     [| 0 < x; 0 < y|]
      ==> preal-of-rat ((x::rat) * y) = preal-of-rat x * preal-of-rat y
apply (unfold preal-of-rat-def preal-mult-def)
apply (simp add: rat-mem-preal)
apply (rule-tac f = Abs-preal in arg-cong)
apply (auto simp add: zero-less-mult-iff mult-strict-mono mult-set-def)
apply (blast dest: preal-of-rat-mult-lemma3)
done
```

lemma *preal-of-rat-less-iff*:
  $[| \ 0 < x; \ 0 < y |] ==> (preal\text{-}of\text{-}rat \ x < preal\text{-}of\text{-}rat \ y) = (x < y)$
by (*force simp add*: *preal-of-rat-def preal-less-def rat-mem-preal*)

lemma *preal-of-rat-le-iff*:
  $[| \ 0 < x; \ 0 < y |] ==> (preal\text{-}of\text{-}rat \ x \leq preal\text{-}of\text{-}rat \ y) = (x \leq y)$
by (*simp add*: *preal-of-rat-less-iff linorder-not-less* [*symmetric*])

lemma *preal-of-rat-eq-iff*:
  $[| \ 0 < x; \ 0 < y |] ==> (preal\text{-}of\text{-}rat \ x = preal\text{-}of\text{-}rat \ y) = (x = y)$
by (*simp add*: *preal-of-rat-le-iff order-eq-iff*)

end

6 RealDef: Defining the Reals from the Positive Reals

theory *RealDef*
imports *PReal*
uses (*real-arith.ML*)
begin

definition
  realrel :: ((*preal* ∗ *preal*) ∗ (*preal* ∗ *preal*)) *set* **where**
  $realrel = \{p. \ \exists x1 \ y1 \ x2 \ y2. \ p = ((x1,y1),(x2,y2)) \ \& \ x1+y2 = x2+y1\}$

typedef (*Real*) *real* = *UNIV*//*realrel*
  by (*auto simp add*: *quotient-def*)

definition

  real-of-preal :: *preal* => *real* **where**
  real-of-preal m = *Abs-Real*(*realrel*``{(*m* + *1*, *1*)})

instance *real* :: *zero*
  real-zero-def: *0* == *Abs-Real*(*realrel*``{(*1*, *1*)}) ..
lemmas [*code func del*] = *real-zero-def*

instance *real* :: *one*
  real-one-def: *1* == *Abs-Real*(*realrel*``{(*1* + *1*, *1*)}) ..
lemmas [*code func del*] = *real-one-def*

instance *real* :: *plus*
  real-add-def: *z* + *w* ==
    contents ($\bigcup$(*x*,*y*) ∈ *Rep-Real*(*z*). $\bigcup$(*u*,*v*) ∈ *Rep-Real*(*w*).
      { *Abs-Real*(*realrel*``{(*x*+*u*, *y*+*v*)}) }) ..

lemmas [*code func del*] = *real-add-def*

instance *real* :: *minus*
real-minus-def: $- r ==$ *contents* $(\bigcup (x,y) \in$ *Rep-Real*(r). { *Abs-Real*(*realrel*``$\{(y,x)\}$)
})
real-diff-def: $r - (s$::*real*$) == r + - s$ **..**
lemmas [*code func del*] = *real-minus-def real-diff-def*

instance *real* :: *times*
real-mult-def:
$z * w ==$
contents $(\bigcup (x,y) \in$ *Rep-Real*(z). $\bigcup (u,v) \in$ *Rep-Real*(w).
{ *Abs-Real*(*realrel*``$\{(x*u + y*v,\ x*v + y*u)\}$) }) **..**
lemmas [*code func del*] = *real-mult-def*

instance *real* :: *inverse*
real-inverse-def: *inverse* $(R$::*real*$) ==$ (*THE S*. $(R = 0\ \&\ S = 0) \mid S * R = 1)$
real-divide-def: $R\ /\ (S$::*real*$) == R *$ *inverse S* **..**
lemmas [*code func del*] = *real-inverse-def real-divide-def*

instance *real* :: *ord*
real-le-def: $z \le (w$::*real*$) ==$
$\exists x\ y\ u\ v.\ x+v \le u+y\ \&\ (x,y) \in$ *Rep-Real* $z\ \&\ (u,v) \in$ *Rep-Real* w
real-less-def: $(x < (y$::*real*$)) == (x \le y\ \&\ x \ne y)$ **..**
lemmas [*code func del*] = *real-le-def real-less-def*

instance *real* :: *abs*
real-abs-def: *abs* $(r$::*real*$) ==$ (*if* $r < 0$ *then* $- r$ *else* r) **..**

instance *real* :: *sgn*
real-sgn-def: *sgn* $x ==$ (*if* $x=0$ *then* 0 *else if* $0<x$ *then* 1 *else* $- 1$) **..**

6.1 Equivalence relation over positive reals

lemma *preal-trans-lemma*:
assumes $x + y1 = x1 + y$
and $x + y2 = x2 + y$
shows $x1 + y2 = x2 + (y1$::*preal*$)$
proof −
have $(x1 + y2) + x = (x + y2) + x1$ **by** (*simp add*: *add-ac*)
also have ... $= (x2 + y) + x1$ **by** (*simp add*: *prems*)
also have ... $= x2 + (x1 + y)$ **by** (*simp add*: *add-ac*)
also have ... $= x2 + (x + y1)$ **by** (*simp add*: *prems*)
also have ... $= (x2 + y1) + x$ **by** (*simp add*: *add-ac*)
finally have $(x1 + y2) + x = (x2 + y1) + x$ **.**
thus *?thesis* **by** (*rule add-right-imp-eq*)
qed

lemma *realrel-iff* [*simp*]: $(((x1,y1),(x2,y2)) \in realrel) = (x1 + y2 = x2 + y1)$
by (*simp add*: *realrel-def*)

lemma *equiv-realrel*: *equiv UNIV realrel*
apply (*auto simp add*: *equiv-def refl-def sym-def trans-def realrel-def*)
apply (*blast dest*: *preal-trans-lemma*)
done

Reduces equality of equivalence classes to the *realrel* relation: $(realrel \text{ `` } \{x\} = realrel \text{ `` } \{y\}) = ((x, y) \in realrel)$

lemmas *equiv-realrel-iff* =
eq-equiv-class-iff [*OF equiv-realrel UNIV-I UNIV-I*]

declare *equiv-realrel-iff* [*simp*]

lemma *realrel-in-real* [*simp*]: $realrel``\{(x,y)\}$: *Real*
by (*simp add*: *Real-def realrel-def quotient-def*, *blast*)

declare *Abs-Real-inject* [*simp*]
declare *Abs-Real-inverse* [*simp*]

Case analysis on the representation of a real number as an equivalence class of pairs of positive reals.

lemma *eq-Abs-Real* [*case-names Abs-Real*, *cases type*: *real*]:
(!!*x y*. $z = Abs\text{-}Real(realrel``\{(x,y)\})$ ==> *P*) ==> *P*
apply (*rule Rep-Real* [*of z*, *unfolded Real-def*, *THEN quotientE*])
apply (*drule arg-cong* [**where** *f*=*Abs-Real*])
apply (*auto simp add*: *Rep-Real-inverse*)
done

6.2 Addition and Subtraction

lemma *real-add-congruent2-lemma*:
[|*a* + *ba* = *aa* + *b*; *ab* + *bc* = *ac* + *bb*|]
==> *a* + *ab* + (*ba* + *bc*) = *aa* + *ac* + (*b* + (*bb*::*preal*))
apply (*simp add*: *add-assoc*)
apply (*rule add-left-commute* [*of ab*, *THEN ssubst*])
apply (*simp add*: *add-assoc* [*symmetric*])
apply (*simp add*: *add-ac*)
done

lemma *real-add*:
Abs-Real ($realrel``\{(x,y)\}$) + *Abs-Real* ($realrel``\{(u,v)\}$) =
Abs-Real ($realrel``\{(x+u, y+v)\}$)
proof −
have $(\lambda z\ w.\ (\lambda(x,y).\ (\lambda(u,v).\ \{Abs\text{-}Real\ (realrel \text{ `` } \{(x+u,\ y+v)\})\})\ w)\ z)$
respects2 realrel
by (*simp add*: *congruent2-def*, *blast intro*: *real-add-congruent2-lemma*)

```
  thus ?thesis
    by (simp add: real-add-def UN-UN-split-split-eq
                  UN-equiv-class2 [OF equiv-realrel equiv-realrel])
qed

lemma real-minus: − Abs-Real(realrel``{(x,y)}) = Abs-Real(realrel `` {(y,x)})
proof −
  have (λ(x,y). {Abs-Real (realrel``{(y,x)})}) respects realrel
    by (simp add: congruent-def add-commute)
  thus ?thesis
    by (simp add: real-minus-def UN-equiv-class [OF equiv-realrel])
qed

instance real :: ab-group-add
proof
  fix x y z :: real
  show (x + y) + z = x + (y + z)
    by (cases x, cases y, cases z, simp add: real-add add-assoc)
  show x + y = y + x
    by (cases x, cases y, simp add: real-add add-commute)
  show 0 + x = x
    by (cases x, simp add: real-add real-zero-def add-ac)
  show − x + x = 0
    by (cases x, simp add: real-minus real-add real-zero-def add-commute)
  show x − y = x + − y
    by (simp add: real-diff-def)
qed
```

6.3 Multiplication

```
lemma real-mult-congruent2-lemma:
     !!(x1::preal). [| x1 + y2 = x2 + y1 |] ==>
          x * x1 + y * y1 + (x * y2 + y * x2) =
          x * x2 + y * y2 + (x * y1 + y * x1)
apply (simp add: add-left-commute add-assoc [symmetric])
apply (simp add: add-assoc right-distrib [symmetric])
apply (simp add: add-commute)
done

lemma real-mult-congruent2:
    (%p1 p2.
        (%(x1,y1). (%(x2,y2).
          { Abs-Real (realrel``{(x1*x2 + y1*y2, x1*y2+y1*x2)}) }) p2) p1)
     respects2 realrel
apply (rule congruent2-commuteI [OF equiv-realrel], clarify)
apply (simp add: mult-commute add-commute)
apply (auto simp add: real-mult-congruent2-lemma)
done
```

```
lemma real-mult:
    Abs-Real((realrel``{(x1,y1)})) * Abs-Real((realrel``{(x2,y2)})) =
     Abs-Real(realrel `` {(x1*x2+y1*y2,x1*y2+y1*x2)})
by (simp add: real-mult-def UN-UN-split-split-eq
        UN-equiv-class2 [OF equiv-realrel equiv-realrel real-mult-congruent2])

lemma real-mult-commute: (z::real) * w = w * z
by (cases z, cases w, simp add: real-mult add-ac mult-ac)

lemma real-mult-assoc: ((z1::real) * z2) * z3 = z1 * (z2 * z3)
apply (cases z1, cases z2, cases z3)
apply (simp add: real-mult right-distrib add-ac mult-ac)
done

lemma real-mult-1: (1::real) * z = z
apply (cases z)
apply (simp add: real-mult real-one-def right-distrib
                mult-1-right mult-ac add-ac)
done

lemma real-add-mult-distrib: ((z1::real) + z2) * w = (z1 * w) + (z2 * w)
apply (cases z1, cases z2, cases w)
apply (simp add: real-add real-mult right-distrib add-ac mult-ac)
done
```

one and zero are distinct

```
lemma real-zero-not-eq-one: 0 ≠ (1::real)
proof −
  have (1::preal) < 1 + 1
    by (simp add: preal-self-less-add-left)
  thus ?thesis
    by (simp add: real-zero-def real-one-def)
qed

instance real :: comm-ring-1
proof
  fix x y z :: real
  show (x * y) * z = x * (y * z) by (rule real-mult-assoc)
  show x * y = y * x by (rule real-mult-commute)
  show 1 * x = x by (rule real-mult-1)
  show (x + y) * z = x * z + y * z by (rule real-add-mult-distrib)
  show 0 ≠ (1::real) by (rule real-zero-not-eq-one)
qed
```

6.4 Inverse and Division

```
lemma real-zero-iff: Abs-Real (realrel `` {(x, x)}) = 0
by (simp add: real-zero-def add-commute)
```

Instead of using an existential quantifier and constructing the inverse within the proof, we could define the inverse explicitly.

```
lemma real-mult-inverse-left-ex: x ≠ 0 ==> ∃ y. y∗x = (1::real)
apply (simp add: real-zero-def real-one-def, cases x)
apply (cut-tac x = xa and y = y in linorder-less-linear)
apply (auto dest!: less-add-left-Ex simp add: real-zero-iff)
apply (rule-tac
        x = Abs-Real (realrel``{(1, inverse (D) + 1)})
       in exI)
apply (rule-tac [2]
        x = Abs-Real (realrel``{(inverse (D) + 1, 1)})
       in exI)
apply (auto simp add: real-mult preal-mult-inverse-right ring-simps)
done

lemma real-mult-inverse-left: x ≠ 0 ==> inverse(x)∗x = (1::real)
apply (simp add: real-inverse-def)
apply (drule real-mult-inverse-left-ex, safe)
apply (rule theI, assumption, rename-tac z)
apply (subgoal-tac (z ∗ x) ∗ y = z ∗ (x ∗ y))
apply (simp add: mult-commute)
apply (rule mult-assoc)
done
```

6.5 The Real Numbers form a Field

```
instance real :: field
proof
  fix x y z :: real
  show x ≠ 0 ==> inverse x ∗ x = 1 by (rule real-mult-inverse-left)
  show x / y = x ∗ inverse y by (simp add: real-divide-def)
qed
```

Inverse of zero! Useful to simplify certain equations

```
lemma INVERSE-ZERO: inverse 0 = (0::real)
by (simp add: real-inverse-def)

instance real :: division-by-zero
proof
  show inverse 0 = (0::real) by (rule INVERSE-ZERO)
qed
```

6.6 The $\leq$ Ordering

```
lemma real-le-refl: w ≤ (w::real)
by (cases w, force simp add: real-le-def)
```

The arithmetic decision procedure is not set up for type preal. This lemma is currently unused, but it could simplify the proofs of the following two lemmas.

```
lemma preal-eq-le-imp-le:
  assumes eq: a+b = c+d and le: c ≤ a
  shows b ≤ (d::preal)
proof −
  have c+d ≤ a+d by (simp add: prems)
  hence a+b ≤ a+d by (simp add: prems)
  thus b ≤ d by simp
qed

lemma real-le-lemma:
  assumes l: u1 + v2 ≤ u2 + v1
      and x1 + v1 = u1 + y1
      and x2 + v2 = u2 + y2
  shows x1 + y2 ≤ x2 + (y1::preal)
proof −
  have (x1+v1) + (u2+y2) = (u1+y1) + (x2+v2) by (simp add: prems)
  hence (x1+y2) + (u2+v1) = (x2+y1) + (u1+v2) by (simp add: add-ac)
  also have ... ≤ (x2+y1) + (u2+v1) by (simp add: prems)
  finally show ?thesis by simp
qed

lemma real-le:
     (Abs-Real(realrel``{(x1,y1)}) ≤ Abs-Real(realrel``{(x2,y2)})) =
      (x1 + y2 ≤ x2 + y1)
apply (simp add: real-le-def)
apply (auto intro: real-le-lemma)
done

lemma real-le-anti-sym: [| z ≤ w; w ≤ z |] ==> z = (w::real)
by (cases z, cases w, simp add: real-le)

lemma real-trans-lemma:
  assumes x + v ≤ u + y
      and u + v' ≤ u' + v
      and x2 + v2 = u2 + y2
  shows x + v' ≤ u' + (y::preal)
proof −
  have (x+v') + (u+v) = (x+v) + (u+v') by (simp add: add-ac)
  also have ... ≤ (u+y) + (u+v') by (simp add: prems)
  also have ... ≤ (u+y) + (u'+v) by (simp add: prems)
  also have ... = (u'+y) + (u+v)  by (simp add: add-ac)
  finally show ?thesis by simp
qed

lemma real-le-trans: [| i ≤ j; j ≤ k |] ==> i ≤ (k::real)
apply (cases i, cases j, cases k)
apply (simp add: real-le)
apply (blast intro: real-trans-lemma)
done
```

lemma *real-less-le*: $((w::real) < z) = (w \leq z \ \& \ w \neq z)$
by (*simp add*: *real-less-def*)

instance *real* :: *order*
proof qed
(*assumption* |
rule real-le-refl real-le-trans real-le-anti-sym real-less-le)+

lemma *real-le-linear*: $(z::real) \leq w \mid w \leq z$
apply (*cases* z, *cases* w)
apply (*auto simp add*: *real-le real-zero-def add-ac*)
done

instance *real* :: *linorder*
by (*intro-classes*, *rule real-le-linear*)

lemma *real-le-eq-diff*: $(x \leq y) = (x-y \leq (0::real))$
apply (*cases* x, *cases* y)
apply (*auto simp add*: *real-le real-zero-def real-diff-def real-add real-minus add-ac*)
apply (*simp-all add*: *add-assoc* [*symmetric*])
done

lemma *real-add-left-mono*:
assumes *le*: $x \leq y$ **shows** $z + x \leq z + (y::real)$
proof −
have $z + x - (z + y) = (z + -z) + (x - y)$
by (*simp add*: *diff-minus add-ac*)
with *le* **show** *?thesis*
by (*simp add*: *real-le-eq-diff* [*of* x] *real-le-eq-diff* [*of* $z+x$] *diff-minus*)
qed

lemma *real-sum-gt-zero-less*: $(0 < S + (-W::real)) ==> (W < S)$
by (*simp add*: *linorder-not-le* [*symmetric*] *real-le-eq-diff* [*of* S] *diff-minus*)

lemma *real-less-sum-gt-zero*: $(W < S) ==> (0 < S + (-W::real))$
by (*simp add*: *linorder-not-le* [*symmetric*] *real-le-eq-diff* [*of* S] *diff-minus*)

lemma *real-mult-order*: $[| \ 0 < x; \ 0 < y \ |] ==> (0::real) < x * y$
apply (*cases* x, *cases* y)
apply (*simp add*: *linorder-not-le* [**where** $'a$ = *real*, *symmetric*]
linorder-not-le [**where** $'a$ = *preal*]
real-zero-def real-le real-mult)
— Reduce to the (simpler) $\leq$ relation

apply (*auto dest*!: *less-add-left-Ex*
simp add: *add-ac mult-ac*
right-distrib preal-self-less-add-left)
done

lemma *real-mult-less-mono2*: $[| (0::real) < z; x < y |] ==> z * x < z * y$
apply (*rule real-sum-gt-zero-less*)
apply (*drule real-less-sum-gt-zero* [*of x y*])
apply (*drule real-mult-order*, *assumption*)
apply (*simp add*: *right-distrib*)
done

instance *real* :: *distrib-lattice*
inf x y ≡ *min x y*
sup x y ≡ *max x y*
by *default* (*auto simp add*: *inf-real-def sup-real-def min-max.sup-inf-distrib1*)

6.7 The Reals Form an Ordered Field

instance *real* :: *ordered-field*
proof
fix *x y z* :: *real*
show $x \le y ==> z + x \le z + y$ **by** (*rule real-add-left-mono*)
show $x < y ==> 0 < z ==> z * x < z * y$ **by** (*rule real-mult-less-mono2*)
show $|x| = (\textit{if } x < 0 \textit{ then } -x \textit{ else } x)$ **by** (*simp only*: *real-abs-def*)
show $sgn\ x = (\textit{if } x{=}0 \textit{ then } 0 \textit{ else if } 0{<}x \textit{ then } 1 \textit{ else } -1)$
by (*simp only*: *real-sgn-def*)
qed

instance *real* :: *lordered-ab-group-add* **..**

The function *real-of-preal* requires many proofs, but it seems to be essential for proving completeness of the reals from that of the positive reals.

lemma *real-of-preal-add*:
real-of-preal ((*x*::*preal*) + *y*) = *real-of-preal x* + *real-of-preal y*
by (*simp add*: *real-of-preal-def real-add left-distrib add-ac*)

lemma *real-of-preal-mult*:
real-of-preal ((*x*::*preal*) ∗ *y*) = *real-of-preal x*∗ *real-of-preal y*
by (*simp add*: *real-of-preal-def real-mult right-distrib add-ac mult-ac*)

Gleason prop 9-4.4 p 127

lemma *real-of-preal-trichotomy*:
∃ *m*. (*x*::*real*) = *real-of-preal m* | *x* = *0* | *x* = −(*real-of-preal m*)
apply (*simp add*: *real-of-preal-def real-zero-def*, *cases x*)
apply (*auto simp add*: *real-minus add-ac*)
apply (*cut-tac x* = *x* **and** *y* = *y* **in** *linorder-less-linear*)
apply (*auto dest*!: *less-add-left-Ex simp add*: *add-assoc* [*symmetric*])
done

```
lemma real-of-preal-leD:
      real-of-preal m1 ≤ real-of-preal m2 ==> m1 ≤ m2
by (simp add: real-of-preal-def real-le)

lemma real-of-preal-lessI: m1 < m2 ==> real-of-preal m1 < real-of-preal m2
by (auto simp add: real-of-preal-leD linorder-not-le [symmetric])

lemma real-of-preal-lessD:
      real-of-preal m1 < real-of-preal m2 ==> m1 < m2
by (simp add: real-of-preal-def real-le linorder-not-le [symmetric])

lemma real-of-preal-less-iff [simp]:
     (real-of-preal m1 < real-of-preal m2) = (m1 < m2)
by (blast intro: real-of-preal-lessI real-of-preal-lessD)

lemma real-of-preal-le-iff:
     (real-of-preal m1 ≤ real-of-preal m2) = (m1 ≤ m2)
by (simp add: linorder-not-less [symmetric])

lemma real-of-preal-zero-less: 0 < real-of-preal m
apply (insert preal-self-less-add-left [of 1 m])
apply (auto simp add: real-zero-def real-of-preal-def
                      real-less-def real-le-def add-ac)
apply (rule-tac x=m + 1 in exI, rule-tac x=1 in exI)
apply (simp add: add-ac)
done

lemma real-of-preal-minus-less-zero: − real-of-preal m < 0
by (simp add: real-of-preal-zero-less)

lemma real-of-preal-not-minus-gt-zero: ~ 0 < − real-of-preal m
proof −
  from real-of-preal-minus-less-zero
  show ?thesis by (blast dest: order-less-trans)
qed
```

6.8 Theorems About the Ordering

```
lemma real-gt-zero-preal-Ex: (0 < x) = (∃ y. x = real-of-preal y)
apply (auto simp add: real-of-preal-zero-less)
apply (cut-tac x = x in real-of-preal-trichotomy)
apply (blast elim!: real-of-preal-not-minus-gt-zero [THEN notE])
done

lemma real-gt-preal-preal-Ex:
     real-of-preal z < x ==> ∃ y. x = real-of-preal y
by (blast dest!: real-of-preal-zero-less [THEN order-less-trans]
            intro: real-gt-zero-preal-Ex [THEN iffD1])
```

lemma real-ge-preal-preal-Ex:
real-of-preal z ≤ x ==> ∃ y. x = real-of-preal y
by (blast dest: order-le-imp-less-or-eq real-gt-preal-preal-Ex)

lemma real-less-all-preal: y ≤ 0 ==> ∀ x. y < real-of-preal x
by (auto elim: order-le-imp-less-or-eq [THEN disjE]
intro: real-of-preal-zero-less [THEN [2] order-less-trans]
simp add: real-of-preal-zero-less)

lemma real-less-all-real2: ~ 0 < y ==> ∀ x. y < real-of-preal x
by (blast intro!: real-less-all-preal linorder-not-less [THEN iffD1])

6.9 More Lemmas

lemma real-mult-left-cancel: (c::real) ≠ 0 ==> (c∗a=c∗b) = (a=b)
by auto

lemma real-mult-right-cancel: (c::real) ≠ 0 ==> (a∗c=b∗c) = (a=b)
by auto

lemma real-mult-less-iff1 [simp]: (0::real) < z ==> (x∗z < y∗z) = (x < y)
by (force elim: order-less-asym
simp add: Ring-and-Field.mult-less-cancel-right)

lemma real-mult-le-cancel-iff1 [simp]: (0::real) < z ==> (x∗z ≤ y∗z) = (x≤y)
apply (simp add: mult-le-cancel-right)
apply (blast intro: elim: order-less-asym)
done

lemma real-mult-le-cancel-iff2 [simp]: (0::real) < z ==> (z∗x ≤ z∗y) = (x≤y)
by(simp add:mult-commute)

lemma real-inverse-gt-one: [| (0::real) < x; x < 1 |] ==> 1 < inverse x
by (simp add: one-less-inverse-iff)

6.10 Embedding numbers into the Reals

abbreviation
real-of-nat :: nat ⇒ real
where
real-of-nat ≡ of-nat

abbreviation
real-of-int :: int ⇒ real
where
real-of-int ≡ of-int

abbreviation
real-of-rat :: rat ⇒ real

where
real-of-rat ≡ *of-rat*

consts

real :: *'a* => *real*

defs (**overloaded**)
real-of-nat-def [*code inline*]: *real* == *real-of-nat*
real-of-int-def [*code inline*]: *real* == *real-of-int*

lemma *real-eq-of-nat*: *real* = *of-nat*
unfolding *real-of-nat-def* **..**

lemma *real-eq-of-int*: *real* = *of-int*
unfolding *real-of-int-def* **..**

lemma *real-of-int-zero* [*simp*]: *real* (*0*::*int*) = *0*
by (*simp add*: *real-of-int-def*)

lemma *real-of-one* [*simp*]: *real* (*1*::*int*) = (*1*::*real*)
by (*simp add*: *real-of-int-def*)

lemma *real-of-int-add* [*simp*]: *real*(*x* + *y*) = *real* (*x*::*int*) + *real y*
by (*simp add*: *real-of-int-def*)

lemma *real-of-int-minus* [*simp*]: *real*(−*x*) = −*real* (*x*::*int*)
by (*simp add*: *real-of-int-def*)

lemma *real-of-int-diff* [*simp*]: *real*(*x* − *y*) = *real* (*x*::*int*) − *real y*
by (*simp add*: *real-of-int-def*)

lemma *real-of-int-mult* [*simp*]: *real*(*x* ∗ *y*) = *real* (*x*::*int*) ∗ *real y*
by (*simp add*: *real-of-int-def*)

lemma *real-of-int-setsum* [*simp*]: *real* ((*SUM x*:*A*. *f x*)::*int*) = (*SUM x*:*A*. *real*(*f* *x*))
apply (*subst real-eq-of-int*)+
apply (*rule of-int-setsum*)
done

lemma *real-of-int-setprod* [*simp*]: *real* ((*PROD x*:*A*. *f x*)::*int*) =
(*PROD x*:*A*. *real*(*f x*))
apply (*subst real-eq-of-int*)+
apply (*rule of-int-setprod*)
done

lemma *real-of-int-zero-cancel* [*simp*]: (*real x* = *0*) = (*x* = (*0*::*int*))
by (*simp add*: *real-of-int-def*)

lemma *real-of-int-inject* [*iff*]: (*real* (*x*::*int*) = *real* *y*) = (*x* = *y*)
by (*simp add*: *real-of-int-def*)

lemma *real-of-int-less-iff* [*iff*]: (*real* (*x*::*int*) < *real* *y*) = (*x* < *y*)
by (*simp add*: *real-of-int-def*)

lemma *real-of-int-le-iff* [*simp*]: (*real* (*x*::*int*) ≤ *real* *y*) = (*x* ≤ *y*)
by (*simp add*: *real-of-int-def*)

lemma *real-of-int-gt-zero-cancel-iff* [*simp*]: (*0* < *real* (*n*::*int*)) = (*0* < *n*)
by (*simp add*: *real-of-int-def*)

lemma *real-of-int-ge-zero-cancel-iff* [*simp*]: (*0* <= *real* (*n*::*int*)) = (*0* <= *n*)
by (*simp add*: *real-of-int-def*)

lemma *real-of-int-lt-zero-cancel-iff* [*simp*]: (*real* (*n*::*int*) < *0*) = (*n* < *0*)
by (*simp add*: *real-of-int-def*)

lemma *real-of-int-le-zero-cancel-iff* [*simp*]: (*real* (*n*::*int*) <= *0*) = (*n* <= *0*)
by (*simp add*: *real-of-int-def*)

lemma *real-of-int-abs* [*simp*]: *real* (*abs x*) = *abs*(*real* (*x*::*int*))
by (*auto simp add*: *abs-if*)

lemma *int-less-real-le*: ((*n*::*int*) < *m*) = (*real n* + *1* <= *real m*)
  apply (*subgoal-tac real n* + *1* = *real* (*n* + *1*))
  apply (*simp del*: *real-of-int-add*)
  apply *auto*
done

lemma *int-le-real-less*: ((*n*::*int*) <= *m*) = (*real n* < *real m* + *1*)
  apply (*subgoal-tac real m* + *1* = *real* (*m* + *1*))
  apply (*simp del*: *real-of-int-add*)
  apply *simp*
done

lemma *real-of-int-div-aux*: *d* ~= *0* ==> (*real* (*x*::*int*)) / (*real d*) =
    real (*x div d*) + (*real* (*x mod d*)) / (*real d*)
proof −
  assume *d* ~= *0*
  have *x* = (*x div d*) ∗ *d* + *x mod d*
    by *auto*
  then have *real x* = *real* (*x div d*) ∗ *real d* + *real*(*x mod d*)
    by (*simp only*: *real-of-int-mult* [*THEN sym*] *real-of-int-add* [*THEN sym*])
  then have *real x* / *real d* = ... / *real d*
    by *simp*
  then show *?thesis*
    by (*auto simp add*: *add-divide-distrib ring-simps prems*)

qed

lemma *real-of-int-div*: (*d*::*int*) ~= *0* ==> *d dvd n* ==>
real(*n div d*) = *real n* / *real d*
apply (*frule real-of-int-div-aux* [*of d n*])
apply *simp*
apply (*simp add*: *zdvd-iff-zmod-eq-0*)
done

lemma *real-of-int-div2*:
0 <= *real* (*n*::*int*) / *real* (*x*) − *real* (*n div x*)
apply (*case-tac x* = *0*)
apply *simp*
apply (*case-tac 0* < *x*)
apply (*simp add*: *compare-rls*)
apply (*subst real-of-int-div-aux*)
apply *simp*
apply *simp*
apply (*subst zero-le-divide-iff*)
apply *auto*
apply (*simp add*: *compare-rls*)
apply (*subst real-of-int-div-aux*)
apply *simp*
apply *simp*
apply (*subst zero-le-divide-iff*)
apply *auto*
done

lemma *real-of-int-div3*:
real (*n*::*int*) / *real* (*x*) − *real* (*n div x*) <= *1*
apply(*case-tac x* = *0*)
apply *simp*
apply (*simp add*: *compare-rls*)
apply (*subst real-of-int-div-aux*)
apply *assumption*
apply *simp*
apply (*subst divide-le-eq*)
apply *clarsimp*
apply (*rule conjI*)
apply (*rule impI*)
apply (*rule order-less-imp-le*)
apply *simp*
apply (*rule impI*)
apply (*rule order-less-imp-le*)
apply *simp*
done

lemma *real-of-int-div4*: *real* (*n div x*) <= *real* (*n*::*int*) / *real x*
by (*insert real-of-int-div2* [*of n x*], *simp*)

6.11 Embedding the Naturals into the Reals

lemma *real-of-nat-zero* [*simp*]: *real* (*0*::*nat*) = *0*
by (*simp add*: *real-of-nat-def*)

lemma *real-of-nat-one* [*simp*]: *real* (*Suc 0*) = (*1*::*real*)
by (*simp add*: *real-of-nat-def*)

lemma *real-of-nat-add* [*simp*]: *real* (*m* + *n*) = *real* (*m*::*nat*) + *real n*
by (*simp add*: *real-of-nat-def*)

lemma *real-of-nat-Suc*: *real* (*Suc n*) = *real n* + (*1*::*real*)
by (*simp add*: *real-of-nat-def*)

lemma *real-of-nat-less-iff* [*iff*]:
  (*real* (*n*::*nat*) < *real m*) = (*n* < *m*)
by (*simp add*: *real-of-nat-def*)

lemma *real-of-nat-le-iff* [*iff*]: (*real* (*n*::*nat*) ≤ *real m*) = (*n* ≤ *m*)
by (*simp add*: *real-of-nat-def*)

lemma *real-of-nat-ge-zero* [*iff*]: *0* ≤ *real* (*n*::*nat*)
by (*simp add*: *real-of-nat-def zero-le-imp-of-nat*)

lemma *real-of-nat-Suc-gt-zero*: *0* < *real* (*Suc n*)
by (*simp add*: *real-of-nat-def del*: *of-nat-Suc*)

lemma *real-of-nat-mult* [*simp*]: *real* (*m* * *n*) = *real* (*m*::*nat*) * *real n*
by (*simp add*: *real-of-nat-def of-nat-mult*)

lemma *real-of-nat-setsum* [*simp*]: *real* ((*SUM x*:*A*. *f x*)::*nat*) =
  (*SUM x*:*A*. *real*(*f x*))
 apply (*subst real-eq-of-nat*)+
 apply (*rule of-nat-setsum*)
done

lemma *real-of-nat-setprod* [*simp*]: *real* ((*PROD x*:*A*. *f x*)::*nat*) =
  (*PROD x*:*A*. *real*(*f x*))
 apply (*subst real-eq-of-nat*)+
 apply (*rule of-nat-setprod*)
done

lemma *real-of-card*: *real* (*card A*) = *setsum* (%*x*.*1*) *A*
 apply (*subst card-eq-setsum*)
 apply (*subst real-of-nat-setsum*)
 apply *simp*
done

lemma *real-of-nat-inject* [*iff*]: (*real* (*n*::*nat*) = *real m*) = (*n* = *m*)

```
by (simp add: real-of-nat-def)

lemma real-of-nat-zero-iff [iff]: (real (n::nat) = 0) = (n = 0)
by (simp add: real-of-nat-def)

lemma real-of-nat-diff: n ≤ m ==> real (m − n) = real (m::nat) − real n
by (simp add: add: real-of-nat-def of-nat-diff)

lemma real-of-nat-gt-zero-cancel-iff [simp]: (0 < real (n::nat)) = (0 < n)
by (auto simp: real-of-nat-def)

lemma real-of-nat-le-zero-cancel-iff [simp]: (real (n::nat) ≤ 0) = (n = 0)
by (simp add: add: real-of-nat-def)

lemma not-real-of-nat-less-zero [simp]: ~ real (n::nat) < 0
by (simp add: add: real-of-nat-def)

lemma real-of-nat-ge-zero-cancel-iff [simp]: (0 ≤ real (n::nat))
by (simp add: add: real-of-nat-def)

lemma nat-less-real-le: ((n::nat) < m) = (real n + 1 <= real m)
  apply (subgoal-tac real n + 1 = real (Suc n))
  apply simp
  apply (auto simp add: real-of-nat-Suc)
done

lemma nat-le-real-less: ((n::nat) <= m) = (real n < real m + 1)
  apply (subgoal-tac real m + 1 = real (Suc m))
  apply (simp add: less-Suc-eq-le)
  apply (simp add: real-of-nat-Suc)
done

lemma real-of-nat-div-aux: 0 < d ==> (real (x::nat)) / (real d) =
    real (x div d) + (real (x mod d)) / (real d)
proof −
  assume 0 < d
  have x = (x div d) ∗ d + x mod d
    by auto
  then have real x = real (x div d) ∗ real d + real(x mod d)
    by (simp only: real-of-nat-mult [THEN sym] real-of-nat-add [THEN sym])
  then have real x / real d = ... / real d
    by simp
  then show ?thesis
    by (auto simp add: add-divide-distrib ring-simps prems)
qed

lemma real-of-nat-div: 0 < (d::nat) ==> d dvd n ==>
    real(n div d) = real n / real d
  apply (frule real-of-nat-div-aux [of d n])
```

apply *simp*
apply (*subst dvd-eq-mod-eq-0* [*THEN sym*])
apply *assumption*
done

lemma *real-of-nat-div2*:
$0 <= real\ (n::nat)\ /\ real\ (x) - real\ (n\ div\ x)$
apply(*case-tac* $x = 0$)
apply (*simp*)
apply (*simp add*: *compare-rls*)
apply (*subst real-of-nat-div-aux*)
apply *simp*
apply *simp*
apply (*subst zero-le-divide-iff*)
apply *simp*
done

lemma *real-of-nat-div3*:
$real\ (n::nat)\ /\ real\ (x) - real\ (n\ div\ x) <= 1$
apply(*case-tac* $x = 0$)
apply (*simp*)
apply (*simp add*: *compare-rls*)
apply (*subst real-of-nat-div-aux*)
apply *simp*
apply *simp*
done

lemma *real-of-nat-div4*: $real\ (n\ div\ x) <= real\ (n::nat)\ /\ real\ x$
by (*insert real-of-nat-div2* [*of n x*], *simp*)

lemma *real-of-int-real-of-nat*: $real\ (int\ n) = real\ n$
by (*simp add*: *real-of-nat-def real-of-int-def int-eq-of-nat*)

lemma *real-of-int-of-nat-eq* [*simp*]: $real\ (of\text{-}nat\ n :: int) = real\ n$
by (*simp add*: *real-of-int-def real-of-nat-def*)

lemma *real-nat-eq-real* [*simp*]: $0 <= x ==> real(nat\ x) = real\ x$
apply (*subgoal-tac* $real(int(nat\ x)) = real(nat\ x)$)
apply *force*
apply (*simp only*: *real-of-int-real-of-nat*)
done

6.12 Numerals and Arithmetic

instance *real* :: *number-ring*
real-number-of-def: *number-of w* $\equiv$ *real-of-int w*
by *intro-classes* (*simp add*: *real-number-of-def*)

lemma [*code*, *code unfold*]:

number-of k = *real-of-int* (*number-of k*)
unfolding *number-of-is-id real-number-of-def* **..**

Collapse applications of *real* to *number-of*

lemma *real-number-of* [*simp*]: *real* (*number-of v* :: *int*) = *number-of v*
by (*simp add*: *real-of-int-def of-int-number-of-eq*)

lemma *real-of-nat-number-of* [*simp*]:
real (*number-of v* :: *nat*) =
(*if neg* (*number-of v* :: *int*) *then 0*
else (*number-of v* :: *real*))
by (*simp add*: *real-of-int-real-of-nat* [*symmetric*] *int-nat-number-of*)

use *real-arith.ML*
declaration ⟪ *K real-arith-setup* ⟫

6.13 Simprules combining x+y and 0: ARE THEY NEEDED?

Needed in this non-standard form by Hyperreal/Transcendental

lemma *real-0-le-divide-iff*:
((*0*::*real*) ≤ *x*/*y*) = ((*x* ≤ *0* | *0* ≤ *y*) & (*0* ≤ *x* | *y* ≤ *0*))
by (*simp add*: *real-divide-def zero-le-mult-iff*, *auto*)

lemma *real-add-minus-iff* [*simp*]: (*x* + − *a* = (*0*::*real*)) = (*x*=*a*)
by *arith*

lemma *real-add-eq-0-iff*: (*x*+*y* = (*0*::*real*)) = (*y* = −*x*)
by *auto*

lemma *real-add-less-0-iff*: (*x*+*y* < (*0*::*real*)) = (*y* < −*x*)
by *auto*

lemma *real-0-less-add-iff*: ((*0*::*real*) < *x*+*y*) = (−*x* < *y*)
by *auto*

lemma *real-add-le-0-iff*: (*x*+*y* ≤ (*0*::*real*)) = (*y* ≤ −*x*)
by *auto*

lemma *real-0-le-add-iff*: ((*0*::*real*) ≤ *x*+*y*) = (−*x* ≤ *y*)
by *auto*

6.13.1 Density of the Reals

lemma *real-lbound-gt-zero*:
[| (*0*::*real*) < *d1*; *0* < *d2* |] ==> ∃ *e*. *0* < *e* & *e* < *d1* & *e* < *d2*
apply (*rule-tac x* = (*min d1 d2*) /*2* **in** *exI*)
apply (*simp add*: *min-def*)
done

Similar results are proved in *Ring-and-Field*

lemma *real-less-half-sum*: $x < y ==> x < (x+y) \ / \ (2::real)$
by *auto*

lemma *real-gt-half-sum*: $x < y ==> (x+y)/(2::real) < y$
by *auto*

6.14 Absolute Value Function for the Reals

lemma *abs-minus-add-cancel*: $abs(x + (-y)) = abs\ (y + (-(x::real)))$
by (*simp add*: *abs-if*)

lemma *abs-le-interval-iff*: $(abs\ x \leq r) = (-r \leq x\ \&\ x \leq (r::real))$
by (*force simp add*: *OrderedGroup.abs-le-iff*)

lemma *abs-add-one-gt-zero* [*simp*]: $(0::real) < 1 + abs(x)$
by (*simp add*: *abs-if*)

lemma *abs-real-of-nat-cancel* [*simp*]: $abs\ (real\ x) = real\ (x::nat)$
by (*rule abs-of-nonneg* [*OF real-of-nat-ge-zero*])

lemma *abs-add-one-not-less-self* [*simp*]: $\sim\ abs(x) + (1::real) < x$
by *simp*

lemma *abs-sum-triangle-ineq*: $abs\ ((x::real) + y + (-l + -m)) \leq abs(x + -l) + abs(y + -m)$
by *simp*

6.15 Implementation of rational real numbers as pairs of integers

definition
Ratreal :: $int \times int \Rightarrow real$
where
Ratreal = *INum*

code-datatype *Ratreal*

lemma *Ratreal-simp*:
Ratreal (k, l) = *real-of-int* k / *real-of-int* l
unfolding *Ratreal-def INum-def* **by** *simp*

lemma *Ratreal-zero* [*simp*]: *Ratreal* $0_N = 0$
by (*simp add*: *Ratreal-simp*)

lemma *Ratreal-lit* [*simp*]: *Ratreal* i_N = *real-of-int* i
by (*simp add*: *Ratreal-simp*)

lemma *zero-real-code* [*code*, *code unfold*]:
$0 = Ratreal\ 0_N$ **by** *simp*

lemma *one-real-code* [*code*, *code unfold*]:
$1 = Ratreal\ 1_N$ **by** *simp*

instance *real* :: *eq* **..**

lemma *real-eq-code* [*code*]: $Ratreal\ x = Ratreal\ y \longleftrightarrow normNum\ x = normNum\ y$
unfolding *Ratreal-def INum-normNum-iff* **..**

lemma *real-less-eq-code* [*code*]: $Ratreal\ x \leq Ratreal\ y \longleftrightarrow normNum\ x \leq_N normNum\ y$
proof −
have $normNum\ x \leq_N normNum\ y \longleftrightarrow Ratreal\ (normNum\ x) \leq Ratreal\ (normNum\ y)$
by (*simp add*: *Ratreal-def del*: *normNum*)
also have $\ldots = (Ratreal\ x \leq Ratreal\ y)$ **by** (*simp add*: *Ratreal-def*)
finally show *?thesis* **by** *simp*
qed

lemma *real-less-code* [*code*]: $Ratreal\ x < Ratreal\ y \longleftrightarrow normNum\ x <_N normNum\ y$
proof −
have $normNum\ x <_N normNum\ y \longleftrightarrow Ratreal\ (normNum\ x) < Ratreal\ (normNum\ y)$
by (*simp add*: *Ratreal-def del*: *normNum*)
also have $\ldots = (Ratreal\ x < Ratreal\ y)$ **by** (*simp add*: *Ratreal-def*)
finally show *?thesis* **by** *simp*
qed

lemma *real-add-code* [*code*]: $Ratreal\ x + Ratreal\ y = Ratreal\ (x +_N y)$
unfolding *Ratreal-def* **by** *simp*

lemma *real-mul-code* [*code*]: $Ratreal\ x * Ratreal\ y = Ratreal\ (x *_N y)$
unfolding *Ratreal-def* **by** *simp*

lemma *real-neg-code* [*code*]: $-\ Ratreal\ x = Ratreal\ (\sim_N x)$
unfolding *Ratreal-def* **by** *simp*

lemma *real-sub-code* [*code*]: $Ratreal\ x - Ratreal\ y = Ratreal\ (x -_N y)$
unfolding *Ratreal-def* **by** *simp*

lemma *real-inv-code* [*code*]: $inverse\ (Ratreal\ x) = Ratreal\ (Ninv\ x)$
unfolding *Ratreal-def Ninv real-divide-def* **by** *simp*

lemma *real-div-code* [*code*]: $Ratreal\ x\ /\ Ratreal\ y = Ratreal\ (x \div_N y)$
unfolding *Ratreal-def* **by** *simp*

Setup for SML code generator

types-code
  real ((*int* */ *int*))
attach (*term-of*) ⟪
fun term-of-real (*p*, *q*) =
  let
    val rT = *HOLogic.realT*
  in
    if q = *1 orelse p* = *0 then HOLogic.mk-number rT p*
    else @{*term op* / :: *real* ⇒ *real* ⇒ *real*} $
      HOLogic.mk-number rT p $ *HOLogic.mk-number rT q*
  end;
⟫
attach (*test*) ⟪
fun gen-real i =
  let
    val p = *random-range 0 i*;
    val q = *random-range 1* (*i* + *1*);
    val g = *Integer.gcd p q*;
    val p′ = *p div g*;
    val q′ = *q div g*;
  in
    (*if one-of* [*true*, *false*] *then p′ else* ~ *p′*,
     if p′ = *0 then 0 else q′*)
  end;
⟫

consts-code
  Ratreal ((-))

consts-code
  of-int :: *int* ⇒ *real* (⟨**module**⟩*real′-of′-int*)
attach ⟪
fun real-of-int 0 = (*0*, *0*)
  | *real-of-int i* = (*i*, *1*);
⟫

declare *real-of-int-of-nat-eq* [*symmetric*, *code*]

end

7 RComplete: Completeness of the Reals; Floor and Ceiling Functions

theory *RComplete*
imports *Lubs RealDef*
begin

```
lemma real-sum-of-halves: x/2 + x/2 = (x::real)
  by simp
```

7.1 Completeness of Positive Reals

Supremum property for the set of positive reals

Let P be a non-empty set of positive reals, with an upper bound y. Then P has a least upper bound (written S).

FIXME: Can the premise be weakened to $\forall x \in P.\ x \le y$?

```
lemma posreal-complete:
  assumes positive-P: ∀ x ∈ P. (0::real) < x
    and not-empty-P: ∃ x. x ∈ P
    and upper-bound-Ex: ∃ y. ∀ x ∈ P. x<y
  shows ∃ S. ∀ y. (∃ x ∈ P. y < x) = (y < S)
proof (rule exI, rule allI)
  fix y
  let ?pP = {w. real-of-preal w ∈ P}

  show (∃ x∈P. y < x) = (y < real-of-preal (psup ?pP))
  proof (cases 0 < y)
    assume neg-y: ¬ 0 < y
    show ?thesis
    proof
      assume ∃ x∈P. y < x
      have ∀ x. y < real-of-preal x
        using neg-y by (rule real-less-all-real2)
      thus y < real-of-preal (psup ?pP) ..
    next
      assume y < real-of-preal (psup ?pP)
      obtain x where x-in-P: x ∈ P using not-empty-P ..
      hence 0 < x using positive-P by simp
      hence y < x using neg-y by simp
      thus ∃ x ∈ P. y < x using x-in-P ..
    qed
  next
    assume pos-y: 0 < y

    then obtain py where y-is-py: y = real-of-preal py
      by (auto simp add: real-gt-zero-preal-Ex)

    obtain a where a ∈ P using not-empty-P ..
    with positive-P have a-pos: 0 < a ..
    then obtain pa where a = real-of-preal pa
      by (auto simp add: real-gt-zero-preal-Ex)
    hence pa ∈ ?pP using ‹a ∈ P› by auto
    hence pP-not-empty: ?pP ≠ {} by auto

    obtain sup where sup: ∀ x ∈ P. x < sup
```

```
      using upper-bound-Ex ..
    from this and ‹a ∈ P› have a < sup ..
    hence 0 < sup using a-pos by arith
    then obtain possup where sup = real-of-preal possup
      by (auto simp add: real-gt-zero-preal-Ex)
    hence ∀ X ∈ ?pP. X ≤ possup
      using sup by (auto simp add: real-of-preal-lessI)
    with pP-not-empty have psup: ⋀Z. (∃ X ∈ ?pP. Z < X) = (Z < psup ?pP)
      by (rule preal-complete)

    show ?thesis
    proof
      assume ∃ x ∈ P. y < x
      then obtain x where x-in-P: x ∈ P and y-less-x: y < x ..
      hence 0 < x using pos-y by arith
      then obtain px where x-is-px: x = real-of-preal px
        by (auto simp add: real-gt-zero-preal-Ex)

      have py-less-X: ∃ X ∈ ?pP. py < X
      proof
        show py < px using y-is-py and x-is-px and y-less-x
          by (simp add: real-of-preal-lessI)
        show px ∈ ?pP using x-in-P and x-is-px by simp
      qed

      have (∃ X ∈ ?pP. py < X) ==> (py < psup ?pP)
        using psup by simp
      hence py < psup ?pP using py-less-X by simp
      thus y < real-of-preal (psup {w. real-of-preal w ∈ P})
        using y-is-py and pos-y by (simp add: real-of-preal-lessI)
    next
      assume y-less-psup: y < real-of-preal (psup ?pP)

      hence py < psup ?pP using y-is-py
        by (simp add: real-of-preal-lessI)
      then obtain X where py-less-X: py < X and X-in-pP: X ∈ ?pP
        using psup by auto
      then obtain x where x-is-X: x = real-of-preal X
        by (simp add: real-gt-zero-preal-Ex)
      hence y < x using py-less-X and y-is-py
        by (simp add: real-of-preal-lessI)

      moreover have x ∈ P using x-is-X and X-in-pP by simp

      ultimately show ∃ x ∈ P. y < x ..
    qed
  qed
qed
```

Completeness properties using *isUb*, *isLub* etc.

```
lemma real-isLub-unique: [| isLub R S x; isLub R S y |] ==> x = (y::real)
  apply (frule isLub-isUb)
  apply (frule-tac x = y in isLub-isUb)
  apply (blast intro!: order-antisym dest!: isLub-le-isUb)
  done
```

Completeness theorem for the positive reals (again).

```
lemma posreals-complete:
  assumes positive-S: ∀ x ∈ S. 0 < x
    and not-empty-S: ∃ x. x ∈ S
    and upper-bound-Ex: ∃ u. isUb (UNIV::real set) S u
  shows ∃ t. isLub (UNIV::real set) S t
proof
  let ?pS = {w. real-of-preal w ∈ S}

  obtain u where isUb UNIV S u using upper-bound-Ex ..
  hence sup: ∀ x ∈ S. x ≤ u by (simp add: isUb-def setle-def)

  obtain x where x-in-S: x ∈ S using not-empty-S ..
  hence x-gt-zero: 0 < x using positive-S by simp
  have x ≤ u using sup and x-in-S ..
  hence 0 < u using x-gt-zero by arith

  then obtain pu where u-is-pu: u = real-of-preal pu
    by (auto simp add: real-gt-zero-preal-Ex)

  have pS-less-pu: ∀ pa ∈ ?pS. pa ≤ pu
  proof
    fix pa
    assume pa ∈ ?pS
    then obtain a where a ∈ S and a = real-of-preal pa
      by simp
    moreover hence a ≤ u using sup by simp
    ultimately show pa ≤ pu
      using sup and u-is-pu by (simp add: real-of-preal-le-iff)
  qed

  have ∀ y ∈ S. y ≤ real-of-preal (psup ?pS)
  proof
    fix y
    assume y-in-S: y ∈ S
    hence 0 < y using positive-S by simp
    then obtain py where y-is-py: y = real-of-preal py
      by (auto simp add: real-gt-zero-preal-Ex)
    hence py-in-pS: py ∈ ?pS using y-in-S by simp
    with pS-less-pu have py ≤ psup ?pS
      by (rule preal-psup-le)
```

```
  thus y ≤ real-of-preal (psup ?pS)
    using y-is-py by (simp add: real-of-preal-le-iff)
 qed

 moreover {
  fix x
  assume x-ub-S: ∀ y∈S. y ≤ x
  have real-of-preal (psup ?pS) ≤ x
  proof −
    obtain s where s-in-S: s ∈ S using not-empty-S ..
    hence s-pos: 0 < s using positive-S by simp

    hence ∃ ps. s = real-of-preal ps by (simp add: real-gt-zero-preal-Ex)
    then obtain ps where s-is-ps: s = real-of-preal ps ..
    hence ps-in-pS: ps ∈ {w. real-of-preal w ∈ S} using s-in-S by simp

    from x-ub-S have s ≤ x using s-in-S ..
    hence 0 < x using s-pos by simp
    hence ∃ px. x = real-of-preal px by (simp add: real-gt-zero-preal-Ex)
    then obtain px where x-is-px: x = real-of-preal px ..

    have ∀ pe ∈ ?pS. pe ≤ px
    proof
      fix pe
      assume pe ∈ ?pS
      hence real-of-preal pe ∈ S by simp
      hence real-of-preal pe ≤ x using x-ub-S by simp
      thus pe ≤ px using x-is-px by (simp add: real-of-preal-le-iff)
    qed

    moreover have ?pS ≠ {} using ps-in-pS by auto
    ultimately have (psup ?pS) ≤ px by (simp add: psup-le-ub)
   thus real-of-preal (psup ?pS) ≤ x using x-is-px by (simp add: real-of-preal-le-iff)
  qed
 }
 ultimately show isLub UNIV S (real-of-preal (psup ?pS))
  by (simp add: isLub-def leastP-def isUb-def setle-def setge-def)
qed
```

reals Completeness (again!)

```
lemma reals-complete:
 assumes notempty-S: ∃ X. X ∈ S
  and exists-Ub: ∃ Y. isUb (UNIV::real set) S Y
 shows ∃ t. isLub (UNIV :: real set) S t
proof −
 obtain X where X-in-S: X ∈ S using notempty-S ..
 obtain Y where Y-isUb: isUb (UNIV::real set) S Y
  using exists-Ub ..
 let ?SHIFT = {z. ∃ x ∈S. z = x + (−X) + 1} ∩ {x. 0 < x}
```

```
{
  fix x
  assume isUb (UNIV::real set) S x
  hence S-le-x: ∀ y ∈ S. y <= x
    by (simp add: isUb-def setle-def)
  {
    fix s
    assume s ∈ {z. ∃x∈S. z = x + − X + 1}
    hence ∃ x ∈ S. s = x + −X + 1 ..
    then obtain x1 where x1 ∈ S and s = x1 + (−X) + 1 ..
    moreover hence x1 ≤ x using S-le-x by simp
    ultimately have s ≤ x + − X + 1 by arith
  }
  then have isUb (UNIV::real set) ?SHIFT (x + (−X) + 1)
    by (auto simp add: isUb-def setle-def)
} note S-Ub-is-SHIFT-Ub = this

hence isUb UNIV ?SHIFT (Y + (−X) + 1) using Y-isUb by simp
hence ∃Z. isUb UNIV ?SHIFT Z ..
moreover have ∀y ∈ ?SHIFT. 0 < y by auto
moreover have shifted-not-empty: ∃u. u ∈ ?SHIFT
  using X-in-S and Y-isUb by auto
ultimately obtain t where t-is-Lub: isLub UNIV ?SHIFT t
  using posreals-complete [of ?SHIFT] by blast

show ?thesis
proof
  show isLub UNIV S (t + X + (−1))
  proof (rule isLubI2)
    {
      fix x
      assume isUb (UNIV::real set) S x
      hence isUb (UNIV::real set) (?SHIFT) (x + (−X) + 1)
        using S-Ub-is-SHIFT-Ub by simp
      hence t ≤ (x + (−X) + 1)
        using t-is-Lub by (simp add: isLub-le-isUb)
      hence t + X + −1 ≤ x by arith
    }
    then show (t + X + −1) <=* Collect (isUb UNIV S)
      by (simp add: setgeI)
  next
    show isUb UNIV S (t + X + −1)
    proof −
      {
        fix y
        assume y-in-S: y ∈ S
        have y ≤ t + X + −1
        proof −
```

```
          obtain u where u-in-shift: u ∈ ?SHIFT using shifted-not-empty ..
          hence ∃ x ∈ S. u = x + − X + 1 by simp
          then obtain x where x-and-u: u = x + − X + 1 ..
         have u-le-t: u ≤ t using u-in-shift and t-is-Lub by (simp add: isLubD2)

          show ?thesis
          proof cases
            assume y ≤ x
            moreover have x = u + X + − 1 using x-and-u by arith
            moreover have u + X + − 1  ≤ t + X + −1 using u-le-t by arith
            ultimately show y  ≤ t + X + −1 by arith
          next
            assume ~(y ≤ x)
            hence x-less-y: x < y by arith

           have x + (−X) + 1 ∈ ?SHIFT using x-and-u and u-in-shift by simp
            hence 0 < x + (−X) + 1 by simp
            hence 0 < y + (−X) + 1 using x-less-y by arith
            hence y + (−X) + 1 ∈ ?SHIFT using y-in-S by simp
            hence y + (−X) + 1 ≤ t using t-is-Lub  by (simp add: isLubD2)
            thus ?thesis by simp
          qed
        qed
      }
      then show ?thesis by (simp add: isUb-def setle-def)
    qed
  qed
 qed
qed
```

7.2 The Archimedean Property of the Reals

```
theorem reals-Archimedean:
  assumes x-pos: 0 < x
  shows ∃ n. inverse (real (Suc n)) < x
proof (rule ccontr)
  assume contr: ¬ ?thesis
  have ∀ n. x ∗ real (Suc n) <= 1
  proof
    fix n
    from contr have x ≤ inverse (real (Suc n))
      by (simp add: linorder-not-less)
    hence x ≤ (1 / (real (Suc n)))
      by (simp add: inverse-eq-divide)
    moreover have 0 ≤ real (Suc n)
      by (rule real-of-nat-ge-zero)
    ultimately have x ∗ real (Suc n) ≤ (1 / real (Suc n)) ∗ real (Suc n)
      by (rule mult-right-mono)
    thus x ∗ real (Suc n) ≤ 1 by simp
```

qed
hence $\{z.\ \exists\, n.\ z = x * (\mathit{real}\ (\mathit{Suc}\ n))\}\ {*}{<=}\ 1$
by $(\mathit{simp\ add}\colon \mathit{setle\text{-}def},\ \mathit{safe},\ \mathit{rule\ spec})$
hence $\mathit{isUb}\ (\mathit{UNIV}{::}\mathit{real\ set})\ \{z.\ \exists\, n.\ z = x * (\mathit{real}\ (\mathit{Suc}\ n))\}\ 1$
by $(\mathit{simp\ add}\colon \mathit{isUbI})$
hence $\exists\, Y.\ \mathit{isUb}\ (\mathit{UNIV}{::}\mathit{real\ set})\ \{z.\ \exists\, n.\ z = x{*}\ (\mathit{real}\ (\mathit{Suc}\ n))\}\ Y$ **..**
moreover have $\exists\, X.\ X \in \{z.\ \exists\, n.\ z = x{*}\ (\mathit{real}\ (\mathit{Suc}\ n))\}$ **by** *auto*
ultimately have $\exists\, t.\ \mathit{isLub}\ \mathit{UNIV}\ \{z.\ \exists\, n.\ z = x * \mathit{real}\ (\mathit{Suc}\ n)\}\ t$
by $(\mathit{simp\ add}\colon \mathit{reals\text{-}complete})$
then obtain t **where**
$\mathit{t\text{-}is\text{-}Lub}\colon \mathit{isLub}\ \mathit{UNIV}\ \{z.\ \exists\, n.\ z = x * \mathit{real}\ (\mathit{Suc}\ n)\}\ t$ **..**

have $\forall\, n{::}\mathit{nat}.\ x * \mathit{real}\ n \le t + -\ x$
proof
fix n
from *t-is-Lub* **have** $x * \mathit{real}\ (\mathit{Suc}\ n) \le t$
by $(\mathit{simp\ add}\colon \mathit{isLubD2})$
hence $x * (\mathit{real}\ n) + x \le t$
by $(\mathit{simp\ add}\colon \mathit{right\text{-}distrib}\ \mathit{real\text{-}of\text{-}nat\text{-}Suc})$
thus $x * (\mathit{real}\ n) \le t + -\ x$ **by** *arith*
qed

hence $\forall\, m.\ x * \mathit{real}\ (\mathit{Suc}\ m) \le t + -\ x$ **by** *simp*
hence $\{z.\ \exists\, n.\ z = x * (\mathit{real}\ (\mathit{Suc}\ n))\}\ {*}{<=}\ (t + -\ x)$
by $(\mathit{auto\ simp\ add}\colon \mathit{setle\text{-}def})$
hence $\mathit{isUb}\ (\mathit{UNIV}{::}\mathit{real\ set})\ \{z.\ \exists\, n.\ z = x * (\mathit{real}\ (\mathit{Suc}\ n))\}\ (t + (-x))$
by $(\mathit{simp\ add}\colon \mathit{isUbI})$
hence $t \le t + -\ x$
using *t-is-Lub* **by** $(\mathit{simp\ add}\colon \mathit{isLub\text{-}le\text{-}isUb})$
thus *False* **using** *x-pos* **by** *arith*
qed

There must be other proofs, e.g. *Suc* of the largest integer in the cut representing x.

lemma $\mathit{reals\text{-}Archimedean2}\colon \exists\, n.\ (x{::}\mathit{real}) < \mathit{real}\ (n{::}\mathit{nat})$
proof *cases*
assume $x \le 0$
hence $x < \mathit{real}\ (1{::}\mathit{nat})$ **by** *simp*
thus *?thesis* **..**
next
assume $\neg\ x \le 0$
hence $\mathit{x\text{-}greater\text{-}zero}\colon 0 < x$ **by** *simp*
hence $0 < \mathit{inverse}\ x$ **by** *simp*
then obtain n **where** $\mathit{inverse}\ (\mathit{real}\ (\mathit{Suc}\ n)) < \mathit{inverse}\ x$
using *reals-Archimedean* **by** *blast*
hence $\mathit{inverse}\ (\mathit{real}\ (\mathit{Suc}\ n)) * x < \mathit{inverse}\ x * x$
using *x-greater-zero* **by** $(\mathit{rule\ mult\text{-}strict\text{-}right\text{-}mono})$
hence $\mathit{inverse}\ (\mathit{real}\ (\mathit{Suc}\ n)) * x < 1$
using *x-greater-zero* **by** *simp*

hence *real* (*Suc n*) $*$ (*inverse* (*real* (*Suc n*)) $*$ *x*) $<$ *real* (*Suc n*) $*$ *1*
  by (*rule mult-strict-left-mono*) *simp*
hence $x <$ *real* (*Suc n*)
  by (*simp add*: *ring-simps*)
thus $\exists$ (*n*::*nat*). $x <$ *real n* **..**
qed

lemma *reals-Archimedean3*:
assumes *x-greater-zero*: $0 < x$
shows $\forall$ (*y*::*real*). $\exists$ (*n*::*nat*). $y <$ *real n* $*$ *x*
proof
fix *y*
have *x-not-zero*: $x \neq 0$ **using** *x-greater-zero* **by** *simp*
obtain *n* **where** $y *$ *inverse x* $<$ *real* (*n*::*nat*)
  using *reals-Archimedean2* **..**
hence $y *$ *inverse x* $* x <$ *real n* $* x$
  using *x-greater-zero* **by** (*simp add*: *mult-strict-right-mono*)
hence $x *$ *inverse x* $* y < x *$ *real n*
  by (*simp add*: *ring-simps*)
hence $y <$ *real* (*n*::*nat*) $* x$
  using *x-not-zero* **by** (*simp add*: *ring-simps*)
thus $\exists$ (*n*::*nat*). $y <$ *real n* $* x$ **..**
qed

lemma *reals-Archimedean6*:
$0 \leq r$ ==> $\exists$ (*n*::*nat*). *real* $(n - 1) \leq r$ & $r <$ *real* (*n*)
apply (*insert reals-Archimedean2* [*of r*], *safe*)
apply (*subgoal-tac* $\exists$ *x*::*nat*. $r <$ *real x* $\wedge$ ($\forall y$. $r <$ *real y* $\longrightarrow x \leq y$), *auto*)
apply (*rule-tac* $x = x$ **in** *exI*)
apply (*case-tac x*, *simp*)
apply (*rename-tac* x')
apply (*drule-tac* $x = x'$ **in** *spec*, *simp*)
apply (*rule-tac x*=*LEAST n*. $r <$ *real n* **in** *exI*, *safe*)
apply (*erule LeastI*, *erule Least-le*)
done

lemma *reals-Archimedean6a*: $0 \leq r$ ==> $\exists n$. *real* (*n*) $\leq r$ & $r <$ *real* (*Suc n*)
  by (*drule reals-Archimedean6*) *auto*

lemma *reals-Archimedean-6b-int*:
$0 \leq r$ ==> $\exists$ *n*::*int*. *real n* $\leq r$ & $r <$ *real* (n+*1*)
apply (*drule reals-Archimedean6a*, *auto*)
apply (*rule-tac* $x =$ *int n* **in** *exI*)
apply (*simp add*: *real-of-int-real-of-nat real-of-nat-Suc*)
done

lemma *reals-Archimedean-6c-int*:
$r < 0$ ==> $\exists$ *n*::*int*. *real n* $\leq r$ & $r <$ *real* (n+*1*)
apply (*rule reals-Archimedean-6b-int* [*of* $-r$, *THEN exE*], *simp*, *auto*)

apply (*rename-tac n*)
apply (*drule order-le-imp-less-or-eq*, *auto*)
apply (*rule-tac x* = − *n* − *1* **in** *exI*)
apply (*rule-tac* [*2*] *x* = − *n* **in** *exI*, *auto*)
done

7.3 Floor and Ceiling Functions from the Reals to the Integers

definition
floor :: *real* => *int* **where**
floor r = (*LEAST n*::*int*. *r* < *real* (*n*+*1*))

definition
ceiling :: *real* => *int* **where**
ceiling r = − *floor* (− *r*)

notation (*xsymbols*)
floor (⌊-⌋) **and**
ceiling (⌈-⌉)

notation (*HTML* **output**)
floor (⌊-⌋) **and**
ceiling (⌈-⌉)

lemma *number-of-less-real-of-int-iff* [*simp*]:
((*number-of n*) < *real* (*m*::*int*)) = (*number-of n* < *m*)
apply *auto*
apply (*rule real-of-int-less-iff* [*THEN iffD1*])
apply (*drule-tac* [*2*] *real-of-int-less-iff* [*THEN iffD2*], *auto*)
done

lemma *number-of-less-real-of-int-iff2* [*simp*]:
(*real* (*m*::*int*) < (*number-of n*)) = (*m* < *number-of n*)
apply *auto*
apply (*rule real-of-int-less-iff* [*THEN iffD1*])
apply (*drule-tac* [*2*] *real-of-int-less-iff* [*THEN iffD2*], *auto*)
done

lemma *number-of-le-real-of-int-iff* [*simp*]:
((*number-of n*) ≤ *real* (*m*::*int*)) = (*number-of n* ≤ *m*)
by (*simp add*: *linorder-not-less* [*symmetric*])

lemma *number-of-le-real-of-int-iff2* [*simp*]:
(*real* (*m*::*int*) ≤ (*number-of n*)) = (*m* ≤ *number-of n*)
by (*simp add*: *linorder-not-less* [*symmetric*])

lemma *floor-zero* [*simp*]: *floor 0* = *0*

apply (*simp add*: *floor-def del*: *real-of-int-add*)
apply (*rule Least-equality*)
apply *simp-all*
done

lemma *floor-real-of-nat-zero* [*simp*]: *floor* (*real* (*0*::*nat*)) = *0*
by *auto*

lemma *floor-real-of-nat* [*simp*]: *floor* (*real* (*n*::*nat*)) = *int n*
apply (*simp only*: *floor-def*)
apply (*rule Least-equality*)
apply (*drule-tac* [*2*] *real-of-int-of-nat-eq* [*THEN ssubst*])
apply (*drule-tac* [*2*] *real-of-int-less-iff* [*THEN iffD1*])
apply *simp-all*
done

lemma *floor-minus-real-of-nat* [*simp*]: *floor* (− *real* (*n*::*nat*)) = − *int n*
apply (*simp only*: *floor-def*)
apply (*rule Least-equality*)
apply (*drule-tac* [*2*] *real-of-int-of-nat-eq* [*THEN ssubst*])
apply (*drule-tac* [*2*] *real-of-int-minus* [*THEN sym*, *THEN subst*])
apply (*drule-tac* [*2*] *real-of-int-less-iff* [*THEN iffD1*])
apply *simp-all*
done

lemma *floor-real-of-int* [*simp*]: *floor* (*real* (*n*::*int*)) = *n*
apply (*simp only*: *floor-def*)
apply (*rule Least-equality*)
apply *auto*
done

lemma *floor-minus-real-of-int* [*simp*]: *floor* (− *real* (*n*::*int*)) = − *n*
apply (*simp only*: *floor-def*)
apply (*rule Least-equality*)
apply (*drule-tac* [*2*] *real-of-int-minus* [*THEN sym*, *THEN subst*])
apply *auto*
done

lemma *real-lb-ub-int*: ∃ *n*::*int*. *real n* ≤ *r* & *r* < *real* (*n*+*1*)
apply (*case-tac r* < *0*)
apply (*blast intro*: *reals-Archimedean-6c-int*)
apply (*simp only*: *linorder-not-less*)
apply (*blast intro*: *reals-Archimedean-6b-int reals-Archimedean-6c-int*)
done

lemma *lemma-floor*:
  assumes *a1*: *real m* ≤ *r* **and** *a2*: *r* < *real n* + *1*
  shows *m* ≤ (*n*::*int*)
proof −

```
  have real m < real n + 1 using a1 a2 by (rule order-le-less-trans)
  also have ... = real (n + 1) by simp
  finally have m < n + 1 by (simp only: real-of-int-less-iff)
  thus ?thesis by arith
qed

lemma real-of-int-floor-le [simp]: real (floor r) ≤ r
apply (simp add: floor-def Least-def)
apply (insert real-lb-ub-int [of r], safe)
apply (rule theI2)
apply auto
done

lemma floor-mono: x < y ==> floor x ≤ floor y
apply (simp add: floor-def Least-def)
apply (insert real-lb-ub-int [of x])
apply (insert real-lb-ub-int [of y], safe)
apply (rule theI2)
apply (rule-tac [3] theI2)
apply simp
apply (erule conjI)
apply (auto simp add: order-eq-iff int-le-real-less)
done

lemma floor-mono2: x ≤ y ==> floor x ≤ floor y
by (auto dest: order-le-imp-less-or-eq simp add: floor-mono)

lemma lemma-floor2: real n < real (x::int) + 1 ==> n ≤ x
by (auto intro: lemma-floor)

lemma real-of-int-floor-cancel [simp]:
  (real (floor x) = x) = (∃ n::int. x = real n)
apply (simp add: floor-def Least-def)
apply (insert real-lb-ub-int [of x], erule exE)
apply (rule theI2)
apply (auto intro: lemma-floor)
done

lemma floor-eq: [| real n < x; x < real n + 1 |] ==> floor x = n
apply (simp add: floor-def)
apply (rule Least-equality)
apply (auto intro: lemma-floor)
done

lemma floor-eq2: [| real n ≤ x; x < real n + 1 |] ==> floor x = n
apply (simp add: floor-def)
apply (rule Least-equality)
apply (auto intro: lemma-floor)
done
```

lemma *floor-eq3*: [| *real n* $<$ *x*; *x* $<$ *real* (*Suc n*) |] ==> *nat*(*floor x*) = *n*
apply (*rule inj-int* [*THEN injD*])
apply (*simp add*: *real-of-nat-Suc*)
apply (*simp add*: *real-of-nat-Suc floor-eq floor-eq* [**where** *n* = *int n*])
done

lemma *floor-eq4*: [| *real n* $\le$ *x*; *x* $<$ *real* (*Suc n*) |] ==> *nat*(*floor x*) = *n*
apply (*drule order-le-imp-less-or-eq*)
apply (*auto intro*: *floor-eq3*)
done

lemma *floor-number-of-eq* [*simp*]:
  floor(*number-of n* :: *real*) = (*number-of n* :: *int*)
apply (*subst real-number-of* [*symmetric*])
apply (*rule floor-real-of-int*)
done

lemma *floor-one* [*simp*]: *floor 1* = *1*
  apply (*rule trans*)
  prefer *2*
  apply (*rule floor-real-of-int*)
  apply *simp*
done

lemma *real-of-int-floor-ge-diff-one* [*simp*]: *r* − *1* $\le$ *real*(*floor r*)
apply (*simp add*: *floor-def Least-def*)
apply (*insert real-lb-ub-int* [*of r*], *safe*)
apply (*rule theI2*)
apply (*auto intro*: *lemma-floor*)
done

lemma *real-of-int-floor-gt-diff-one* [*simp*]: *r* − *1* $<$ *real*(*floor r*)
apply (*simp add*: *floor-def Least-def*)
apply (*insert real-lb-ub-int* [*of r*], *safe*)
apply (*rule theI2*)
apply (*auto intro*: *lemma-floor*)
done

lemma *real-of-int-floor-add-one-ge* [*simp*]: *r* $\le$ *real*(*floor r*) + *1*
apply (*insert real-of-int-floor-ge-diff-one* [*of r*])
apply (*auto simp del*: *real-of-int-floor-ge-diff-one*)
done

lemma *real-of-int-floor-add-one-gt* [*simp*]: *r* $<$ *real*(*floor r*) + *1*
apply (*insert real-of-int-floor-gt-diff-one* [*of r*])
apply (*auto simp del*: *real-of-int-floor-gt-diff-one*)
done

```
lemma le-floor: real a <= x ==> a <= floor x
  apply (subgoal-tac a < floor x + 1)
  apply arith
  apply (subst real-of-int-less-iff [THEN sym])
  apply simp
  apply (insert real-of-int-floor-add-one-gt [of x])
  apply arith
done

lemma real-le-floor: a <= floor x ==> real a <= x
  apply (rule order-trans)
  prefer 2
  apply (rule real-of-int-floor-le)
  apply (subst real-of-int-le-iff)
  apply assumption
done

lemma le-floor-eq: (a <= floor x) = (real a <= x)
  apply (rule iffI)
  apply (erule real-le-floor)
  apply (erule le-floor)
done

lemma le-floor-eq-number-of [simp]:
    (number-of n <= floor x) = (number-of n <= x)
by (simp add: le-floor-eq)

lemma le-floor-eq-zero [simp]: (0 <= floor x) = (0 <= x)
by (simp add: le-floor-eq)

lemma le-floor-eq-one [simp]: (1 <= floor x) = (1 <= x)
by (simp add: le-floor-eq)

lemma floor-less-eq: (floor x < a) = (x < real a)
  apply (subst linorder-not-le [THEN sym])+
  apply simp
  apply (rule le-floor-eq)
done

lemma floor-less-eq-number-of [simp]:
    (floor x < number-of n) = (x < number-of n)
by (simp add: floor-less-eq)

lemma floor-less-eq-zero [simp]: (floor x < 0) = (x < 0)
by (simp add: floor-less-eq)

lemma floor-less-eq-one [simp]: (floor x < 1) = (x < 1)
by (simp add: floor-less-eq)
```

```
lemma less-floor-eq: (a < floor x) = (real a + 1 <= x)
  apply (insert le-floor-eq [of a + 1 x])
  apply auto
done

lemma less-floor-eq-number-of [simp]:
    (number-of n < floor x) = (number-of n + 1 <= x)
by (simp add: less-floor-eq)

lemma less-floor-eq-zero [simp]: (0 < floor x) = (1 <= x)
by (simp add: less-floor-eq)

lemma less-floor-eq-one [simp]: (1 < floor x) = (2 <= x)
by (simp add: less-floor-eq)

lemma floor-le-eq: (floor x <= a) = (x < real a + 1)
  apply (insert floor-less-eq [of x a + 1])
  apply auto
done

lemma floor-le-eq-number-of [simp]:
    (floor x <= number-of n) = (x < number-of n + 1)
by (simp add: floor-le-eq)

lemma floor-le-eq-zero [simp]: (floor x <= 0) = (x < 1)
by (simp add: floor-le-eq)

lemma floor-le-eq-one [simp]: (floor x <= 1) = (x < 2)
by (simp add: floor-le-eq)

lemma floor-add [simp]: floor (x + real a) = floor x + a
  apply (subst order-eq-iff)
  apply (rule conjI)
  prefer 2
  apply (subgoal-tac floor x + a < floor (x + real a) + 1)
  apply arith
  apply (subst real-of-int-less-iff [THEN sym])
  apply simp
  apply (subgoal-tac x + real a < real(floor(x + real a)) + 1)
  apply (subgoal-tac real (floor x) <= x)
  apply arith
  apply (rule real-of-int-floor-le)
  apply (rule real-of-int-floor-add-one-gt)
  apply (subgoal-tac floor (x + real a) < floor x + a + 1)
  apply arith
  apply (subst real-of-int-less-iff [THEN sym])
  apply simp
  apply (subgoal-tac real(floor(x + real a)) <= x + real a)
  apply (subgoal-tac x < real(floor x) + 1)
```

apply *arith*
apply (*rule real-of-int-floor-add-one-gt*)
apply (*rule real-of-int-floor-le*)
done

lemma *floor-add-number-of* [*simp*]:
floor (*x* + *number-of n*) = *floor x* + *number-of n*
apply (*subst floor-add* [*THEN sym*])
apply *simp*
done

lemma *floor-add-one* [*simp*]: *floor* (*x* + *1*) = *floor x* + *1*
apply (*subst floor-add* [*THEN sym*])
apply *simp*
done

lemma *floor-subtract* [*simp*]: *floor* (*x* − *real a*) = *floor x* − *a*
apply (*subst diff-minus*)+
apply (*subst real-of-int-minus* [*THEN sym*])
apply (*rule floor-add*)
done

lemma *floor-subtract-number-of* [*simp*]: *floor* (*x* − *number-of n*) =
floor x − *number-of n*
apply (*subst floor-subtract* [*THEN sym*])
apply *simp*
done

lemma *floor-subtract-one* [*simp*]: *floor* (*x* − *1*) = *floor x* − *1*
apply (*subst floor-subtract* [*THEN sym*])
apply *simp*
done

lemma *ceiling-zero* [*simp*]: *ceiling 0* = *0*
by (*simp add*: *ceiling-def*)

lemma *ceiling-real-of-nat* [*simp*]: *ceiling* (*real* (*n*::*nat*)) = *int n*
by (*simp add*: *ceiling-def*)

lemma *ceiling-real-of-nat-zero* [*simp*]: *ceiling* (*real* (*0*::*nat*)) = *0*
by *auto*

lemma *ceiling-floor* [*simp*]: *ceiling* (*real* (*floor r*)) = *floor r*
by (*simp add*: *ceiling-def*)

lemma *floor-ceiling* [*simp*]: *floor* (*real* (*ceiling r*)) = *ceiling r*
by (*simp add*: *ceiling-def*)

lemma *real-of-int-ceiling-ge* [*simp*]: *r* ≤ *real* (*ceiling r*)

apply (*simp add*: *ceiling-def*)
apply (*subst le-minus-iff*, *simp*)
done

lemma *ceiling-mono*: $x < y$ ==> *ceiling* $x \leq$ *ceiling* y
by (*simp add*: *floor-mono ceiling-def*)

lemma *ceiling-mono2*: $x \leq y$ ==> *ceiling* $x \leq$ *ceiling* y
by (*simp add*: *floor-mono2 ceiling-def*)

lemma *real-of-int-ceiling-cancel* [*simp*]:
(*real* (*ceiling* x) = x) = ($\exists$ *n*::*int*. x = *real* n)
apply (*auto simp add*: *ceiling-def*)
apply (*drule arg-cong* [**where** f = *uminus*], *auto*)
apply (*rule-tac* $x = -n$ **in** *exI*, *auto*)
done

lemma *ceiling-eq*: [| *real* $n < x$; $x <$ *real* $n + 1$ |] ==> *ceiling* $x = n + 1$
apply (*simp add*: *ceiling-def*)
apply (*rule minus-equation-iff* [*THEN iffD1*])
apply (*simp add*: *floor-eq* [**where** $n = -(n+1)$])
done

lemma *ceiling-eq2*: [| *real* $n < x$; $x \leq$ *real* $n + 1$ |] ==> *ceiling* $x = n + 1$
by (*simp add*: *ceiling-def floor-eq2* [**where** $n = -(n+1)$])

lemma *ceiling-eq3*: [| *real* $n - 1 < x$; $x \leq$ *real* n |] ==> *ceiling* $x = n$
by (*simp add*: *ceiling-def floor-eq2* [**where** $n = -n$])

lemma *ceiling-real-of-int* [*simp*]: *ceiling* (*real* (*n*::*int*)) = n
by (*simp add*: *ceiling-def*)

lemma *ceiling-number-of-eq* [*simp*]:
ceiling (*number-of* n :: *real*) = (*number-of* n)
apply (*subst real-number-of* [*symmetric*])
apply (*rule ceiling-real-of-int*)
done

lemma *ceiling-one* [*simp*]: *ceiling* 1 = 1
by (*unfold ceiling-def*, *simp*)

lemma *real-of-int-ceiling-diff-one-le* [*simp*]: *real* (*ceiling* r) $- 1 \leq r$
apply (*rule neg-le-iff-le* [*THEN iffD1*])
apply (*simp add*: *ceiling-def diff-minus*)
done

lemma *real-of-int-ceiling-le-add-one* [*simp*]: *real* (*ceiling* r) $\leq r + 1$
apply (*insert real-of-int-ceiling-diff-one-le* [*of* r])
apply (*simp del*: *real-of-int-ceiling-diff-one-le*)

done

lemma *ceiling-le*: *x <= real a ==> ceiling x <= a*
  apply (*unfold ceiling-def*)
  apply (*subgoal-tac −a <= floor(− x)*)
  apply *simp*
  apply (*rule le-floor*)
  apply *simp*
done

lemma *ceiling-le-real*: *ceiling x <= a ==> x <= real a*
  apply (*unfold ceiling-def*)
  apply (*subgoal-tac real(− a) <= − x*)
  apply *simp*
  apply (*rule real-le-floor*)
  apply *simp*
done

lemma *ceiling-le-eq*: (*ceiling x <= a*) = (*x <= real a*)
  apply (*rule iffI*)
  apply (*erule ceiling-le-real*)
  apply (*erule ceiling-le*)
done

lemma *ceiling-le-eq-number-of* [*simp*]:
    (*ceiling x <= number-of n*) = (*x <= number-of n*)
by (*simp add*: *ceiling-le-eq*)

lemma *ceiling-le-zero-eq* [*simp*]: (*ceiling x <= 0*) = (*x <= 0*)
by (*simp add*: *ceiling-le-eq*)

lemma *ceiling-le-eq-one* [*simp*]: (*ceiling x <= 1*) = (*x <= 1*)
by (*simp add*: *ceiling-le-eq*)

lemma *less-ceiling-eq*: (*a < ceiling x*) = (*real a < x*)
  apply (*subst linorder-not-le* [*THEN sym*])+
  apply *simp*
  apply (*rule ceiling-le-eq*)
done

lemma *less-ceiling-eq-number-of* [*simp*]:
    (*number-of n < ceiling x*) = (*number-of n < x*)
by (*simp add*: *less-ceiling-eq*)

lemma *less-ceiling-eq-zero* [*simp*]: (*0 < ceiling x*) = (*0 < x*)
by (*simp add*: *less-ceiling-eq*)

lemma *less-ceiling-eq-one* [*simp*]: (*1 < ceiling x*) = (*1 < x*)
by (*simp add*: *less-ceiling-eq*)

lemma *ceiling-less-eq*: (*ceiling* x < a) = (x <= *real* a − *1*)
  apply (*insert ceiling-le-eq* [*of* x a − *1*])
  apply *auto*
done

lemma *ceiling-less-eq-number-of* [*simp*]:
  (*ceiling* x < *number-of* n) = (x <= *number-of* n − *1*)
by (*simp add*: *ceiling-less-eq*)

lemma *ceiling-less-eq-zero* [*simp*]: (*ceiling* x < *0*) = (x <= −*1*)
by (*simp add*: *ceiling-less-eq*)

lemma *ceiling-less-eq-one* [*simp*]: (*ceiling* x < *1*) = (x <= *0*)
by (*simp add*: *ceiling-less-eq*)

lemma *le-ceiling-eq*: (a <= *ceiling* x) = (*real* a − *1* < x)
  apply (*insert less-ceiling-eq* [*of* a − *1* x])
  apply *auto*
done

lemma *le-ceiling-eq-number-of* [*simp*]:
  (*number-of* n <= *ceiling* x) = (*number-of* n − *1* < x)
by (*simp add*: *le-ceiling-eq*)

lemma *le-ceiling-eq-zero* [*simp*]: (*0* <= *ceiling* x) = (−*1* < x)
by (*simp add*: *le-ceiling-eq*)

lemma *le-ceiling-eq-one* [*simp*]: (*1* <= *ceiling* x) = (*0* < x)
by (*simp add*: *le-ceiling-eq*)

lemma *ceiling-add* [*simp*]: *ceiling* (x + *real* a) = *ceiling* x + a
  apply (*unfold ceiling-def*, *simp*)
  apply (*subst real-of-int-minus* [*THEN sym*])
  apply (*subst floor-add*)
  apply *simp*
done

lemma *ceiling-add-number-of* [*simp*]: *ceiling* (x + *number-of* n) =
  ceiling x + *number-of* n
  apply (*subst ceiling-add* [*THEN sym*])
  apply *simp*
done

lemma *ceiling-add-one* [*simp*]: *ceiling* (x + *1*) = *ceiling* x + *1*
  apply (*subst ceiling-add* [*THEN sym*])
  apply *simp*
done

lemma *ceiling-subtract* [*simp*]: *ceiling* $(x - real\ a) = ceiling\ x - a$
  apply (*subst diff-minus*)+
  apply (*subst real-of-int-minus* [*THEN sym*])
  apply (*rule ceiling-add*)
done

lemma *ceiling-subtract-number-of* [*simp*]: *ceiling* $(x - number\text{-}of\ n) =$
    ceiling $x - number\text{-}of\ n$
  apply (*subst ceiling-subtract* [*THEN sym*])
  apply *simp*
done

lemma *ceiling-subtract-one* [*simp*]: *ceiling* $(x - 1) = ceiling\ x - 1$
  apply (*subst ceiling-subtract* [*THEN sym*])
  apply *simp*
done

7.4 Versions for the natural numbers

definition
  natfloor :: *real* => *nat* **where**
  natfloor $x = nat(floor\ x)$

definition
  natceiling :: *real* => *nat* **where**
  natceiling $x = nat(ceiling\ x)$

lemma *natfloor-zero* [*simp*]: *natfloor* $0 = 0$
  by (*unfold natfloor-def*, *simp*)

lemma *natfloor-one* [*simp*]: *natfloor* $1 = 1$
  by (*unfold natfloor-def*, *simp*)

lemma *zero-le-natfloor* [*simp*]: $0 <= natfloor\ x$
  by (*unfold natfloor-def*, *simp*)

lemma *natfloor-number-of-eq* [*simp*]: *natfloor* $(number\text{-}of\ n) = number\text{-}of\ n$
  by (*unfold natfloor-def*, *simp*)

lemma *natfloor-real-of-nat* [*simp*]: *natfloor*$(real\ n) = n$
  by (*unfold natfloor-def*, *simp*)

lemma *real-natfloor-le*: $0 <= x ==> real(natfloor\ x) <= x$
  by (*unfold natfloor-def*, *simp*)

lemma *natfloor-neg*: $x <= 0 ==> natfloor\ x = 0$
  apply (*unfold natfloor-def*)
  apply (*subgoal-tac floor* $x <= floor\ 0$)
  apply *simp*

```
  apply (erule floor-mono2)
done

lemma natfloor-mono: x <= y ==> natfloor x <= natfloor y
  apply (case-tac 0 <= x)
  apply (subst natfloor-def)+
  apply (subst nat-le-eq-zle)
  apply force
  apply (erule floor-mono2)
  apply (subst natfloor-neg)
  apply simp
  apply simp
done

lemma le-natfloor: real x <= a ==> x <= natfloor a
  apply (unfold natfloor-def)
  apply (subst nat-int [THEN sym])
  apply (subst nat-le-eq-zle)
  apply simp
  apply (rule le-floor)
  apply simp
done

lemma le-natfloor-eq: 0 <= x ==> (a <= natfloor x) = (real a <= x)
  apply (rule iffI)
  apply (rule order-trans)
  prefer 2
  apply (erule real-natfloor-le)
  apply (subst real-of-nat-le-iff)
  apply assumption
  apply (erule le-natfloor)
done

lemma le-natfloor-eq-number-of [simp]:
    ~ neg((number-of n)::int) ==> 0 <= x ==>
      (number-of n <= natfloor x) = (number-of n <= x)
  apply (subst le-natfloor-eq, assumption)
  apply simp
done

lemma le-natfloor-eq-one [simp]: (1 <= natfloor x) = (1 <= x)
  apply (case-tac 0 <= x)
  apply (subst le-natfloor-eq, assumption, simp)
  apply (rule iffI)
  apply (subgoal-tac natfloor x <= natfloor 0)
  apply simp
  apply (rule natfloor-mono)
  apply simp
  apply simp
```

```
done

lemma natfloor-eq: real n <= x ==> x < real n + 1 ==> natfloor x = n
  apply (unfold natfloor-def)
  apply (subst nat-int [THEN sym])back
  apply (subst eq-nat-nat-iff)
  apply simp
  apply simp
  apply (rule floor-eq2)
  apply auto
done

lemma real-natfloor-add-one-gt: x < real(natfloor x) + 1
  apply (case-tac 0 <= x)
  apply (unfold natfloor-def)
  apply simp
  apply simp-all
done

lemma real-natfloor-gt-diff-one: x - 1 < real(natfloor x)
  apply (simp add: compare-rls)
  apply (rule real-natfloor-add-one-gt)
done

lemma ge-natfloor-plus-one-imp-gt: natfloor z + 1 <= n ==> z < real n
  apply (subgoal-tac z < real(natfloor z) + 1)
  apply arith
  apply (rule real-natfloor-add-one-gt)
done

lemma natfloor-add [simp]: 0 <= x ==> natfloor (x + real a) = natfloor x + a
  apply (unfold natfloor-def)
  apply (subgoal-tac real a = real (int a))
  apply (erule ssubst)
  apply (simp add: nat-add-distrib del: real-of-int-of-nat-eq)
  apply simp
done

lemma natfloor-add-number-of [simp]:
    ~neg ((number-of n)::int) ==> 0 <= x ==>
      natfloor (x + number-of n) = natfloor x + number-of n
  apply (subst natfloor-add [THEN sym])
  apply simp-all
done

lemma natfloor-add-one: 0 <= x ==> natfloor(x + 1) = natfloor x + 1
  apply (subst natfloor-add [THEN sym])
  apply assumption
  apply simp
```

done

lemma *natfloor-subtract* [*simp*]: *real a* <= *x* ==>
  natfloor(*x* − *real a*) = *natfloor x* − *a*
 apply (*unfold natfloor-def*)
 apply (*subgoal-tac real a* = *real* (*int a*))
 apply (*erule ssubst*)
 apply (*simp del*: *real-of-int-of-nat-eq*)
 apply *simp*
done

lemma *natceiling-zero* [*simp*]: *natceiling 0* = *0*
 by (*unfold natceiling-def*, *simp*)

lemma *natceiling-one* [*simp*]: *natceiling 1* = *1*
 by (*unfold natceiling-def*, *simp*)

lemma *zero-le-natceiling* [*simp*]: *0* <= *natceiling x*
 by (*unfold natceiling-def*, *simp*)

lemma *natceiling-number-of-eq* [*simp*]: *natceiling* (*number-of n*) = *number-of n*
 by (*unfold natceiling-def*, *simp*)

lemma *natceiling-real-of-nat* [*simp*]: *natceiling*(*real n*) = *n*
 by (*unfold natceiling-def*, *simp*)

lemma *real-natceiling-ge*: *x* <= *real*(*natceiling x*)
 apply (*unfold natceiling-def*)
 apply (*case-tac x* < *0*)
 apply *simp*
 apply (*subst real-nat-eq-real*)
 apply (*subgoal-tac ceiling 0* <= *ceiling x*)
 apply *simp*
 apply (*rule ceiling-mono2*)
 apply *simp*
 apply *simp*
done

lemma *natceiling-neg*: *x* <= *0* ==> *natceiling x* = *0*
 apply (*unfold natceiling-def*)
 apply *simp*
done

lemma *natceiling-mono*: *x* <= *y* ==> *natceiling x* <= *natceiling y*
 apply (*case-tac 0* <= *x*)
 apply (*subst natceiling-def*)+
 apply (*subst nat-le-eq-zle*)
 apply (*rule disjI2*)
 apply (*subgoal-tac real* (*0*::*int*) <= *real*(*ceiling y*))

```
  apply simp
  apply (rule order-trans)
  apply simp
  apply (erule order-trans)
  apply simp
  apply (erule ceiling-mono2)
  apply (subst natceiling-neg)
  apply simp-all
done

lemma natceiling-le: x <= real a ==> natceiling x <= a
  apply (unfold natceiling-def)
  apply (case-tac x < 0)
  apply simp
  apply (subst nat-int [THEN sym])back
  apply (subst nat-le-eq-zle)
  apply simp
  apply (rule ceiling-le)
  apply simp
done

lemma natceiling-le-eq: 0 <= x ==> (natceiling x <= a) = (x <= real a)
  apply (rule iffI)
  apply (rule order-trans)
  apply (rule real-natceiling-ge)
  apply (subst real-of-nat-le-iff)
  apply assumption
  apply (erule natceiling-le)
done

lemma natceiling-le-eq-number-of [simp]:
    ~ neg((number-of n)::int) ==> 0 <= x ==>
      (natceiling x <= number-of n) = (x <= number-of n)
  apply (subst natceiling-le-eq, assumption)
  apply simp
done

lemma natceiling-le-eq-one: (natceiling x <= 1) = (x <= 1)
  apply (case-tac 0 <= x)
  apply (subst natceiling-le-eq)
  apply assumption
  apply simp
  apply (subst natceiling-neg)
  apply simp
  apply simp
done

lemma natceiling-eq: real n < x ==> x <= real n + 1 ==> natceiling x = n +
1
```

```
  apply (unfold natceiling-def)
  apply (simplesubst nat-int [THEN sym]) back back
  apply (subgoal-tac nat(int n) + 1 = nat(int n + 1))
  apply (erule ssubst)
  apply (subst eq-nat-nat-iff)
  apply (subgoal-tac ceiling 0 <= ceiling x)
  apply simp
  apply (rule ceiling-mono2)
  apply force
  apply force
  apply (rule ceiling-eq2)
  apply (simp, simp)
  apply (subst nat-add-distrib)
  apply auto
done

lemma natceiling-add [simp]: 0 <= x ==>
    natceiling (x + real a) = natceiling x + a
  apply (unfold natceiling-def)
  apply (subgoal-tac real a = real (int a))
  apply (erule ssubst)
  apply (simp del: real-of-int-of-nat-eq)
  apply (subst nat-add-distrib)
  apply (subgoal-tac 0 = ceiling 0)
  apply (erule ssubst)
  apply (erule ceiling-mono2)
  apply simp-all
done

lemma natceiling-add-number-of [simp]:
    ~ neg ((number-of n)::int) ==> 0 <= x ==>
      natceiling (x + number-of n) = natceiling x + number-of n
  apply (subst natceiling-add [THEN sym])
  apply simp-all
done

lemma natceiling-add-one: 0 <= x ==> natceiling(x + 1) = natceiling x + 1
  apply (subst natceiling-add [THEN sym])
  apply assumption
  apply simp
done

lemma natceiling-subtract [simp]: real a <= x ==>
    natceiling(x - real a) = natceiling x - a
  apply (unfold natceiling-def)
  apply (subgoal-tac real a = real (int a))
  apply (erule ssubst)
  apply (simp del: real-of-int-of-nat-eq)
  apply simp
```

```
done

lemma natfloor-div-nat: 1 <= x ==> y > 0 ==>
  natfloor (x / real y) = natfloor x div y
proof -
  assume 1 <= (x::real) and (y::nat) > 0
  have natfloor x = (natfloor x) div y * y + (natfloor x) mod y
    by simp
  then have a: real(natfloor x) = real ((natfloor x) div y) * real y +
    real((natfloor x) mod y)
    by (simp only: real-of-nat-add [THEN sym] real-of-nat-mult [THEN sym])
  have x = real(natfloor x) + (x - real(natfloor x))
    by simp
  then have x = real ((natfloor x) div y) * real y +
      real((natfloor x) mod y) + (x - real(natfloor x))
    by (simp add: a)
  then have x / real y = ... / real y
    by simp
  also have ... = real((natfloor x) div y) + real((natfloor x) mod y) /
    real y + (x - real(natfloor x)) / real y
    by (auto simp add: ring-simps add-divide-distrib
      diff-divide-distrib prems)
  finally have natfloor (x / real y) = natfloor(...) by simp
  also have ... = natfloor(real((natfloor x) mod y) /
    real y + (x - real(natfloor x)) / real y + real((natfloor x) div y))
    by (simp add: add-ac)
  also have ... = natfloor(real((natfloor x) mod y) /
    real y + (x - real(natfloor x)) / real y) + (natfloor x) div y
    apply (rule natfloor-add)
    apply (rule add-nonneg-nonneg)
    apply (rule divide-nonneg-pos)
    apply simp
    apply (simp add: prems)
    apply (rule divide-nonneg-pos)
    apply (simp add: compare-rls)
    apply (rule real-natfloor-le)
    apply (insert prems, auto)
    done
  also have natfloor(real((natfloor x) mod y) /
    real y + (x - real(natfloor x)) / real y) = 0
    apply (rule natfloor-eq)
    apply simp
    apply (rule add-nonneg-nonneg)
    apply (rule divide-nonneg-pos)
    apply force
    apply (force simp add: prems)
    apply (rule divide-nonneg-pos)
    apply (simp add: compare-rls)
    apply (rule real-natfloor-le)
```

```
  apply (auto simp add: prems)
  apply (insert prems, arith)
  apply (simp add: add-divide-distrib [THEN sym])
  apply (subgoal-tac real y = real y − 1 + 1)
  apply (erule ssubst)
  apply (rule add-le-less-mono)
  apply (simp add: compare-rls)
  apply (subgoal-tac real(natfloor x mod y) + 1 =
    real(natfloor x mod y + 1))
  apply (erule ssubst)
  apply (subst real-of-nat-le-iff)
  apply (subgoal-tac natfloor x mod y < y)
  apply arith
  apply (rule mod-less-divisor)
  apply auto
  apply (simp add: compare-rls)
  apply (subst add-commute)
  apply (rule real-natfloor-add-one-gt)
  done
 finally show ?thesis by simp
qed

end
```

8 ContNotDenum: Non-denumerability of the Continuum.

```
theory ContNotDenum
imports RComplete
begin
```

8.1 Abstract

The following document presents a proof that the Continuum is uncountable. It is formalised in the Isabelle/Isar theorem proving system.

Theorem: The Continuum $\mathbb{R}$ is not denumerable. In other words, there does not exist a function f:$\mathbb{N}\Rightarrow\mathbb{R}$ such that f is surjective.

Outline: An elegant informal proof of this result uses Cantor's Diagonalisation argument. The proof presented here is not this one. First we formalise some properties of closed intervals, then we prove the Nested Interval Property. This property relies on the completeness of the Real numbers and is the foundation for our argument. Informally it states that an intersection of countable closed intervals (where each successive interval is a subset of the last) is non-empty. We then assume a surjective function f:$\mathbb{N}\Rightarrow\mathbb{R}$ exists and find a real x such that x is not in the range of f by generating a sequence of

closed intervals then using the NIP.

8.2 Closed Intervals

This section formalises some properties of closed intervals.

8.2.1 Definition

definition
  closed-int :: *real* ⇒ *real* ⇒ *real set* **where**
  $closed\text{-}int\ x\ y = \{z.\ x \leq z \wedge z \leq y\}$

8.2.2 Properties

lemma *closed-int-subset*:
  assumes *xy*: $x1 \geq x0\ y1 \leq y0$
  shows $closed\text{-}int\ x1\ y1 \subseteq closed\text{-}int\ x0\ y0$
proof −
  {
    fix *x*::*real*
    assume $x \in closed\text{-}int\ x1\ y1$
    hence $x \geq x1 \wedge x \leq y1$ **by** (*simp add*: *closed-int-def*)
    with *xy* **have** $x \geq x0 \wedge x \leq y0$ **by** *auto*
    hence $x \in closed\text{-}int\ x0\ y0$ **by** (*simp add*: *closed-int-def*)
  }
  thus *?thesis* **by** *auto*
qed

lemma *closed-int-least*:
  assumes *a*: $a \leq b$
  shows $a \in closed\text{-}int\ a\ b \wedge (\forall x \in closed\text{-}int\ a\ b.\ a \leq x)$
proof
  from *a* **have** $a \in \{x.\ a \leq x \wedge x \leq b\}$ **by** *simp*
  thus $a \in closed\text{-}int\ a\ b$ **by** (*unfold closed-int-def*)
next
  have $\forall x \in \{x.\ a \leq x \wedge x \leq b\}.\ a \leq x$ **by** *simp*
  thus $\forall x \in closed\text{-}int\ a\ b.\ a \leq x$ **by** (*unfold closed-int-def*)
qed

lemma *closed-int-most*:
  assumes *a*: $a \leq b$
  shows $b \in closed\text{-}int\ a\ b \wedge (\forall x \in closed\text{-}int\ a\ b.\ x \leq b)$
proof
  from *a* **have** $b \in \{x.\ a \leq x \wedge x \leq b\}$ **by** *simp*
  thus $b \in closed\text{-}int\ a\ b$ **by** (*unfold closed-int-def*)
next
  have $\forall x \in \{x.\ a \leq x \wedge x \leq b\}.\ x \leq b$ **by** *simp*
  thus $\forall x \in closed\text{-}int\ a\ b.\ x \leq b$ **by** (*unfold closed-int-def*)
qed

```
lemma closed-not-empty:
  shows a ≤ b ⟹ ∃ x. x ∈ closed-int a b
  by (auto dest: closed-int-least)

lemma closed-mem:
  assumes a ≤ c and c ≤ b
  shows c ∈ closed-int a b
  using assms unfolding closed-int-def by auto

lemma closed-subset:
  assumes ac: a ≤ b  c ≤ d
  assumes closed: closed-int a b ⊆ closed-int c d
  shows b ≥ c
proof −
  from closed have ∀ x∈closed-int a b. x∈closed-int c d by auto
  hence ∀ x. a≤x ∧ x≤b ⟶ c≤x ∧ x≤d by (unfold closed-int-def, auto)
  with ac have c≤b ∧ b≤d by simp
  thus ?thesis by auto
qed
```

8.3 Nested Interval Property

```
theorem NIP:
  fixes f::nat ⇒ real set
  assumes subset: ∀ n. f (Suc n) ⊆ f n
  and closed: ∀ n. ∃ a b. f n = closed-int a b ∧ a ≤ b
  shows (⋂ n. f n) ≠ {}
proof −
  let ?g = λn. (SOME c. c∈(f n) ∧ (∀ x∈(f n). c ≤ x))
  have ne: ∀ n. ∃ x. x∈(f n)
  proof
    fix n
    from closed have ∃ a b. f n = closed-int a b ∧ a ≤ b by simp
    then obtain a and b where fn: f n = closed-int a b ∧ a ≤ b by auto
    hence a ≤ b ..
    with closed-not-empty have ∃ x. x∈closed-int a b by simp
    with fn show ∃ x. x∈(f n) by simp
  qed

  have gdef: ∀ n. (?g n)∈(f n) ∧ (∀ x∈(f n). (?g n)≤x)
  proof
    fix n
    from closed have ∃ a b. f n = closed-int a b ∧ a ≤ b ..
    then obtain a and b where ff: f n = closed-int a b and a ≤ b by auto
    hence a ≤ b by simp
   hence a∈closed-int a b ∧ (∀ x∈closed-int a b. a ≤ x) by (rule closed-int-least)
    with ff have a∈(f n) ∧ (∀ x∈(f n). a ≤ x) by simp
    hence ∃ c. c∈(f n) ∧ (∀ x∈(f n). c ≤ x) ..
```

```
  thus (?g n)∈(f n) ∧ (∀x∈(f n). (?g n)≤x) by (rule someI-ex)
qed

— A denotes the set of all left-most points of all the intervals ...
moreover obtain A where Adef: A = ?g ‘ ℕ by simp
ultimately have ∃x. x∈A
proof −
  have (0::nat) ∈ ℕ by simp
  moreover have ?g 0 = ?g 0 by simp
  ultimately have ?g 0 ∈ ?g ‘ ℕ by (rule  rev-image-eqI)
  with Adef have ?g 0 ∈ A by simp
  thus ?thesis ..
qed

— Now show that A is bounded above ...
moreover have ∃y. isUb (UNIV::real set) A y
proof −
  {
    fix n
    from ne have ex: ∃x. x∈(f n) ..
    from gdef have (?g n)∈(f n) ∧ (∀x∈(f n). (?g n)≤x) by simp
    moreover
    from closed have ∃a b. f n = closed-int a b ∧ a ≤ b ..
    then obtain a and b where f n = closed-int a b ∧ a ≤ b by auto
    hence b∈(f n) ∧ (∀x∈(f n). x ≤ b) using closed-int-most by blast
    ultimately have ∀x∈(f n). (?g n) ≤ b by simp
    with ex have (?g n) ≤ b by auto
    hence ∃b. (?g n) ≤ b by auto
  }
  hence aux: ∀n. ∃b. (?g n) ≤ b ..

  have fs: ∀n::nat. f n ⊆ f 0
  proof (rule allI, induct-tac n)
    show f 0 ⊆ f 0 by simp
  next
    fix n
    assume f n ⊆ f 0
    moreover from subset have f (Suc n) ⊆ f n ..
    ultimately show f (Suc n) ⊆ f 0 by simp
  qed
  have ∀n. (?g n)∈(f 0)
  proof
    fix n
    from gdef have (?g n)∈(f n) ∧ (∀x∈(f n). (?g n)≤x) by simp
    hence ?g n ∈ f n ..
    with fs show ?g n ∈ f 0 by auto
  qed
  moreover from closed
    obtain a and b where f 0 = closed-int a b and alb: a ≤ b by blast
```

ultimately have $\forall n.\ ?g\ n \in closed\text{-}int\ a\ b$ **by** *auto*
with *alb* **have** $\forall n.\ ?g\ n \leq b$ **using** *closed-int-most* **by** *blast*
with *Adef* **have** $\forall y{\in}A.\ y{\leq}b$ **by** *auto*
hence $A *{<=} b$ **by** (*unfold setle-def*)
moreover have $b \in (UNIV::real\ set)$ **by** *simp*
ultimately have $A *{<=} b \wedge b \in (UNIV::real\ set)$ **by** *simp*
hence *isUb* $(UNIV::real\ set)\ A\ b$ **by** (*unfold isUb-def*)
thus *?thesis* **by** *auto*
qed
— by the Axiom Of Completeness, A has a least upper bound ...
ultimately have $\exists t.\ isLub\ UNIV\ A\ t$ **by** (*rule reals-complete*)

— denote this least upper bound as t ...
then obtain t **where** *tdef*: $isLub\ UNIV\ A\ t$ **..**

— and finally show that this least upper bound is in all the intervals...
have $\forall n.\ t \in f\ n$
proof
fix $n::nat$
from *closed* **obtain** a **and** b **where**
int: $f\ n = closed\text{-}int\ a\ b$ **and** *alb*: $a \leq b$ **by** *blast*

have $t \geq a$
proof −
have $a \in A$
proof −

from *alb int* **have** *ain*: $a{\in}f\ n \wedge (\forall x{\in}f\ n.\ a \leq x)$
using *closed-int-least* **by** *blast*
moreover have $\forall e.\ e{\in}f\ n \wedge (\forall x{\in}f\ n.\ e \leq x) \longrightarrow e = a$
proof *clarsimp*
fix e
assume *ein*: $e \in f\ n$ **and** *lt*: $\forall x{\in}f\ n.\ e \leq x$
from *lt ain* **have** *aux*: $\forall x{\in}f\ n.\ a \leq x \wedge e \leq x$ **by** *auto*

from *ein aux* **have** $a \leq e \wedge e \leq e$ **by** *auto*
moreover from *ain aux* **have** $a \leq a \wedge e \leq a$ **by** *auto*
ultimately show $e = a$ **by** *simp*
qed
hence $\bigwedge e.\ e{\in}f\ n \wedge (\forall x{\in}f\ n.\ e \leq x) \Longrightarrow e = a$ **by** *simp*
ultimately have $(?g\ n) = a$ **by** (*rule some-equality*)
moreover
{
have $n = of\text{-}nat\ n$ **by** *simp*
moreover have $of\text{-}nat\ n \in \mathbb{N}$ **by** *simp*
ultimately have $n \in \mathbb{N}$
apply −
apply (*subst*(*asm*) *eq-sym-conv*)
apply (*erule subst*)

```
        .
      }
      with Adef have (?g n) ∈ A by auto
      ultimately show ?thesis by simp
    qed
    with tdef show a ≤ t by (rule isLubD2)
  qed
  moreover have t ≤ b
  proof −
    have isUb UNIV A b
    proof −
      {
        from alb int have
          ain: b∈f n ∧ (∀x∈f n. x ≤ b) using closed-int-most by blast

        have subsetd: ∀m. ∀n. f (n + m) ⊆ f n
        proof (rule allI, induct-tac m)
          show ∀n. f (n + 0) ⊆ f n by simp
        next
          fix m n
          assume pp: ∀p. f (p + n) ⊆ f p
          {
            fix p
            from pp have f (p + n) ⊆ f p by simp
            moreover from subset have f (Suc (p + n)) ⊆ f (p + n) by auto
            hence f (p + (Suc n)) ⊆ f (p + n) by simp
            ultimately have f (p + (Suc n)) ⊆ f p by simp
          }
          thus ∀p. f (p + Suc n) ⊆ f p ..
        qed
        have subsetm: ∀α β. α ≥ β ⟶ (f α) ⊆ (f β)
        proof ((rule allI)+, rule impI)
          fix α::nat and β::nat
          assume β ≤ α
          hence ∃k. α = β + k by (simp only: le-iff-add)
          then obtain k where α = β + k ..
          moreover
          from subsetd have f (β + k) ⊆ f β by simp
          ultimately show f α ⊆ f β by auto
        qed

        fix m
        {
          assume m ≥ n
          with subsetm have f m ⊆ f n by simp
          with ain have ∀x∈f m. x ≤ b by auto
          moreover
          from gdef have ?g m ∈ f m ∧ (∀x∈f m. ?g m ≤ x) by simp
          ultimately have ?g m ≤ b by auto
```

```
        }
        moreover
        {
          assume ¬(m ≥ n)
          hence m < n by simp
          with subsetm have sub: (f n) ⊆ (f m) by simp
          from closed obtain ma and mb where
            f m = closed-int ma mb ∧ ma ≤ mb by blast
          hence one: ma ≤ mb and fm: f m = closed-int ma mb by auto
          from one alb sub fm int have ma ≤ b using closed-subset by blast
          moreover have (?g m) = ma
          proof -
            from gdef have ?g m ∈ f m ∧ (∀ x∈f m. ?g m ≤ x) ..
            moreover from one have
              ma ∈ closed-int ma mb ∧ (∀ x∈closed-int ma mb. ma ≤ x)
              by (rule closed-int-least)
            with fm have ma∈f m ∧ (∀ x∈f m. ma ≤ x) by simp
            ultimately have ma ≤ ?g m ∧ ?g m ≤ ma by auto
            thus ?g m = ma by auto
          qed
          ultimately have ?g m ≤ b by simp
        }
        ultimately have ?g m ≤ b by (rule case-split)
      }
      with Adef have ∀ y∈A. y≤b by auto
      hence A *<= b by (unfold setle-def)
      moreover have b ∈ (UNIV::real set) by simp
      ultimately have A *<= b ∧ b ∈ (UNIV::real set) by simp
      thus isUb (UNIV::real set) A b by (unfold isUb-def)
    qed
    with tdef show t ≤ b by (rule isLub-le-isUb)
  qed
  ultimately have t ∈ closed-int a b by (rule closed-mem)
  with int show t ∈ f n by simp
 qed
 hence t ∈ (⋂ n. f n) by auto
 thus ?thesis by auto
qed
```

8.4 Generating the intervals

8.4.1 Existence of non-singleton closed intervals

This lemma asserts that given any non-singleton closed interval (a,b) and any element c, there exists a closed interval that is a subset of (a,b) and that does not contain c and is a non-singleton itself.

```
lemma closed-subset-ex:
  fixes c::real
  assumes alb: a < b
```

```
 shows
   ∃ka kb. ka < kb ∧ closed-int ka kb ⊆ closed-int a b ∧ c ∉ (closed-int ka kb)
proof −
 {
   assume clb: c < b
   {
     assume cla: c < a
     from alb cla clb have c ∉ closed-int a b by (unfold closed-int-def, auto)
     with alb have
       a < b ∧ closed-int a b ⊆ closed-int a b ∧ c ∉ closed-int a b
       by auto
     hence
       ∃ka kb. ka < kb ∧ closed-int ka kb ⊆ closed-int a b ∧ c ∉ (closed-int ka kb)
       by auto
   }
   moreover
   {
     assume ncla: ¬(c < a)
     with clb have cdef: a ≤ c ∧ c < b by simp
     obtain ka where kadef: ka = (c + b)/2 by blast

     from kadef clb have kalb: ka < b by auto
     moreover from kadef cdef have kagc: ka > c by simp
     ultimately have c∉(closed-int ka b) by (unfold closed-int-def, auto)
     moreover from cdef kagc have ka ≥ a by simp
     hence closed-int ka b ⊆ closed-int a b by (unfold closed-int-def, auto)
     ultimately have
       ka < b  ∧ closed-int ka b ⊆ closed-int a b ∧ c ∉ closed-int ka b
       using kalb by auto
     hence
       ∃ka kb. ka < kb ∧ closed-int ka kb ⊆ closed-int a b ∧ c ∉ (closed-int ka kb)
       by auto

   }
   ultimately have
     ∃ka kb. ka < kb ∧ closed-int ka kb ⊆ closed-int a b ∧ c ∉ (closed-int ka kb)
     by (rule case-split)
 }
 moreover
 {
   assume ¬ (c < b)
   hence cgeb: c ≥ b by simp

   obtain kb where kbdef: kb = (a + b)/2 by blast
   with alb have kblb: kb < b by auto
   with kbdef cgeb have a < kb ∧ kb < c by auto
   moreover hence c ∉ (closed-int a kb) by (unfold closed-int-def, auto)
   moreover from kblb have
     closed-int a kb ⊆ closed-int a b by (unfold closed-int-def, auto)
```

ultimately have
$a < kb \wedge$ *closed-int a kb* $\subseteq$ *closed-int a b* $\wedge$ $c \notin$ *closed-int a kb*
by *simp*
hence
$\exists$ *ka kb*. *ka* $<$ *kb* $\wedge$ *closed-int ka kb* $\subseteq$ *closed-int a b* $\wedge$ $c \notin$ (*closed-int ka kb*)
by *auto*
}
ultimately show *?thesis* **by** (*rule case-split*)
qed

8.5 newInt: Interval generation

Given a function f:$\mathbb{N}\Rightarrow\mathbb{R}$, newInt (Suc n) f returns a closed interval such that *newInt* (*Suc n*) *f* $\subseteq$ *newInt n f* and does not contain *f* (*Suc n*). With the base case defined such that (*f 0*)$\notin$*newInt 0 f*.

8.5.1 Definition

consts *newInt* :: *nat* $\Rightarrow$ (*nat* $\Rightarrow$ *real*) $\Rightarrow$ (*real set*)
primrec
newInt 0 f = *closed-int* (*f 0* + *1*) (*f 0* + *2*)
newInt (*Suc n*) *f* =
(*SOME e*. ($\exists$ *e1 e2*.
e1 $<$ *e2* $\wedge$
e = *closed-int e1 e2* $\wedge$
e $\subseteq$ (*newInt n f*) $\wedge$
(*f* (*Suc n*)) $\notin$ *e*)
)

8.5.2 Properties

We now show that every application of newInt returns an appropriate interval.

lemma *newInt-ex*:
$\exists$ *a b*. *a* $<$ *b* $\wedge$
newInt (*Suc n*) *f* = *closed-int a b* $\wedge$
newInt (*Suc n*) *f* $\subseteq$ *newInt n f* $\wedge$
f (*Suc n*) $\notin$ *newInt* (*Suc n*) *f*
proof (*induct n*)
case *0*

let *?e* = *SOME e*. $\exists$ *e1 e2*.
e1 $<$ *e2* $\wedge$
e = *closed-int e1 e2* $\wedge$
e $\subseteq$ *closed-int* (*f 0* + *1*) (*f 0* + *2*) $\wedge$
f (*Suc 0*) $\notin$ *e*

have *newInt* (*Suc 0*) *f* = *?e* **by** *auto*

```
  moreover
  have f 0 + 1 < f 0 + 2 by simp
  with closed-subset-ex have
    ∃ka kb. ka < kb ∧ closed-int ka kb ⊆ closed-int (f 0 + 1) (f 0 + 2) ∧
     f (Suc 0) ∉ (closed-int ka kb) .
  hence
    ∃e. ∃ka kb. ka < kb ∧ e = closed-int ka kb ∧
     e ⊆ closed-int (f 0 + 1) (f 0 + 2) ∧ f (Suc 0) ∉ e by simp
  hence
    ∃ka kb. ka < kb ∧ ?e = closed-int ka kb ∧
     ?e ⊆ closed-int (f 0 + 1) (f 0 + 2) ∧ f (Suc 0) ∉ ?e
    by (rule someI-ex)
  ultimately have ∃e1 e2. e1 < e2 ∧
   newInt (Suc 0) f = closed-int e1 e2 ∧
   newInt (Suc 0) f ⊆ closed-int (f 0 + 1) (f 0 + 2) ∧
   f (Suc 0) ∉ newInt (Suc 0) f by simp
  thus
    ∃a b. a < b ∧ newInt (Suc 0) f = closed-int a b ∧
     newInt (Suc 0) f ⊆ newInt 0 f ∧ f (Suc 0) ∉ newInt (Suc 0) f
    by simp
next
  case (Suc n)
  hence ∃a b.
   a < b ∧
   newInt (Suc n) f = closed-int a b ∧
   newInt (Suc n) f ⊆ newInt n f ∧
   f (Suc n) ∉ newInt (Suc n) f by simp
  then obtain a and b where ab: a < b ∧
   newInt (Suc n) f = closed-int a b ∧
   newInt (Suc n) f ⊆ newInt n f ∧
   f (Suc n) ∉ newInt (Suc n) f by auto
  hence cab: closed-int a b = newInt (Suc n) f by simp

  let ?e = SOME e. ∃e1 e2.
    e1 < e2 ∧
    e = closed-int e1 e2 ∧
    e ⊆ closed-int a b ∧
    f (Suc (Suc n)) ∉ e
  from cab have ni: newInt (Suc (Suc n)) f = ?e by auto

  from ab have a < b by simp
  with closed-subset-ex have
    ∃ka kb. ka < kb ∧ closed-int ka kb ⊆ closed-int a b ∧
     f (Suc (Suc n)) ∉ closed-int ka kb .
  hence
    ∃e. ∃ka kb. ka < kb ∧ e = closed-int ka kb ∧
     closed-int ka kb ⊆ closed-int a b ∧ f (Suc (Suc n)) ∉ closed-int ka kb
    by simp
  hence
```

```
    ∃ e.  ∃ ka kb. ka < kb ∧ e = closed-int ka kb ∧
     e ⊆ closed-int a b ∧ f (Suc (Suc n)) ∉ e by simp
  hence
    ∃ ka kb. ka < kb ∧ ?e = closed-int ka kb ∧
     ?e ⊆ closed-int a b ∧ f (Suc (Suc n)) ∉ ?e by (rule someI-ex)
  with ab ni show
    ∃ ka kb. ka < kb ∧
     newInt (Suc (Suc n)) f = closed-int ka kb ∧
     newInt (Suc (Suc n)) f ⊆ newInt (Suc n) f ∧
     f (Suc (Suc n)) ∉ newInt (Suc (Suc n)) f by auto
qed

lemma newInt-subset:
  newInt (Suc n) f ⊆ newInt n f
  using newInt-ex by auto
```

Another fundamental property is that no element in the range of f is in the intersection of all closed intervals generated by newInt.

```
lemma newInt-inter:
  ∀ n. f n ∉ (⋂ n. newInt n f)
proof
  fix n::nat
  {
    assume n0: n = 0
    moreover have newInt 0 f = closed-int (f 0 + 1) (f 0 + 2) by simp
    ultimately have f n ∉ newInt n f by (unfold closed-int-def, simp)
  }
  moreover
  {
    assume ¬ n = 0
    hence n > 0 by simp
    then obtain m where ndef: n = Suc m by (auto simp add: gr0-conv-Suc)

    from newInt-ex have
      ∃ a b. a < b ∧ (newInt (Suc m) f) = closed-int a b ∧
       newInt (Suc m) f ⊆ newInt m f ∧ f (Suc m) ∉ newInt (Suc m) f .
    then have f (Suc m) ∉ newInt (Suc m) f by auto
    with ndef have f n ∉ newInt n f by simp
  }
  ultimately have f n ∉ newInt n f by (rule case-split)
  thus f n ∉ (⋂ n. newInt n f) by auto
qed


lemma newInt-notempty:
  (⋂ n. newInt n f) ≠ {}
proof −
  let ?g = λn. newInt n f
  have ∀ n. ?g (Suc n) ⊆ ?g n
```

```
  proof
    fix n
    show ?g (Suc n) ⊆ ?g n by (rule newInt-subset)
  qed
  moreover have ∀ n. ∃ a b. ?g n = closed-int a b ∧ a ≤ b
  proof
    fix n::nat
    {
      assume n = 0
      then have
        ?g n = closed-int (f 0 + 1) (f 0 + 2) ∧ (f 0 + 1 ≤ f 0 + 2)
        by simp
      hence ∃ a b. ?g n = closed-int a b ∧ a ≤ b by blast
    }
    moreover
    {
      assume ¬ n = 0
      then have n > 0 by simp
      then obtain m where nd: n = Suc m by (auto simp add: gr0-conv-Suc)

      have
        ∃ a b. a < b ∧ (newInt (Suc m) f) = closed-int a b ∧
        (newInt (Suc m) f) ⊆ (newInt m f) ∧ (f (Suc m)) ∉ (newInt (Suc m) f)
        by (rule newInt-ex)
      then obtain a and b where
        a < b ∧ (newInt (Suc m) f) = closed-int a b by auto
      with nd have ?g n = closed-int a b ∧ a ≤ b by auto
      hence ∃ a b. ?g n = closed-int a b ∧ a ≤ b by blast
    }
    ultimately show ∃ a b. ?g n = closed-int a b ∧ a ≤ b by (rule case-split)
  qed
  ultimately show ?thesis by (rule NIP)
qed
```

8.6 Final Theorem

```
theorem real-non-denum:
  shows ¬ (∃ f::nat⇒real. surj f)
proof — by contradiction
  assume ∃ f::nat⇒real. surj f
  then obtain f::nat⇒real where surj f by auto
  hence rangeF: range f = UNIV by (rule surj-range)
  — We now produce a real number x that is not in the range of f, using the
properties of newInt.
  have ∃ x. x ∈ (⋂ n. newInt n f) using newInt-notempty by blast
  moreover have ∀ n. f n ∉ (⋂ n. newInt n f) by (rule newInt-inter)
  ultimately obtain x where x ∈ (⋂ n. newInt n f) and ∀ n. f n ≠ x by blast
  moreover from rangeF have x ∈ range f by simp
  ultimately show False by blast
```

qed

end

9 RealPow: Natural powers theory

theory *RealPow*
imports *RealDef*
begin

declare *abs-mult-self* [*simp*]

instance *real* :: *power* **..**

primrec (*realpow*)
  realpow-0: $r \ \hat{} \ 0 \quad = 1$
  realpow-Suc: $r \ \hat{} \ (Suc\ n) = (r{::}real) * (r \ \hat{} \ n)$

instance *real* :: *recpower*
proof
  fix z :: *real*
  fix n :: *nat*
  show $z\hat{}0 = 1$ **by** *simp*
  show $z\hat{}(Suc\ n) = z * (z\hat{}n)$ **by** *simp*
qed

lemma *two-realpow-ge-one* [*simp*]: $(1{::}real) \le 2 \ \hat{} \ n$
by (*rule power-increasing*[*of 0 n 2::real*, *simplified*])

lemma *two-realpow-gt* [*simp*]: $real\ (n{::}nat) < 2 \ \hat{} \ n$
apply (*induct n*)
apply (*auto simp add*: *real-of-nat-Suc*)
apply (*subst mult-2*)
apply (*rule add-less-le-mono*)
apply (*auto simp add*: *two-realpow-ge-one*)
done

lemma *realpow-Suc-le-self*: $[|\ 0 \le r;\ r \le (1{::}real)\ |] ==> r \ \hat{} \ Suc\ n \le r$
by (*insert power-decreasing* [*of 1 Suc n r*], *simp*)

lemma *realpow-minus-mult* [*rule-format*]:
  $0 < n \ {-}{-}{>} \ (x{::}real) \ \hat{} \ (n - 1) * x = x \ \hat{} \ n$
apply (*simp split add*: *nat-diff-split*)
done

lemma *realpow-two-mult-inverse* [*simp*]:

$r \neq 0 ==> r * \mathit{inverse}\ r\ \hat{}\mathit{Suc}\ (\mathit{Suc}\ 0) = \mathit{inverse}\ (r{::}\mathit{real})$
by (*simp add*: *real-mult-assoc* [*symmetric*])

lemma *realpow-two-minus* [*simp*]: $(-x)\ \hat{}\mathit{Suc}\ (\mathit{Suc}\ 0) = (x{::}\mathit{real})\ \hat{}\mathit{Suc}\ (\mathit{Suc}\ 0)$
by *simp*

lemma *realpow-two-diff*:
$(x{::}\mathit{real})\ \hat{}\mathit{Suc}\ (\mathit{Suc}\ 0) - y\ \hat{}\mathit{Suc}\ (\mathit{Suc}\ 0) = (x - y) * (x + y)$
apply (*unfold real-diff-def*)
apply (*simp add*: *ring-simps*)
done

lemma *realpow-two-disj*:
$((x{::}\mathit{real})\ \hat{}\mathit{Suc}\ (\mathit{Suc}\ 0) = y\ \hat{}\mathit{Suc}\ (\mathit{Suc}\ 0)) = (x = y \mid x = -y)$
apply (*cut-tac* $x = x$ **and** $y = y$ **in** *realpow-two-diff*)
apply (*auto simp del*: *realpow-Suc*)
done

lemma *realpow-real-of-nat*: $\mathit{real}\ (m{::}\mathit{nat})\ \hat{}\ n = \mathit{real}\ (m\ \hat{}\ n)$
apply (*induct n*)
apply (*auto simp add*: *real-of-nat-one real-of-nat-mult*)
done

lemma *realpow-real-of-nat-two-pos* [*simp*] : $0 < \mathit{real}\ (\mathit{Suc}\ (\mathit{Suc}\ 0)\ \hat{}\ n)$
apply (*induct n*)
apply (*auto simp add*: *real-of-nat-mult zero-less-mult-iff*)
done

lemma *realpow-increasing*:
$[|(0{::}\mathit{real}) \leq x;\ 0 \leq y;\ x\ \hat{}\ \mathit{Suc}\ n \leq y\ \hat{}\ \mathit{Suc}\ n|] ==> x \leq y$
by (*rule power-le-imp-le-base*)

9.1 Literal Arithmetic Involving Powers, Type *real*

lemma *real-of-int-power*: $\mathit{real}\ (x{::}\mathit{int})\ \hat{}\ n = \mathit{real}\ (x\ \hat{}\ n)$
apply (*induct n*)
apply (*simp-all add*: *nat-mult-distrib*)
done
declare *real-of-int-power* [*symmetric*, *simp*]

lemma *power-real-number-of*:
$(\mathit{number\text{-}of}\ v :: \mathit{real})\ \hat{}\ n = \mathit{real}\ ((\mathit{number\text{-}of}\ v :: \mathit{int})\ \hat{}\ n)$
by (*simp only*: *real-number-of* [*symmetric*] *real-of-int-power*)

declare *power-real-number-of* [*of - number-of w*, *standard*, *simp*]

9.2 Properties of Squares

lemma *sum-squares-ge-zero*:

fixes $x\ y$ :: $'a$::*ordered-ring-strict*
shows $0 \leq x * x + y * y$
by (*intro add-nonneg-nonneg zero-le-square*)

lemma *not-sum-squares-lt-zero*:
fixes $x\ y$ :: $'a$::*ordered-ring-strict*
shows $\neg\ x * x + y * y < 0$
by (*simp add*: *linorder-not-less sum-squares-ge-zero*)

lemma *sum-nonneg-eq-zero-iff*:
fixes $x\ y$ :: $'a$::*pordered-ab-group-add*
assumes x: $0 \leq x$ **and** y: $0 \leq y$
shows $(x + y = 0) = (x = 0 \wedge y = 0)$
proof (*auto*)
from y **have** $x + 0 \leq x + y$ **by** (*rule add-left-mono*)
also assume $x + y = 0$
finally have $x \leq 0$ **by** *simp*
thus $x = 0$ **using** x **by** (*rule order-antisym*)
next
from x **have** $0 + y \leq x + y$ **by** (*rule add-right-mono*)
also assume $x + y = 0$
finally have $y \leq 0$ **by** *simp*
thus $y = 0$ **using** y **by** (*rule order-antisym*)
qed

lemma *sum-squares-eq-zero-iff*:
fixes $x\ y$ :: $'a$::*ordered-ring-strict*
shows $(x * x + y * y = 0) = (x = 0 \wedge y = 0)$
by (*simp add*: *sum-nonneg-eq-zero-iff*)

lemma *sum-squares-le-zero-iff*:
fixes $x\ y$ :: $'a$::*ordered-ring-strict*
shows $(x * x + y * y \leq 0) = (x = 0 \wedge y = 0)$
by (*simp add*: *order-le-less not-sum-squares-lt-zero sum-squares-eq-zero-iff*)

lemma *sum-squares-gt-zero-iff*:
fixes $x\ y$ :: $'a$::*ordered-ring-strict*
shows $(0 < x * x + y * y) = (x \neq 0 \vee y \neq 0)$
by (*simp add*: *order-less-le sum-squares-ge-zero sum-squares-eq-zero-iff*)

lemma *sum-power2-ge-zero*:
fixes $x\ y$ :: $'a$::{*ordered-idom*,*recpower*}
shows $0 \leq x^2 + y^2$
unfolding *power2-eq-square* **by** (*rule sum-squares-ge-zero*)

lemma *not-sum-power2-lt-zero*:
fixes $x\ y$ :: $'a$::{*ordered-idom*,*recpower*}
shows $\neg\ x^2 + y^2 < 0$
unfolding *power2-eq-square* **by** (*rule not-sum-squares-lt-zero*)

lemma *sum-power2-eq-zero-iff*:
  fixes $x\ y$ :: $'a$::{*ordered-idom*,*recpower*}
  shows $(x^2 + y^2 = 0) = (x = 0 \wedge y = 0)$
unfolding *power2-eq-square* **by** (*rule sum-squares-eq-zero-iff*)

lemma *sum-power2-le-zero-iff*:
  fixes $x\ y$ :: $'a$::{*ordered-idom*,*recpower*}
  shows $(x^2 + y^2 \leq 0) = (x = 0 \wedge y = 0)$
unfolding *power2-eq-square* **by** (*rule sum-squares-le-zero-iff*)

lemma *sum-power2-gt-zero-iff*:
  fixes $x\ y$ :: $'a$::{*ordered-idom*,*recpower*}
  shows $(0 < x^2 + y^2) = (x \neq 0 \vee y \neq 0)$
unfolding *power2-eq-square* **by** (*rule sum-squares-gt-zero-iff*)

9.3 Squares of Reals

lemma *real-two-squares-add-zero-iff* [*simp*]:
  $(x * x + y * y = 0) = ((x::real) = 0 \wedge y = 0)$
by (*rule sum-squares-eq-zero-iff*)

lemma *real-sum-squares-cancel*: $x * x + y * y = 0 ==> x = (0::real)$
by *simp*

lemma *real-sum-squares-cancel2*: $x * x + y * y = 0 ==> y = (0::real)$
by *simp*

lemma *real-mult-self-sum-ge-zero*: $(0::real) \leq x*x + y*y$
by (*rule sum-squares-ge-zero*)

lemma *real-sum-squares-cancel-a*: $x * x = -(y * y) ==> x = (0::real)$ & $y=0$
by (*simp add*: *real-add-eq-0-iff* [*symmetric*])

lemma *real-squared-diff-one-factored*: $x*x - (1::real) = (x + 1)*(x - 1)$
by (*simp add*: *left-distrib right-diff-distrib*)

lemma *real-mult-is-one* [*simp*]: $(x*x = (1::real)) = (x = 1 \mid x = - 1)$
apply *auto*
apply (*drule right-minus-eq* [*THEN iffD2*])
apply (*auto simp add*: *real-squared-diff-one-factored*)
done

lemma *real-sum-squares-not-zero*: $x \sim= 0 ==> x * x + y * y \sim= (0::real)$
by *simp*

lemma *real-sum-squares-not-zero2*: $y \sim= 0 ==> x * x + y * y \sim= (0::real)$
by *simp*

lemma *realpow-two-sum-zero-iff* [*simp*]:
  $(x \hat{} 2 + y \hat{} 2 = (0::real)) = (x = 0 \ \& \ y = 0)$
by (*rule sum-power2-eq-zero-iff*)

lemma *realpow-two-le-add-order* [*simp*]: $(0::real) \le u \hat{} 2 + v \hat{} 2$
by (*rule sum-power2-ge-zero*)

lemma *realpow-two-le-add-order2* [*simp*]: $(0::real) \le u \hat{} 2 + v \hat{} 2 + w \hat{} 2$
by (*intro add-nonneg-nonneg zero-le-power2*)

lemma *real-sum-square-gt-zero*: $x \ \tilde{}{=}\ 0 ==> (0::real) < x * x + y * y$
by (*simp add*: *sum-squares-gt-zero-iff*)

lemma *real-sum-square-gt-zero2*: $y \ \tilde{}{=}\ 0 ==> (0::real) < x * x + y * y$
by (*simp add*: *sum-squares-gt-zero-iff*)

lemma *real-minus-mult-self-le* [*simp*]: $-(u * u) \le (x * (x::real))$
by (*rule-tac* $j = 0$ **in** *real-le-trans*, *auto*)

lemma *realpow-square-minus-le* [*simp*]: $-(u \hat{} 2) \le (x::real) \hat{} 2$
by (*auto simp add*: *power2-eq-square*)

lemma *real-sq-order*:
 fixes $x::real$
 assumes *xgt0*: $0 \le x$ **and** *ygt0*: $0 \le y$ **and** *sq*: $x\hat{}2 \le y\hat{}2$
 shows $x \le y$
proof −
 from *sq* **have** $x \hat{} Suc\ (Suc\ 0) \le y \hat{} Suc\ (Suc\ 0)$
  by (*simp only*: *numeral-2-eq-2*)
 thus $x \le y$ **using** *ygt0*
  by (*rule power-le-imp-le-base*)
qed

9.4 Various Other Theorems

lemma *real-le-add-half-cancel*: $(x + y/2 \le (y::real)) = (x \le y\ /2)$
by *auto*

lemma *real-minus-half-eq* [*simp*]: $(x::real) - x/2 = x/2$
by *auto*

lemma *real-mult-inverse-cancel*:
  $[|(0::real) < x;\ 0 < x1;\ x1 * y < x * u\ |]$
  $==> inverse\ x * y < inverse\ x1 * u$
apply (*rule-tac c=x* **in** *mult-less-imp-less-left*)
apply (*auto simp add*: *real-mult-assoc* [*symmetric*])
apply (*simp* (*no-asm*) *add*: *mult-ac*)
apply (*rule-tac c=x1* **in** *mult-less-imp-less-right*)

apply (*auto simp add*: *mult-ac*)
done

lemma *real-mult-inverse-cancel2*:
$[|(0::real) < x;0 < x1;\ x1 * y < x * u\ |] ==> y * inverse\ x < u * inverse\ x1$
apply (*auto dest*: *real-mult-inverse-cancel simp add*: *mult-ac*)
done

lemma *inverse-real-of-nat-gt-zero* [*simp*]: $0 < inverse\ (real\ (Suc\ n))$
by *simp*

lemma *inverse-real-of-nat-ge-zero* [*simp*]: $0 \leq inverse\ (real\ (Suc\ n))$
by *simp*

lemma *realpow-num-eq-if*: $(m::real)\ \hat{}\ n = (if\ n{=}0\ then\ 1\ else\ m * m\ \hat{}\ (n - 1))$
by (*case-tac n*, *auto*)

end

10 RealVector: Vector Spaces and Algebras over the Reals

theory *RealVector*
imports *RealPow*
begin

10.1 Locale for additive functions

locale *additive* =
fixes $f :: {'a}::ab\text{-}group\text{-}add \Rightarrow {'b}::ab\text{-}group\text{-}add$
assumes *add*: $f\ (x + y) = f\ x + f\ y$

lemma (**in** *additive*) *zero*: $f\ 0 = 0$
proof −
have $f\ 0 = f\ (0 + 0)$ **by** *simp*
also have $\ldots = f\ 0 + f\ 0$ **by** (*rule add*)
finally show $f\ 0 = 0$ **by** *simp*
qed

lemma (**in** *additive*) *minus*: $f\ (- x) = - f\ x$
proof −
have $f\ (- x) + f\ x = f\ (- x + x)$ **by** (*rule add* [*symmetric*])
also have $\ldots = - f\ x + f\ x$ **by** (*simp add*: *zero*)
finally show $f\ (- x) = - f\ x$ **by** (*rule add-right-imp-eq*)
qed

lemma (**in** *additive*) *diff*: $f\ (x - y) = f\ x - f\ y$

by (*simp add*: *diff-def add minus*)

lemma (**in** *additive*) *setsum*: f (*setsum* g A) = ($\sum x{\in}A.\ f\ (g\ x)$)
apply (*cases finite A*)
apply (*induct set*: *finite*)
apply (*simp add*: *zero*)
apply (*simp add*: *add*)
apply (*simp add*: *zero*)
done

10.2 Real vector spaces

class *scaleR* = *type* +
  fixes *scaleR* :: *real* ⇒ *'a* ⇒ *'a* (**infixr** $*_R$ *75*)
begin

abbreviation
  divideR :: *'a* ⇒ *real* ⇒ *'a* (**infixl** $'/_R$ *70*)
where
  $x\ /_R\ r$ == *scaleR* (*inverse r*) *x*

end

instance *real* :: *scaleR*
  real-scaleR-def [*simp*]: *scaleR a x* ≡ *a* ∗ *x* **..**

class *real-vector* = *scaleR* + *ab-group-add* +
  assumes *scaleR-right-distrib*: *scaleR a* (*x* + *y*) = *scaleR a x* + *scaleR a y*
  and *scaleR-left-distrib*: *scaleR* (*a* + *b*) *x* = *scaleR a x* + *scaleR b x*
  and *scaleR-scaleR* [*simp*]: *scaleR a* (*scaleR b x*) = *scaleR* (*a* ∗ *b*) *x*
  and *scaleR-one* [*simp*]: *scaleR 1 x* = *x*

class *real-algebra* = *real-vector* + *ring* +
  assumes *mult-scaleR-left* [*simp*]: *scaleR a x* ∗ *y* = *scaleR a* (*x* ∗ *y*)
  and *mult-scaleR-right* [*simp*]: *x* ∗ *scaleR a y* = *scaleR a* (*x* ∗ *y*)

class *real-algebra-1* = *real-algebra* + *ring-1*

class *real-div-algebra* = *real-algebra-1* + *division-ring*

class *real-field* = *real-div-algebra* + *field*

instance *real* :: *real-field*
apply (*intro-classes*, *unfold real-scaleR-def*)
apply (*rule right-distrib*)
apply (*rule left-distrib*)
apply (*rule mult-assoc* [*symmetric*])
apply (*rule mult-1-left*)
apply (*rule mult-assoc*)

apply (*rule mult-left-commute*)
done

lemma *scaleR-left-commute*:
  fixes x :: $'a$::*real-vector*
  shows $scaleR\ a\ (scaleR\ b\ x) = scaleR\ b\ (scaleR\ a\ x)$
by (*simp add*: *mult-commute*)

interpretation *scaleR-left*: *additive* $[(\lambda a.\ scaleR\ a\ x{::}'a{::}real\text{-}vector)]$
by *unfold-locales* (*rule scaleR-left-distrib*)

interpretation *scaleR-right*: *additive* $[(\lambda x.\ scaleR\ a\ x{::}'a{::}real\text{-}vector)]$
by *unfold-locales* (*rule scaleR-right-distrib*)

lemmas *scaleR-zero-left* [*simp*] = *scaleR-left.zero*

lemmas *scaleR-zero-right* [*simp*] = *scaleR-right.zero*

lemmas *scaleR-minus-left* [*simp*] = *scaleR-left.minus*

lemmas *scaleR-minus-right* [*simp*] = *scaleR-right.minus*

lemmas *scaleR-left-diff-distrib* = *scaleR-left.diff*

lemmas *scaleR-right-diff-distrib* = *scaleR-right.diff*

lemma *scaleR-eq-0-iff* [*simp*]:
  fixes x :: $'a$::*real-vector*
  shows $(scaleR\ a\ x = 0) = (a = 0 \lor x = 0)$
proof *cases*
  assume $a = 0$ **thus** *?thesis* **by** *simp*
next
  assume *anz* [*simp*]: $a \neq 0$
  { assume $scaleR\ a\ x = 0$
    hence $scaleR\ (inverse\ a)\ (scaleR\ a\ x) = 0$ **by** *simp*
    hence $x = 0$ **by** *simp* **}**
  thus *?thesis* **by** *force*
qed

lemma *scaleR-left-imp-eq*:
  fixes $x\ y$ :: $'a$::*real-vector*
  shows $[\![a \neq 0;\ scaleR\ a\ x = scaleR\ a\ y]\!] \Longrightarrow x = y$
proof −
  assume *nonzero*: $a \neq 0$
  assume $scaleR\ a\ x = scaleR\ a\ y$
  hence $scaleR\ a\ (x - y) = 0$
    by (*simp add*: *scaleR-right-diff-distrib*)
  hence $x - y = 0$ **by** (*simp add*: *nonzero*)
  thus $x = y$ **by** *simp*

qed

lemma *scaleR-right-imp-eq*:
fixes $x\ y :: 'a::real\text{-}vector$
shows $[\![x \neq 0;\ scaleR\ a\ x = scaleR\ b\ x]\!] \Longrightarrow a = b$
proof −
assume *nonzero*: $x \neq 0$
assume $scaleR\ a\ x = scaleR\ b\ x$
hence $scaleR\ (a - b)\ x = 0$
by (*simp add*: *scaleR-left-diff-distrib*)
hence $a - b = 0$ **by** (*simp add*: *nonzero*)
thus $a = b$ **by** *simp*
qed

lemma *scaleR-cancel-left*:
fixes $x\ y :: 'a::real\text{-}vector$
shows $(scaleR\ a\ x = scaleR\ a\ y) = (x = y \lor a = 0)$
by (*auto intro*: *scaleR-left-imp-eq*)

lemma *scaleR-cancel-right*:
fixes $x\ y :: 'a::real\text{-}vector$
shows $(scaleR\ a\ x = scaleR\ b\ x) = (a = b \lor x = 0)$
by (*auto intro*: *scaleR-right-imp-eq*)

lemma *nonzero-inverse-scaleR-distrib*:
fixes $x :: 'a::real\text{-}div\text{-}algebra$ **shows**
$[\![a \neq 0;\ x \neq 0]\!] \Longrightarrow inverse\ (scaleR\ a\ x) = scaleR\ (inverse\ a)\ (inverse\ x)$
by (*rule inverse-unique*, *simp*)

lemma *inverse-scaleR-distrib*:
fixes $x :: 'a::\{real\text{-}div\text{-}algebra,division\text{-}by\text{-}zero\}$
shows $inverse\ (scaleR\ a\ x) = scaleR\ (inverse\ a)\ (inverse\ x)$
apply (*case-tac* $a = 0$, *simp*)
apply (*case-tac* $x = 0$, *simp*)
apply (*erule* (*1*) *nonzero-inverse-scaleR-distrib*)
done

10.3 Embedding of the Reals into any *real-algebra-1*: *of-real*

definition
$of\text{-}real :: real \Rightarrow 'a::real\text{-}algebra\text{-}1$ **where**
$of\text{-}real\ r = scaleR\ r\ 1$

lemma *scaleR-conv-of-real*: $scaleR\ r\ x = of\text{-}real\ r * x$
by (*simp add*: *of-real-def*)

lemma *of-real-0* [*simp*]: $of\text{-}real\ 0 = 0$
by (*simp add*: *of-real-def*)

lemma *of-real-1* [*simp*]: *of-real 1* = *1*
by (*simp add*: *of-real-def*)

lemma *of-real-add* [*simp*]: *of-real* (*x* + *y*) = *of-real x* + *of-real y*
by (*simp add*: *of-real-def scaleR-left-distrib*)

lemma *of-real-minus* [*simp*]: *of-real* (− *x*) = − *of-real x*
by (*simp add*: *of-real-def*)

lemma *of-real-diff* [*simp*]: *of-real* (*x* − *y*) = *of-real x* − *of-real y*
by (*simp add*: *of-real-def scaleR-left-diff-distrib*)

lemma *of-real-mult* [*simp*]: *of-real* (*x* ∗ *y*) = *of-real x* ∗ *of-real y*
by (*simp add*: *of-real-def mult-commute*)

lemma *nonzero-of-real-inverse*:
$x \neq 0 \Longrightarrow$ *of-real* (*inverse x*) =
inverse (*of-real x* :: *'a*::*real-div-algebra*)
by (*simp add*: *of-real-def nonzero-inverse-scaleR-distrib*)

lemma *of-real-inverse* [*simp*]:
of-real (*inverse x*) =
inverse (*of-real x* :: *'a*::{*real-div-algebra*,*division-by-zero*})
by (*simp add*: *of-real-def inverse-scaleR-distrib*)

lemma *nonzero-of-real-divide*:
$y \neq 0 \Longrightarrow$ *of-real* (*x* / *y*) =
(*of-real x* / *of-real y* :: *'a*::*real-field*)
by (*simp add*: *divide-inverse nonzero-of-real-inverse*)

lemma *of-real-divide* [*simp*]:
of-real (*x* / *y*) =
(*of-real x* / *of-real y* :: *'a*::{*real-field*,*division-by-zero*})
by (*simp add*: *divide-inverse*)

lemma *of-real-power* [*simp*]:
of-real (*x* ˆ *n*) = (*of-real x* :: *'a*::{*real-algebra-1*,*recpower*}) ˆ *n*
by (*induct n*) (*simp-all add*: *power-Suc*)

lemma *of-real-eq-iff* [*simp*]: (*of-real x* = *of-real y*) = (*x* = *y*)
by (*simp add*: *of-real-def scaleR-cancel-right*)

lemmas *of-real-eq-0-iff* [*simp*] = *of-real-eq-iff* [*of - 0*, *simplified*]

lemma *of-real-eq-id* [*simp*]: *of-real* = (*id* :: *real* ⇒ *real*)
proof
fix *r*
show *of-real r* = *id r*
by (*simp add*: *of-real-def*)

qed

Collapse nested embeddings

lemma *of-real-of-nat-eq* [*simp*]: *of-real* (*of-nat n*) = *of-nat n*
by (*induct n*) *auto*

lemma *of-real-of-int-eq* [*simp*]: *of-real* (*of-int z*) = *of-int z*
by (*cases z rule*: *int-diff-cases*, *simp*)

lemma *of-real-number-of-eq*:
of-real (*number-of w*) = (*number-of w* :: '*a*::{*number-ring*,*real-algebra-1*})
by (*simp add*: *number-of-eq*)

Every real algebra has characteristic zero

instance *real-algebra-1* < *ring-char-0*
proof
fix *m n* :: *nat*
have (*of-real* (*of-nat m*) = (*of-real* (*of-nat n*)::'*a*)) = (*m* = *n*)
by (*simp only*: *of-real-eq-iff of-nat-eq-iff*)
thus (*of-nat m* = (*of-nat n*::'*a*)) = (*m* = *n*)
by (*simp only*: *of-real-of-nat-eq*)
qed

10.4 The Set of Real Numbers

definition
Reals :: '*a*::*real-algebra-1 set* **where**
Reals ≡ *range of-real*

notation (*xsymbols*)
Reals (ℝ)

lemma *Reals-of-real* [*simp*]: *of-real r* ∈ *Reals*
by (*simp add*: *Reals-def*)

lemma *Reals-of-int* [*simp*]: *of-int z* ∈ *Reals*
by (*subst of-real-of-int-eq* [*symmetric*], *rule Reals-of-real*)

lemma *Reals-of-nat* [*simp*]: *of-nat n* ∈ *Reals*
by (*subst of-real-of-nat-eq* [*symmetric*], *rule Reals-of-real*)

lemma *Reals-number-of* [*simp*]:
(*number-of w*::'*a*::{*number-ring*,*real-algebra-1*}) ∈ *Reals*
by (*subst of-real-number-of-eq* [*symmetric*], *rule Reals-of-real*)

lemma *Reals-0* [*simp*]: *0* ∈ *Reals*
apply (*unfold Reals-def*)
apply (*rule range-eqI*)
apply (*rule of-real-0* [*symmetric*])

done

lemma *Reals-1* [*simp*]: *1* ∈ *Reals*
apply (*unfold Reals-def*)
apply (*rule range-eqI*)
apply (*rule of-real-1* [*symmetric*])
done

lemma *Reals-add* [*simp*]: ⟦*a* ∈ *Reals*; *b* ∈ *Reals*⟧ ⟹ *a* + *b* ∈ *Reals*
apply (*auto simp add*: *Reals-def*)
apply (*rule range-eqI*)
apply (*rule of-real-add* [*symmetric*])
done

lemma *Reals-minus* [*simp*]: *a* ∈ *Reals* ⟹ − *a* ∈ *Reals*
apply (*auto simp add*: *Reals-def*)
apply (*rule range-eqI*)
apply (*rule of-real-minus* [*symmetric*])
done

lemma *Reals-diff* [*simp*]: ⟦*a* ∈ *Reals*; *b* ∈ *Reals*⟧ ⟹ *a* − *b* ∈ *Reals*
apply (*auto simp add*: *Reals-def*)
apply (*rule range-eqI*)
apply (*rule of-real-diff* [*symmetric*])
done

lemma *Reals-mult* [*simp*]: ⟦*a* ∈ *Reals*; *b* ∈ *Reals*⟧ ⟹ *a* * *b* ∈ *Reals*
apply (*auto simp add*: *Reals-def*)
apply (*rule range-eqI*)
apply (*rule of-real-mult* [*symmetric*])
done

lemma *nonzero-Reals-inverse*:
  fixes *a* :: *'a*::*real-div-algebra*
  shows ⟦*a* ∈ *Reals*; *a* ≠ *0*⟧ ⟹ *inverse a* ∈ *Reals*
apply (*auto simp add*: *Reals-def*)
apply (*rule range-eqI*)
apply (*erule nonzero-of-real-inverse* [*symmetric*])
done

lemma *Reals-inverse* [*simp*]:
  fixes *a* :: *'a*::{*real-div-algebra*,*division-by-zero*}
  shows *a* ∈ *Reals* ⟹ *inverse a* ∈ *Reals*
apply (*auto simp add*: *Reals-def*)
apply (*rule range-eqI*)
apply (*rule of-real-inverse* [*symmetric*])
done

lemma *nonzero-Reals-divide*:

fixes $a\ b :: 'a::real\text{-}field$
shows $[\![a \in Reals;\ b \in Reals;\ b \neq 0]\!] \Longrightarrow a\ /\ b \in Reals$
apply (*auto simp add*: *Reals-def*)
apply (*rule range-eqI*)
apply (*erule nonzero-of-real-divide* [*symmetric*])
done

lemma *Reals-divide* [*simp*]:
fixes $a\ b :: 'a::\{real\text{-}field, division\text{-}by\text{-}zero\}$
shows $[\![a \in Reals;\ b \in Reals]\!] \Longrightarrow a\ /\ b \in Reals$
apply (*auto simp add*: *Reals-def*)
apply (*rule range-eqI*)
apply (*rule of-real-divide* [*symmetric*])
done

lemma *Reals-power* [*simp*]:
fixes $a :: 'a::\{real\text{-}algebra\text{-}1, recpower\}$
shows $a \in Reals \Longrightarrow a\ \hat{}\ n \in Reals$
apply (*auto simp add*: *Reals-def*)
apply (*rule range-eqI*)
apply (*rule of-real-power* [*symmetric*])
done

lemma *Reals-cases* [*cases set*: *Reals*]:
assumes $q \in \mathbb{R}$
obtains (*of-real*) *r* **where** $q = of\text{-}real\ r$
unfolding *Reals-def*
proof −
from $\langle q \in \mathbb{R} \rangle$ **have** $q \in range\ of\text{-}real$ **unfolding** *Reals-def* **.**
then obtain *r* **where** $q = of\text{-}real\ r$ **..**
then show *thesis* **..**
qed

lemma *Reals-induct* [*case-names of-real*, *induct set*: *Reals*]:
$q \in \mathbb{R} \Longrightarrow (\bigwedge r.\ P\ (of\text{-}real\ r)) \Longrightarrow P\ q$
by (*rule Reals-cases*) *auto*

10.5 Real normed vector spaces

class $norm = type\ +$
fixes $norm :: 'a \Rightarrow real$

instance $real :: norm$
real-norm-def [*simp*]: $norm\ r \equiv |r|$ **..**

class $sgn\text{-}div\text{-}norm = scaleR + norm + sgn\ +$
assumes *sgn-div-norm*: $sgn\ x = x\ /_R\ norm\ x$

class $real\text{-}normed\text{-}vector = real\text{-}vector + sgn\text{-}div\text{-}norm\ +$

```
  assumes norm-ge-zero [simp]: 0 ≤ norm x
  and norm-eq-zero [simp]: norm x = 0 ⟷ x = 0
  and norm-triangle-ineq: norm (x + y) ≤ norm x + norm y
  and norm-scaleR: norm (scaleR a x) = |a| * norm x

class real-normed-algebra = real-algebra + real-normed-vector +
  assumes norm-mult-ineq: norm (x * y) ≤ norm x * norm y

class real-normed-algebra-1 = real-algebra-1 + real-normed-algebra +
  assumes norm-one [simp]: norm 1 = 1

class real-normed-div-algebra = real-div-algebra + real-normed-vector +
  assumes norm-mult: norm (x * y) = norm x * norm y

class real-normed-field = real-field + real-normed-div-algebra

instance real-normed-div-algebra < real-normed-algebra-1
proof
  fix x y :: 'a
  show norm (x * y) ≤ norm x * norm y
    by (simp add: norm-mult)
next
  have norm (1 * 1::'a) = norm (1::'a) * norm (1::'a)
    by (rule norm-mult)
  thus norm (1::'a) = 1 by simp
qed

instance real :: real-normed-field
apply (intro-classes, unfold real-norm-def real-scaleR-def)
apply (simp add: real-sgn-def)
apply (rule abs-ge-zero)
apply (rule abs-eq-0)
apply (rule abs-triangle-ineq)
apply (rule abs-mult)
apply (rule abs-mult)
done

lemma norm-zero [simp]: norm (0::'a::real-normed-vector) = 0
by simp

lemma zero-less-norm-iff [simp]:
  fixes x :: 'a::real-normed-vector
  shows (0 < norm x) = (x ≠ 0)
by (simp add: order-less-le)

lemma norm-not-less-zero [simp]:
  fixes x :: 'a::real-normed-vector
  shows ¬ norm x < 0
by (simp add: linorder-not-less)
```

lemma *norm-le-zero-iff* [*simp*]:
  fixes x :: $'a$::*real-normed-vector*
  shows (*norm* $x \leq 0$) = ($x = 0$)
by (*simp add*: *order-le-less*)

lemma *norm-minus-cancel* [*simp*]:
  fixes x :: $'a$::*real-normed-vector*
  shows *norm* $(- x)$ = *norm* x
proof −
  have *norm* $(- x)$ = *norm* (*scaleR* $(- 1)$ x)
    by (*simp only*: *scaleR-minus-left scaleR-one*)
  also have ... = $|- 1|$ ∗ *norm* x
    by (*rule norm-scaleR*)
  finally show *?thesis* **by** *simp*
qed

lemma *norm-minus-commute*:
  fixes a b :: $'a$::*real-normed-vector*
  shows *norm* $(a - b)$ = *norm* $(b - a)$
proof −
  have *norm* $(- (b - a))$ = *norm* $(b - a)$
    by (*rule norm-minus-cancel*)
  thus *?thesis* **by** *simp*
qed

lemma *norm-triangle-ineq2*:
  fixes a b :: $'a$::*real-normed-vector*
  shows *norm* a − *norm* b ≤ *norm* $(a - b)$
proof −
  have *norm* $(a - b + b)$ ≤ *norm* $(a - b)$ + *norm* b
    by (*rule norm-triangle-ineq*)
  thus *?thesis* **by** *simp*
qed

lemma *norm-triangle-ineq3*:
  fixes a b :: $'a$::*real-normed-vector*
  shows |*norm* a − *norm* b| ≤ *norm* $(a - b)$
apply (*subst abs-le-iff*)
apply *auto*
apply (*rule norm-triangle-ineq2*)
apply (*subst norm-minus-commute*)
apply (*rule norm-triangle-ineq2*)
done

lemma *norm-triangle-ineq4*:
  fixes a b :: $'a$::*real-normed-vector*
  shows *norm* $(a - b)$ ≤ *norm* a + *norm* b
proof −

```
  have norm (a + - b) ≤ norm a + norm (- b)
    by (rule norm-triangle-ineq)
  thus ?thesis
    by (simp only: diff-minus norm-minus-cancel)
qed

lemma norm-diff-ineq:
  fixes a b :: 'a::real-normed-vector
  shows norm a - norm b ≤ norm (a + b)
proof -
  have norm a - norm (- b) ≤ norm (a - - b)
    by (rule norm-triangle-ineq2)
  thus ?thesis by simp
qed

lemma norm-diff-triangle-ineq:
  fixes a b c d :: 'a::real-normed-vector
  shows norm ((a + b) - (c + d)) ≤ norm (a - c) + norm (b - d)
proof -
  have norm ((a + b) - (c + d)) = norm ((a - c) + (b - d))
    by (simp add: diff-minus add-ac)
  also have ... ≤ norm (a - c) + norm (b - d)
    by (rule norm-triangle-ineq)
  finally show ?thesis .
qed

lemma abs-norm-cancel [simp]:
  fixes a :: 'a::real-normed-vector
  shows |norm a| = norm a
by (rule abs-of-nonneg [OF norm-ge-zero])

lemma norm-add-less:
  fixes x y :: 'a::real-normed-vector
  shows ⟦norm x < r; norm y < s⟧ ⟹ norm (x + y) < r + s
by (rule order-le-less-trans [OF norm-triangle-ineq add-strict-mono])

lemma norm-mult-less:
  fixes x y :: 'a::real-normed-algebra
  shows ⟦norm x < r; norm y < s⟧ ⟹ norm (x * y) < r * s
apply (rule order-le-less-trans [OF norm-mult-ineq])
apply (simp add: mult-strict-mono')
done

lemma norm-of-real [simp]:
  norm (of-real r :: 'a::real-normed-algebra-1) = |r|
unfolding of-real-def by (simp add: norm-scaleR)

lemma norm-number-of [simp]:
  norm (number-of w::'a::{number-ring,real-normed-algebra-1})
```

```
  = |number-of w|
by (subst of-real-number-of-eq [symmetric], rule norm-of-real)

lemma norm-of-int [simp]:
  norm (of-int z::'a::real-normed-algebra-1) = |of-int z|
by (subst of-real-of-int-eq [symmetric], rule norm-of-real)

lemma norm-of-nat [simp]:
  norm (of-nat n::'a::real-normed-algebra-1) = of-nat n
apply (subst of-real-of-nat-eq [symmetric])
apply (subst norm-of-real, simp)
done

lemma nonzero-norm-inverse:
  fixes a :: 'a::real-normed-div-algebra
  shows a ≠ 0 ⟹ norm (inverse a) = inverse (norm a)
apply (rule inverse-unique [symmetric])
apply (simp add: norm-mult [symmetric])
done

lemma norm-inverse:
  fixes a :: 'a::{real-normed-div-algebra,division-by-zero}
  shows norm (inverse a) = inverse (norm a)
apply (case-tac a = 0, simp)
apply (erule nonzero-norm-inverse)
done

lemma nonzero-norm-divide:
  fixes a b :: 'a::real-normed-field
  shows b ≠ 0 ⟹ norm (a / b) = norm a / norm b
by (simp add: divide-inverse norm-mult nonzero-norm-inverse)

lemma norm-divide:
  fixes a b :: 'a::{real-normed-field,division-by-zero}
  shows norm (a / b) = norm a / norm b
by (simp add: divide-inverse norm-mult norm-inverse)

lemma norm-power-ineq:
  fixes x :: 'a::{real-normed-algebra-1,recpower}
  shows norm (x ^ n) ≤ norm x ^ n
proof (induct n)
  case 0 show norm (x ^ 0) ≤ norm x ^ 0 by simp
next
  case (Suc n)
  have norm (x * x ^ n) ≤ norm x * norm (x ^ n)
    by (rule norm-mult-ineq)
  also from Suc have ... ≤ norm x * norm x ^ n
    using norm-ge-zero by (rule mult-left-mono)
  finally show norm (x ^ Suc n) ≤ norm x ^ Suc n
```

by (*simp add*: *power-Suc*)
qed

lemma *norm-power*:
fixes x :: $'a$::{*real-normed-div-algebra*,*recpower*}
shows *norm* (x ^ n) = *norm* x ^ n
by (*induct n*) (*simp-all add*: *power-Suc norm-mult*)

10.6 Sign function

lemma *norm-sgn*:
norm (*sgn*(x::$'a$::*real-normed-vector*)) = (*if* $x = 0$ *then* 0 *else* 1)
by (*simp add*: *sgn-div-norm norm-scaleR*)

lemma *sgn-zero* [*simp*]: *sgn*(0::$'a$::*real-normed-vector*) = 0
by (*simp add*: *sgn-div-norm*)

lemma *sgn-zero-iff*: (*sgn*(x::$'a$::*real-normed-vector*) = 0) = ($x = 0$)
by (*simp add*: *sgn-div-norm*)

lemma *sgn-minus*: *sgn* ($-$ x) = $-$ *sgn*(x::$'a$::*real-normed-vector*)
by (*simp add*: *sgn-div-norm*)

lemma *sgn-scaleR*:
sgn (*scaleR r x*) = *scaleR* (*sgn r*) (*sgn*(x::$'a$::*real-normed-vector*))
by (*simp add*: *sgn-div-norm norm-scaleR mult-ac*)

lemma *sgn-one* [*simp*]: *sgn* (1::$'a$::*real-normed-algebra-1*) = 1
by (*simp add*: *sgn-div-norm*)

lemma *sgn-of-real*:
sgn (*of-real r*::$'a$::*real-normed-algebra-1*) = *of-real* (*sgn r*)
unfolding *of-real-def* **by** (*simp only*: *sgn-scaleR sgn-one*)

lemma *sgn-mult*:
fixes x y :: $'a$::*real-normed-div-algebra*
shows *sgn* ($x * y$) = *sgn* x $*$ *sgn* y
by (*simp add*: *sgn-div-norm norm-mult mult-commute*)

lemma *real-sgn-eq*: *sgn* (x::*real*) = x / $|x|$
by (*simp add*: *sgn-div-norm divide-inverse*)

lemma *real-sgn-pos*: $0 <$ (x::*real*) $\Longrightarrow$ *sgn* $x = 1$
unfolding *real-sgn-eq* **by** *simp*

lemma *real-sgn-neg*: (x::*real*) $< 0 \Longrightarrow$ *sgn* $x = -1$
unfolding *real-sgn-eq* **by** *simp*

10.7 Bounded Linear and Bilinear Operators

```
locale bounded-linear = additive +
  constrains f :: 'a::real-normed-vector ⇒ 'b::real-normed-vector
  assumes scaleR: f (scaleR r x) = scaleR r (f x)
  assumes bounded: ∃ K. ∀ x. norm (f x) ≤ norm x ∗ K

lemma (in bounded-linear) pos-bounded:
  ∃ K>0. ∀ x. norm (f x) ≤ norm x ∗ K
proof −
  obtain K where K: ⋀x. norm (f x) ≤ norm x ∗ K
    using bounded by fast
  show ?thesis
  proof (intro exI impI conjI allI)
    show 0 < max 1 K
      by (rule order-less-le-trans [OF zero-less-one le-maxI1])
  next
    fix x
    have norm (f x) ≤ norm x ∗ K using K .
    also have . . . ≤ norm x ∗ max 1 K
      by (rule mult-left-mono [OF le-maxI2 norm-ge-zero])
    finally show norm (f x) ≤ norm x ∗ max 1 K .
  qed
qed

lemma (in bounded-linear) nonneg-bounded:
  ∃ K≥0. ∀ x. norm (f x) ≤ norm x ∗ K
proof −
  from pos-bounded
  show ?thesis by (auto intro: order-less-imp-le)
qed

locale bounded-bilinear =
  fixes prod :: ['a::real-normed-vector, 'b::real-normed-vector]
              ⇒ 'c::real-normed-vector
    (infixl ∗∗ 70)
  assumes add-left: prod (a + a') b = prod a b + prod a' b
  assumes add-right: prod a (b + b') = prod a b + prod a b'
  assumes scaleR-left: prod (scaleR r a) b = scaleR r (prod a b)
  assumes scaleR-right: prod a (scaleR r b) = scaleR r (prod a b)
  assumes bounded: ∃ K. ∀ a b. norm (prod a b) ≤ norm a ∗ norm b ∗ K

lemma (in bounded-bilinear) pos-bounded:
  ∃ K>0. ∀ a b. norm (a ∗∗ b) ≤ norm a ∗ norm b ∗ K
apply (cut-tac bounded, erule exE)
apply (rule-tac x=max 1 K in exI, safe)
apply (rule order-less-le-trans [OF zero-less-one le-maxI1])
apply (drule spec, drule spec, erule order-trans)
apply (rule mult-left-mono [OF le-maxI2])
apply (intro mult-nonneg-nonneg norm-ge-zero)
```

done

lemma (**in** *bounded-bilinear*) *nonneg-bounded*:
$\exists K \geq 0.\ \forall a\ b.\ norm\ (a ** b) \leq norm\ a * norm\ b * K$
proof −
from *pos-bounded*
show *?thesis* **by** (*auto intro*: *order-less-imp-le*)
qed

lemma (**in** *bounded-bilinear*) *additive-right*: *additive* ($\lambda b.\ prod\ a\ b$)
by (*rule additive.intro*, *rule add-right*)

lemma (**in** *bounded-bilinear*) *additive-left*: *additive* ($\lambda a.\ prod\ a\ b$)
by (*rule additive.intro*, *rule add-left*)

lemma (**in** *bounded-bilinear*) *zero-left*: $prod\ 0\ b = 0$
by (*rule additive.zero* [*OF additive-left*])

lemma (**in** *bounded-bilinear*) *zero-right*: $prod\ a\ 0 = 0$
by (*rule additive.zero* [*OF additive-right*])

lemma (**in** *bounded-bilinear*) *minus-left*: $prod\ (-\ a)\ b = -\ prod\ a\ b$
by (*rule additive.minus* [*OF additive-left*])

lemma (**in** *bounded-bilinear*) *minus-right*: $prod\ a\ (-\ b) = -\ prod\ a\ b$
by (*rule additive.minus* [*OF additive-right*])

lemma (**in** *bounded-bilinear*) *diff-left*:
$prod\ (a - a')\ b = prod\ a\ b - prod\ a'\ b$
by (*rule additive.diff* [*OF additive-left*])

lemma (**in** *bounded-bilinear*) *diff-right*:
$prod\ a\ (b - b') = prod\ a\ b - prod\ a\ b'$
by (*rule additive.diff* [*OF additive-right*])

lemma (**in** *bounded-bilinear*) *bounded-linear-left*:
bounded-linear ($\lambda a.\ a ** b$)
apply (*unfold-locales*)
apply (*rule add-left*)
apply (*rule scaleR-left*)
apply (*cut-tac bounded*, *safe*)
apply (*rule-tac* $x{=}norm\ b * K$ **in** *exI*)
apply (*simp add*: *mult-ac*)
done

lemma (**in** *bounded-bilinear*) *bounded-linear-right*:
bounded-linear ($\lambda b.\ a ** b$)
apply (*unfold-locales*)
apply (*rule add-right*)

apply (*rule scaleR-right*)
apply (*cut-tac bounded, safe*)
apply (*rule-tac x=norm a* $*$ *K* **in** *exI*)
apply (*simp add: mult-ac*)
done

lemma (**in** *bounded-bilinear*) *prod-diff-prod*:
$(x ** y - a ** b) = (x - a) ** (y - b) + (x - a) ** b + a ** (y - b)$
by (*simp add: diff-left diff-right*)

interpretation *mult*:
bounded-bilinear [$op * :: 'a \Rightarrow 'a \Rightarrow 'a$::*real-normed-algebra*]
apply (*rule bounded-bilinear.intro*)
apply (*rule left-distrib*)
apply (*rule right-distrib*)
apply (*rule mult-scaleR-left*)
apply (*rule mult-scaleR-right*)
apply (*rule-tac x=1* **in** *exI*)
apply (*simp add: norm-mult-ineq*)
done

interpretation *mult-left*:
bounded-linear [(λx::'a::*real-normed-algebra*. $x * y$)]
by (*rule mult.bounded-linear-left*)

interpretation *mult-right*:
bounded-linear [(λy::'a::*real-normed-algebra*. $x * y$)]
by (*rule mult.bounded-linear-right*)

interpretation *divide*:
bounded-linear [(λx::'a::*real-normed-field*. x / y)]
unfolding *divide-inverse* **by** (*rule mult.bounded-linear-left*)

interpretation *scaleR*: *bounded-bilinear* [*scaleR*]
apply (*rule bounded-bilinear.intro*)
apply (*rule scaleR-left-distrib*)
apply (*rule scaleR-right-distrib*)
apply *simp*
apply (*rule scaleR-left-commute*)
apply (*rule-tac x=1* **in** *exI*)
apply (*simp add: norm-scaleR*)
done

interpretation *scaleR-left*: *bounded-linear* [λr. *scaleR r x*]
by (*rule scaleR.bounded-linear-left*)

interpretation *scaleR-right*: *bounded-linear* [λx. *scaleR r x*]
by (*rule scaleR.bounded-linear-right*)

interpretation *of-real*: *bounded-linear* [$\lambda r.$ *of-real r*]
unfolding *of-real-def* **by** (*rule scaleR.bounded-linear-left*)

end

theory *Real*
imports *ContNotDenum RealVector*
begin
end

11 Float: Floating Point Representation of the Reals

theory *Float*
imports *Real Parity*
uses ~~/*src*/*Tools*/*float.ML* (*float-arith.ML*)
begin

definition
  pow2 :: *int* ⇒ *real* **where**
  pow2 a = (*if* (*0* <= *a*) *then* (*2*^(*nat a*)) *else* (*inverse* (*2*^(*nat* (−*a*)))))

definition
  float :: *int* ∗ *int* ⇒ *real* **where**
  float x = *real* (*fst x*) ∗ *pow2* (*snd x*)

lemma *pow2-0*[*simp*]: *pow2 0* = *1*
by (*simp add*: *pow2-def*)

lemma *pow2-1*[*simp*]: *pow2 1* = *2*
by (*simp add*: *pow2-def*)

lemma *pow2-neg*: *pow2 x* = *inverse* (*pow2* (−*x*))
by (*simp add*: *pow2-def*)

lemma *pow2-add1*: *pow2* (*1* + *a*) = *2* ∗ (*pow2 a*)
proof −
  have *h*: ! *n*. *nat* (*2* + *int n*) − *Suc 0* = *nat* (*1* + *int n*) **by** *arith*
  have *g*: ! *a b*. *a* − −*1* = *a* + (*1*::*int*) **by** *arith*
  have *pos*: ! *n*. *pow2* (*int n* + *1*) = *2* ∗ *pow2* (*int n*)
    apply (*auto*, *induct-tac n*)
    apply (*simp-all add*: *pow2-def*)
   apply (*rule-tac m1*=*2* **and** *n1*=*nat* (*2* + *int na*) **in** *ssubst*[*OF realpow-num-eq-if*])
    by (*auto simp add*: *h*)

```
  show ?thesis
  proof (induct a)
    case (1 n)
    from pos show ?case by (simp add: ring-simps)
  next
    case (2 n)
    show ?case
      apply (auto)
      apply (subst pow2-neg[of − int n])
      apply (subst pow2-neg[of −1 − int n])
      apply (auto simp add: g pos)
      done
  qed
qed

lemma pow2-add: pow2 (a+b) = (pow2 a) ∗ (pow2 b)
proof (induct b)
  case (1 n)
  show ?case
  proof (induct n)
    case 0
    show ?case by simp
  next
    case (Suc m)
    show ?case by (auto simp add: ring-simps pow2-add1 prems)
  qed
next
  case (2 n)
  show ?case
  proof (induct n)
    case 0
    show ?case
      apply (auto)
      apply (subst pow2-neg[of a + −1])
      apply (subst pow2-neg[of −1])
      apply (simp)
      apply (insert pow2-add1[of −a])
      apply (simp add: ring-simps)
      apply (subst pow2-neg[of −a])
      apply (simp)
      done
    case (Suc m)
    have a: int m − (a + −2) =  1 + (int m − a + 1) by arith
    have b: int m − −2 = 1 + (int m + 1) by arith
    show ?case
      apply (auto)
      apply (subst pow2-neg[of a + (−2 − int m)])
      apply (subst pow2-neg[of −2 − int m])
      apply (auto simp add: ring-simps)
```

```
    apply (subst a)
    apply (subst b)
    apply (simp only: pow2-add1)
    apply (subst pow2-neg[of int m - a + 1])
    apply (subst pow2-neg[of int m + 1])
    apply auto
    apply (insert prems)
    apply (auto simp add: ring-simps)
    done
  qed
qed

lemma float (a, e) + float (b, e) = float (a + b, e)
by (simp add: float-def ring-simps)

definition
  int-of-real :: real ⇒ int where
  int-of-real x = (SOME y. real y = x)

definition
  real-is-int :: real ⇒ bool where
  real-is-int x = (EX (u::int). x = real u)

lemma real-is-int-def2: real-is-int x = (x = real (int-of-real x))
by (auto simp add: real-is-int-def int-of-real-def)

lemma float-transfer: real-is-int ((real a)*(pow2 c)) ⟹ float (a, b) = float (int-of-real
((real a)*(pow2 c)), b - c)
by (simp add: float-def real-is-int-def2 pow2-add[symmetric])

lemma pow2-int: pow2 (int c) = (2::real)^c
by (simp add: pow2-def)

lemma float-transfer-nat: float (a, b) = float (a * 2^c, b - int c)
by (simp add: float-def pow2-int[symmetric] pow2-add[symmetric])

lemma real-is-int-real[simp]: real-is-int (real (x::int))
by (auto simp add: real-is-int-def int-of-real-def)

lemma int-of-real-real[simp]: int-of-real (real x) = x
by (simp add: int-of-real-def)

lemma real-int-of-real[simp]: real-is-int x ⟹ real (int-of-real x) = x
by (auto simp add: int-of-real-def real-is-int-def)

lemma real-is-int-add-int-of-real: real-is-int a ⟹ real-is-int b ⟹ (int-of-real
(a+b)) = (int-of-real a) + (int-of-real b)
by (auto simp add: int-of-real-def real-is-int-def)
```

lemma *real-is-int-add*[*simp*]: *real-is-int a* $\Longrightarrow$ *real-is-int b* $\Longrightarrow$ *real-is-int* (*a*+*b*)
apply (*subst real-is-int-def2*)
apply (*simp add*: *real-is-int-add-int-of-real real-int-of-real*)
done

lemma *int-of-real-sub*: *real-is-int a* $\Longrightarrow$ *real-is-int b* $\Longrightarrow$ (*int-of-real* (*a*−*b*)) = (*int-of-real a*) − (*int-of-real b*)
by (*auto simp add*: *int-of-real-def real-is-int-def*)

lemma *real-is-int-sub*[*simp*]: *real-is-int a* $\Longrightarrow$ *real-is-int b* $\Longrightarrow$ *real-is-int* (*a*−*b*)
apply (*subst real-is-int-def2*)
apply (*simp add*: *int-of-real-sub real-int-of-real*)
done

lemma *real-is-int-rep*: *real-is-int x* $\Longrightarrow$ *?*! (*a*::*int*). *real a* = *x*
by (*auto simp add*: *real-is-int-def*)

lemma *int-of-real-mult*:
  assumes *real-is-int a real-is-int b*
  shows (*int-of-real* (*a*∗*b*)) = (*int-of-real a*) ∗ (*int-of-real b*)
proof −
  from *prems* **have** *a*: *?*! (*a*′::*int*). *real a*′ = *a* **by** (*rule-tac real-is-int-rep*, *auto*)
  from *prems* **have** *b*: *?*! (*b*′::*int*). *real b*′ = *b* **by** (*rule-tac real-is-int-rep*, *auto*)
  from *a* **obtain** *a*′::*int* **where** *a*′:*a* = *real a*′ **by** *auto*
  from *b* **obtain** *b*′::*int* **where** *b*′:*b* = *real b*′ **by** *auto*
  have *r*: *real a*′ ∗ *real b*′ = *real* (*a*′ ∗ *b*′) **by** *auto*
  show *?thesis*
    apply (*simp add*: *a*′ *b*′)
    apply (*subst r*)
    apply (*simp only*: *int-of-real-real*)
    done
qed

lemma *real-is-int-mult*[*simp*]: *real-is-int a* $\Longrightarrow$ *real-is-int b* $\Longrightarrow$ *real-is-int* (*a*∗*b*)
apply (*subst real-is-int-def2*)
apply (*simp add*: *int-of-real-mult*)
done

lemma *real-is-int-0*[*simp*]: *real-is-int* (*0*::*real*)
by (*simp add*: *real-is-int-def int-of-real-def*)

lemma *real-is-int-1*[*simp*]: *real-is-int* (*1*::*real*)
proof −
  have *real-is-int* (*1*::*real*) = *real-is-int*(*real* (*1*::*int*)) **by** *auto*
  also have ... = *True* **by** (*simp only*: *real-is-int-real*)
  ultimately show *?thesis* **by** *auto*
qed

lemma *real-is-int-n1*: *real-is-int* (−*1*::*real*)

```
proof −
  have real-is-int (−1::real) = real-is-int(real (−1::int)) by auto
  also have ... = True by (simp only: real-is-int-real)
  ultimately show ?thesis by auto
qed

lemma real-is-int-number-of[simp]: real-is-int ((number-of :: int ⇒ real) x)
proof −
  have neg1: real-is-int (−1::real)
  proof −
    have real-is-int (−1::real) = real-is-int(real (−1::int)) by auto
    also have ... = True by (simp only: real-is-int-real)
    ultimately show ?thesis by auto
  qed

  {
    fix x :: int
    have real-is-int ((number-of :: int ⇒ real) x)
      unfolding number-of-eq
      apply (induct x)
      apply (induct-tac n)
      apply (simp)
      apply (simp)
      apply (induct-tac n)
      apply (simp add: neg1)
    proof −
      fix n :: nat
      assume rn: (real-is-int (of-int (− (int (Suc n)))))
      have s: −(int (Suc (Suc n))) = −1 + − (int (Suc n)) by simp
      show real-is-int (of-int (− (int (Suc (Suc n)))))
        apply (simp only: s of-int-add)
        apply (rule real-is-int-add)
        apply (simp add: neg1)
        apply (simp only: rn)
        done
    qed
  }
  note Abs-Bin = this
  {
    fix x :: int
    have ? u. x = u
      apply (rule exI[where x = x])
      apply (simp)
      done
  }
  then obtain u::int where x = u by auto
  with Abs-Bin show ?thesis by auto
qed
```

lemma *int-of-real-0*[*simp*]: *int-of-real* (*0*::*real*) = (*0*::*int*)
by (*simp add*: *int-of-real-def*)

lemma *int-of-real-1*[*simp*]: *int-of-real* (*1*::*real*) = (*1*::*int*)
proof −
  have *1*: (*1*::*real*) = *real* (*1*::*int*) **by** *auto*
  show *?thesis* **by** (*simp only*: *1 int-of-real-real*)
qed

lemma *int-of-real-number-of*[*simp*]: *int-of-real* (*number-of b*) = *number-of b*
proof −
  have *real-is-int* (*number-of b*) **by** *simp*
  then have *uu*: *?! u*::*int*. *number-of b* = *real u* **by** (*auto simp add*: *real-is-int-rep*)
  then obtain *u*::*int* **where** *u*:*number-of b* = *real u* **by** *auto*
  have *number-of b* = *real* ((*number-of b*)::*int*)
    by (*simp add*: *number-of-eq real-of-int-def*)
  have *ub*: *number-of b* = *real* ((*number-of b*)::*int*)
    by (*simp add*: *number-of-eq real-of-int-def*)
  from *uu u ub* **have** *unb*: *u* = *number-of b*
    by *blast*
  have *int-of-real* (*number-of b*) = *u* **by** (*simp add*: *u*)
  with *unb* **show** *?thesis* **by** *simp*
qed

lemma *float-transfer-even*: *even a* ⟹ *float* (*a*, *b*) = *float* (*a div 2*, *b*+*1*)
  apply (*subst float-transfer*[**where** *a*=*a* **and** *b*=*b* **and** *c*=−*1*, *simplified*])
  apply (*simp-all add*: *pow2-def even-def real-is-int-def ring-simps*)
  apply (*auto*)
proof −
  fix *q*::*int*
  have *a*:*b* − (−*1*::*int*) = (*1*::*int*) + *b* **by** *arith*
  show (*float* (*q*, (*b* − (−*1*::*int*)))) = (*float* (*q*, ((*1*::*int*) + *b*)))
    by (*simp add*: *a*)
qed

consts
  norm-float :: *int*∗*int* ⇒ *int*∗*int*

lemma *int-div-zdiv*: *int* (*a div b*) = (*int a*) *div* (*int b*)
by (*rule zdiv-int*)

lemma *int-mod-zmod*: *int* (*a mod b*) = (*int a*) *mod* (*int b*)
by (*rule zmod-int*)

lemma *abs-div-2-less*: *a* ≠ *0* ⟹ *a* ≠ −*1* ⟹ *abs*((*a*::*int*) *div 2*) < *abs a*
by *arith*

lemma *terminating-norm-float*: ∀ *a*. (*a*::*int*) ≠ *0* ∧ *even a* ⟶ *a* ≠ *0* ∧ |*a div 2*| < |*a*|

```
apply (auto)
apply (rule abs-div-2-less)
apply (auto)
done

declare [[simp-depth-limit = 2]]
recdef norm-float measure (% (a,b). nat (abs a))
  norm-float (a,b) = (if (a ≠ 0) & (even a) then norm-float (a div 2, b+1) else
(if a=0 then (0,0) else (a,b)))
(hints simp: even-def terminating-norm-float)
declare [[simp-depth-limit = 100]]

lemma norm-float: float x = float (norm-float x)
proof −
  {
    fix a b :: int
    have norm-float-pair: float (a,b) = float (norm-float (a,b))
    proof (induct a b rule: norm-float.induct)
      case (1 u v)
      show ?case
      proof cases
        assume u: u ≠ 0 ∧ even u
        with prems have ind: float (u div 2, v + 1) = float (norm-float (u div 2,
v + 1)) by auto
      with u have float (u,v) = float (u div 2, v+1) by (simp add: float-transfer-even)
        then show ?thesis
          apply (subst norm-float.simps)
          apply (simp add: ind)
          done
      next
        assume ~(u ≠ 0 ∧ even u)
        then show ?thesis
          by (simp add: prems float-def)
      qed
    qed
  }
  note helper = this
  have ? a b. x = (a,b) by auto
  then obtain a b where x = (a, b) by blast
  then show ?thesis by (simp only: helper)
qed

lemma pow2-int: pow2 (int n) = 2^n
  by (simp add: pow2-def)

lemma float-add-l0: float (0, e) + x = x
  by (simp add: float-def)

lemma float-add-r0: x + float (0, e) = x
```

by (*simp add*: *float-def*)

lemma *float-add*:
float (*a1*, *e1*) + *float* (*a2*, *e2*) =
(*if e1*<=*e2 then float* (*a1*+*a2***2*^(*nat*(*e2*−*e1*)), *e1*)
else float (*a1***2*^(*nat* (*e1*−*e2*))+*a2*, *e2*))
apply (*simp add*: *float-def ring-simps*)
apply (*auto simp add*: *pow2-int*[*symmetric*] *pow2-add*[*symmetric*])
done

lemma *float-add-assoc1*:
(*x* + *float* (*y1*, *e1*)) + *float* (*y2*, *e2*) = (*float* (*y1*, *e1*) + *float* (*y2*, *e2*)) + *x*
by *simp*

lemma *float-add-assoc2*:
(*float* (*y1*, *e1*) + *x*) + *float* (*y2*, *e2*) = (*float* (*y1*, *e1*) + *float* (*y2*, *e2*)) + *x*
by *simp*

lemma *float-add-assoc3*:
float (*y1*, *e1*) + (*x* + *float* (*y2*, *e2*)) = (*float* (*y1*, *e1*) + *float* (*y2*, *e2*)) + *x*
by *simp*

lemma *float-add-assoc4*:
float (*y1*, *e1*) + (*float* (*y2*, *e2*) + *x*) = (*float* (*y1*, *e1*) + *float* (*y2*, *e2*)) + *x*
by *simp*

lemma *float-mult-l0*: *float* (*0*, *e*) * *x* = *float* (*0*, *0*)
by (*simp add*: *float-def*)

lemma *float-mult-r0*: *x* * *float* (*0*, *e*) = *float* (*0*, *0*)
by (*simp add*: *float-def*)

definition
lbound :: *real* ⇒ *real*
where
lbound x = *min 0 x*

definition
ubound :: *real* ⇒ *real*
where
ubound x = *max 0 x*

lemma *lbound*: *lbound x* ≤ *x*
by (*simp add*: *lbound-def*)

lemma *ubound*: *x* ≤ *ubound x*
by (*simp add*: *ubound-def*)

lemma *float-mult*:

```
  float (a1, e1) * float (a2, e2) =
  (float (a1 * a2, e1 + e2))
  by (simp add: float-def pow2-add)

lemma float-minus:
  - (float (a,b)) = float (-a, b)
  by (simp add: float-def)

lemma zero-less-pow2:
  0 < pow2 x
proof -
  {
    fix y
    have 0 <= y ⟹ 0 < pow2 y
      by (induct y, induct-tac n, simp-all add: pow2-add)
  }
  note helper=this
  show ?thesis
    apply (case-tac 0 <= x)
    apply (simp add: helper)
    apply (subst pow2-neg)
    apply (simp add: helper)
    done
qed

lemma zero-le-float:
  (0 <= float (a,b)) = (0 <= a)
  apply (auto simp add: float-def)
  apply (auto simp add: zero-le-mult-iff zero-less-pow2)
  apply (insert zero-less-pow2[of b])
  apply (simp-all)
  done

lemma float-le-zero:
  (float (a,b) <= 0) = (a <= 0)
  apply (auto simp add: float-def)
  apply (auto simp add: mult-le-0-iff)
  apply (insert zero-less-pow2[of b])
  apply auto
  done

lemma float-abs:
  abs (float (a,b)) = (if 0 <= a then (float (a,b)) else (float (-a,b)))
  apply (auto simp add: abs-if)
  apply (simp-all add: zero-le-float[symmetric, of a b] float-minus)
  done

lemma float-zero:
  float (0, b) = 0
```

by (*simp add*: *float-def*)

lemma *float-pprt*:
pprt (*float* (*a*, *b*)) = (*if 0* <= *a then* (*float* (*a*,*b*)) *else* (*float* (*0*, *b*)))
by (*auto simp add*: *zero-le-float float-le-zero float-zero*)

lemma *pprt-lbound*: *pprt* (*lbound x*) = *float* (*0*, *0*)
apply (*simp add*: *float-def*)
apply (*rule pprt-eq-0*)
apply (*simp add*: *lbound-def*)
done

lemma *nprt-ubound*: *nprt* (*ubound x*) = *float* (*0*, *0*)
apply (*simp add*: *float-def*)
apply (*rule nprt-eq-0*)
apply (*simp add*: *ubound-def*)
done

lemma *float-nprt*:
nprt (*float* (*a*, *b*)) = (*if 0* <= *a then* (*float* (*0*,*b*)) *else* (*float* (*a*, *b*)))
by (*auto simp add*: *zero-le-float float-le-zero float-zero*)

lemma *norm-0-1*: (*0*::-::*number-ring*) = *Numeral0* & (*1*::-::*number-ring*) = *Numeral1*
by *auto*

lemma *add-left-zero*: *0* + *a* = (*a*::'*a*::*comm-monoid-add*)
by *simp*

lemma *add-right-zero*: *a* + *0* = (*a*::'*a*::*comm-monoid-add*)
by *simp*

lemma *mult-left-one*: *1* ∗ *a* = (*a*::'*a*::*semiring-1*)
by *simp*

lemma *mult-right-one*: *a* ∗ *1* = (*a*::'*a*::*semiring-1*)
by *simp*

lemma *int-pow-0*: (*a*::*int*)^(*Numeral0*) = *1*
by *simp*

lemma *int-pow-1*: (*a*::*int*)^(*Numeral1*) = *a*
by *simp*

lemma *zero-eq-Numeral0-nring*: (*0*::'*a*::*number-ring*) = *Numeral0*
by *simp*

lemma *one-eq-Numeral1-nring*: (*1*::'*a*::*number-ring*) = *Numeral1*
by *simp*

lemma *zero-eq-Numeral0-nat*: $(0::nat) = Numeral0$
  by *simp*

lemma *one-eq-Numeral1-nat*: $(1::nat) = Numeral1$
  by *simp*

lemma *zpower-Pls*: $(z::int)\hat{}Numeral0 = Numeral1$
  by *simp*

lemma *zpower-Min*: $(z::int)\hat{}((-1)::nat) = Numeral1$
proof −
  have $1:((-1)::nat) = 0$
    by *simp*
  show *?thesis* **by** (*simp add*: *1*)
qed

lemma *fst-cong*: $a=a' \Longrightarrow fst\ (a,b) = fst\ (a',b)$
  by *simp*

lemma *snd-cong*: $b=b' \Longrightarrow snd\ (a,b) = snd\ (a,b')$
  by *simp*

lemma *lift-bool*: $x \Longrightarrow x=True$
  by *simp*

lemma *nlift-bool*: $\sim x \Longrightarrow x=False$
  by *simp*

lemma *not-false-eq-true*: $(\sim\ False) = True$ **by** *simp*

lemma *not-true-eq-false*: $(\sim\ True) = False$ **by** *simp*

lemmas *binarith* =
  Pls-0-eq Min-1-eq
  pred-Pls pred-Min pred-1 pred-0
  succ-Pls succ-Min succ-1 succ-0
  add-Pls add-Min add-BIT-0 add-BIT-10
  add-BIT-11 minus-Pls minus-Min minus-1
  minus-0 mult-Pls mult-Min mult-num1 mult-num0
  add-Pls-right add-Min-right

lemma *int-eq-number-of-eq*:
  $(((number\text{-}of\ v)::int)=(number\text{-}of\ w)) = iszero\ ((number\text{-}of\ (v + uminus\ w))::int)$
  by *simp*

lemma *int-iszero-number-of-Pls*: $iszero\ (Numeral0::int)$
  by (*simp only*: *iszero-number-of-Pls*)

lemma *int-nonzero-number-of-Min*: $\sim(iszero\ ((-1)::int))$

by *simp*

lemma *int-iszero-number-of-0*: *iszero* ((*number-of* (*w BIT bit.B0*))::*int*) = *iszero* ((*number-of w*)::*int*)
by *simp*

lemma *int-iszero-number-of-1*: ¬ *iszero* ((*number-of* (*w BIT bit.B1*))::*int*)
by *simp*

lemma *int-less-number-of-eq-neg*: (((*number-of x*)::*int*) < *number-of y*) = *neg* ((*number-of* (*x* + (*uminus y*)))::*int*)
by *simp*

lemma *int-not-neg-number-of-Pls*: ¬ (*neg* (*Numeral0*::*int*))
by *simp*

lemma *int-neg-number-of-Min*: *neg* (−*1*::*int*)
by *simp*

lemma *int-neg-number-of-BIT*: *neg* ((*number-of* (*w BIT x*))::*int*) = *neg* ((*number-of w*)::*int*)
by *simp*

lemma *int-le-number-of-eq*: (((*number-of x*)::*int*) ≤ *number-of y*) = (¬ *neg* ((*number-of* (*y* + (*uminus x*)))::*int*))
by *simp*

lemmas *intarithrel* =
int-eq-number-of-eq
lift-bool[*OF int-iszero-number-of-Pls*] *nlift-bool*[*OF int-nonzero-number-of-Min*] *int-iszero-number-of-0*
lift-bool[*OF int-iszero-number-of-1*] *int-less-number-of-eq-neg nlift-bool*[*OF int-not-neg-number-of-Pls*] *lift-bool*[*OF int-neg-number-of-Min*]
int-neg-number-of-BIT int-le-number-of-eq

lemma *int-number-of-add-sym*: ((*number-of v*)::*int*) + *number-of w* = *number-of* (*v* + *w*)
by *simp*

lemma *int-number-of-diff-sym*: ((*number-of v*)::*int*) − *number-of w* = *number-of* (*v* + (*uminus w*))
by *simp*

lemma *int-number-of-mult-sym*: ((*number-of v*)::*int*) ∗ *number-of w* = *number-of* (*v* ∗ *w*)
by *simp*

lemma *int-number-of-minus-sym*: − ((*number-of v*)::*int*) = *number-of* (*uminus v*)
by *simp*

lemmas *intarith* = *int-number-of-add-sym int-number-of-minus-sym int-number-of-diff-sym int-number-of-mult-sym*

lemmas *natarith* = *add-nat-number-of diff-nat-number-of mult-nat-number-of eq-nat-number-of less-nat-number-of*

lemmas *powerarith* = *nat-number-of zpower-number-of-even zpower-number-of-odd*[*simplified zero-eq-Numeral0-nring one-eq-Numeral1-nring*] *zpower-Pls zpower-Min*

lemmas *floatarith*[*simplified norm-0-1*] = *float-add float-add-l0 float-add-r0 float-mult float-mult-l0 float-mult-r0*
float-minus float-abs zero-le-float float-pprt float-nprt pprt-lbound nprt-ubound

lemmas *arith* = *binarith intarith intarithrel natarith powerarith floatarith not-false-eq-true not-true-eq-false*

use *float-arith.ML*

end

12 SEQ: Sequences and Convergence

theory *SEQ*
imports *../Real/Real*
begin

definition
Zseq :: [*nat* ⇒ ′*a*::*real-normed-vector*] ⇒ *bool* **where**
— Standard definition of sequence converging to zero
Zseq X = (∀ *r*>*0*. ∃ *no*. ∀ *n*≥*no*. *norm* (*X n*) < *r*)

definition
LIMSEQ :: [*nat* => ′*a*::*real-normed-vector*, ′*a*] => *bool*
(((-)/ ----> (-)) [*60*, *60*] *60*) **where**
— Standard definition of convergence of sequence
X ----> *L* = (∀ *r*. *0* < *r* --> (∃ *no*. ∀ *n*. *no* ≤ *n* --> *norm* (*X n* − *L*) < *r*))

definition
lim :: (*nat* => ′*a*::*real-normed-vector*) => ′*a* **where**
— Standard definition of limit using choice operator
lim X = (*THE L*. *X* ----> *L*)

definition
convergent :: (*nat* => ′*a*::*real-normed-vector*) => *bool* **where**

— Standard definition of convergence
convergent $X = (\exists L.\ X \ {-}{-}{-}{-}{>}\ L)$

definition
Bseq :: $(nat => 'a::real\text{-}normed\text{-}vector) => bool$ **where**
— Standard definition for bounded sequence
Bseq $X = (\exists K{>}0. \forall n.\ norm\ (X\ n) \le K)$

definition
monoseq :: $(nat{=>}real){=>}bool$ **where**
— Definition for monotonicity
monoseq $X = ((\forall m.\ \forall n{\ge}m.\ X\ m \le X\ n) \mid (\forall m.\ \forall n{\ge}m.\ X\ n \le X\ m))$

definition
subseq :: $(nat => nat) => bool$ **where**
— Definition of subsequence
subseq $f = (\forall m.\ \forall n{>}m.\ (f\ m) < (f\ n))$

definition
Cauchy :: $(nat => 'a::real\text{-}normed\text{-}vector) => bool$ **where**
— Standard definition of the Cauchy condition
Cauchy $X = (\forall e{>}0.\ \exists M.\ \forall m \ge M.\ \forall n \ge M.\ norm\ (X\ m - X\ n) < e)$

12.1 Bounded Sequences

lemma *BseqI*: **assumes** K: $\bigwedge n.\ norm\ (X\ n) \le K$ **shows** *Bseq X*
unfolding *Bseq-def*
proof (*intro exI conjI allI*)
show $0 < max\ K\ 1$ **by** *simp*
next
fix $n{::}nat$
have $norm\ (X\ n) \le K$ **by** (*rule K*)
thus $norm\ (X\ n) \le max\ K\ 1$ **by** *simp*
qed

lemma *BseqD*: $Bseq\ X \Longrightarrow \exists K{>}0.\ \forall n.\ norm\ (X\ n) \le K$
unfolding *Bseq-def* **by** *simp*

lemma *BseqE*: $[\![Bseq\ X;\ \bigwedge K.\ [\![0 < K;\ \forall n.\ norm\ (X\ n) \le K]\!] \Longrightarrow Q]\!] \Longrightarrow Q$
unfolding *Bseq-def* **by** *auto*

lemma *BseqI2*: **assumes** K: $\forall n{\ge}N.\ norm\ (X\ n) \le K$ **shows** *Bseq X*
proof (*rule BseqI*)
let $?A = norm\ `\ X\ `\ \{..N\}$
have 1: *finite ?A* **by** *simp*
have 2: $?A \ne \{\}$ **by** *auto*
fix $n{::}nat$
show $norm\ (X\ n) \le max\ K\ (Max\ ?A)$
proof (*cases rule*: *linorder-le-cases*)

```
    assume n ≥ N
    hence norm (X n) ≤ K using K by simp
    thus norm (X n) ≤ max K (Max ?A) by simp
  next
    assume n ≤ N
    hence norm (X n) ∈ ?A by simp
    with 1 2 have norm (X n) ≤ Max ?A by (rule Max-ge)
    thus norm (X n) ≤ max K (Max ?A) by simp
  qed
qed

lemma Bseq-ignore-initial-segment: Bseq X ⟹ Bseq (λn. X (n + k))
unfolding Bseq-def by auto

lemma Bseq-offset: Bseq (λn. X (n + k)) ⟹ Bseq X
apply (erule BseqE)
apply (rule-tac N=k and K=K in BseqI2)
apply clarify
apply (drule-tac x=n − k in spec, simp)
done
```

12.2 Sequences That Converge to Zero

```
lemma ZseqI:
  (⋀r. 0 < r ⟹ ∃no. ∀n≥no. norm (X n) < r) ⟹ Zseq X
unfolding Zseq-def by simp

lemma ZseqD:
  ⟦Zseq X; 0 < r⟧ ⟹ ∃no. ∀n≥no. norm (X n) < r
unfolding Zseq-def by simp

lemma Zseq-zero: Zseq (λn. 0)
unfolding Zseq-def by simp

lemma Zseq-const-iff: Zseq (λn. k) = (k = 0)
unfolding Zseq-def by force

lemma Zseq-norm-iff: Zseq (λn. norm (X n)) = Zseq (λn. X n)
unfolding Zseq-def by simp

lemma Zseq-imp-Zseq:
  assumes X: Zseq X
  assumes Y: ⋀n. norm (Y n) ≤ norm (X n) * K
  shows Zseq (λn. Y n)
proof (cases)
  assume K: 0 < K
  show ?thesis
  proof (rule ZseqI)
    fix r::real assume 0 < r
```

```
    hence 0 < r / K
      using K by (rule divide-pos-pos)
    then obtain N where ∀ n≥N. norm (X n) < r / K
      using ZseqD [OF X] by fast
    hence ∀ n≥N. norm (X n) ∗ K < r
      by (simp add: pos-less-divide-eq K)
    hence ∀ n≥N. norm (Y n) < r
      by (simp add: order-le-less-trans [OF Y])
    thus ∃ N. ∀ n≥N. norm (Y n) < r ..
  qed
next
  assume ¬ 0 < K
  hence K: K ≤ 0 by (simp only: linorder-not-less)
  {
    fix n::nat
    have norm (Y n) ≤ norm (X n) ∗ K by (rule Y)
    also have . . . ≤ norm (X n) ∗ 0
      using K norm-ge-zero by (rule mult-left-mono)
    finally have norm (Y n) = 0 by simp
  }
  thus ?thesis by (simp add: Zseq-zero)
qed

lemma Zseq-le: [[Zseq Y; ∀ n. norm (X n) ≤ norm (Y n)]] ⟹ Zseq X
by (erule-tac K=1 in Zseq-imp-Zseq, simp)

lemma Zseq-add:
  assumes X: Zseq X
  assumes Y: Zseq Y
  shows Zseq (λn. X n + Y n)
proof (rule ZseqI)
  fix r::real assume 0 < r
  hence r: 0 < r / 2 by simp
  obtain M where M: ∀ n≥M. norm (X n) < r/2
    using ZseqD [OF X r] by fast
  obtain N where N: ∀ n≥N. norm (Y n) < r/2
    using ZseqD [OF Y r] by fast
  show ∃ N. ∀ n≥N. norm (X n + Y n) < r
  proof (intro exI allI impI)
    fix n assume n: max M N ≤ n
    have norm (X n + Y n) ≤ norm (X n) + norm (Y n)
      by (rule norm-triangle-ineq)
    also have . . . < r/2 + r/2
    proof (rule add-strict-mono)
      from M n show norm (X n) < r/2 by simp
      from N n show norm (Y n) < r/2 by simp
    qed
    finally show norm (X n + Y n) < r by simp
  qed
```

qed

lemma *Zseq-minus*: *Zseq X* $\Longrightarrow$ *Zseq* ($\lambda n. - X\ n$)
unfolding *Zseq-def* **by** *simp*

lemma *Zseq-diff*: $[\![$*Zseq X*; *Zseq Y*$]\!] \Longrightarrow$ *Zseq* ($\lambda n. X\ n - Y\ n$)
by (*simp only*: *diff-minus Zseq-add Zseq-minus*)

lemma (**in** *bounded-linear*) *Zseq*:
  assumes *X*: *Zseq X*
  shows *Zseq* ($\lambda n. f\ (X\ n)$)
proof −
  obtain *K* **where** $\bigwedge x.$ *norm* $(f\ x) \leq$ *norm* $x * K$
    using *bounded* **by** *fast*
  with *X* **show** *?thesis*
    by (*rule Zseq-imp-Zseq*)
qed

lemma (**in** *bounded-bilinear*) *Zseq*:
  assumes *X*: *Zseq X*
  assumes *Y*: *Zseq Y*
  shows *Zseq* ($\lambda n. X\ n ** Y\ n$)
proof (*rule ZseqI*)
  fix *r*::*real* **assume** *r*: $0 < r$
  obtain *K* **where** *K*: $0 < K$
    and *norm-le*: $\bigwedge x\ y.$ *norm* $(x ** y) \leq$ *norm* $x *$ *norm* $y * K$
    using *pos-bounded* **by** *fast*
  from *K* **have** *K'*: $0 <$ *inverse K*
    by (*rule positive-imp-inverse-positive*)
  obtain *M* **where** *M*: $\forall n \geq M.$ *norm* $(X\ n) < r$
    using *ZseqD* [*OF X r*] **by** *fast*
  obtain *N* **where** *N*: $\forall n \geq N.$ *norm* $(Y\ n) <$ *inverse K*
    using *ZseqD* [*OF Y K'*] **by** *fast*
  show $\exists N. \forall n \geq N.$ *norm* $(X\ n ** Y\ n) < r$
  proof (*intro exI allI impI*)
    fix *n* **assume** *n*: *max M N* $\leq n$
    have *norm* $(X\ n ** Y\ n) \leq$ *norm* $(X\ n) *$ *norm* $(Y\ n) * K$
      by (*rule norm-le*)
    also have *norm* $(X\ n) *$ *norm* $(Y\ n) * K < r *$ *inverse* $K * K$
    proof (*intro mult-strict-right-mono mult-strict-mono' norm-ge-zero K*)
      from *M n* **show** *Xn*: *norm* $(X\ n) < r$ **by** *simp*
      from *N n* **show** *Yn*: *norm* $(Y\ n) <$ *inverse K* **by** *simp*
    qed
    also from *K* **have** $r *$ *inverse* $K * K = r$ **by** *simp*
    finally show *norm* $(X\ n ** Y\ n) < r$ **.**
  qed
qed

lemma (**in** *bounded-bilinear*) *Zseq-prod-Bseq*:

```
  assumes X: Zseq X
  assumes Y: Bseq Y
  shows Zseq (λn. X n ** Y n)
proof -
  obtain K where K: 0 ≤ K
    and norm-le: ⋀x y. norm (x ** y) ≤ norm x * norm y * K
    using nonneg-bounded by fast
  obtain B where B: 0 < B
    and norm-Y: ⋀n. norm (Y n) ≤ B
    using Y [unfolded Bseq-def] by fast
  from X show ?thesis
  proof (rule Zseq-imp-Zseq)
    fix n::nat
    have norm (X n ** Y n) ≤ norm (X n) * norm (Y n) * K
      by (rule norm-le)
    also have ... ≤ norm (X n) * B * K
      by (intro mult-mono' order-refl norm-Y norm-ge-zero
              mult-nonneg-nonneg K)
    also have ... = norm (X n) * (B * K)
      by (rule mult-assoc)
    finally show norm (X n ** Y n) ≤ norm (X n) * (B * K) .
  qed
qed

lemma (in bounded-bilinear) Bseq-prod-Zseq:
  assumes X: Bseq X
  assumes Y: Zseq Y
  shows Zseq (λn. X n ** Y n)
proof -
  obtain K where K: 0 ≤ K
    and norm-le: ⋀x y. norm (x ** y) ≤ norm x * norm y * K
    using nonneg-bounded by fast
  obtain B where B: 0 < B
    and norm-X: ⋀n. norm (X n) ≤ B
    using X [unfolded Bseq-def] by fast
  from Y show ?thesis
  proof (rule Zseq-imp-Zseq)
    fix n::nat
    have norm (X n ** Y n) ≤ norm (X n) * norm (Y n) * K
      by (rule norm-le)
    also have ... ≤ B * norm (Y n) * K
      by (intro mult-mono' order-refl norm-X norm-ge-zero
              mult-nonneg-nonneg K)
    also have ... = norm (Y n) * (B * K)
      by (simp only: mult-ac)
    finally show norm (X n ** Y n) ≤ norm (Y n) * (B * K) .
  qed
qed
```

lemma (**in** *bounded-bilinear*) *Zseq-left*:
$Zseq\ X \Longrightarrow Zseq\ (\lambda n.\ X\ n\ **\ a)$
by (*rule bounded-linear-left* [*THEN bounded-linear.Zseq*])

lemma (**in** *bounded-bilinear*) *Zseq-right*:
$Zseq\ X \Longrightarrow Zseq\ (\lambda n.\ a\ **\ X\ n)$
by (*rule bounded-linear-right* [*THEN bounded-linear.Zseq*])

lemmas *Zseq-mult* = *mult.Zseq*
lemmas *Zseq-mult-right* = *mult.Zseq-right*
lemmas *Zseq-mult-left* = *mult.Zseq-left*

12.3 Limits of Sequences

lemma *LIMSEQ-iff*:
$(X\ {-}{-}{-}{-}{>}\ L) = (\forall r{>}0.\ \exists no.\ \forall n \geq no.\ norm\ (X\ n - L) < r)$
by (*rule LIMSEQ-def*)

lemma *LIMSEQ-Zseq-iff*: $((\lambda n.\ X\ n)\ {-}{-}{-}{-}{>}\ L) = Zseq\ (\lambda n.\ X\ n - L)$
by (*simp only*: *LIMSEQ-def Zseq-def*)

lemma *LIMSEQ-I*:
$(\bigwedge r.\ 0 < r \Longrightarrow \exists no.\ \forall n{\geq}no.\ norm\ (X\ n - L) < r) \Longrightarrow X\ {-}{-}{-}{-}{>}\ L$
by (*simp add*: *LIMSEQ-def*)

lemma *LIMSEQ-D*:
$[\![X\ {-}{-}{-}{-}{>}\ L;\ 0 < r]\!] \Longrightarrow \exists no.\ \forall n{\geq}no.\ norm\ (X\ n - L) < r$
by (*simp add*: *LIMSEQ-def*)

lemma *LIMSEQ-const*: $(\lambda n.\ k)\ {-}{-}{-}{-}{>}\ k$
by (*simp add*: *LIMSEQ-def*)

lemma *LIMSEQ-const-iff*: $(\lambda n.\ k)\ {-}{-}{-}{-}{>}\ l = (k = l)$
by (*simp add*: *LIMSEQ-Zseq-iff Zseq-const-iff*)

lemma *LIMSEQ-norm*: $X\ {-}{-}{-}{-}{>}\ a \Longrightarrow (\lambda n.\ norm\ (X\ n))\ {-}{-}{-}{-}{>}\ norm\ a$
apply (*simp add*: *LIMSEQ-def*, *safe*)
apply (*drule-tac x=r* **in** *spec*, *safe*)
apply (*rule-tac x=no* **in** *exI*, *safe*)
apply (*drule-tac x=n* **in** *spec*, *safe*)
apply (*erule order-le-less-trans* [*OF norm-triangle-ineq3*])
done

lemma *LIMSEQ-ignore-initial-segment*:
$f\ {-}{-}{-}{-}{>}\ a \Longrightarrow (\lambda n.\ f\ (n + k))\ {-}{-}{-}{-}{>}\ a$
apply (*rule LIMSEQ-I*)
apply (*drule* (*1*) *LIMSEQ-D*)
apply (*erule exE*, *rename-tac N*)
apply (*rule-tac x=N* **in** *exI*)

apply *simp*
done

lemma *LIMSEQ-offset*:
 $(\lambda n.\ f\ (n + k))$ ----> $a \Longrightarrow f$ ----> a
apply (*rule LIMSEQ-I*)
apply (*drule* (*1*) *LIMSEQ-D*)
apply (*erule exE*, *rename-tac N*)
apply (*rule-tac x=N + k* **in** *exI*)
apply *clarify*
apply (*drule-tac x=n − k* **in** *spec*)
apply (*simp add*: *le-diff-conv2*)
done

lemma *LIMSEQ-Suc*: f ----> $l \Longrightarrow (\lambda n.\ f\ (Suc\ n))$ ----> l
by (*drule-tac k=1* **in** *LIMSEQ-ignore-initial-segment*, *simp*)

lemma *LIMSEQ-imp-Suc*: $(\lambda n.\ f\ (Suc\ n))$ ----> $l \Longrightarrow f$ ----> l
by (*rule-tac k=1* **in** *LIMSEQ-offset*, *simp*)

lemma *LIMSEQ-Suc-iff*: $(\lambda n.\ f\ (Suc\ n))$ ----> $l = f$ ----> l
by (*blast intro*: *LIMSEQ-imp-Suc LIMSEQ-Suc*)

lemma *add-diff-add*:
 fixes *a b c d* :: *'a::ab-group-add*
 shows $(a + c) - (b + d) = (a - b) + (c - d)$
by *simp*

lemma *minus-diff-minus*:
 fixes *a b* :: *'a::ab-group-add*
 shows $(- a) - (- b) = - (a - b)$
by *simp*

lemma *LIMSEQ-add*: $[\![X$ ----> $a;\ Y$ ----> $b]\!] \Longrightarrow (\lambda n.\ X\ n + Y\ n)$ ----> $a + b$
by (*simp only*: *LIMSEQ-Zseq-iff add-diff-add Zseq-add*)

lemma *LIMSEQ-minus*: X ----> $a \Longrightarrow (\lambda n. - X\ n)$ ----> $- a$
by (*simp only*: *LIMSEQ-Zseq-iff minus-diff-minus Zseq-minus*)

lemma *LIMSEQ-minus-cancel*: $(\lambda n. - X\ n)$ ----> $- a \Longrightarrow X$ ----> a
by (*drule LIMSEQ-minus*, *simp*)

lemma *LIMSEQ-diff*: $[\![X$ ----> $a;\ Y$ ----> $b]\!] \Longrightarrow (\lambda n.\ X\ n - Y\ n)$ ----> $a - b$
by (*simp add*: *diff-minus LIMSEQ-add LIMSEQ-minus*)

lemma *LIMSEQ-unique*: $[\![X$ ----> $a;\ X$ ----> $b]\!] \Longrightarrow a = b$
by (*drule* (*1*) *LIMSEQ-diff*, *simp add*: *LIMSEQ-const-iff*)

```
lemma (in bounded-linear) LIMSEQ:
  X ----> a ⟹ (λn. f (X n)) ----> f a
by (simp only: LIMSEQ-Zseq-iff diff [symmetric] Zseq)

lemma (in bounded-bilinear) LIMSEQ:
  ⟦X ----> a; Y ----> b⟧ ⟹ (λn. X n ** Y n) ----> a ** b
by (simp only: LIMSEQ-Zseq-iff prod-diff-prod
              Zseq-add Zseq Zseq-left Zseq-right)

lemma LIMSEQ-mult:
  fixes a b :: 'a::real-normed-algebra
  shows [| X ----> a; Y ----> b |] ==> (%n. X n * Y n) ----> a * b
by (rule mult.LIMSEQ)

lemma inverse-diff-inverse:
  ⟦(a::'a::division-ring) ≠ 0; b ≠ 0⟧
   ⟹ inverse a − inverse b = − (inverse a * (a − b) * inverse b)
by (simp add: ring-simps)

lemma Bseq-inverse-lemma:
  fixes x :: 'a::real-normed-div-algebra
  shows ⟦r ≤ norm x; 0 < r⟧ ⟹ norm (inverse x) ≤ inverse r
apply (subst nonzero-norm-inverse, clarsimp)
apply (erule (1) le-imp-inverse-le)
done

lemma Bseq-inverse:
  fixes a :: 'a::real-normed-div-algebra
  assumes X: X ----> a
  assumes a: a ≠ 0
  shows Bseq (λn. inverse (X n))
proof −
  from a have 0 < norm a by simp
  hence ∃r>0. r < norm a by (rule dense)
  then obtain r where r1: 0 < r and r2: r < norm a by fast
  obtain N where N: ⋀n. N ≤ n ⟹ norm (X n − a) < r
    using LIMSEQ-D [OF X r1] by fast
  show ?thesis
  proof (rule BseqI2 [rule-format])
    fix n assume n: N ≤ n
    hence 1: norm (X n − a) < r by (rule N)
    hence 2: X n ≠ 0 using r2 by auto
    hence norm (inverse (X n)) = inverse (norm (X n))
      by (rule nonzero-norm-inverse)
    also have ... ≤ inverse (norm a − r)
    proof (rule le-imp-inverse-le)
      show 0 < norm a − r using r2 by simp
    next
```

have *norm a* $-$ *norm* $(X\ n) \leq$ *norm* $(a - X\ n)$
by (*rule norm-triangle-ineq2*)
also have $\ldots =$ *norm* $(X\ n - a)$
by (*rule norm-minus-commute*)
also have $\ldots < r$ **using** *1* **.**
finally show *norm a* $- r \leq$ *norm* $(X\ n)$ **by** *simp*
qed
finally show *norm* (*inverse* $(X\ n)$) $\leq$ *inverse* (*norm a* $- r$) **.**
qed
qed

lemma *LIMSEQ-inverse-lemma*:
fixes *a* :: *'a::real-normed-div-algebra*
shows $[\![X$ ----> $a;\ a \neq 0;\ \forall n.\ X\ n \neq 0]\!]$
$\Longrightarrow$ ($\lambda n.$ *inverse* $(X\ n)$) ----> *inverse a*
apply (*subst LIMSEQ-Zseq-iff*)
apply (*simp add*: *inverse-diff-inverse nonzero-imp-inverse-nonzero*)
apply (*rule Zseq-minus*)
apply (*rule Zseq-mult-left*)
apply (*rule mult.Bseq-prod-Zseq*)
apply (*erule* (*1*) *Bseq-inverse*)
apply (*simp add*: *LIMSEQ-Zseq-iff*)
done

lemma *LIMSEQ-inverse*:
fixes *a* :: *'a::real-normed-div-algebra*
assumes *X*: *X* ----> *a*
assumes *a*: $a \neq 0$
shows ($\lambda n.$ *inverse* $(X\ n)$) ----> *inverse a*
proof $-$
from *a* **have** $0 <$ *norm a* **by** *simp*
then obtain *k* **where** $\forall n \geq k.$ *norm* $(X\ n - a) <$ *norm a*
using *LIMSEQ-D* [*OF X*] **by** *fast*
hence $\forall n \geq k.\ X\ n \neq 0$ **by** *auto*
hence *k*: $\forall n.\ X\ (n + k) \neq 0$ **by** *simp*

from *X* **have** ($\lambda n.\ X\ (n + k)$) ----> *a*
by (*rule LIMSEQ-ignore-initial-segment*)
hence ($\lambda n.$ *inverse* $(X\ (n + k))$) ----> *inverse a*
using *a k* **by** (*rule LIMSEQ-inverse-lemma*)
thus ($\lambda n.$ *inverse* $(X\ n)$) ----> *inverse a*
by (*rule LIMSEQ-offset*)
qed

lemma *LIMSEQ-divide*:
fixes *a b* :: *'a::real-normed-field*
shows $[\![X$ ----> $a;\ Y$ ----> $b;\ b \neq 0]\!] \Longrightarrow$ ($\lambda n.\ X\ n\ /\ Y\ n$) ----> *a*
/ *b*
by (*simp add*: *LIMSEQ-mult LIMSEQ-inverse divide-inverse*)

lemma *LIMSEQ-pow*:
  fixes a :: $'a$::{*real-normed-algebra*,*recpower*}
  shows X $---->$ $a \Longrightarrow (\lambda n.\ (X\ n)\ \hat{}\ m)$ $---->$ $a\ \hat{}\ m$
by (*induct m*) (*simp-all add*: *power-Suc LIMSEQ-const LIMSEQ-mult*)

lemma *LIMSEQ-setsum*:
  assumes n: $\bigwedge n.\ n \in S \Longrightarrow X\ n$ $---->$ $L\ n$
  shows $(\lambda m.\ \sum n{\in}S.\ X\ n\ m)$ $---->$ $(\sum n{\in}S.\ L\ n)$
proof (*cases finite S*)
  case *True*
  thus *?thesis* **using** n
  proof (*induct*)
    case *empty*
    show *?case*
      by (*simp add*: *LIMSEQ-const*)
  next
    case *insert*
    thus *?case*
      by (*simp add*: *LIMSEQ-add*)
  qed
next
  case *False*
  thus *?thesis*
    by (*simp add*: *LIMSEQ-const*)
qed

lemma *LIMSEQ-setprod*:
  fixes L :: $'a \Rightarrow 'b$::{*real-normed-algebra*,*comm-ring-1*}
  assumes n: $\bigwedge n.\ n \in S \Longrightarrow X\ n$ $---->$ $L\ n$
  shows $(\lambda m.\ \prod n{\in}S.\ X\ n\ m)$ $---->$ $(\prod n{\in}S.\ L\ n)$
proof (*cases finite S*)
  case *True*
  thus *?thesis* **using** n
  proof (*induct*)
    case *empty*
    show *?case*
      by (*simp add*: *LIMSEQ-const*)
  next
    case *insert*
    thus *?case*
      by (*simp add*: *LIMSEQ-mult*)
  qed
next
  case *False*
  thus *?thesis*
    by (*simp add*: *setprod-def LIMSEQ-const*)
qed

lemma *LIMSEQ-add-const*: f ––––> a ==> (%n.(f n + b)) ––––> a + b
by (*simp add*: *LIMSEQ-add LIMSEQ-const*)

lemma *LIMSEQ-add-minus*:
[| X ––––> a; Y ––––> b |] ==> (%n. X n + $-Y$ n) ––––> a + $-b$
by (*simp only*: *LIMSEQ-add LIMSEQ-minus*)

lemma *LIMSEQ-diff-const*: f ––––> a ==> (%n.(f n − b)) ––––> a − b
by (*simp add*: *LIMSEQ-diff LIMSEQ-const*)

lemma *LIMSEQ-diff-approach-zero*:
g ––––> L ==> (%x. f x − g x) ––––> *0* ==>
f ––––> L
apply (*drule LIMSEQ-add*)
apply *assumption*
apply *simp*
done

lemma *LIMSEQ-diff-approach-zero2*:
f ––––> L ==> (%x. f x − g x) ––––> *0* ==>
g ––––> L
apply (*drule LIMSEQ-diff*)
apply *assumption*
apply *simp*
done

A sequence tends to zero iff its abs does

lemma *LIMSEQ-norm-zero*: ((λn. *norm* (X n)) ––––> *0*) = (X ––––> *0*)
by (*simp add*: *LIMSEQ-def*)

lemma *LIMSEQ-rabs-zero*: ((%n. $|f\ n|$) ––––> *0*) = (f ––––> (*0*::*real*))
by (*simp add*: *LIMSEQ-def*)

lemma *LIMSEQ-imp-rabs*: f ––––> (l::*real*) ==> (%n. $|f\ n|$) ––––> $|l|$
by (*drule LIMSEQ-norm*, *simp*)

An unbounded sequence's inverse tends to 0

lemma *LIMSEQ-inverse-zero*:
$\forall$ r::*real*. $\exists$ N. $\forall$ $n \geq N$. r < X n $\Longrightarrow$ (λn. *inverse* (X n)) ––––> *0*
apply (*rule LIMSEQ-I*)
apply (*drule-tac x=inverse r* **in** *spec*, *safe*)
apply (*rule-tac x=N* **in** *exI*, *safe*)
apply (*drule-tac x=n* **in** *spec*, *safe*)
apply (*frule positive-imp-inverse-positive*)
apply (*frule* (*1*) *less-imp-inverse-less*)
apply (*subgoal-tac 0* < X n, *simp*)
apply (*erule* (*1*) *order-less-trans*)
done

The sequence $(1::'a)$ / n tends to 0 as n tends to infinity

lemma *LIMSEQ-inverse-real-of-nat*: (%n. *inverse*(*real*(*Suc n*))) ----> 0
apply (*rule LIMSEQ-inverse-zero*, *safe*)
apply (*cut-tac x* = *r* **in** *reals-Archimedean2*)
apply (*safe*, *rule-tac x* = *n* **in** *exI*)
apply (*auto simp add*: *real-of-nat-Suc*)
done

The sequence r + $(1::'a)$ / n tends to r as n tends to infinity is now easily proved

lemma *LIMSEQ-inverse-real-of-nat-add*:
(%n. *r* + *inverse*(*real*(*Suc n*))) ----> *r*
by (*cut-tac LIMSEQ-add* [*OF LIMSEQ-const LIMSEQ-inverse-real-of-nat*], *auto*)

lemma *LIMSEQ-inverse-real-of-nat-add-minus*:
(%n. *r* + −*inverse*(*real*(*Suc n*))) ----> *r*
by (*cut-tac LIMSEQ-add-minus* [*OF LIMSEQ-const LIMSEQ-inverse-real-of-nat*], *auto*)

lemma *LIMSEQ-inverse-real-of-nat-add-minus-mult*:
(%n. *r*∗(1 + −*inverse*(*real*(*Suc n*)))) ----> *r*
by (*cut-tac b*=1 **in**
LIMSEQ-mult [*OF LIMSEQ-const LIMSEQ-inverse-real-of-nat-add-minus*], *auto*)

lemma *LIMSEQ-le-const*:
⟦X ----> (x::real); ∃ N. ∀ n≥N. a ≤ X n⟧ ⟹ a ≤ x
apply (*rule ccontr*, *simp only*: *linorder-not-le*)
apply (*drule-tac r*=*a* − *x* **in** *LIMSEQ-D*, *simp*)
apply *clarsimp*
apply (*drule-tac x*=*max N no* **in** *spec*, *drule mp*, *rule le-maxI1*)
apply (*drule-tac x*=*max N no* **in** *spec*, *drule mp*, *rule le-maxI2*)
apply *simp*
done

lemma *LIMSEQ-le-const2*:
⟦X ----> (x::real); ∃ N. ∀ n≥N. X n ≤ a⟧ ⟹ x ≤ a
apply (*subgoal-tac* − *a* ≤ − *x*, *simp*)
apply (*rule LIMSEQ-le-const*)
apply (*erule LIMSEQ-minus*)
apply *simp*
done

lemma *LIMSEQ-le*:
⟦X ----> x; Y ----> y; ∃ N. ∀ n≥N. X n ≤ Y n⟧ ⟹ x ≤ (y::real)
apply (*subgoal-tac* 0 ≤ *y* − *x*, *simp*)
apply (*rule LIMSEQ-le-const*)
apply (*erule* (1) *LIMSEQ-diff*)
apply (*simp add*: *le-diff-eq*)

done

12.4 Convergence

lemma *limI*: X ----> L ==> $lim\ X = L$
apply (*simp add*: *lim-def*)
apply (*blast intro*: *LIMSEQ-unique*)
done

lemma *convergentD*: *convergent* X ==> $\exists L.$ (X ----> L)
by (*simp add*: *convergent-def*)

lemma *convergentI*: (X ----> L) ==> *convergent* X
by (*auto simp add*: *convergent-def*)

lemma *convergent-LIMSEQ-iff*: *convergent* X = (X ----> *lim* X)
by (*auto intro*: *theI LIMSEQ-unique simp add*: *convergent-def lim-def*)

lemma *convergent-minus-iff*: (*convergent* X) = (*convergent* (%n. $-(X\ n)$))
apply (*simp add*: *convergent-def*)
apply (*auto dest*: *LIMSEQ-minus*)
apply (*drule LIMSEQ-minus*, *auto*)
done

12.5 Bounded Monotonic Sequences

Subsequence (alternative definition, (e.g. Hoskins)

lemma *subseq-Suc-iff*: *subseq* f = ($\forall n.$ ($f\ n$) < (f (*Suc* n)))
apply (*simp add*: *subseq-def*)
apply (*auto dest*!: *less-imp-Suc-add*)
apply (*induct-tac k*)
apply (*auto intro*: *less-trans*)
done

lemma *monoseq-Suc*:
  monoseq X = (($\forall n.$ $X\ n \le X$ (*Suc* n))
    | ($\forall n.$ X (*Suc* n) $\le X\ n$))
apply (*simp add*: *monoseq-def*)
apply (*auto dest*!: *le-imp-less-or-eq*)
apply (*auto intro*!: *lessI* [*THEN less-imp-le*] *dest*!: *less-imp-Suc-add*)
apply (*induct-tac ka*)
apply (*auto intro*: *order-trans*)
apply (*erule contrapos-np*)
apply (*induct-tac k*)
apply (*auto intro*: *order-trans*)
done

lemma *monoI1*: $\forall m.\ \forall\ n \ge m.\ X\ m \le X\ n$ ==> *monoseq* X
by (*simp add*: *monoseq-def*)

lemma *monoI2*: $\forall\, m.\ \forall\ \ n \geq m.\ X\ n \leq X\ m$ ==> *monoseq X*
by (*simp add*: *monoseq-def*)

lemma *mono-SucI1*: $\forall\, n.\ X\ n \leq X\ (Suc\ n)$ ==> *monoseq X*
by (*simp add*: *monoseq-Suc*)

lemma *mono-SucI2*: $\forall\, n.\ X\ (Suc\ n) \leq X\ n$ ==> *monoseq X*
by (*simp add*: *monoseq-Suc*)

Bounded Sequence

lemma *BseqD*: *Bseq X* ==> $\exists\, K.\ 0 < K$ & $(\forall\, n.\ norm\ (X\ n) \leq K)$
by (*simp add*: *Bseq-def*)

lemma *BseqI*: [| $0 < K$; $\forall\, n.\ norm\ (X\ n) \leq K$ |] ==> *Bseq X*
by (*auto simp add*: *Bseq-def*)

lemma *lemma-NBseq-def*:
    $(\exists\, K > 0.\ \forall\, n.\ norm\ (X\ n) \leq K) =$
    $(\exists\, N.\ \forall\, n.\ norm\ (X\ n) \leq real(Suc\ N))$
apply *auto*
 prefer *2* **apply** *force*
apply (*cut-tac x* = *K* **in** *reals-Archimedean2*, *clarify*)
apply (*rule-tac x* = *n* **in** *exI*, *clarify*)
apply (*drule-tac x* = *na* **in** *spec*)
apply (*auto simp add*: *real-of-nat-Suc*)
done

alternative definition for Bseq

lemma *Bseq-iff*: *Bseq X* = $(\exists\, N.\ \forall\, n.\ norm\ (X\ n) \leq real(Suc\ N))$
apply (*simp add*: *Bseq-def*)
apply (*simp* (*no-asm*) *add*: *lemma-NBseq-def*)
done

lemma *lemma-NBseq-def2*:
    $(\exists\, K > 0.\ \forall\, n.\ norm\ (X\ n) \leq K) = (\exists\, N.\ \forall\, n.\ norm\ (X\ n) < real(Suc\ N))$
apply (*subst lemma-NBseq-def*, *auto*)
apply (*rule-tac x* = *Suc N* **in** *exI*)
apply (*rule-tac* [*2*] *x* = *N* **in** *exI*)
apply (*auto simp add*: *real-of-nat-Suc*)
 prefer *2* **apply** (*blast intro*: *order-less-imp-le*)
apply (*drule-tac x* = *n* **in** *spec*, *simp*)
done

lemma *Bseq-iff1a*: *Bseq X* = $(\exists\, N.\ \forall\, n.\ norm\ (X\ n) < real(Suc\ N))$
by (*simp add*: *Bseq-def lemma-NBseq-def2*)

12.5.1 Upper Bounds and Lubs of Bounded Sequences

lemma *Bseq-isUb*:
!!(X::*nat*=>*real*). *Bseq X* ==> $\exists$ *U*. *isUb* (*UNIV*::*real set*) {*x*. $\exists$ *n*. *X n* = *x*} *U*
by (*auto intro*: *isUbI setleI simp add*: *Bseq-def abs-le-iff*)

Use completeness of reals (supremum property) to show that any bounded sequence has a least upper bound

lemma *Bseq-isLub*:
!!(X::*nat*=>*real*). *Bseq X* ==>
$\exists$ *U*. *isLub* (*UNIV*::*real set*) {*x*. $\exists$ *n*. *X n* = *x*} *U*
by (*blast intro*: *reals-complete Bseq-isUb*)

12.5.2 A Bounded and Monotonic Sequence Converges

lemma *lemma-converg1*:
!!(X::*nat*=>*real*). [| $\forall$ *m*. $\forall$ *n* $\geq$ *m*. *X m* $\leq$ *X n*;
isLub (*UNIV*::*real set*) {*x*. $\exists$ *n*. *X n* = *x*} (*X ma*)
|] ==> $\forall$ *n* $\geq$ *ma*. *X n* = *X ma*
apply *safe*
apply (*drule-tac y* = *X n* **in** *isLubD2*)
apply (*blast dest*: *order-antisym*)+
done

The best of both worlds: Easier to prove this result as a standard theorem and then use equivalence to "transfer" it into the equivalent nonstandard form if needed!

lemma *Bmonoseq-LIMSEQ*: $\forall$ *n*. *m* $\leq$ *n* --> *X n* = *X m* ==> $\exists$ *L*. (*X* ----> *L*)
apply (*simp add*: *LIMSEQ-def*)
apply (*rule-tac x* = *X m* **in** *exI*, *safe*)
apply (*rule-tac x* = *m* **in** *exI*, *safe*)
apply (*drule spec*, *erule impE*, *auto*)
done

lemma *lemma-converg2*:
!!(X::*nat*=>*real*).
[| $\forall$ *m*. *X m* ~= *U*; *isLub UNIV* {*x*. $\exists$ *n*. *X n* = *x*} *U* |] ==> $\forall$ *m*. *X m* < *U*
apply *safe*
apply (*drule-tac y* = *X m* **in** *isLubD2*)
apply (*auto dest*!: *order-le-imp-less-or-eq*)
done

lemma *lemma-converg3*: !!(X ::*nat*=>*real*). $\forall$ *m*. *X m* $\leq$ *U* ==> *isUb UNIV* {*x*. $\exists$ *n*. *X n* = *x*} *U*
by (*rule setleI* [*THEN isUbI*], *auto*)

FIXME: $U - T < U$ is redundant

lemma *lemma-converg4*: !!(X::*nat*=> *real*).

[| $\forall m.$ *X m* ~= *U*;
isLub UNIV $\{x. \exists n.$ *X n* $= x\}$ *U*;
0 $<$ *T*;
U $+ -$ *T* $<$ *U*
|] ==> $\exists m.$ *U* $+ -T <$ *X m* & *X m* $<$ *U*
apply (*drule lemma-converg2*, *assumption*)
apply (*rule ccontr*, *simp*)
apply (*simp add*: *linorder-not-less*)
apply (*drule lemma-converg3*)
apply (*drule isLub-le-isUb*, *assumption*)
apply (*auto dest*: *order-less-le-trans*)
done

A standard proof of the theorem for monotone increasing sequence

lemma *Bseq-mono-convergent*:
[| *Bseq X*; $\forall m. \forall n \geq m.$ *X m* $\leq$ *X n* |] ==> *convergent* (*X*::*nat*=>*real*)
apply (*simp add*: *convergent-def*)
apply (*frule Bseq-isLub*, *safe*)
apply (*case-tac* $\exists m.$ *X m* $=$ *U*, *auto*)
apply (*blast dest*: *lemma-converg1 Bmonoseq-LIMSEQ*)

apply (*rule-tac x* $=$ *U* **in** *exI*)
apply (*subst LIMSEQ-iff*, *safe*)
apply (*frule lemma-converg2*, *assumption*)
apply (*drule lemma-converg4*, *auto*)
apply (*rule-tac x* $=$ *m* **in** *exI*, *safe*)
apply (*subgoal-tac X m* $\leq$ *X n*)
prefer *2* **apply** *blast*
apply (*drule-tac x*=*n* **and** *P*=%*m*. *X m* $<$ *U* **in** *spec*, *arith*)
done

lemma *Bseq-minus-iff*: *Bseq* (%*n*. $-$(*X n*)) = *Bseq X*
by (*simp add*: *Bseq-def*)

Main monotonicity theorem

lemma *Bseq-monoseq-convergent*: [| *Bseq X*; *monoseq X* |] ==> *convergent X*
apply (*simp add*: *monoseq-def*, *safe*)
apply (*rule-tac* [*2*] *convergent-minus-iff* [*THEN ssubst*])
apply (*drule-tac* [*2*] *Bseq-minus-iff* [*THEN ssubst*])
apply (*auto intro*!: *Bseq-mono-convergent*)
done

12.5.3 A Few More Equivalence Theorems for Boundedness

alternative formulation for boundedness

lemma *Bseq-iff2*: *Bseq X* $= (\exists k > 0. \exists x. \forall n.$ *norm* $(X(n) + -x) \leq k)$
apply (*unfold Bseq-def*, *safe*)
apply (*rule-tac* [*2*] *x* $=$ *k* $+$ *norm x* **in** *exI*)

apply (*rule-tac* $x = K$ **in** *exI*, *simp*)
apply (*rule exI* [**where** $x = 0$], *auto*)
apply (*erule order-less-le-trans*, *simp*)
apply (*drule-tac* $x=n$ **in** *spec*, *fold diff-def*)
apply (*drule order-trans* [*OF norm-triangle-ineq2*])
apply *simp*
done

alternative formulation for boundedness

lemma *Bseq-iff3*: $Bseq\ X = (\exists\, k > 0.\ \exists\, N.\ \forall\, n.\ norm(X(n) + -X(N)) \leq k)$
apply *safe*
apply (*simp add*: *Bseq-def*, *safe*)
apply (*rule-tac* $x = K + norm\ (X\ N)$ **in** *exI*)
apply *auto*
apply (*erule order-less-le-trans*, *simp*)
apply (*rule-tac* $x = N$ **in** *exI*, *safe*)
apply (*drule-tac* $x = n$ **in** *spec*)
apply (*rule order-trans* [*OF norm-triangle-ineq*], *simp*)
apply (*auto simp add*: *Bseq-iff2*)
done

lemma *BseqI2*: $(\forall\, n.\ k \leq f\ n\ \&\ f\ n \leq (K{::}real))$ ==> $Bseq\ f$
apply (*simp add*: *Bseq-def*)
apply (*rule-tac* $x = (|k| + |K|) + 1$ **in** *exI*, *auto*)
apply (*drule-tac* $x = n$ **in** *spec*, *arith*)
done

12.6 Cauchy Sequences

lemma *CauchyI*:
  $(\bigwedge e.\ 0 < e \Longrightarrow \exists\, M.\ \forall\, m{\geq}M.\ \forall\, n{\geq}M.\ norm\ (X\ m - X\ n) < e) \Longrightarrow Cauchy\ X$
by (*simp add*: *Cauchy-def*)

lemma *CauchyD*:
  $[\![Cauchy\ X;\ 0 < e]\!] \Longrightarrow \exists\, M.\ \forall\, m{\geq}M.\ \forall\, n{\geq}M.\ norm\ (X\ m - X\ n) < e$
by (*simp add*: *Cauchy-def*)

12.6.1 Cauchy Sequences are Bounded

A Cauchy sequence is bounded – this is the standard proof mechanization rather than the nonstandard proof

lemma *lemmaCauchy*: $\forall\, n \geq M.\ norm\ (X\ M - X\ n) < (1{::}real)$
  ==> $\forall\, n \geq M.\ norm\ (X\ n :: 'a{::}real\text{-}normed\text{-}vector) < 1 + norm\ (X\ M)$
apply (*clarify*, *drule spec*, *drule* (*1*) *mp*)
apply (*simp only*: *norm-minus-commute*)
apply (*drule order-le-less-trans* [*OF norm-triangle-ineq2*])
apply *simp*
done

lemma *Cauchy-Bseq*: *Cauchy X ==> Bseq X*
apply (*simp add*: *Cauchy-def*)
apply (*drule spec*, *drule mp*, *rule zero-less-one*, *safe*)
apply (*drule-tac x=M* **in** *spec*, *simp*)
apply (*drule lemmaCauchy*)
apply (*rule-tac k=M* **in** *Bseq-offset*)
apply (*simp add*: *Bseq-def*)
apply (*rule-tac x=1 + norm (X M)* **in** *exI*)
apply (*rule conjI*, *rule order-less-le-trans* [*OF zero-less-one*], *simp*)
apply (*simp add*: *order-less-imp-le*)
done

12.6.2 Cauchy Sequences are Convergent

axclass *banach* $\subseteq$ *real-normed-vector*
Cauchy-convergent: *Cauchy X* $\Longrightarrow$ *convergent X*

theorem *LIMSEQ-imp-Cauchy*:
assumes *X*: *X ----> a* **shows** *Cauchy X*
proof (*rule CauchyI*)
fix *e*::*real* **assume** *0 < e*
hence *0 < e/2* **by** *simp*
with *X* **have** $\exists N. \forall n \geq N.$ *norm* $(X\ n - a) < e/2$ **by** (*rule LIMSEQ-D*)
then obtain *N* **where** *N*: $\forall n \geq N.$ *norm* $(X\ n - a) < e/2$ **..**
show $\exists N. \forall m \geq N. \forall n \geq N.$ *norm* $(X\ m - X\ n) < e$
proof (*intro exI allI impI*)
fix *m* **assume** $N \leq m$
hence *m*: *norm* $(X\ m - a) < e/2$ **using** *N* **by** *fast*
fix *n* **assume** $N \leq n$
hence *n*: *norm* $(X\ n - a) < e/2$ **using** *N* **by** *fast*
have *norm* $(X\ m - X\ n) =$ *norm* $((X\ m - a) - (X\ n - a))$ **by** *simp*
also have $\ldots \leq$ *norm* $(X\ m - a) +$ *norm* $(X\ n - a)$
by (*rule norm-triangle-ineq4*)
also from *m n* **have** $\ldots < e$ **by**(*simp add*:*field-simps*)
finally show *norm* $(X\ m - X\ n) < e$ **.**
qed
qed

lemma *convergent-Cauchy*: *convergent X* $\Longrightarrow$ *Cauchy X*
unfolding *convergent-def*
by (*erule exE*, *erule LIMSEQ-imp-Cauchy*)

Proof that Cauchy sequences converge based on the one from http://pirate.shu.edu/ wachsmut/ira/nu

If sequence X is Cauchy, then its limit is the lub of $\{r.\ \exists N.\ \forall n \geq N.\ r < X\ n\}$

lemma *isUb-UNIV-I*: $(\bigwedge y.\ y \in S \Longrightarrow y \leq u) \Longrightarrow$ *isUb UNIV S u*
by (*simp add*: *isUbI setleI*)

lemma *real-abs-diff-less-iff*:
  $(|x - a| < (r::real)) = (a - r < x \wedge x < a + r)$
by *auto*

locale (**open**) *real-Cauchy* =
  fixes $X :: nat \Rightarrow real$
  assumes X: *Cauchy X*
  fixes $S :: real\ set$
  defines *S-def*: $S \equiv \{x::real.\ \exists N.\ \forall n \geq N.\ x < X\ n\}$

lemma (**in** *real-Cauchy*) *mem-S*: $\forall n \geq N.\ x < X\ n \Longrightarrow x \in S$
by (*unfold S-def*, *auto*)

lemma (**in** *real-Cauchy*) *bound-isUb*:
  assumes N: $\forall n \geq N.\ X\ n < x$
  shows *isUb UNIV S x*
proof (*rule isUb-UNIV-I*)
  fix $y::real$ **assume** $y \in S$
  hence $\exists M.\ \forall n \geq M.\ y < X\ n$
    by (*simp add*: *S-def*)
  then obtain M **where** $\forall n \geq M.\ y < X\ n$ **..**
  hence $y < X\ (max\ M\ N)$ **by** *simp*
  also have $\ldots < x$ **using** N **by** *simp*
  finally show $y \leq x$
    by (*rule order-less-imp-le*)
qed

lemma (**in** *real-Cauchy*) *isLub-ex*: $\exists u.$ *isLub UNIV S u*
proof (*rule reals-complete*)
  obtain N **where** $\forall m \geq N.\ \forall n \geq N.\ norm\ (X\ m - X\ n) < 1$
    using *CauchyD* [*OF X zero-less-one*] **by** *fast*
  hence N: $\forall n \geq N.\ norm\ (X\ n - X\ N) < 1$ **by** *simp*
  show $\exists x.\ x \in S$
  proof
    from N **have** $\forall n \geq N.\ X\ N - 1 < X\ n$
      by (*simp add*: *real-abs-diff-less-iff*)
    thus $X\ N - 1 \in S$ **by** (*rule mem-S*)
  qed
  show $\exists u.$ *isUb UNIV S u*
  proof
    from N **have** $\forall n \geq N.\ X\ n < X\ N + 1$
      by (*simp add*: *real-abs-diff-less-iff*)
    thus *isUb UNIV S* $(X\ N + 1)$
      by (*rule bound-isUb*)
  qed
qed

lemma (**in** *real-Cauchy*) *isLub-imp-LIMSEQ*:
  assumes x: *isLub UNIV S x*

```
  shows X ----> x
proof (rule LIMSEQ-I)
  fix r::real assume 0 < r
  hence r: 0 < r/2 by simp
  obtain N where ∀n≥N. ∀m≥N. norm (X n − X m) < r/2
    using CauchyD [OF X r] by fast
  hence ∀n≥N. norm (X n − X N) < r/2 by simp
  hence N: ∀n≥N. X N − r/2 < X n ∧ X n < X N + r/2
    by (simp only: real-norm-def real-abs-diff-less-iff)

  from N have ∀n≥N. X N − r/2 < X n by fast
  hence X N − r/2 ∈ S by (rule mem-S)
  hence 1: X N − r/2 ≤ x using x isLub-isUb isUbD by fast

  from N have ∀n≥N. X n < X N + r/2 by fast
  hence isUb UNIV S (X N + r/2) by (rule bound-isUb)
  hence 2: x ≤ X N + r/2 using x isLub-le-isUb by fast

  show ∃N. ∀n≥N. norm (X n − x) < r
  proof (intro exI allI impI)
    fix n assume n: N ≤ n
    from N n have X n < X N + r/2 and X N − r/2 < X n by simp+
    thus norm (X n − x) < r using 1 2
      by (simp add: real-abs-diff-less-iff)
  qed
qed

lemma (in real-Cauchy) LIMSEQ-ex: ∃x. X ----> x
proof −
  obtain x where isLub UNIV S x
    using isLub-ex by fast
  hence X ----> x
    by (rule isLub-imp-LIMSEQ)
  thus ?thesis ..
qed

lemma real-Cauchy-convergent:
  fixes X :: nat ⇒ real
  shows Cauchy X ⟹ convergent X
unfolding convergent-def by (rule real-Cauchy.LIMSEQ-ex)

instance real :: banach
by intro-classes (rule real-Cauchy-convergent)

lemma Cauchy-convergent-iff:
  fixes X :: nat ⇒ 'a::banach
  shows Cauchy X = convergent X
by (fast intro: Cauchy-convergent convergent-Cauchy)
```

12.7 Power Sequences

The sequence $x \;\hat{}\; n$ tends to 0 if $(0::'a) \leq x$ and $x < (1::'a)$. Proof will use (NS) Cauchy equivalence for convergence and also fact that bounded and monotonic sequence converges.

lemma *Bseq-realpow*: [| $0 \leq (x::real)$; $x \leq 1$ |] ==> *Bseq* (%n. $x \;\hat{}\; n$)
apply (*simp add*: *Bseq-def*)
apply (*rule-tac* $x = 1$ **in** *exI*)
apply (*simp add*: *power-abs*)
apply (*auto dest*: *power-mono*)
done

lemma *monoseq-realpow*: [| $0 \leq x$; $x \leq 1$ |] ==> *monoseq* (%n. $x \;\hat{}\; n$)
apply (*clarify intro*!: *mono-SucI2*)
apply (*cut-tac* $n = n$ **and** $N = Suc\ n$ **and** $a = x$ **in** *power-decreasing*, *auto*)
done

lemma *convergent-realpow*:
  [| $0 \leq (x::real)$; $x \leq 1$ |] ==> *convergent* (%n. $x \;\hat{}\; n$)
by (*blast intro*!: *Bseq-monoseq-convergent Bseq-realpow monoseq-realpow*)

lemma *LIMSEQ-inverse-realpow-zero-lemma*:
  fixes x :: *real*
  assumes x: $0 \leq x$
  shows *real* $n * x + 1 \leq (x + 1) \;\hat{}\; n$
apply (*induct* n)
apply *simp*
apply *simp*
apply (*rule order-trans*)
prefer *2*
apply (*erule mult-left-mono*)
apply (*rule add-increasing* [*OF* x], *simp*)
apply (*simp add*: *real-of-nat-Suc*)
apply (*simp add*: *ring-distribs*)
apply (*simp add*: *mult-nonneg-nonneg* x)
done

lemma *LIMSEQ-inverse-realpow-zero*:
  $1 < (x::real) \Longrightarrow (\lambda n.\ inverse\ (x \;\hat{}\; n))$ ----> 0
proof (*rule LIMSEQ-inverse-zero* [*rule-format*])
  fix y :: *real*
  assume x: $1 < x$
  hence $0 < x - 1$ **by** *simp*
  hence $\forall y.\ \exists N::nat.\ y < real\ N * (x - 1)$
    by (*rule reals-Archimedean3*)
  hence $\exists N::nat.\ y < real\ N * (x - 1)$ **..**
  then obtain N::*nat* **where** $y < real\ N * (x - 1)$ **..**
  also have $\ldots \leq real\ N * (x - 1) + 1$ **by** *simp*
  also have $\ldots \leq (x - 1 + 1) \;\hat{}\; N$

```
    by (rule LIMSEQ-inverse-realpow-zero-lemma, cut-tac x, simp)
  also have ... = x ^ N by simp
  finally have y < x ^ N .
  hence ∀n≥N. y < x ^ n
    apply clarify
    apply (erule order-less-le-trans)
    apply (erule power-increasing)
    apply (rule order-less-imp-le [OF x])
    done
  thus ∃N. ∀n≥N. y < x ^ n ..
qed

lemma LIMSEQ-realpow-zero:
  ⟦0 ≤ (x::real); x < 1⟧ ⟹ (λn. x ^ n) ----> 0
proof (cases)
  assume x = 0
  hence (λn. x ^ Suc n) ----> 0 by (simp add: LIMSEQ-const)
  thus ?thesis by (rule LIMSEQ-imp-Suc)
next
  assume 0 ≤ x and x ≠ 0
  hence x0: 0 < x by simp
  assume x1: x < 1
  from x0 x1 have 1 < inverse x
    by (rule real-inverse-gt-one)
  hence (λn. inverse (inverse x ^ n)) ----> 0
    by (rule LIMSEQ-inverse-realpow-zero)
  thus ?thesis by (simp add: power-inverse)
qed

lemma LIMSEQ-power-zero:
  fixes x :: 'a::{real-normed-algebra-1,recpower}
  shows norm x < 1 ⟹ (λn. x ^ n) ----> 0
apply (drule LIMSEQ-realpow-zero [OF norm-ge-zero])
apply (simp only: LIMSEQ-Zseq-iff, erule Zseq-le)
apply (simp add: power-abs norm-power-ineq)
done

lemma LIMSEQ-divide-realpow-zero:
  1 < (x::real) ==> (%n. a / (x ^ n)) ----> 0
apply (cut-tac a = a and x1 = inverse x in
       LIMSEQ-mult [OF LIMSEQ-const LIMSEQ-realpow-zero])
apply (auto simp add: divide-inverse power-inverse)
apply (simp add: inverse-eq-divide pos-divide-less-eq)
done
```

Limit of c ^ n for $|c| < (1::'a)$

```
lemma LIMSEQ-rabs-realpow-zero: |c| < (1::real) ==> (%n. |c| ^ n) ----> 0
by (rule LIMSEQ-realpow-zero [OF abs-ge-zero])
```

lemma *LIMSEQ-rabs-realpow-zero2*: |*c*| < (*1*::*real*) ==> (%*n*. *c* ^ *n*) ----> *0*
apply (*rule LIMSEQ-rabs-zero* [*THEN iffD1*])
apply (*auto intro*: *LIMSEQ-rabs-realpow-zero simp add*: *power-abs*)
done

end

13 Lim: Limits and Continuity

theory *Lim*
imports *SEQ*
begin

Standard Definitions

definition
LIM :: ['*a*::*real-normed-vector* => '*b*::*real-normed-vector*, '*a*, '*b*] => *bool*
(((-)/ -- (-)/ --> (-)) [*60*, *0*, *60*] *60*) **where**
f -- *a* --> *L* =
(∀ *r* > *0*. ∃ *s* > *0*. ∀ *x*. *x* ≠ *a* & *norm* (*x* − *a*) < *s*
--> *norm* (*f x* − *L*) < *r*)

definition
isCont :: ['*a*::*real-normed-vector* => '*b*::*real-normed-vector*, '*a*] => *bool* **where**
isCont f a = (*f* -- *a* --> (*f a*))

definition
isUCont :: ['*a*::*real-normed-vector* => '*b*::*real-normed-vector*] => *bool* **where**
isUCont f = (∀ *r*>*0*. ∃ *s*>*0*. ∀ *x y*. *norm* (*x* − *y*) < *s* ⟶ *norm* (*f x* − *f y*) < *r*)

13.1 Limits of Functions

13.1.1 Purely standard proofs

lemma *LIM-eq*:
f -- *a* --> *L* =
(∀ *r*>*0*.∃ *s*>*0*.∀ *x*. *x* ≠ *a* & *norm* (*x*−*a*) < *s* --> *norm* (*f x* − *L*) < *r*)
by (*simp add*: *LIM-def diff-minus*)

lemma *LIM-I*:
(!!*r*. *0*<*r* ==> ∃ *s*>*0*.∀ *x*. *x* ≠ *a* & *norm* (*x*−*a*) < *s* --> *norm* (*f x* − *L*) <
r)
==> *f* -- *a* --> *L*
by (*simp add*: *LIM-eq*)

lemma *LIM-D*:
[| *f* -- *a* --> *L*; *0*<*r* |]
==> ∃ *s*>*0*.∀ *x*. *x* ≠ *a* & *norm* (*x*−*a*) < *s* --> *norm* (*f x* − *L*) < *r*
by (*simp add*: *LIM-eq*)

lemma *LIM-offset*: f −− a −−> $L \Longrightarrow (\lambda x.\ f\ (x + k))$ −− $a - k$ −−> L
apply (*rule LIM-I*)
apply (*drule-tac r=r* **in** *LIM-D*, *safe*)
apply (*rule-tac x=s* **in** *exI*, *safe*)
apply (*drule-tac x=x + k* **in** *spec*)
apply (*simp add*: *compare-rls*)
done

lemma *LIM-offset-zero*: f −− a −−> $L \Longrightarrow (\lambda h.\ f\ (a + h))$ −− 0 −−> L
by (*drule-tac k=a* **in** *LIM-offset*, *simp add*: *add-commute*)

lemma *LIM-offset-zero-cancel*: $(\lambda h.\ f\ (a + h))$ −− 0 −−> $L \Longrightarrow f$ −− a −−>
L
by (*drule-tac k=− a* **in** *LIM-offset*, *simp*)

lemma *LIM-const* [*simp*]: (%x. k) −− x −−> k
by (*simp add*: *LIM-def*)

lemma *LIM-add*:
  fixes $f\ g$ :: $'a$::*real-normed-vector* $\Rightarrow$ $'b$::*real-normed-vector*
  assumes f: f −− a −−> L **and** g: g −− a −−> M
  shows (%x. $f\ x + g(x)$) −− a −−> $(L + M)$
proof (*rule LIM-I*)
  fix r :: *real*
  assume r: $0 < r$
  from *LIM-D* [*OF f half-gt-zero* [*OF r*]]
  obtain *fs*
    where *fs*: $0 <$ *fs*
      and *fs-lt*: $\forall x.\ x \neq a$ & *norm* $(x-a) <$ *fs* −−> *norm* $(f\ x - L) < r/2$
  by *blast*
  from *LIM-D* [*OF g half-gt-zero* [*OF r*]]
  obtain *gs*
    where *gs*: $0 <$ *gs*
      and *gs-lt*: $\forall x.\ x \neq a$ & *norm* $(x-a) <$ *gs* −−> *norm* $(g\ x - M) < r/2$
  by *blast*
  show $\exists s>0.\forall x.\ x \neq a \wedge$ *norm* $(x-a) < s \longrightarrow$ *norm* $(f\ x + g\ x - (L + M)) <$
r
  proof (*intro exI conjI strip*)
    show $0 <$ *min fs gs* **by** (*simp add*: *fs gs*)
    fix x :: $'a$
    assume $x \neq a \wedge$ *norm* $(x-a) <$ *min fs gs*
    hence $x \neq a \wedge$ *norm* $(x-a) <$ *fs* $\wedge$ *norm* $(x-a) <$ *gs* **by** *simp*
    with *fs-lt gs-lt*
    have *norm* $(f\ x - L) < r/2$ **and** *norm* $(g\ x - M) < r/2$ **by** *blast+*
    hence *norm* $(f\ x - L) +$ *norm* $(g\ x - M) < r$ **by** *arith*
    thus *norm* $(f\ x + g\ x - (L + M)) < r$
      by (*blast intro*: *norm-diff-triangle-ineq order-le-less-trans*)
  qed

qed

lemma *LIM-add-zero*:
$[\![f \;\text{--}\; a \;\text{-->}\; 0;\; g \;\text{--}\; a \;\text{-->}\; 0]\!] \Longrightarrow (\lambda x.\ f\,x + g\,x) \;\text{--}\; a \;\text{-->}\; 0$
by (*drule* (*1*) *LIM-add*, *simp*)

lemma *minus-diff-minus*:
fixes *a b* :: *'a*::*ab-group-add*
shows $(-\ a) - (-\ b) = -\ (a - b)$
by *simp*

lemma *LIM-minus*: $f \;\text{--}\; a \;\text{-->}\; L \;\text{==>}\; (\%x.\ -f(x)) \;\text{--}\; a \;\text{-->}\; -L$
by (*simp only*: *LIM-eq minus-diff-minus norm-minus-cancel*)

lemma *LIM-add-minus*:
$[|\ f \;\text{--}\; x \;\text{-->}\; l;\; g \;\text{--}\; x \;\text{-->}\; m\ |] \;\text{==>}\; (\%x.\ f(x) + -g(x)) \;\text{--}\; x \;\text{-->}\; (l + -m)$
by (*intro LIM-add LIM-minus*)

lemma *LIM-diff*:
$[|\ f \;\text{--}\; x \;\text{-->}\; l;\; g \;\text{--}\; x \;\text{-->}\; m\ |] \;\text{==>}\; (\%x.\ f(x) - g(x)) \;\text{--}\; x \;\text{-->}\; l{-}m$
by (*simp only*: *diff-minus LIM-add LIM-minus*)

lemma *LIM-zero*: $f \;\text{--}\; a \;\text{-->}\; l \Longrightarrow (\lambda x.\ f\,x - l) \;\text{--}\; a \;\text{-->}\; 0$
by (*simp add*: *LIM-def*)

lemma *LIM-zero-cancel*: $(\lambda x.\ f\,x - l) \;\text{--}\; a \;\text{-->}\; 0 \Longrightarrow f \;\text{--}\; a \;\text{-->}\; l$
by (*simp add*: *LIM-def*)

lemma *LIM-zero-iff*: $(\lambda x.\ f\,x - l) \;\text{--}\; a \;\text{-->}\; 0 = f \;\text{--}\; a \;\text{-->}\; l$
by (*simp add*: *LIM-def*)

lemma *LIM-imp-LIM*:
assumes *f*: $f \;\text{--}\; a \;\text{-->}\; l$
assumes *le*: $\bigwedge x.\ x \neq a \Longrightarrow norm\ (g\ x - m) \leq norm\ (f\,x - l)$
shows $g \;\text{--}\; a \;\text{-->}\; m$
apply (*rule LIM-I*, *drule LIM-D* [*OF f*], *safe*)
apply (*rule-tac x=s* **in** *exI*, *safe*)
apply (*drule-tac x=x* **in** *spec*, *safe*)
apply (*erule* (*1*) *order-le-less-trans* [*OF le*])
done

lemma *LIM-norm*: $f \;\text{--}\; a \;\text{-->}\; l \Longrightarrow (\lambda x.\ norm\ (f\,x)) \;\text{--}\; a \;\text{-->}\; norm\ l$
by (*erule LIM-imp-LIM*, *simp add*: *norm-triangle-ineq3*)

lemma *LIM-norm-zero*: $f \;\text{--}\; a \;\text{-->}\; 0 \Longrightarrow (\lambda x.\ norm\ (f\,x)) \;\text{--}\; a \;\text{-->}\; 0$
by (*drule LIM-norm*, *simp*)

lemma *LIM-norm-zero-cancel*: $(\lambda x.\ norm\ (f\,x)) \;\text{--}\; a \;\text{-->}\; 0 \Longrightarrow f \;\text{--}\; a \;\text{-->}$

0
by (*erule LIM-imp-LIM*, *simp*)

lemma *LIM-norm-zero-iff*: $(\lambda x.\ norm\ (f\ x))\ {-}{-}\ a\ {-}{-}{>}\ 0 = f\ {-}{-}\ a\ {-}{-}{>}\ 0$
by (*rule iffI* [*OF LIM-norm-zero-cancel LIM-norm-zero*])

lemma *LIM-rabs*: $f\ {-}{-}\ a\ {-}{-}{>}\ (l{::}real) \Longrightarrow (\lambda x.\ |f\ x|)\ {-}{-}\ a\ {-}{-}{>}\ |l|$
by (*fold real-norm-def*, *rule LIM-norm*)

lemma *LIM-rabs-zero*: $f\ {-}{-}\ a\ {-}{-}{>}\ (0{::}real) \Longrightarrow (\lambda x.\ |f\ x|)\ {-}{-}\ a\ {-}{-}{>}\ 0$
by (*fold real-norm-def*, *rule LIM-norm-zero*)

lemma *LIM-rabs-zero-cancel*: $(\lambda x.\ |f\ x|)\ {-}{-}\ a\ {-}{-}{>}\ (0{::}real) \Longrightarrow f\ {-}{-}\ a\ {-}{-}{>}$
0
by (*fold real-norm-def*, *rule LIM-norm-zero-cancel*)

lemma *LIM-rabs-zero-iff*: $(\lambda x.\ |f\ x|)\ {-}{-}\ a\ {-}{-}{>}\ (0{::}real) = f\ {-}{-}\ a\ {-}{-}{>}\ 0$
by (*fold real-norm-def*, *rule LIM-norm-zero-iff*)

lemma *LIM-const-not-eq*:
  fixes $a :: 'a{::}real\text{-}normed\text{-}algebra\text{-}1$
  shows $k \neq L \Longrightarrow \neg\ (\lambda x.\ k)\ {-}{-}\ a\ {-}{-}{>}\ L$
apply (*simp add*: *LIM-eq*)
apply (*rule-tac x=norm* $(k - L)$ **in** *exI*, *simp*, *safe*)
apply (*rule-tac x=a +* *of-real* $(s/2)$ **in** *exI*, *simp add*: *norm-of-real*)
done

lemmas *LIM-not-zero* = *LIM-const-not-eq* [**where** $L = 0$]

lemma *LIM-const-eq*:
  fixes $a :: 'a{::}real\text{-}normed\text{-}algebra\text{-}1$
  shows $(\lambda x.\ k)\ {-}{-}\ a\ {-}{-}{>}\ L \Longrightarrow k = L$
apply (*rule ccontr*)
apply (*blast dest*: *LIM-const-not-eq*)
done

lemma *LIM-unique*:
  fixes $a :: 'a{::}real\text{-}normed\text{-}algebra\text{-}1$
  shows $[\![f\ {-}{-}\ a\ {-}{-}{>}\ L;\ f\ {-}{-}\ a\ {-}{-}{>}\ M]\!] \Longrightarrow L = M$
apply (*drule* (*1*) *LIM-diff*)
apply (*auto dest*!: *LIM-const-eq*)
done

lemma *LIM-ident* [*simp*]: $(\lambda x.\ x)\ {-}{-}\ a\ {-}{-}{>}\ a$
by (*auto simp add*: *LIM-def*)

Limits are equal for functions equal except at limit point

lemma *LIM-equal*:
    $[|\ \forall x.\ x \neq a\ {-}{-}{>}\ (f\ x = g\ x)\ |]\ {=}{=}{>}\ (f\ {-}{-}\ a\ {-}{-}{>}\ l) = (g\ {-}{-}\ a\ {-}{-}{>}\ l)$

by (*simp add*: *LIM-def*)

lemma *LIM-cong*:
  $[\![a = b; \bigwedge x.\ x \neq b \Longrightarrow f\,x = g\,x;\ l = m]\!]$
  $\Longrightarrow ((\lambda x.\ f\,x)\ \text{--}\ a\ \text{-->}\ l) = ((\lambda x.\ g\,x)\ \text{--}\ b\ \text{-->}\ m)$
by (*simp add*: *LIM-def*)

lemma *LIM-equal2*:
  assumes *1*: $0 < R$
  assumes *2*: $\bigwedge x.\ [\![x \neq a;\ norm\ (x - a) < R]\!] \Longrightarrow f\,x = g\,x$
  shows $g\ \text{--}\ a\ \text{-->}\ l \Longrightarrow f\ \text{--}\ a\ \text{-->}\ l$
apply (*unfold LIM-def*, *safe*)
apply (*drule-tac x=r* **in** *spec*, *safe*)
apply (*rule-tac x=min s R* **in** *exI*, *safe*)
apply (*simp add*: *1*)
apply (*simp add*: *2*)
done

Two uses in Hyperreal/Transcendental.ML

lemma *LIM-trans*:
    $[|\ (\%x.\ f(x) + -g(x))\ \text{--}\ a\ \text{-->}\ 0;\ \ g\ \text{--}\ a\ \text{-->}\ l\ |] \text{==>}\ f\ \text{--}\ a\ \text{-->}\ l$
apply (*drule LIM-add*, *assumption*)
apply (*auto simp add*: *add-assoc*)
done

lemma *LIM-compose*:
  assumes *g*: $g\ \text{--}\ l\ \text{-->}\ g\ l$
  assumes *f*: $f\ \text{--}\ a\ \text{-->}\ l$
  shows $(\lambda x.\ g\ (f\,x))\ \text{--}\ a\ \text{-->}\ g\ l$
proof (*rule LIM-I*)
  fix *r*::*real* **assume** *r*: $0 < r$
  obtain *s* **where** *s*: $0 < s$
    and *less-r*: $\bigwedge y.\ [\![y \neq l;\ norm\ (y - l) < s]\!] \Longrightarrow norm\ (g\ y - g\ l) < r$
    using *LIM-D* [*OF g r*] **by** *fast*
  obtain *t* **where** *t*: $0 < t$
    and *less-s*: $\bigwedge x.\ [\![x \neq a;\ norm\ (x - a) < t]\!] \Longrightarrow norm\ (f\,x - l) < s$
    using *LIM-D* [*OF f s*] **by** *fast*

  show $\exists\,t{>}0.\ \forall\,x.\ x \neq a \wedge norm\ (x - a) < t \longrightarrow norm\ (g\ (f\,x) - g\ l) < r$
  proof (*rule exI*, *safe*)
    show $0 < t$ **using** *t* **.**
  next
    fix *x* **assume** $x \neq a$ **and** $norm\ (x - a) < t$
    hence $norm\ (f\,x - l) < s$ **by** (*rule less-s*)
    thus $norm\ (g\ (f\,x) - g\ l) < r$
      using *r less-r* **by** (*case-tac* $f\,x = l$, *simp-all*)
  qed
qed

lemma *LIM-compose2*:
assumes f: f $--$ a $-->$ b
assumes g: g $--$ b $-->$ c
assumes *inj*: $\exists\, d{>}0.\ \forall\, x.\ x \neq a \wedge \mathit{norm}\ (x - a) < d \longrightarrow f\,x \neq b$
shows $(\lambda x.\ g\ (f\,x))$ $--$ a $-->$ c
proof (*rule LIM-I*)
fix r :: *real*
assume r: $0 < r$
obtain s **where** s: $0 < s$
and *less-r*: $\bigwedge y.\ [\![y \neq b;\ \mathit{norm}\ (y - b) < s]\!] \Longrightarrow \mathit{norm}\ (g\ y - c) < r$
using *LIM-D* [*OF g r*] **by** *fast*
obtain t **where** t: $0 < t$
and *less-s*: $\bigwedge x.\ [\![x \neq a;\ \mathit{norm}\ (x - a) < t]\!] \Longrightarrow \mathit{norm}\ (f\,x - b) < s$
using *LIM-D* [*OF f s*] **by** *fast*
obtain d **where** d: $0 < d$
and *neq-b*: $\bigwedge x.\ [\![x \neq a;\ \mathit{norm}\ (x - a) < d]\!] \Longrightarrow f\,x \neq b$
using *inj* **by** *fast*

show $\exists\, t{>}0.\ \forall\, x.\ x \neq a \wedge \mathit{norm}\ (x - a) < t \longrightarrow \mathit{norm}\ (g\ (f\,x) - c) < r$
proof (*safe intro*!: *exI*)
show $0 < \mathit{min}\ d\ t$ **using** d t **by** *simp*
next
fix x
assume $x \neq a$ **and** $\mathit{norm}\ (x - a) < \mathit{min}\ d\ t$
hence $f\,x \neq b$ **and** $\mathit{norm}\ (f\,x - b) < s$
using *neq-b less-s* **by** *simp-all*
thus $\mathit{norm}\ (g\ (f\,x) - c) < r$
by (*rule less-r*)
qed
qed

lemma *LIM-o*: $[\![g$ $--$ l $-->$ $g\ l;\ f$ $--$ a $-->$ $l]\!] \Longrightarrow (g \circ f)$ $--$ a $-->$ $g\ l$
unfolding *o-def* **by** (*rule LIM-compose*)

lemma *real-LIM-sandwich-zero*:
fixes $f\ g$ :: *'a::real-normed-vector* $\Rightarrow$ *real*
assumes f: f $--$ a $-->$ 0
assumes *1*: $\bigwedge x.\ x \neq a \Longrightarrow 0 \leq g\ x$
assumes *2*: $\bigwedge x.\ x \neq a \Longrightarrow g\ x \leq f\,x$
shows g $--$ a $-->$ 0
proof (*rule LIM-imp-LIM* [*OF f*])
fix x **assume** x: $x \neq a$
have $\mathit{norm}\ (g\ x - 0) = g\ x$ **by** (*simp add*: *1 x*)
also have $g\ x \leq f\,x$ **by** (*rule 2* [*OF x*])
also have $f\,x \leq |f\,x|$ **by** (*rule abs-ge-self*)
also have $|f\,x| = \mathit{norm}\ (f\,x - 0)$ **by** *simp*
finally show $\mathit{norm}\ (g\ x - 0) \leq \mathit{norm}\ (f\,x - 0)$ **.**
qed

Bounded Linear Operators

lemma (**in** *bounded-linear*) *cont*: f −− a −−> $f\ a$
proof (*rule LIM-I*)
fix r::*real* **assume** r: $0 < r$
obtain K **where** K: $0 < K$ **and** *norm-le*: $\bigwedge x.\ norm\ (f\ x) \leq norm\ x * K$
using *pos-bounded* **by** *fast*
show $\exists s>0.\ \forall x.\ x \neq a \wedge norm\ (x - a) < s \longrightarrow norm\ (f\ x - f\ a) < r$
proof (*rule exI*, *safe*)
from r K **show** $0 < r\ /\ K$ **by** (*rule divide-pos-pos*)
next
fix x **assume** x: $norm\ (x - a) < r\ /\ K$
have $norm\ (f\ x - f\ a) = norm\ (f\ (x - a))$ **by** (*simp only*: *diff*)
also have $\ldots \leq norm\ (x - a) * K$ **by** (*rule norm-le*)
also from K x **have** $\ldots < r$ **by** (*simp only*: *pos-less-divide-eq*)
finally show $norm\ (f\ x - f\ a) < r$ **.**
qed
qed

lemma (**in** *bounded-linear*) *LIM*:
g −− a −−> $l \Longrightarrow (\lambda x.\ f\ (g\ x))$ −− a −−> $f\ l$
by (*rule LIM-compose* [*OF cont*])

lemma (**in** *bounded-linear*) *LIM-zero*:
g −− a −−> $0 \Longrightarrow (\lambda x.\ f\ (g\ x))$ −− a −−> 0
by (*drule LIM*, *simp only*: *zero*)

Bounded Bilinear Operators

lemma (**in** *bounded-bilinear*) *LIM-prod-zero*:
assumes f: f −− a −−> 0
assumes g: g −− a −−> 0
shows $(\lambda x.\ f\ x ** g\ x)$ −− a −−> 0
proof (*rule LIM-I*)
fix r::*real* **assume** r: $0 < r$
obtain K **where** K: $0 < K$
and *norm-le*: $\bigwedge x\ y.\ norm\ (x ** y) \leq norm\ x * norm\ y * K$
using *pos-bounded* **by** *fast*
from K **have** K': $0 < inverse\ K$
by (*rule positive-imp-inverse-positive*)
obtain s **where** s: $0 < s$
and *norm-f*: $\bigwedge x.\ [\![x \neq a;\ norm\ (x - a) < s]\!] \Longrightarrow norm\ (f\ x) < r$
using *LIM-D* [*OF f r*] **by** *auto*
obtain t **where** t: $0 < t$
and *norm-g*: $\bigwedge x.\ [\![x \neq a;\ norm\ (x - a) < t]\!] \Longrightarrow norm\ (g\ x) < inverse\ K$
using *LIM-D* [*OF g K'*] **by** *auto*
show $\exists s>0.\ \forall x.\ x \neq a \wedge norm\ (x - a) < s \longrightarrow norm\ (f\ x ** g\ x - 0) < r$
proof (*rule exI*, *safe*)
from s t **show** $0 < min\ s\ t$ **by** *simp*
next
fix x **assume** x: $x \neq a$
assume $norm\ (x - a) < min\ s\ t$

hence *xs*: *norm* (*x* − *a*) < *s* **and** *xt*: *norm* (*x* − *a*) < *t* **by** *simp-all*
from *x xs* **have** *1*: *norm* (*f x*) < *r* **by** (*rule norm-f*)
from *x xt* **have** *2*: *norm* (*g x*) < *inverse K* **by** (*rule norm-g*)
have *norm* (*f x* ** *g x*) ≤ *norm* (*f x*) ∗ *norm* (*g x*) ∗ *K* **by** (*rule norm-le*)
also from *1 2 K* **have** ... < *r* ∗ *inverse K* ∗ *K*
by (*intro mult-strict-right-mono mult-strict-mono′ norm-ge-zero*)
also from *K* **have** *r* ∗ *inverse K* ∗ *K* = *r* **by** *simp*
finally show *norm* (*f x* ** *g x* − *0*) < *r* **by** *simp*
qed
qed

lemma (**in** *bounded-bilinear*) *LIM-left-zero*:
f −− *a* −−> *0* ⟹ (λ*x*. *f x* ** *c*) −− *a* −−> *0*
by (*rule bounded-linear.LIM-zero* [*OF bounded-linear-left*])

lemma (**in** *bounded-bilinear*) *LIM-right-zero*:
f −− *a* −−> *0* ⟹ (λ*x*. *c* ** *f x*) −− *a* −−> *0*
by (*rule bounded-linear.LIM-zero* [*OF bounded-linear-right*])

lemma (**in** *bounded-bilinear*) *LIM*:
⟦*f* −− *a* −−> *L*; *g* −− *a* −−> *M*⟧ ⟹ (λ*x*. *f x* ** *g x*) −− *a* −−> *L* ** *M*
apply (*drule LIM-zero*)
apply (*drule LIM-zero*)
apply (*rule LIM-zero-cancel*)
apply (*subst prod-diff-prod*)
apply (*rule LIM-add-zero*)
apply (*rule LIM-add-zero*)
apply (*erule* (*1*) *LIM-prod-zero*)
apply (*erule LIM-left-zero*)
apply (*erule LIM-right-zero*)
done

lemmas *LIM-mult* = *mult.LIM*

lemmas *LIM-mult-zero* = *mult.LIM-prod-zero*

lemmas *LIM-mult-left-zero* = *mult.LIM-left-zero*

lemmas *LIM-mult-right-zero* = *mult.LIM-right-zero*

lemmas *LIM-scaleR* = *scaleR.LIM*

lemmas *LIM-of-real* = *of-real.LIM*

lemma *LIM-power*:
fixes *f* :: ′*a*::*real-normed-vector* ⇒ ′*b*::{*recpower*,*real-normed-algebra*}
assumes *f*: *f* −− *a* −−> *l*
shows (λ*x*. *f x* ˆ *n*) −− *a* −−> *l* ˆ *n*
by (*induct n*, *simp*, *simp add*: *power-Suc LIM-mult f*)

13.1.2 Derived theorems about *LIM*

```
lemma LIM-inverse-lemma:
  fixes x :: 'a::real-normed-div-algebra
  assumes r: 0 < r
  assumes x: norm (x − 1) < min (1/2) (r/2)
  shows norm (inverse x − 1) < r
proof −
  from r have r2: 0 < r/2 by simp
  from x have 0: x ≠ 0 by clarsimp
  from x have x': norm (1 − x) < min (1/2) (r/2)
    by (simp only: norm-minus-commute)
  hence less1: norm (1 − x) < r/2 by simp
  have norm (1::'a) − norm x ≤ norm (1 − x) by (rule norm-triangle-ineq2)
  also from x' have norm (1 − x) < 1/2 by simp
  finally have 1/2 < norm x by simp
  hence inverse (norm x) < inverse (1/2)
    by (rule less-imp-inverse-less, simp)
  hence less2: norm (inverse x) < 2
    by (simp add: nonzero-norm-inverse 0)
  from less1 less2 r2 norm-ge-zero
  have norm (1 − x) ∗ norm (inverse x) < (r/2) ∗ 2
    by (rule mult-strict-mono)
  thus norm (inverse x − 1) < r
    by (simp only: norm-mult [symmetric] left-diff-distrib, simp add: 0)
qed

lemma LIM-inverse-fun:
  assumes a: a ≠ (0::'a::real-normed-div-algebra)
  shows inverse −− a −−> inverse a
proof (rule LIM-equal2)
  from a show 0 < norm a by simp
next
  fix x assume norm (x − a) < norm a
  hence x ≠ 0 by auto
  with a show inverse x = inverse (inverse a ∗ x) ∗ inverse a
    by (simp add: nonzero-inverse-mult-distrib
                  nonzero-imp-inverse-nonzero
                  nonzero-inverse-inverse-eq mult-assoc)
next
  have 1: inverse −− 1 −−> inverse (1::'a)
    apply (rule LIM-I)
    apply (rule-tac x=min (1/2) (r/2) in exI)
    apply (simp add: LIM-inverse-lemma)
    done
  have (λx. inverse a ∗ x) −− a −−> inverse a ∗ a
    by (intro LIM-mult LIM-ident LIM-const)
  hence (λx. inverse a ∗ x) −− a −−> 1
    by (simp add: a)
  with 1 have (λx. inverse (inverse a ∗ x)) −− a −−> inverse 1
```

  by (*rule LIM-compose*)
 hence ($\lambda x.$ *inverse* (*inverse a* $*$ *x*)) $--$ *a* $-->$ *1*
  by *simp*
 hence ($\lambda x.$ *inverse* (*inverse a* $*$ *x*) $*$ *inverse a*) $--$ *a* $-->$ *1* $*$ *inverse a*
  by (*intro LIM-mult LIM-const*)
 thus ($\lambda x.$ *inverse* (*inverse a* $*$ *x*) $*$ *inverse a*) $--$ *a* $-->$ *inverse a*
  by *simp*
qed

lemma *LIM-inverse*:
 fixes *L* :: *'a::real-normed-div-algebra*
 shows $[\![$*f* $--$ *a* $-->$ *L*; *L* $\neq$ *0*$]\!]$ $\Longrightarrow$ ($\lambda x.$ *inverse* (*f x*)) $--$ *a* $-->$ *inverse L*
by (*rule LIM-inverse-fun* [*THEN LIM-compose*])

13.2 Continuity

13.2.1 Purely standard proofs

lemma *LIM-isCont-iff*: (*f* $--$ *a* $-->$ *f a*) = (($\lambda h.$ *f* (*a* + *h*)) $--$ *0* $-->$ *f a*)
by (*rule iffI* [*OF LIM-offset-zero LIM-offset-zero-cancel*])

lemma *isCont-iff*: *isCont f x* = ($\lambda h.$ *f* (*x* + *h*)) $--$ *0* $-->$ *f x*
by (*simp add*: *isCont-def LIM-isCont-iff*)

lemma *isCont-ident* [*simp*]: *isCont* ($\lambda x.$ *x*) *a*
 unfolding *isCont-def* **by** (*rule LIM-ident*)

lemma *isCont-const* [*simp*]: *isCont* ($\lambda x.$ *k*) *a*
 unfolding *isCont-def* **by** (*rule LIM-const*)

lemma *isCont-norm*: *isCont f a* $\Longrightarrow$ *isCont* ($\lambda x.$ *norm* (*f x*)) *a*
 unfolding *isCont-def* **by** (*rule LIM-norm*)

lemma *isCont-rabs*: *isCont f a* $\Longrightarrow$ *isCont* ($\lambda x.$ $|$*f x* :: *real*$|$) *a*
 unfolding *isCont-def* **by** (*rule LIM-rabs*)

lemma *isCont-add*: $[\![$*isCont f a*; *isCont g a*$]\!]$ $\Longrightarrow$ *isCont* ($\lambda x.$ *f x* + *g x*) *a*
 unfolding *isCont-def* **by** (*rule LIM-add*)

lemma *isCont-minus*: *isCont f a* $\Longrightarrow$ *isCont* ($\lambda x.$ $-$ *f x*) *a*
 unfolding *isCont-def* **by** (*rule LIM-minus*)

lemma *isCont-diff*: $[\![$*isCont f a*; *isCont g a*$]\!]$ $\Longrightarrow$ *isCont* ($\lambda x.$ *f x* $-$ *g x*) *a*
 unfolding *isCont-def* **by** (*rule LIM-diff*)

lemma *isCont-mult*:
 fixes *f g* :: *'a::real-normed-vector* $\Rightarrow$ *'b::real-normed-algebra*
 shows $[\![$*isCont f a*; *isCont g a*$]\!]$ $\Longrightarrow$ *isCont* ($\lambda x.$ *f x* $*$ *g x*) *a*
 unfolding *isCont-def* **by** (*rule LIM-mult*)

lemma *isCont-inverse*:
fixes f :: $'a$::*real-normed-vector* $\Rightarrow$ $'b$::*real-normed-div-algebra*
shows $[\![isCont\ f\ a;\ f\ a \neq 0]\!] \Longrightarrow isCont\ (\lambda x.\ inverse\ (f\ x))\ a$
unfolding *isCont-def* **by** (*rule LIM-inverse*)

lemma *isCont-LIM-compose*:
$[\![isCont\ g\ l;\ f\ {-}{-}\ a\ {-}{-}{>}\ l]\!] \Longrightarrow (\lambda x.\ g\ (f\ x))\ {-}{-}\ a\ {-}{-}{>}\ g\ l$
unfolding *isCont-def* **by** (*rule LIM-compose*)

lemma *isCont-LIM-compose2*:
assumes f [*unfolded isCont-def*]: *isCont f a*
assumes g: $g\ {-}{-}\ f\ a\ {-}{-}{>}\ l$
assumes *inj*: $\exists d{>}0.\ \forall x.\ x \neq a \wedge norm\ (x - a) < d \longrightarrow f\ x \neq f\ a$
shows $(\lambda x.\ g\ (f\ x))\ {-}{-}\ a\ {-}{-}{>}\ l$
by (*rule LIM-compose2* [*OF f g inj*])

lemma *isCont-o2*: $[\![isCont\ f\ a;\ isCont\ g\ (f\ a)]\!] \Longrightarrow isCont\ (\lambda x.\ g\ (f\ x))\ a$
unfolding *isCont-def* **by** (*rule LIM-compose*)

lemma *isCont-o*: $[\![isCont\ f\ a;\ isCont\ g\ (f\ a)]\!] \Longrightarrow isCont\ (g\ o\ f)\ a$
unfolding *o-def* **by** (*rule isCont-o2*)

lemma (**in** *bounded-linear*) *isCont*: *isCont f a*
unfolding *isCont-def* **by** (*rule cont*)

lemma (**in** *bounded-bilinear*) *isCont*:
$[\![isCont\ f\ a;\ isCont\ g\ a]\!] \Longrightarrow isCont\ (\lambda x.\ f\ x\ {*}{*}\ g\ x)\ a$
unfolding *isCont-def* **by** (*rule LIM*)

lemmas *isCont-scaleR* = *scaleR.isCont*

lemma *isCont-of-real*:
$isCont\ f\ a \Longrightarrow isCont\ (\lambda x.\ of\text{-}real\ (f\ x))\ a$
unfolding *isCont-def* **by** (*rule LIM-of-real*)

lemma *isCont-power*:
fixes f :: $'a$::*real-normed-vector* $\Rightarrow$ $'b$::{*recpower*,*real-normed-algebra*}
shows $isCont\ f\ a \Longrightarrow isCont\ (\lambda x.\ f\ x\ \hat{}\ n)\ a$
unfolding *isCont-def* **by** (*rule LIM-power*)

lemma *isCont-abs* [*simp*]: *isCont abs* (*a*::*real*)
by (*rule isCont-rabs* [*OF isCont-ident*])

13.3 Uniform Continuity

lemma *isUCont-isCont*: *isUCont f* ==> *isCont f x*
by (*simp add*: *isUCont-def isCont-def LIM-def*, *force*)

lemma *isUCont-Cauchy*:

⟦*isUCont f*; *Cauchy X*⟧ ⟹ *Cauchy* (λ*n*. *f* (*X n*))
unfolding *isUCont-def*
apply (*rule CauchyI*)
apply (*drule-tac x=e* **in** *spec*, *safe*)
apply (*drule-tac e=s* **in** *CauchyD*, *safe*)
apply (*rule-tac x=M* **in** *exI*, *simp*)
done

lemma (**in** *bounded-linear*) *isUCont*: *isUCont f*
unfolding *isUCont-def*
proof (*intro allI impI*)
  fix *r*::*real* **assume** *r*: *0* < *r*
  obtain *K* **where** *K*: *0* < *K* **and** *norm-le*: ⋀*x*. *norm* (*f x*) ≤ *norm x* ∗ *K*
    using *pos-bounded* **by** *fast*
  show ∃ *s*>*0*. ∀ *x y*. *norm* (*x* − *y*) < *s* ⟶ *norm* (*f x* − *f y*) < *r*
  proof (*rule exI*, *safe*)
    from *r K* **show** *0* < *r* / *K* **by** (*rule divide-pos-pos*)
  next
    fix *x y* :: ′*a*
    assume *xy*: *norm* (*x* − *y*) < *r* / *K*
    have *norm* (*f x* − *f y*) = *norm* (*f* (*x* − *y*)) **by** (*simp only*: *diff*)
    also have . . . ≤ *norm* (*x* − *y*) ∗ *K* **by** (*rule norm-le*)
    also from *K xy* **have** . . . < *r* **by** (*simp only*: *pos-less-divide-eq*)
    finally show *norm* (*f x* − *f y*) < *r* **.**
  qed
qed

lemma (**in** *bounded-linear*) *Cauchy*: *Cauchy X* ⟹ *Cauchy* (λ*n*. *f* (*X n*))
by (*rule isUCont* [*THEN isUCont-Cauchy*])

13.4 Relation of LIM and LIMSEQ

lemma *LIMSEQ-SEQ-conv1*:
  fixes *a* :: ′*a*::*real-normed-vector*
  assumes *X*: *X* −− *a* −−> *L*
  shows ∀ *S*. (∀ *n*. *S n* ≠ *a*) ∧ *S* −−−−> *a* ⟶ (λ*n*. *X* (*S n*)) −−−−> *L*
proof (*safe intro*!: *LIMSEQ-I*)
  fix *S* :: *nat* ⇒ ′*a*
  fix *r* :: *real*
  assume *rgz*: *0* < *r*
  assume *as*: ∀ *n*. *S n* ≠ *a*
  assume *S*: *S* −−−−> *a*
  from *LIM-D* [*OF X rgz*] **obtain** *s*
    where *sgz*: *0* < *s*
    and *aux*: ⋀*x*. ⟦*x* ≠ *a*; *norm* (*x* − *a*) < *s*⟧ ⟹ *norm* (*X x* − *L*) < *r*
    by *fast*
  from *LIMSEQ-D* [*OF S sgz*]
  obtain *no* **where** ∀ *n*≥*no*. *norm* (*S n* − *a*) < *s* **by** *blast*
  hence ∀ *n*≥*no*. *norm* (*X* (*S n*) − *L*) < *r* **by** (*simp add*: *aux as*)

thus $\exists\, no.\ \forall\, n \geq no.\ norm\ (X\ (S\ n) - L) < r$ **..**
qed

lemma *LIMSEQ-SEQ-conv2*:
fixes $a :: real$
assumes $\forall\, S.\ (\forall\, n.\ S\ n \neq a) \wedge S\ {-}{-}{-}{-}{>}\ a \longrightarrow (\lambda n.\ X\ (S\ n))\ {-}{-}{-}{-}{>}\ L$
shows $X\ {-}{-}\ a\ {-}{-}{>}\ L$
proof (*rule ccontr*)
assume $\neg\ (X\ {-}{-}\ a\ {-}{-}{>}\ L)$
hence $\neg\ (\forall\, r > 0.\ \exists\, s > 0.\ \forall\, x.\ x \neq a\ \&\ norm\ (x - a) < s\ {-}{-}{>}\ norm\ (X\ x - L) < r)$ **by** (*unfold LIM-def*)
hence $\exists\, r > 0.\ \forall\, s > 0.\ \exists\, x.\ \neg(x \neq a \wedge |x - a| < s\ {-}{-}{>}\ norm\ (X\ x - L) < r)$ **by** *simp*
hence $\exists\, r > 0.\ \forall\, s > 0.\ \exists\, x.\ (x \neq a \wedge |x - a| < s \wedge norm\ (X\ x - L) \geq r)$ **by** (*simp add*: *linorder-not-less*)
then obtain r **where** *rdef*: $r > 0 \wedge (\forall\, s > 0.\ \exists\, x.\ (x \neq a \wedge |x - a| < s \wedge norm\ (X\ x - L) \geq r))$ **by** *auto*

let $?F = \lambda n{::}nat.\ SOME\ x.\ x{\neq}a \wedge |x - a| < inverse\ (real\ (Suc\ n)) \wedge norm\ (X\ x - L) \geq r$
have $\bigwedge n.\ \exists\, x.\ x{\neq}a \wedge |x - a| < inverse\ (real\ (Suc\ n)) \wedge norm\ (X\ x - L) \geq r$
using *rdef* **by** *simp*
hence *F*: $\bigwedge n.\ ?F\ n \neq a \wedge |?F\ n - a| < inverse\ (real\ (Suc\ n)) \wedge norm\ (X\ (?F\ n) - L) \geq r$
by (*rule someI-ex*)
hence *F1*: $\bigwedge n.\ ?F\ n \neq a$
and *F2*: $\bigwedge n.\ |?F\ n - a| < inverse\ (real\ (Suc\ n))$
and *F3*: $\bigwedge n.\ norm\ (X\ (?F\ n) - L) \geq r$
by *fast+*

have $?F\ {-}{-}{-}{-}{>}\ a$
proof (*rule LIMSEQ-I*, *unfold real-norm-def*)
fix $e{::}real$
assume $0 < e$

then have $\exists\, no.\ inverse\ (real\ (Suc\ no)) < e$ **by** (*rule reals-Archimedean*)
then obtain m **where** *nodef*: $inverse\ (real\ (Suc\ m)) < e$ **by** *auto*
show $\exists\, no.\ \forall\, n.\ no \leq n\ {-}{-}{>}\ |?F\ n - a| < e$
proof (*intro exI allI impI*)
fix n
assume *mlen*: $m \leq n$
have $|?F\ n - a| < inverse\ (real\ (Suc\ n))$
by (*rule F2*)
also have $inverse\ (real\ (Suc\ n)) \leq inverse\ (real\ (Suc\ m))$
using *mlen* **by** *auto*
also from *nodef* **have**
$inverse\ (real\ (Suc\ m)) < e$ **.**
finally show $|?F\ n - a| < e$ **.**
qed

```
  qed

  moreover have ∀ n. ?F n ≠ a
    by (rule allI) (rule F1)

  moreover from prems have ∀ S. (∀ n. S n ≠ a) ∧ S ----> a ⟶ (λn. X (S
n)) ----> L by simp
  ultimately have (λn. X (?F n)) ----> L by simp

  moreover have ¬ ((λn. X (?F n)) ----> L)
  proof -
    {
      fix no::nat
      obtain n where n = no + 1 by simp
      then have nolen: no ≤ n by simp

      have norm (X (?F n) - L) ≥ r
        by (rule F3)
      with nolen have ∃ n. no ≤ n ∧ norm (X (?F n) - L) ≥ r by fast
    }
    then have (∀ no. ∃ n. no ≤ n ∧ norm (X (?F n) - L) ≥ r) by simp
    with rdef have ∃ e>0. (∀ no. ∃ n. no ≤ n ∧ norm (X (?F n) - L) ≥ e) by
auto
    thus ?thesis by (unfold LIMSEQ-def, auto simp add: linorder-not-less)
  qed
  ultimately show False by simp
qed

lemma LIMSEQ-SEQ-conv:
  (∀ S. (∀ n. S n ≠ a) ∧ S ----> (a::real) ⟶ (λn. X (S n)) ----> L) =
   (X -- a --> L)
proof
  assume ∀ S. (∀ n. S n ≠ a) ∧ S ----> a ⟶ (λn. X (S n)) ----> L
  thus X -- a --> L by (rule LIMSEQ-SEQ-conv2)
next
  assume (X -- a --> L)
  thus ∀ S. (∀ n. S n ≠ a) ∧ S ----> a ⟶ (λn. X (S n)) ----> L by (rule
LIMSEQ-SEQ-conv1)
qed

end
```

14 Deriv: Differentiation

```
theory Deriv
imports Lim
begin
```

Standard Definitions

definition
deriv :: $['a::real\text{-}normed\text{-}field \Rightarrow 'a, 'a, 'a] \Rightarrow bool$
— Differentiation: D is derivative of function f at x
((*DERIV* (-)/ (-)/ :> (-)) [*1000*, *1000*, *60*] *60*) **where**
DERIV f x :> *D* = ((%h. (f(x + h) − f x) / h) −− 0 −−> D)

definition
differentiable :: $['a::real\text{-}normed\text{-}field \Rightarrow 'a, 'a] \Rightarrow bool$
(**infixl** *differentiable 60*) **where**
f differentiable x = (∃ D. *DERIV f x* :> *D*)

consts
Bolzano-bisect :: [*real*∗*real*=>*bool*, *real*, *real*, *nat*] => (*real*∗*real*)
primrec
Bolzano-bisect P a b 0 = (*a*,*b*)
Bolzano-bisect P a b (*Suc n*) =
(*let* (*x*,*y*) = *Bolzano-bisect P a b n*
in if P(*x*, (*x*+*y*)/*2*) *then* ((*x*+*y*)/*2*, *y*)
else (*x*, (*x*+*y*)/*2*))

14.1 Derivatives

lemma *DERIV-iff*: (*DERIV f x* :> *D*) = ((%h. (f(x + h) − f(x))/h) −− 0 −−> D)
by (*simp add*: *deriv-def*)

lemma *DERIV-D*: *DERIV f x* :> *D* ==> (%h. (f(x + h) − f(x))/h) −− 0 −−> D
by (*simp add*: *deriv-def*)

lemma *DERIV-const* [*simp*]: *DERIV* (λx. k) x :> 0
by (*simp add*: *deriv-def*)

lemma *DERIV-ident* [*simp*]: *DERIV* (λx. x) x :> 1
by (*simp add*: *deriv-def cong*: *LIM-cong*)

lemma *add-diff-add*:
fixes *a b c d* :: $'a::ab\text{-}group\text{-}add$
shows (a + c) − (b + d) = (a − b) + (c − d)
by *simp*

lemma *DERIV-add*:
⟦*DERIV f x* :> *D*; *DERIV g x* :> *E*⟧ ⟹ *DERIV* (λx. f x + g x) x :> D + E
by (*simp only*: *deriv-def add-diff-add add-divide-distrib LIM-add*)

lemma *DERIV-minus*:
DERIV f x :> *D* ⟹ *DERIV* (λx. − f x) x :> − D

```
by (simp only: deriv-def minus-diff-minus divide-minus-left LIM-minus)

lemma DERIV-diff:
  ⟦DERIV f x :> D; DERIV g x :> E⟧ ⟹ DERIV (λx. f x − g x) x :> D − E
by (simp only: diff-def DERIV-add DERIV-minus)

lemma DERIV-add-minus:
  ⟦DERIV f x :> D; DERIV g x :> E⟧ ⟹ DERIV (λx. f x + − g x) x :> D +
− E
by (simp only: DERIV-add DERIV-minus)

lemma DERIV-isCont: DERIV f x :> D ⟹ isCont f x
proof (unfold isCont-iff)
  assume DERIV f x :> D
  hence (λh. (f(x+h) − f(x)) / h) −− 0 −−> D
    by (rule DERIV-D)
  hence (λh. (f(x+h) − f(x)) / h ∗ h) −− 0 −−> D ∗ 0
    by (intro LIM-mult LIM-ident)
  hence (λh. (f(x+h) − f(x)) ∗ (h / h)) −− 0 −−> 0
    by simp
  hence (λh. f(x+h) − f(x)) −− 0 −−> 0
    by (simp cong: LIM-cong)
  thus (λh. f(x+h)) −− 0 −−> f(x)
    by (simp add: LIM-def)
qed

lemma DERIV-mult-lemma:
  fixes a b c d :: 'a::real-field
  shows (a ∗ b − c ∗ d) / h = a ∗ ((b − d) / h) + ((a − c) / h) ∗ d
by (simp add: diff-minus add-divide-distrib [symmetric] ring-distribs)

lemma DERIV-mult':
  assumes f: DERIV f x :> D
  assumes g: DERIV g x :> E
  shows DERIV (λx. f x ∗ g x) x :> f x ∗ E + D ∗ g x
proof (unfold deriv-def)
  from f have isCont f x
    by (rule DERIV-isCont)
  hence (λh. f(x+h)) −− 0 −−> f x
    by (simp only: isCont-iff)
  hence (λh. f(x+h) ∗ ((g(x+h) − g x) / h) +
              ((f(x+h) − f x) / h) ∗ g x)
          −− 0 −−> f x ∗ E + D ∗ g x
    by (intro LIM-add LIM-mult LIM-const DERIV-D f g)
  thus (λh. (f(x+h) ∗ g(x+h) − f x ∗ g x) / h)
         −− 0 −−> f x ∗ E + D ∗ g x
    by (simp only: DERIV-mult-lemma)
qed
```

lemma *DERIV-mult*:
 $[| \ DERIV\, f\, x :> Da;\ DERIV\ g\ x :> Db\ |]$
 $==> DERIV\ (\%x.\ f\ x * g\ x)\ x :> (Da * g(x)) + (Db * f(x))$
by (*drule* (*1*) *DERIV-mult'*, *simp only*: *mult-commute add-commute*)

lemma *DERIV-unique*:
 $[| \ DERIV\, f\, x :> D;\ DERIV\, f\, x :> E\ |] ==> D = E$
apply (*simp add*: *deriv-def*)
apply (*blast intro*: *LIM-unique*)
done

Differentiation of finite sum

lemma *DERIV-sumr* [*rule-format* (*no-asm*)]:
 $(\forall r.\ m \le r\ \&\ r < (m + n) --> DERIV\ (\%x.\ f\ r\ x)\ x :> (f'\ r\ x))$
 $--> DERIV\ (\%x.\ \sum n{=}m..{<}n{::}nat.\ f\ n\ x :: real)\ x :> (\sum r{=}m..{<}n.\ f'\ r$
$x)$
apply (*induct n*)
apply (*auto intro*: *DERIV-add*)
done

Alternative definition for differentiability

lemma *DERIV-LIM-iff*:
 $((\%h.\ (f(a + h) - f(a))\ /\ h) -- 0 --> D) =$
 $((\%x.\ (f(x){-}f(a))\ /\ (x{-}a)) -- a --> D)$
apply (*rule iffI*)
apply (*drule-tac* $k{=}-\ a$ **in** *LIM-offset*)
apply (*simp add*: *diff-minus*)
apply (*drule-tac* $k{=}a$ **in** *LIM-offset*)
apply (*simp add*: *add-commute*)
done

lemma *DERIV-iff2*: $(DERIV\, f\, x :> D) = ((\%z.\ (f(z) - f(x))\ /\ (z{-}x)) -- x$
$--> D)$
by (*simp add*: *deriv-def diff-minus* [*symmetric*] *DERIV-LIM-iff*)

lemma *inverse-diff-inverse*:
 $[\![(a{::}'a{::}division\text{-}ring) \neq 0;\ b \neq 0]\!]$
 $\Longrightarrow inverse\ a - inverse\ b = -\ (inverse\ a * (a - b) * inverse\ b)$
by (*simp add*: *ring-simps*)

lemma *DERIV-inverse-lemma*:
 $[\![a \neq 0;\ b \neq (0{::}'a{::}real\text{-}normed\text{-}field)]\!]$
 $\Longrightarrow (inverse\ a - inverse\ b)\ /\ h$
 $= -\ (inverse\ a * ((a - b)\ /\ h) * inverse\ b)$
by (*simp add*: *inverse-diff-inverse*)

lemma *DERIV-inverse'*:
 assumes *der*: $DERIV\, f\, x :> D$
 assumes *neq*: $f\ x \neq 0$

```
  shows DERIV (λx. inverse (f x)) x :> − (inverse (f x) ∗ D ∗ inverse (f x))
    (is DERIV - - :> ?E)
proof (unfold DERIV-iff2)
  from der have lim-f: f −− x −−> f x
    by (rule DERIV-isCont [unfolded isCont-def])

  from neq have 0 < norm (f x) by simp
  with LIM-D [OF lim-f] obtain s
    where s: 0 < s
    and less-fx: ⋀z. ⟦z ≠ x; norm (z − x) < s⟧
              ⟹ norm (f z − f x) < norm (f x)
    by fast

  show (λz. (inverse (f z) − inverse (f x)) / (z − x)) −− x −−> ?E
  proof (rule LIM-equal2 [OF s])
    fix z
    assume z ≠ x norm (z − x) < s
    hence norm (f z − f x) < norm (f x) by (rule less-fx)
    hence f z ≠ 0 by auto
    thus (inverse (f z) − inverse (f x)) / (z − x) =
          − (inverse (f z) ∗ ((f z − f x) / (z − x)) ∗ inverse (f x))
      using neq by (rule DERIV-inverse-lemma)
  next
    from der have (λz. (f z − f x) / (z − x)) −− x −−> D
      by (unfold DERIV-iff2)
    thus (λz. − (inverse (f z) ∗ ((f z − f x) / (z − x)) ∗ inverse (f x)))
          −− x −−> ?E
      by (intro LIM-mult LIM-inverse LIM-minus LIM-const lim-f neq)
  qed
qed

lemma DERIV-divide:
  ⟦DERIV f x :> D; DERIV g x :> E; g x ≠ 0⟧
   ⟹ DERIV (λx. f x / g x) x :> (D ∗ g x − f x ∗ E) / (g x ∗ g x)
apply (subgoal-tac f x ∗ − (inverse (g x) ∗ E ∗ inverse (g x)) +
         D ∗ inverse (g x) = (D ∗ g x − f x ∗ E) / (g x ∗ g x))
apply (erule subst)
apply (unfold divide-inverse)
apply (erule DERIV-mult′)
apply (erule (1) DERIV-inverse′)
apply (simp add: ring-distribs nonzero-inverse-mult-distrib)
apply (simp add: mult-ac)
done

lemma DERIV-power-Suc:
  fixes f :: ′a ⇒ ′a::{real-normed-field,recpower}
  assumes f: DERIV f x :> D
  shows DERIV (λx. f x ˆ Suc n) x :> (1 + of-nat n) ∗ (D ∗ f x ˆ n)
proof (induct n)
```

```
case 0
  show ?case by (simp add: power-Suc f)
case (Suc k)
  from DERIV-mult' [OF f Suc] show ?case
    apply (simp only: of-nat-Suc ring-distribs mult-1-left)
    apply (simp only: power-Suc right-distrib mult-ac add-ac)
    done
qed

lemma DERIV-power:
  fixes f :: 'a ⇒ 'a::{real-normed-field,recpower}
  assumes f: DERIV f x :> D
  shows DERIV (λx. f x ^ n) x :> of-nat n * (D * f x ^ (n − Suc 0))
by (cases n, simp, simp add: DERIV-power-Suc f)




lemma CARAT-DERIV:
     (DERIV f x :> l) =
      (∃ g. (∀ z. f z − f x = g z * (z−x)) & isCont g x & g x = l)
      (is ?lhs = ?rhs)
proof
  assume der: DERIV f x :> l
  show ∃ g. (∀ z. f z − f x = g z * (z−x)) ∧ isCont g x ∧ g x = l
  proof (intro exI conjI)
    let ?g = (%z. if z = x then l else (f z − f x) / (z−x))
    show ∀ z. f z − f x = ?g z * (z−x) by simp
    show isCont ?g x using der
      by (simp add: isCont-iff DERIV-iff diff-minus
               cong: LIM-equal [rule-format])
    show ?g x = l by simp
  qed
next
  assume ?rhs
  then obtain g where
    (∀ z. f z − f x = g z * (z−x)) and isCont g x and g x = l by blast
  thus (DERIV f x :> l)
     by (auto simp add: isCont-iff DERIV-iff cong: LIM-cong)
qed

lemma DERIV-chain':
  assumes f: DERIV f x :> D
  assumes g: DERIV g (f x) :> E
  shows DERIV (λx. g (f x)) x :> E * D
proof (unfold DERIV-iff2)
  obtain d where d: ∀ y. g y − g (f x) = d y * (y − f x)
```

and *cont-d*: *isCont d* (*f x*) **and** *dfx*: *d* (*f x*) = *E*
using *CARAT-DERIV* [*THEN iffD1*, *OF g*] **by** *fast*
from *f* **have** *f* −− *x* −−> *f x*
by (*rule DERIV-isCont* [*unfolded isCont-def*])
with *cont-d* **have** (λz. *d* (*f z*)) −− *x* −−> *d* (*f x*)
by (*rule isCont-LIM-compose*)
hence (λz. *d* (*f z*) ∗ ((*f z* − *f x*) / (*z* − *x*)))
−− *x* −−> *d* (*f x*) ∗ *D*
by (*rule LIM-mult* [*OF* - *f* [*unfolded DERIV-iff2*]])
thus (λz. (*g* (*f z*) − *g* (*f x*)) / (*z* − *x*)) −− *x* −−> *E* ∗ *D*
by (*simp add*: *d dfx real-scaleR-def*)
qed

lemma *DERIV-cmult*:
DERIV f x :> *D* ==> *DERIV* (%*x*. *c* ∗ *f x*) *x* :> *c*∗*D*
by (*drule DERIV-mult′* [*OF DERIV-const*], *simp*)

lemma *DERIV-chain*: [| *DERIV f* (*g x*) :> *Da*; *DERIV g x* :> *Db* |] ==> *DERIV* (*f o g*) *x* :> *Da* ∗ *Db*
by (*drule* (*1*) *DERIV-chain′*, *simp add*: *o-def real-scaleR-def mult-commute*)

lemma *DERIV-chain2*: [| *DERIV f* (*g x*) :> *Da*; *DERIV g x* :> *Db* |] ==> *DERIV* (%*x*. *f* (*g x*)) *x* :> *Da* ∗ *Db*
by (*auto dest*: *DERIV-chain simp add*: *o-def*)

lemma *DERIV-cmult-Id* [*simp*]: *DERIV* (*op* ∗ *c*) *x* :> *c*
by (*cut-tac c* = *c* **and** *x* = *x* **in** *DERIV-ident* [*THEN DERIV-cmult*], *simp*)

lemma *DERIV-pow*: *DERIV* (%*x*. *x* ^ *n*) *x* :> *real n* ∗ (*x* ^ (*n* − *Suc 0*))
apply (*cut-tac DERIV-power* [*OF DERIV-ident*])
apply (*simp add*: *real-scaleR-def real-of-nat-def*)
done

Power of -1

lemma *DERIV-inverse*:
fixes *x* :: *′a*::{*real-normed-field*,*recpower*}
shows *x* ≠ *0* ==> *DERIV* (%*x*. *inverse*(*x*)) *x* :> (−(*inverse x* ^ *Suc* (*Suc 0*)))
by (*drule DERIV-inverse′* [*OF DERIV-ident*]) (*simp add*: *power-Suc*)

Derivative of inverse

lemma *DERIV-inverse-fun*:
fixes *x* :: *′a*::{*real-normed-field*,*recpower*}
shows [| *DERIV f x* :> *d*; *f*(*x*) ≠ *0* |]
==> *DERIV* (%*x*. *inverse*(*f x*)) *x* :> (− (*d* ∗ *inverse*(*f*(*x*) ^ *Suc* (*Suc 0*))))

by (*drule* (*1*) *DERIV-inverse'*) (*simp add*: *mult-ac power-Suc nonzero-inverse-mult-distrib*)

Derivative of quotient

lemma *DERIV-quotient*:
  fixes *x* :: *'a*::{*real-normed-field*,*recpower*}
  shows [| *DERIV f x* :> *d*; *DERIV g x* :> *e*; $g(x) \neq 0$ |]
    ==> *DERIV* (%*y*. *f*(*y*) / (*g y*)) *x* :> (*d*∗*g*(*x*) − (*e*∗*f*(*x*))) / (*g*(*x*) ˆ *Suc* (*Suc 0*))
by (*drule* (*2*) *DERIV-divide*) (*simp add*: *mult-commute power-Suc*)

14.2 Differentiability predicate

lemma *differentiableD*: *f differentiable x* ==> ∃ *D*. *DERIV f x* :> *D*
by (*simp add*: *differentiable-def*)

lemma *differentiableI*: *DERIV f x* :> *D* ==> *f differentiable x*
by (*force simp add*: *differentiable-def*)

lemma *differentiable-const*: (λ*z*. *a*) *differentiable x*
  apply (*unfold differentiable-def*)
  apply (*rule-tac x*=*0* **in** *exI*)
  apply *simp*
  done

lemma *differentiable-sum*:
  assumes *f differentiable x*
  and *g differentiable x*
  shows (λ*x*. *f x* + *g x*) *differentiable x*
proof −
  from *prems* **have** ∃ *D*. *DERIV f x* :> *D* **by** (*unfold differentiable-def*)
  then obtain *df* **where** *DERIV f x* :> *df* **..**
  moreover from *prems* **have** ∃ *D*. *DERIV g x* :> *D* **by** (*unfold differentiable-def*)
  then obtain *dg* **where** *DERIV g x* :> *dg* **..**
  ultimately have *DERIV* (λ*x*. *f x* + *g x*) *x* :> *df* + *dg* **by** (*rule DERIV-add*)
  hence ∃ *D*. *DERIV* (λ*x*. *f x* + *g x*) *x* :> *D* **by** *auto*
  thus *?thesis* **by** (*fold differentiable-def*)
qed

lemma *differentiable-diff*:
  assumes *f differentiable x*
  and *g differentiable x*
  shows (λ*x*. *f x* − *g x*) *differentiable x*
proof −
  from *prems* **have** *f differentiable x* **by** *simp*
  moreover
  from *prems* **have** ∃ *D*. *DERIV g x* :> *D* **by** (*unfold differentiable-def*)
  then obtain *dg* **where** *DERIV g x* :> *dg* **..**
  then have *DERIV* (λ*x*. − *g x*) *x* :> −*dg* **by** (*rule DERIV-minus*)
  hence ∃ *D*. *DERIV* (λ*x*. − *g x*) *x* :> *D* **by** *auto*

hence ($\lambda x.\ -\ g\ x$) *differentiable* x **by** (*fold differentiable-def*)
ultimately
show *?thesis*
by (*auto simp*: *diff-def dest*: *differentiable-sum*)
qed

lemma *differentiable-mult*:
assumes f *differentiable* x
and g *differentiable* x
shows ($\lambda x.\ f\ x * g\ x$) *differentiable* x
proof $-$
from *prems* **have** $\exists\,D.\ DERIV\ f\ x :> D$ **by** (*unfold differentiable-def*)
then obtain *df* **where** $DERIV\ f\ x :> df$ **..**
moreover from *prems* **have** $\exists\,D.\ DERIV\ g\ x :> D$ **by** (*unfold differentiable-def*)
then obtain *dg* **where** $DERIV\ g\ x :> dg$ **..**
ultimately have $DERIV\ (\lambda x.\ f\ x * g\ x)\ x :> df * g\ x + dg * f\ x$ **by** (*simp add*: *DERIV-mult*)
hence $\exists\,D.\ DERIV\ (\lambda x.\ f\ x * g\ x)\ x :> D$ **by** *auto*
thus *?thesis* **by** (*fold differentiable-def*)
qed

14.3 Nested Intervals and Bisection

Lemmas about nested intervals and proof by bisection (cf.Harrison). All considerably tidied by lcp.

lemma *lemma-f-mono-add* [*rule-format* (*no-asm*)]: ($\forall\,n.\ (f{::}nat{=}{>}real)\ n \le f\ (Suc\ n)) \ {-}{-}{>}\ f\ m \le f\,(m + no)$)
apply (*induct no*)
apply (*auto intro*: *order-trans*)
done

lemma *f-inc-g-dec-Beq-f*: $[|\ \forall\,n.\ f(n) \le f(Suc\ n);$
$\forall\,n.\ g(Suc\ n) \le g(n);$
$\forall\,n.\ f(n) \le g(n)\ |]$
$==>\ Bseq\ (f :: nat \Rightarrow real)$
apply (*rule-tac* $k = f\ 0$ **and** $K = g\ 0$ **in** *BseqI2*, *rule allI*)
apply (*induct-tac n*)
apply (*auto intro*: *order-trans*)
apply (*rule-tac* $y = g\ (Suc\ na)$ **in** *order-trans*)
apply (*induct-tac* [*2*] *na*)
apply (*auto intro*: *order-trans*)
done

lemma *f-inc-g-dec-Beq-g*: $[|\ \forall\,n.\ f(n) \le f(Suc\ n);$
$\forall\,n.\ g(Suc\ n) \le g(n);$
$\forall\,n.\ f(n) \le g(n)\ |]$
$==>\ Bseq\ (g :: nat \Rightarrow real)$
apply (*subst Bseq-minus-iff* [*symmetric*])
apply (*rule-tac* $g = \%x.\ -\ (f\ x)$ **in** *f-inc-g-dec-Beq-f*)

apply *auto*
done

lemma *f-inc-imp-le-lim*:
  fixes *f* :: *nat* ⇒ *real*
  shows ⟦∀ *n*. *f n* ≤ *f* (*Suc n*); *convergent f*⟧ ⟹ *f n* ≤ *lim f*
apply (*rule linorder-not-less* [*THEN iffD1*])
apply (*auto simp add*: *convergent-LIMSEQ-iff LIMSEQ-iff monoseq-Suc*)
apply (*drule real-less-sum-gt-zero*)
apply (*drule-tac x* = *f n* + − *lim f* **in** *spec*, *safe*)
apply (*drule-tac P* = %*na*. *no*≤*na* −−> *?Q na* **and** *x* = *no* + *n* **in** *spec*, *auto*)
apply (*subgoal-tac lim f* ≤ *f* (*no* + *n*))
apply (*drule-tac no*=*no* **and** *m*=*n* **in** *lemma-f-mono-add*)
apply (*auto simp add*: *add-commute*)
apply (*induct-tac no*)
apply *simp*
apply (*auto intro*: *order-trans simp add*: *diff-minus abs-if*)
done

lemma *lim-uminus*: *convergent g* ==> *lim* (%*x*. − *g x*) = − (*lim g*)
apply (*rule LIMSEQ-minus* [*THEN limI*])
apply (*simp add*: *convergent-LIMSEQ-iff*)
done

lemma *g-dec-imp-lim-le*:
  fixes *g* :: *nat* ⇒ *real*
  shows ⟦∀ *n*. *g* (*Suc n*) ≤ *g*(*n*); *convergent g*⟧ ⟹ *lim g* ≤ *g n*
apply (*subgoal-tac* − (*g n*) ≤ − (*lim g*))
apply (*cut-tac* [*2*] *f* = %*x*. − (*g x*) **in** *f-inc-imp-le-lim*)
apply (*auto simp add*: *lim-uminus convergent-minus-iff* [*symmetric*])
done

lemma *lemma-nest*: [| ∀ *n*. *f*(*n*) ≤ *f*(*Suc n*);
        ∀ *n*. *g*(*Suc n*) ≤ *g*(*n*);
        ∀ *n*. *f*(*n*) ≤ *g*(*n*) |]
     ==> ∃ *l m* :: *real*. *l* ≤ *m* & ((∀ *n*. *f*(*n*) ≤ *l*) & *f* −−−−> *l*) &
                  ((∀ *n*. *m* ≤ *g*(*n*)) & *g* −−−−> *m*)
apply (*subgoal-tac monoseq f* & *monoseq g*)
prefer *2* **apply** (*force simp add*: *LIMSEQ-iff monoseq-Suc*)
apply (*subgoal-tac Bseq f* & *Bseq g*)
prefer *2* **apply** (*blast intro*: *f-inc-g-dec-Beq-f f-inc-g-dec-Beq-g*)
apply (*auto dest*!: *Bseq-monoseq-convergent simp add*: *convergent-LIMSEQ-iff*)
apply (*rule-tac x* = *lim f* **in** *exI*)
apply (*rule-tac x* = *lim g* **in** *exI*)
apply (*auto intro*: *LIMSEQ-le*)
apply (*auto simp add*: *f-inc-imp-le-lim g-dec-imp-lim-le convergent-LIMSEQ-iff*)
done

lemma *lemma-nest-unique*: [| ∀ *n*. *f*(*n*) ≤ *f*(*Suc n*);

```
        ∀ n. g(Suc n) ≤ g(n);
        ∀ n. f(n) ≤ g(n);
        (%n. f(n) − g(n)) ----> 0 |]
     ==> ∃ l::real. ((∀ n. f(n) ≤ l) & f ----> l) &
             ((∀ n. l ≤ g(n)) & g ----> l)
apply (drule lemma-nest, auto)
apply (subgoal-tac l = m)
apply (drule-tac [2] X = f in LIMSEQ-diff)
apply (auto intro: LIMSEQ-unique)
done
```

The universal quantifiers below are required for the declaration of *Bolzano-nest-unique* below.

```
lemma Bolzano-bisect-le:
 a ≤ b ==> ∀ n. fst (Bolzano-bisect P a b n) ≤ snd (Bolzano-bisect P a b n)
apply (rule allI)
apply (induct-tac n)
apply (auto simp add: Let-def split-def)
done

lemma Bolzano-bisect-fst-le-Suc: a ≤ b ==>
  ∀ n. fst(Bolzano-bisect P a b n) ≤ fst (Bolzano-bisect P a b (Suc n))
apply (rule allI)
apply (induct-tac n)
apply (auto simp add: Bolzano-bisect-le Let-def split-def)
done

lemma Bolzano-bisect-Suc-le-snd: a ≤ b ==>
  ∀ n. snd(Bolzano-bisect P a b (Suc n)) ≤ snd (Bolzano-bisect P a b n)
apply (rule allI)
apply (induct-tac n)
apply (auto simp add: Bolzano-bisect-le Let-def split-def)
done

lemma eq-divide-2-times-iff: ((x::real) = y / (2 * z)) = (2 * x = y/z)
apply (auto)
apply (drule-tac f = %u. (1/2) *u in arg-cong)
apply (simp)
done

lemma Bolzano-bisect-diff:
     a ≤ b ==>
      snd(Bolzano-bisect P a b n) − fst(Bolzano-bisect P a b n) =
      (b−a) / (2 ^ n)
apply (induct n)
apply (auto simp add: eq-divide-2-times-iff add-divide-distrib Let-def split-def)
done

lemmas Bolzano-nest-unique =
```

```
  lemma-nest-unique
  [OF Bolzano-bisect-fst-le-Suc Bolzano-bisect-Suc-le-snd Bolzano-bisect-le]


lemma not-P-Bolzano-bisect:
 assumes P:   !!a b c. [| P(a,b); P(b,c); a ≤ b; b ≤ c|] ==> P(a,c)
   and notP: ~ P(a,b)
   and le:  a ≤ b
 shows ~ P(fst(Bolzano-bisect P a b n), snd(Bolzano-bisect P a b n))
proof (induct n)
 case 0 show ?case using notP by simp
next
 case (Suc n)
 thus ?case
by (auto simp del: surjective-pairing [symmetric]
         simp add: Let-def split-def Bolzano-bisect-le [OF le]
   P [of fst (Bolzano-bisect P a b n) - snd (Bolzano-bisect P a b n)])
qed


lemma not-P-Bolzano-bisect':
   [| ∀ a b c. P(a,b) & P(b,c) & a ≤ b & b ≤ c --> P(a,c);
      ~ P(a,b);  a ≤ b |] ==>
    ∀ n. ~ P(fst(Bolzano-bisect P a b n), snd(Bolzano-bisect P a b n))
by (blast elim!: not-P-Bolzano-bisect [THEN [2] rev-notE])



lemma lemma-BOLZANO:
   [| ∀ a b c. P(a,b) & P(b,c) & a ≤ b & b ≤ c --> P(a,c);
      ∀ x. ∃ d::real. 0 < d &
           (∀ a b. a ≤ x & x ≤ b & (b−a) < d --> P(a,b));
      a ≤ b |]
    ==> P(a,b)
apply (rule Bolzano-nest-unique [where P1=P, THEN exE], assumption+)
apply (rule LIMSEQ-minus-cancel)
apply (simp (no-asm-simp) add: Bolzano-bisect-diff LIMSEQ-divide-realpow-zero)
apply (rule ccontr)
apply (drule not-P-Bolzano-bisect', assumption+)
apply (rename-tac l)
apply (drule-tac x = l in spec, clarify)
apply (simp add: LIMSEQ-def)
apply (drule-tac P = %r. 0<r --> ?Q r and x = d/2 in spec)
apply (drule-tac P = %r. 0<r --> ?Q r and x = d/2 in spec)
apply (drule real-less-half-sum, auto)
apply (drule-tac x = fst (Bolzano-bisect P a b (no + noa)) in spec)
apply (drule-tac x = snd (Bolzano-bisect P a b (no + noa)) in spec)
apply safe
apply (simp-all (no-asm-simp))
```

apply (*rule-tac* $y = abs$ (*fst* (*Bolzano-bisect* P a b ($no + noa$)) $-$ l) $+$ *abs* (*snd* (*Bolzano-bisect* P a b ($no + noa$)) $-$ l) **in** *order-le-less-trans*)
apply (*simp* (*no-asm-simp*) *add*: *abs-if*)
apply (*rule real-sum-of-halves* [*THEN subst*])
apply (*rule add-strict-mono*)
apply (*simp-all add*: *diff-minus* [*symmetric*])
done

lemma *lemma-BOLZANO2*: (($\forall$ a b c. ($a \leq b$ & $b \leq c$ & $P(a,b)$ & $P(b,c)$) −−> $P(a,c)$) &
($\forall$ x. $\exists$ d::*real*. $0 < d$ &
($\forall$ a b. $a \leq x$ & $x \leq b$ & $(b-a) < d$ −−> $P(a,b)$)))
−−> ($\forall$ a b. $a \leq b$ −−> $P(a,b)$)
apply *clarify*
apply (*blast intro*: *lemma-BOLZANO*)
done

14.4 Intermediate Value Theorem

Prove Contrapositive by Bisection

lemma *IVT*: [| $f(a$::*real*) $\leq$ (y::*real*); $y \leq f(b)$;
$a \leq b$;
($\forall$ x. $a \leq x$ & $x \leq b$ −−> *isCont* f x) |]
==> $\exists$ x. $a \leq x$ & $x \leq b$ & $f(x) = y$
apply (*rule contrapos-pp*, *assumption*)
apply (*cut-tac* P = % (u,v) . $a \leq u$ & $u \leq v$ & $v \leq b$ −−> ~ (f $(u) \leq y$ & $y \leq f$ (v)) **in** *lemma-BOLZANO2*)
apply *safe*
apply *simp-all*
apply (*simp add*: *isCont-iff LIM-def*)
apply (*rule ccontr*)
apply (*subgoal-tac* $a \leq x$ & $x \leq b$)
prefer *2*
apply *simp*
apply (*drule-tac* P = %d. $0<d$ −−> *?P* d **and** x = *1* **in** *spec*, *arith*)
apply (*drule-tac* $x = x$ **in** *spec*)+
apply *simp*
apply (*drule-tac* P = %r. *?P* r −−> ($\exists$ s>*0*. *?Q* r s) **and** $x = |y - f\,x|$ **in** *spec*)
apply *safe*
apply *simp*
apply (*drule-tac* $x = s$ **in** *spec*, *clarify*)
apply (*cut-tac* $x = f\,x$ **and** $y = y$ **in** *linorder-less-linear*, *safe*)
apply (*drule-tac* $x = ba-x$ **in** *spec*)
apply (*simp-all add*: *abs-if*)
apply (*drule-tac* $x = aa-x$ **in** *spec*)
apply (*case-tac* $x \leq aa$, *simp-all*)
done

lemma *IVT2*: [| $f(b::real) \leq (y::real)$; $y \leq f(a)$;
$a \leq b$;
$(\forall x.\ a \leq x \ \&\ x \leq b \ {-}{-}{>}\ isCont\ f\ x)$
|] ==> $\exists x.\ a \leq x \ \&\ x \leq b \ \&\ f(x) = y$
apply (*subgoal-tac* $-\ f\ a \leq -y \ \&\ -y \leq -\ f\ b$, *clarify*)
apply (*drule IVT* [**where** $f = \%x.\ -\ f\ x$], *assumption*)
apply (*auto intro*: *isCont-minus*)
done

lemma *IVT-objl*: $(f(a::real) \leq (y::real) \ \&\ y \leq f(b) \ \&\ a \leq b \ \&$
$(\forall x.\ a \leq x \ \&\ x \leq b \ {-}{-}{>}\ isCont\ f\ x))$
$-{-}{>}\ (\exists x.\ a \leq x \ \&\ x \leq b \ \&\ f(x) = y)$
apply (*blast intro*: *IVT*)
done

lemma *IVT2-objl*: $(f(b::real) \leq (y::real) \ \&\ y \leq f(a) \ \&\ a \leq b \ \&$
$(\forall x.\ a \leq x \ \&\ x \leq b \ {-}{-}{>}\ isCont\ f\ x))$
$-{-}{>}\ (\exists x.\ a \leq x \ \&\ x \leq b \ \&\ f(x) = y)$
apply (*blast intro*: *IVT2*)
done

By bisection, function continuous on closed interval is bounded above

lemma *isCont-bounded*:
[| $a \leq b$; $\forall x.\ a \leq x \ \&\ x \leq b \ {-}{-}{>}\ isCont\ f\ x$ |]
==> $\exists M::real.\ \forall x::real.\ a \leq x \ \&\ x \leq b \ {-}{-}{>}\ f(x) \leq M$
apply (*cut-tac* $P = \%\ (u,v)\ .\ a \leq u \ \&\ u \leq v \ \&\ v \leq b \ {-}{-}{>}\ (\exists M.\ \forall x.\ u \leq x \ \&$ $x \leq v \ {-}{-}{>}\ f\ x \leq M)$ **in** *lemma-BOLZANO2*)
apply *safe*
apply *simp-all*
apply (*rename-tac x xa ya M Ma*)
apply (*cut-tac* $x = M$ **and** $y = Ma$ **in** *linorder-linear*, *safe*)
apply (*rule-tac* $x = Ma$ **in** *exI*, *clarify*)
apply (*cut-tac* $x = xb$ **and** $y = xa$ **in** *linorder-linear*, *force*)
apply (*rule-tac* $x = M$ **in** *exI*, *clarify*)
apply (*cut-tac* $x = xb$ **and** $y = xa$ **in** *linorder-linear*, *force*)
apply (*case-tac* $a \leq x \ \&\ x \leq b$)
apply (*rule-tac* [*2*] $x = 1$ **in** *exI*)
prefer *2* **apply** *force*
apply (*simp add*: *LIM-def isCont-iff*)
apply (*drule-tac* $x = x$ **in** *spec*, *auto*)
apply (*erule-tac* $V = \forall M.\ \exists x.\ a \leq x \ \&\ x \leq b \ \&\ {}^{\sim}\ f\ x \leq M$ **in** *thin-rl*)
apply (*drule-tac* $x = 1$ **in** *spec*, *auto*)
apply (*rule-tac* $x = s$ **in** *exI*, *clarify*)
apply (*rule-tac* $x = |f\ x| + 1$ **in** *exI*, *clarify*)
apply (*drule-tac* $x = xa{-}x$ **in** *spec*)
apply (*auto simp add*: *abs-ge-self*)
done

Refine the above to existence of least upper bound

lemma *lemma-reals-complete*: $((\exists\, x.\ x \in S)\ \&\ (\exists\, y.\ isUb\ UNIV\ S\ (y{::}real)))\ {-}{-}{>}$
$(\exists\, t.\ isLub\ UNIV\ S\ t)$
by (*blast intro*: *reals-complete*)

lemma *isCont-has-Ub*: $[|\ a \le b;\ \forall\, x.\ a \le x\ \&\ x \le b\ {-}{-}{>}\ isCont\ f\ x\ |]$
$=={>}\ \exists\, M{::}real.\ (\forall\, x{::}real.\ a \le x\ \&\ x \le b\ {-}{-}{>}\ f(x) \le M)\ \&$
$(\forall\, N.\ N < M\ {-}{-}{>}\ (\exists\, x.\ a \le x\ \&\ x \le b\ \&\ N < f(x)))$
apply (*cut-tac* $S = Collect\ (\%y.\ \exists\, x.\ a \le x\ \&\ x \le b\ \&\ y = f\ x)$
in *lemma-reals-complete*)
apply *auto*
apply (*drule isCont-bounded*, *assumption*)
apply (*auto simp add*: *isUb-def leastP-def isLub-def setge-def setle-def*)
apply (*rule exI*, *auto*)
apply (*auto dest*!: *spec simp add*: *linorder-not-less*)
done

Now show that it attains its upper bound

lemma *isCont-eq-Ub*:
assumes *le*: $a \le b$
and *con*: $\forall\, x{::}real.\ a \le x\ \&\ x \le b\ {-}{-}{>}\ isCont\ f\ x$
shows $\exists\, M{::}real.\ (\forall\, x.\ a \le x\ \&\ x \le b\ {-}{-}{>}\ f(x) \le M)\ \&$
$(\exists\, x.\ a \le x\ \&\ x \le b\ \&\ f(x) = M)$
proof −
from *isCont-has-Ub* [*OF le con*]
obtain *M* **where** *M1*: $\forall\, x.\ a \le x \wedge x \le b \longrightarrow f\ x \le M$
and *M2*: $!!N.\ N{<}M\ =={>}\ \exists\, x.\ a \le x \wedge x \le b \wedge N < f\ x$ **by** *blast*
show *?thesis*
proof (*intro exI*, *intro conjI*)
show $\forall\, x.\ a \le x \wedge x \le b \longrightarrow f\ x \le M$ **by** (*rule M1*)
show $\exists\, x.\ a \le x \wedge x \le b \wedge f\ x = M$
proof (*rule ccontr*)
assume $\neg\ (\exists\, x.\ a \le x \wedge x \le b \wedge f\ x = M)$
with *M1* **have** *M3*: $\forall\, x.\ a \le x\ \&\ x \le b\ {-}{-}{>}\ f\ x < M$
by (*fastsimp simp add*: *linorder-not-le* [*symmetric*])
hence $\forall\, x.\ a \le x\ \&\ x \le b\ {-}{-}{>}\ isCont\ (\%x.\ inverse\ (M - f\ x))\ x$
by (*auto simp add*: *isCont-inverse isCont-diff con*)
from *isCont-bounded* [*OF le this*]
obtain *k* **where** *k*: $!!x.\ a \le x\ \&\ x \le b\ {-}{-}{>}\ inverse\ (M - f\ x) \le k$ **by** *auto*
have *Minv*: $!!x.\ a \le x\ \&\ x \le b\ {-}{-}{>}\ 0 < inverse\ (M - f\ (x))$
by (*simp add*: *M3 compare-rls*)
have $!!x.\ a \le x\ \&\ x \le b\ {-}{-}{>}\ inverse\ (M - f\ x) < k{+}1$ **using** *k*
by (*auto intro*: *order-le-less-trans* [*of - k*])
with *Minv*
have $!!x.\ a \le x\ \&\ x \le b\ {-}{-}{>}\ inverse(k{+}1) < inverse(inverse(M - f\ x))$
by (*intro strip less-imp-inverse-less*, *simp-all*)
hence *invlt*: $!!x.\ a \le x\ \&\ x \le b\ {-}{-}{>}\ inverse(k{+}1) < M - f\ x$
by *simp*
have $M - inverse\ (k{+}1) < M$ **using** *k* [*of a*] *Minv* [*of a*] *le*
by (*simp*, *arith*)

```
    from M2 [OF this]
    obtain x where ax: a ≤ x & x ≤ b & M − inverse(k+1) < f x ..
    thus False using invlt [of x] by force
   qed
  qed
qed
```

Same theorem for lower bound

```
lemma isCont-eq-Lb: [| a ≤ b; ∀ x. a ≤ x & x ≤ b −−> isCont f x |]
         ==> ∃ M::real. (∀ x::real. a ≤ x & x ≤ b −−> M ≤ f(x)) &
                  (∃ x. a ≤ x & x ≤ b & f(x) = M)
apply (subgoal-tac ∀ x. a ≤ x & x ≤ b −−> isCont (%x. − (f x)) x)
prefer 2 apply (blast intro: isCont-minus)
apply (drule-tac f = (%x. − (f x)) in isCont-eq-Ub)
apply safe
apply auto
done
```

Another version.

```
lemma isCont-Lb-Ub: [|a ≤ b; ∀ x. a ≤ x & x ≤ b −−> isCont f x |]
      ==> ∃ L M::real. (∀ x::real. a ≤ x & x ≤ b −−> L ≤ f(x) & f(x) ≤ M) &
          (∀ y. L ≤ y & y ≤ M −−> (∃ x. a ≤ x & x ≤ b & (f(x) = y)))
apply (frule isCont-eq-Lb)
apply (frule-tac [2] isCont-eq-Ub)
apply (assumption+, safe)
apply (rule-tac x = f x in exI)
apply (rule-tac x = f xa in exI, simp, safe)
apply (cut-tac x = x and y = xa in linorder-linear, safe)
apply (cut-tac f = f and a = x and b = xa and y = y in IVT-objl)
apply (cut-tac [2] f = f and a = xa and b = x and y = y in IVT2-objl, safe)
apply (rule-tac [2] x = xb in exI)
apply (rule-tac [4] x = xb in exI, simp-all)
done
```

If $(0::'a) < f'\ x$ then x is Locally Strictly Increasing At The Right

```
lemma DERIV-left-inc:
  fixes f :: real => real
  assumes der: DERIV f x :> l
    and l:  0 < l
  shows ∃ d > 0. ∀ h > 0. h < d −−> f(x) < f(x + h)
proof −
  from l der [THEN DERIV-D, THEN LIM-D [where r = l]]
  have ∃ s > 0. (∀ z. z ≠ 0 ∧ |z| < s ⟶ |(f(x+z) − f x) / z − l| < l)
    by (simp add: diff-minus)
  then obtain s
        where s:  0 < s
          and all: !!z. z ≠ 0 ∧ |z| < s ⟶ |(f(x+z) − f x) / z − l| < l
    by auto
  thus ?thesis
```

proof (*intro exI conjI strip*)
show $0<s$ **using** s **.**
fix h::*real*
assume $0 < h$ $h < s$
with *all* [*of* h] **show** $f\ x < f\ (x+h)$
proof (*simp add*: *abs-if pos-less-divide-eq diff-minus* [*symmetric*]
split add: *split-if-asm*)
assume $\sim (f\ (x+h) - f\ x)\ /\ h < l$ **and** h: $0 < h$
with l
have $0 < (f\ (x+h) - f\ x)\ /\ h$ **by** *arith*
thus $f\ x < f\ (x+h)$
by (*simp add*: *pos-less-divide-eq h*)
qed
qed
qed

lemma *DERIV-left-dec*:
fixes f :: *real* => *real*
assumes *der*: *DERIV f x* :> l
and l: $l < 0$
shows $\exists\, d > 0.\ \forall\, h > 0.\ h < d \dashrightarrow f(x) < f(x-h)$
proof −
from *l der* [*THEN DERIV-D*, *THEN LIM-D* [**where** $r = -l$]]
have $\exists\, s > 0.\ (\forall\, z.\ z \neq 0 \wedge |z| < s \longrightarrow |(f(x+z) - f\ x)\ /\ z - l| < -l)$
by (*simp add*: *diff-minus*)
then obtain s
where s: $0 < s$
and *all*: $!!z.\ z \neq 0 \wedge |z| < s \longrightarrow |(f(x+z) - f\ x)\ /\ z - l| < -l$
by *auto*
thus *?thesis*
proof (*intro exI conjI strip*)
show $0<s$ **using** s **.**
fix h::*real*
assume $0 < h$ $h < s$
with *all* [*of* $-h$] **show** $f\ x < f\ (x-h)$
proof (*simp add*: *abs-if pos-less-divide-eq diff-minus* [*symmetric*]
split add: *split-if-asm*)
assume $-((f\ (x-h) - f\ x)\ /\ h) < l$ **and** h: $0 < h$
with l
have $0 < (f\ (x-h) - f\ x)\ /\ h$ **by** *arith*
thus $f\ x < f\ (x-h)$
by (*simp add*: *pos-less-divide-eq h*)
qed
qed
qed

lemma *DERIV-local-max*:
fixes f :: *real* => *real*
assumes *der*: *DERIV f x* :> l

```
    and d:   0 < d
    and le:  ∀ y. |x−y| < d −−> f(y) ≤ f(x)
  shows l = 0
proof (cases rule: linorder-cases [of l 0])
  case equal thus ?thesis .
next
  case less
  from DERIV-left-dec [OF der less]
  obtain d' where d': 0 < d'
           and lt: ∀ h > 0. h < d' ⟶ f x < f (x−h) by blast
  from real-lbound-gt-zero [OF d d']
  obtain e where 0 < e ∧ e < d ∧ e < d' ..
  with lt le [THEN spec [where x=x−e]]
  show ?thesis by (auto simp add: abs-if)
next
  case greater
  from DERIV-left-inc [OF der greater]
  obtain d' where d': 0 < d'
           and lt: ∀ h > 0. h < d' ⟶ f x < f (x + h) by blast
  from real-lbound-gt-zero [OF d d']
  obtain e where 0 < e ∧ e < d ∧ e < d' ..
  with lt le [THEN spec [where x=x+e]]
  show ?thesis by (auto simp add: abs-if)
qed
```

Similar theorem for a local minimum

```
lemma DERIV-local-min:
  fixes f :: real => real
  shows [| DERIV f x :> l; 0 < d; ∀ y. |x−y| < d −−> f(x) ≤ f(y) |] ==> l =
0
by (drule DERIV-minus [THEN DERIV-local-max], auto)
```

In particular, if a function is locally flat

```
lemma DERIV-local-const:
  fixes f :: real => real
  shows [| DERIV f x :> l; 0 < d; ∀ y. |x−y| < d −−> f(x) = f(y) |] ==> l =
0
by (auto dest!: DERIV-local-max)
```

Lemma about introducing open ball in open interval

```
lemma lemma-interval-lt:
     [| a < x;  x < b |]
      ==> ∃ d::real. 0 < d & (∀ y. |x−y| < d −−> a < y & y < b)
apply (simp add: abs-less-iff)
apply (insert linorder-linear [of x−a b−x], safe)
apply (rule-tac x = x−a in exI)
apply (rule-tac [2] x = b−x in exI, auto)
done
```

```
lemma lemma-interval: [| a < x;  x < b |] ==>
      ∃ d::real. 0 < d & (∀ y. |x−y| < d −−> a ≤ y & y ≤ b)
apply (drule lemma-interval-lt, auto)
apply (auto intro!: exI)
done
```

Rolle's Theorem. If f is defined and continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b), and $f\ a = f\ b$, then there exists $x0 \in (a,b)$ such that $f'\ x0 = (0{::}'a)$

```
theorem Rolle:
  assumes lt: a < b
    and eq: f(a) = f(b)
    and con: ∀ x. a ≤ x & x ≤ b −−> isCont f x
    and dif [rule-format]: ∀ x. a < x & x < b −−> f differentiable x
  shows ∃ z::real. a < z & z < b & DERIV f z :> 0
proof −
  have le: a ≤ b using lt by simp
  from isCont-eq-Ub [OF le con]
  obtain x where x-max: ∀ z. a ≤ z ∧ z ≤ b ⟶ f z ≤ f x
           and alex: a ≤ x and xleb: x ≤ b
    by blast
  from isCont-eq-Lb [OF le con]
  obtain x' where x'-min: ∀ z. a ≤ z ∧ z ≤ b ⟶ f x' ≤ f z
            and alex': a ≤ x' and x'leb: x' ≤ b
    by blast
  show ?thesis
  proof cases
    assume axb: a < x & x < b
      — f attains its maximum within the interval
    hence ax: a<x and xb: x<b by auto
    from lemma-interval [OF ax xb]
    obtain d where d: 0<d and bound: ∀ y. |x−y| < d ⟶ a ≤ y ∧ y ≤ b
      by blast
    hence bound': ∀ y. |x−y| < d ⟶ f y ≤ f x using x-max
      by blast
    from differentiableD [OF dif [OF axb]]
    obtain l where der: DERIV f x :> l ..
    have l=0 by (rule DERIV-local-max [OF der d bound'])
      — the derivative at a local maximum is zero
    thus ?thesis using ax xb der by auto
  next
    assume notaxb: ~ (a < x & x < b)
    hence xeqab: x=a | x=b using alex xleb by arith
    hence fb-eq-fx: f b = f x by (auto simp add: eq)
    show ?thesis
    proof cases
      assume ax'b: a < x' & x' < b
        — f attains its minimum within the interval
      hence ax': a<x' and x'b: x'<b by auto
```

```
    from lemma-interval [OF ax' x'b]
    obtain d where d: 0<d and bound: ∀ y. |x'−y| < d ⟶ a ≤ y ∧ y ≤ b
  by blast
    hence bound': ∀ y. |x'−y| < d ⟶ f x' ≤ f y using x'-min
  by blast
    from differentiableD [OF dif [OF ax'b]]
    obtain l where der: DERIV f x' :> l ..
    have l=0 by (rule DERIV-local-min [OF der d bound'])
      — the derivative at a local minimum is zero
    thus ?thesis using ax' x'b der by auto
   next
    assume notax'b: ~ (a < x' & x' < b)
      — f is constant througout the interval
    hence x'eqab: x'=a | x'=b using alex' x'leb by arith
    hence fb-eq-fx': f b = f x' by (auto simp add: eq)
    from dense [OF lt]
    obtain r where ar: a < r and rb: r < b by blast
    from lemma-interval [OF ar rb]
    obtain d where d: 0<d and bound: ∀ y. |r−y| < d ⟶ a ≤ y ∧ y ≤ b
  by blast
    have eq-fb: ∀ z. a ≤ z --> z ≤ b --> f z = f b
    proof (clarify)
      fix z::real
      assume az: a ≤ z and zb: z ≤ b
      show f z = f b
      proof (rule order-antisym)
        show f z ≤ f b by (simp add: fb-eq-fx x-max az zb)
        show f b ≤ f z by (simp add: fb-eq-fx' x'-min az zb)
      qed
    qed
    have bound': ∀ y. |r−y| < d ⟶ f r = f y
    proof (intro strip)
      fix y::real
      assume lt: |r−y| < d
      hence f y = f b by (simp add: eq-fb bound)
      thus f r = f y by (simp add: eq-fb ar rb order-less-imp-le)
    qed
    from differentiableD [OF dif [OF conjI [OF ar rb]]]
    obtain l where der: DERIV f r :> l ..
    have l=0 by (rule DERIV-local-const [OF der d bound'])
      — the derivative of a constant function is zero
    thus ?thesis using ar rb der by auto
   qed
  qed
qed
```

14.5 Mean Value Theorem

lemma *lemma-MVT*:

```
    f a − (f b − f a)/(b−a) ∗ a = f b − (f b − f a)/(b−a) ∗ (b::real)
proof cases
  assume a=b thus ?thesis by simp
next
  assume a≠b
  hence ba: b−a ≠ 0 by arith
  show ?thesis
    by (rule real-mult-left-cancel [OF ba, THEN iffD1],
        simp add: right-diff-distrib,
        simp add: left-diff-distrib)
qed

theorem MVT:
  assumes lt:  a < b
      and con: ∀ x. a ≤ x & x ≤ b −−> isCont f x
      and dif [rule-format]: ∀ x. a < x & x < b −−> f differentiable x
  shows ∃ l z::real. a < z & z < b & DERIV f z :> l &
                 (f(b) − f(a) = (b−a) ∗ l)
proof −
  let ?F = %x. f x − ((f b − f a) / (b−a)) ∗ x
  have contF: ∀ x. a ≤ x ∧ x ≤ b ⟶ isCont ?F x using con
    by (fast intro: isCont-diff isCont-const isCont-mult isCont-ident)
  have difF: ∀ x. a < x ∧ x < b ⟶ ?F differentiable x
  proof (clarify)
    fix x::real
    assume ax: a < x and xb: x < b
    from differentiableD [OF dif [OF conjI [OF ax xb]]]
    obtain l where der: DERIV f x :> l ..
    show ?F differentiable x
      by (rule differentiableI [where D = l − (f b − f a)/(b−a)],
          blast intro: DERIV-diff DERIV-cmult-Id der)
  qed
  from Rolle [where f = ?F, OF lt lemma-MVT contF difF]
  obtain z where az: a < z and zb: z < b and der: DERIV ?F z :> 0
    by blast
  have DERIV (%x. ((f b − f a)/(b−a)) ∗ x) z :> (f b − f a)/(b−a)
    by (rule DERIV-cmult-Id)
  hence derF: DERIV (λx. ?F x + (f b − f a) / (b − a) ∗ x) z
                 :> 0 + (f b − f a) / (b − a)
    by (rule DERIV-add [OF der])
  show ?thesis
  proof (intro exI conjI)
    show a < z using az .
    show z < b using zb .
    show f b − f a = (b − a) ∗ ((f b − f a)/(b−a)) by (simp)
    show DERIV f z :> ((f b − f a)/(b−a))  using derF by simp
  qed
qed
```

A function is constant if its derivative is 0 over an interval.

```
lemma DERIV-isconst-end:
  fixes f :: real => real
  shows [| a < b;
         ∀x. a ≤ x & x ≤ b --> isCont f x;
         ∀x. a < x & x < b --> DERIV f x :> 0 |]
        ==> f b = f a
apply (drule MVT, assumption)
apply (blast intro: differentiableI)
apply (auto dest!: DERIV-unique simp add: diff-eq-eq)
done

lemma DERIV-isconst1:
  fixes f :: real => real
  shows [| a < b;
         ∀x. a ≤ x & x ≤ b --> isCont f x;
         ∀x. a < x & x < b --> DERIV f x :> 0 |]
        ==> ∀x. a ≤ x & x ≤ b --> f x = f a
apply safe
apply (drule-tac x = a in order-le-imp-less-or-eq, safe)
apply (drule-tac b = x in DERIV-isconst-end, auto)
done

lemma DERIV-isconst2:
  fixes f :: real => real
  shows [| a < b;
         ∀x. a ≤ x & x ≤ b --> isCont f x;
         ∀x. a < x & x < b --> DERIV f x :> 0;
         a ≤ x; x ≤ b |]
        ==> f x = f a
apply (blast dest: DERIV-isconst1)
done

lemma DERIV-isconst-all:
  fixes f :: real => real
  shows ∀x. DERIV f x :> 0 ==> f(x) = f(y)
apply (rule linorder-cases [of x y])
apply (blast intro: sym DERIV-isCont DERIV-isconst-end)+
done

lemma DERIV-const-ratio-const:
  fixes f :: real => real
  shows [|a ≠ b; ∀x. DERIV f x :> k |] ==> (f(b) - f(a)) = (b-a) * k
apply (rule linorder-cases [of a b], auto)
apply (drule-tac [!] f = f in MVT)
apply (auto dest: DERIV-isCont DERIV-unique simp add: differentiable-def)
apply (auto dest: DERIV-unique simp add: ring-distribs diff-minus)
done

lemma DERIV-const-ratio-const2:
```

fixes f :: *real* => *real*
shows $[|a \neq b; \forall x.\ DERIV\ f\ x :> k\ |] ==> (f(b) - f(a))/(b-a) = k$
apply (*rule-tac c1* = $b-a$ **in** *real-mult-right-cancel* [*THEN iffD1*])
apply (*auto dest*!: *DERIV-const-ratio-const simp add*: *mult-assoc*)
done

lemma *real-average-minus-first* [*simp*]: $((a + b) /2 - a) = (b-a)/(2::real)$
by (*simp*)

lemma *real-average-minus-second* [*simp*]: $((b + a)/2 - a) = (b-a)/(2::real)$
by (*simp*)

Gallileo's "trick": average velocity = av. of end velocities

lemma *DERIV-const-average*:
fixes v :: *real* => *real*
assumes *neq*: $a \neq (b::real)$
and *der*: $\forall x.\ DERIV\ v\ x :> k$
shows $v\ ((a + b)/2) = (v\ a + v\ b)/2$
proof (*cases rule*: *linorder-cases* [*of a b*])
case *equal* **with** *neq* **show** *?thesis* **by** *simp*
next
case *less*
have $(v\ b - v\ a) / (b - a) = k$
by (*rule DERIV-const-ratio-const2* [*OF neq der*])
hence $(b-a) * ((v\ b - v\ a) / (b-a)) = (b-a) * k$ **by** *simp*
moreover have $(v\ ((a + b) / 2) - v\ a) / ((a + b) / 2 - a) = k$
by (*rule DERIV-const-ratio-const2* [*OF - der*], *simp add*: *neq*)
ultimately show *?thesis* **using** *neq* **by** *force*
next
case *greater*
have $(v\ b - v\ a) / (b - a) = k$
by (*rule DERIV-const-ratio-const2* [*OF neq der*])
hence $(b-a) * ((v\ b - v\ a) / (b-a)) = (b-a) * k$ **by** *simp*
moreover have $(v\ ((b + a) / 2) - v\ a) / ((b + a) / 2 - a) = k$
by (*rule DERIV-const-ratio-const2* [*OF - der*], *simp add*: *neq*)
ultimately show *?thesis* **using** *neq* **by** (*force simp add*: *add-commute*)
qed

Dull lemma: an continuous injection on an interval must have a strict maximum at an end point, not in the middle.

lemma *lemma-isCont-inj*:
fixes f :: *real* $\Rightarrow$ *real*
assumes *d*: $0 < d$
and *inj* [*rule-format*]: $\forall z.\ |z-x| \leq d \dashrightarrow g(f\ z) = z$
and *cont*: $\forall z.\ |z-x| \leq d \dashrightarrow isCont\ f\ z$
shows $\exists z.\ |z-x| \leq d\ \&\ f\ x < f\ z$
proof (*rule ccontr*)
assume $\sim (\exists z.\ |z-x| \leq d\ \&\ f\ x < f\ z)$
hence *all* [*rule-format*]: $\forall z.\ |z - x| \leq d \dashrightarrow f\ z \leq f\ x$ **by** *auto*

```
  show False
  proof (cases rule: linorder-le-cases [of f(x−d) f(x+d)])
    case le
    from d cont all [of x+d]
    have flef: f(x+d) ≤ f x
     and xlex: x − d ≤ x
     and cont': ∀z. x − d ≤ z ∧ z ≤ x ⟶ isCont f z
       by (auto simp add: abs-if)
    from IVT [OF le flef xlex cont']
    obtain x' where x−d ≤ x' x' ≤ x f x' = f(x+d) by blast
    moreover
    hence g(f x') = g (f(x+d)) by simp
    ultimately show False using d inj [of x'] inj [of x+d]
      by (simp add: abs-le-iff)
  next
    case ge
    from d cont all [of x−d]
    have flef: f(x−d) ≤ f x
     and xlex: x ≤ x+d
     and cont': ∀z. x ≤ z ∧ z ≤ x+d ⟶ isCont f z
       by (auto simp add: abs-if)
    from IVT2 [OF ge flef xlex cont']
    obtain x' where x ≤ x' x' ≤ x+d f x' = f(x−d) by blast
    moreover
    hence g(f x') = g (f(x−d)) by simp
    ultimately show False using d inj [of x'] inj [of x−d]
      by (simp add: abs-le-iff)
  qed
qed
```

Similar version for lower bound.

```
lemma lemma-isCont-inj2:
  fixes f g :: real ⇒ real
  shows [|0 < d; ∀z. |z−x| ≤ d −−> g(f z) = z;
        ∀z. |z−x| ≤ d −−> isCont f z |]
      ==> ∃z. |z−x| ≤ d & f z < f x
apply (insert lemma-isCont-inj
          [where f = %x. − f x and g = %y. g(−y) and x = x and d = d])
apply (simp add: isCont-minus linorder-not-le)
done
```

Show there's an interval surrounding $f\ x$ in $f[[x - d, x + d]]$.

```
lemma isCont-inj-range:
  fixes f :: real ⇒ real
  assumes d: 0 < d
      and inj: ∀z. |z−x| ≤ d −−> g(f z) = z
      and cont: ∀z. |z−x| ≤ d −−> isCont f z
  shows ∃e>0. ∀y. |y − f x| ≤ e −−> (∃z. |z−x| ≤ d & f z = y)
proof −
```

```
  have x−d ≤ x+d ∀z. x−d ≤ z ∧ z ≤ x+d ⟶ isCont f z using cont d
    by (auto simp add: abs-le-iff)
  from isCont-Lb-Ub [OF this]
  obtain L M
  where all1 [rule-format]: ∀z. x−d ≤ z ∧ z ≤ x+d ⟶ L ≤ f z ∧ f z ≤ M
    and all2 [rule-format]:
        ∀y. L ≤ y ∧ y ≤ M ⟶ (∃z. x−d ≤ z ∧ z ≤ x+d ∧ f z = y)
    by auto
  with d have L ≤ f x & f x ≤ M by simp
  moreover have L ≠ f x
  proof −
    from lemma-isCont-inj2 [OF d inj cont]
    obtain u where |u − x| ≤ d f u < f x  by auto
    thus ?thesis using all1 [of u] by arith
  qed
  moreover have f x ≠ M
  proof −
    from lemma-isCont-inj [OF d inj cont]
    obtain u where |u − x| ≤ d f x < f u  by auto
    thus ?thesis using all1 [of u] by arith
  qed
  ultimately have L < f x & f x < M by arith
  hence 0 < f x − L 0 < M − f x by arith+
  from real-lbound-gt-zero [OF this]
  obtain e where e: 0 < e e < f x − L e < M − f x by auto
  thus ?thesis
  proof (intro exI conjI)
    show 0<e using e(1) .
    show ∀y. |y − f x| ≤ e ⟶ (∃z. |z − x| ≤ d ∧ f z = y)
    proof (intro strip)
      fix y::real
      assume |y − f x| ≤ e
      with e have L ≤ y ∧ y ≤ M by arith
      from all2 [OF this]
      obtain z where x − d ≤ z z ≤ x + d f z = y by blast
      thus ∃z. |z − x| ≤ d ∧ f z = y
        by (force simp add: abs-le-iff)
    qed
  qed
qed
```

Continuity of inverse function

```
lemma isCont-inverse-function:
  fixes f g :: real ⇒ real
  assumes d: 0 < d
      and inj: ∀z. |z−x| ≤ d −−> g(f z) = z
      and cont: ∀z. |z−x| ≤ d −−> isCont f z
  shows isCont g (f x)
proof (simp add: isCont-iff LIM-eq)
```

```
  show ∀r. 0 < r ⟶
        (∃s>0. ∀z. z≠0 ∧ |z| < s ⟶ |g(f x + z) − g(f x)| < r)
  proof (intro strip)
    fix r::real
    assume r: 0<r
    from real-lbound-gt-zero [OF r d]
    obtain e where e: 0 < e and e-lt: e < r ∧ e < d by blast
    with inj cont
    have e-simps: ∀z. |z−x| ≤ e −−> g (f z) = z
                  ∀z. |z−x| ≤ e −−> isCont f z   by auto
    from isCont-inj-range [OF e this]
    obtain e' where e': 0 < e'
        and all: ∀y. |y − f x| ≤ e' ⟶ (∃z. |z − x| ≤ e ∧ f z = y)
          by blast
    show ∃s>0. ∀z. z≠0 ∧ |z| < s ⟶ |g(f x + z) − g(f x)| < r
    proof (intro exI conjI)
      show 0<e' using e' .
      show ∀z. z ≠ 0 ∧ |z| < e' ⟶ |g (f x + z) − g (f x)| < r
      proof (intro strip)
        fix z::real
        assume z: z ≠ 0 ∧ |z| < e'
        with e e-lt e-simps all [rule-format, of f x + z]
        show |g (f x + z) − g (f x)| < r by force
      qed
    qed
  qed
qed
```

Derivative of inverse function

```
lemma DERIV-inverse-function:
  fixes f g :: real ⇒ real
  assumes der: DERIV f (g x) :> D
  assumes neq: D ≠ 0
  assumes a: a < x and b: x < b
  assumes inj: ∀y. a < y ∧ y < b ⟶ f (g y) = y
  assumes cont: isCont g x
  shows DERIV g x :> inverse D
unfolding DERIV-iff2
proof (rule LIM-equal2)
  show 0 < min (x − a) (b − x)
    using a b by simp
next
  fix y
  assume norm (y − x) < min (x − a) (b − x)
  hence a < y and y < b
    by (simp-all add: abs-less-iff)
  thus (g y − g x) / (y − x) =
        inverse ((f (g y) − x) / (g y − g x))
    by (simp add: inj)
```

```
next
  have (λz. (f z − f (g x)) / (z − g x)) −− g x −−> D
    by (rule der [unfolded DERIV-iff2])
  hence 1: (λz. (f z − x) / (z − g x)) −− g x −−> D
    using inj a b by simp
  have 2: ∃ d>0. ∀ y. y ≠ x ∧ norm (y − x) < d ⟶ g y ≠ g x
  proof (safe intro!: exI)
    show 0 < min (x − a) (b − x)
      using a b by simp
  next
    fix y
    assume norm (y − x) < min (x − a) (b − x)
    hence y: a < y y < b
      by (simp-all add: abs-less-iff)
    assume g y = g x
    hence f (g y) = f (g x) by simp
    hence y = x using inj y a b by simp
    also assume y ≠ x
    finally show False by simp
  qed
  have (λy. (f (g y) − x) / (g y − g x)) −− x −−> D
    using cont 1 2 by (rule isCont-LIM-compose2)
  thus (λy. inverse ((f (g y) − x) / (g y − g x)))
        −− x −−> inverse D
    using neq by (rule LIM-inverse)
qed

theorem GMVT:
  fixes a b :: real
  assumes alb: a < b
  and fc: ∀ x. a ≤ x ∧ x ≤ b ⟶ isCont f x
  and fd: ∀ x. a < x ∧ x < b ⟶ f differentiable x
  and gc: ∀ x. a ≤ x ∧ x ≤ b ⟶ isCont g x
  and gd: ∀ x. a < x ∧ x < b ⟶ g differentiable x
  shows ∃ g'c f'c c. DERIV g c :> g'c ∧ DERIV f c :> f'c ∧ a < c ∧ c < b ∧
((f b − f a) ∗ g'c) = ((g b − g a) ∗ f'c)
proof −
  let ?h = λx. (f b − f a)∗(g x) − (g b − g a)∗(f x)
  from prems have a < b by simp
  moreover have ∀ x. a ≤ x ∧ x ≤ b ⟶ isCont ?h x
  proof −
    have ∀ x. a <= x ∧ x <= b ⟶ isCont (λx. f b − f a) x by simp
    with gc have ∀ x. a <= x ∧ x <= b ⟶ isCont (λx. (f b − f a) ∗ g x) x
      by (auto intro: isCont-mult)
    moreover
    have ∀ x. a <= x ∧ x <= b ⟶ isCont (λx. g b − g a) x by simp
    with fc have ∀ x. a <= x ∧ x <= b ⟶ isCont (λx. (g b − g a) ∗ f x) x
      by (auto intro: isCont-mult)
    ultimately show ?thesis
```

by (*fastsimp intro*: *isCont-diff*)
qed
moreover
have $\forall x.\ a < x \wedge x < b \longrightarrow$ *?h differentiable x*
proof –
have $\forall x.\ a < x \wedge x < b \longrightarrow (\lambda x.\ f\ b - f\ a)$ *differentiable x* **by** (*simp add*: *differentiable-const*)
with *gd* **have** $\forall x.\ a < x \wedge x < b \longrightarrow (\lambda x.\ (f\ b - f\ a) * g\ x)$ *differentiable x* **by** (*simp add*: *differentiable-mult*)
moreover
have $\forall x.\ a < x \wedge x < b \longrightarrow (\lambda x.\ g\ b - g\ a)$ *differentiable x* **by** (*simp add*: *differentiable-const*)
with *fd* **have** $\forall x.\ a < x \wedge x < b \longrightarrow (\lambda x.\ (g\ b - g\ a) * f\ x)$ *differentiable x* **by** (*simp add*: *differentiable-mult*)
ultimately show *?thesis* **by** (*simp add*: *differentiable-diff*)
qed
ultimately have $\exists\, l\ z.\ a < z \wedge z < b \wedge$ *DERIV ?h z* :> $l \wedge$ *?h b* – *?h a* $= (b - a) * l$ **by** (*rule MVT*)
then obtain *l* **where** *ldef*: $\exists\, z.\ a < z \wedge z < b \wedge$ *DERIV ?h z* :> $l \wedge$ *?h b* – *?h a* $= (b - a) * l$ **..**
then obtain *c* **where** *cdef*: $a < c \wedge c < b \wedge$ *DERIV ?h c* :> $l \wedge$ *?h b* – *?h a* $= (b - a) * l$ **..**

from *cdef* **have** *cint*: $a < c \wedge c < b$ **by** *auto*
with *gd* **have** *g differentiable c* **by** *simp*
hence $\exists\, D.$ *DERIV g c* :> *D* **by** (*rule differentiableD*)
then obtain *g'c* **where** *g'cdef*: *DERIV g c* :> *g'c* **..**

from *cdef* **have** $a < c \wedge c < b$ **by** *auto*
with *fd* **have** *f differentiable c* **by** *simp*
hence $\exists\, D.$ *DERIV f c* :> *D* **by** (*rule differentiableD*)
then obtain *f'c* **where** *f'cdef*: *DERIV f c* :> *f'c* **..**

from *cdef* **have** *DERIV ?h c* :> *l* **by** *auto*
moreover
{
have *DERIV* $(\lambda x.\ (f\ b - f\ a) * g\ x)$ *c* :> $g'c * (f\ b - f\ a)$
apply (*insert DERIV-const* [**where** $k = f\ b - f\ a$])
apply (*drule meta-spec* [*of* - *c*])
apply (*drule DERIV-mult* [*OF* - *g'cdef*])
by *simp*
moreover have *DERIV* $(\lambda x.\ (g\ b - g\ a) * f\ x)$ *c* :> $f'c * (g\ b - g\ a)$
apply (*insert DERIV-const* [**where** $k = g\ b - g\ a$])
apply (*drule meta-spec* [*of* - *c*])
apply (*drule DERIV-mult* [*OF* - *f'cdef*])
by *simp*
ultimately have *DERIV ?h c* :> $g'c * (f\ b - f\ a) - f'c * (g\ b - g\ a)$
by (*simp add*: *DERIV-diff*)
}

ultimately have *leq*: $l = g'c * (f\ b - f\ a) - f'c * (g\ b - g\ a)$ **by** (*rule DERIV-unique*)

{
from *cdef* **have** $?h\ b - ?h\ a = (b - a) * l$ **by** *auto*
also with *leq* **have** $\ldots = (b - a) * (g'c * (f\ b - f\ a) - f'c * (g\ b - g\ a))$ **by** *simp*
finally have $?h\ b - ?h\ a = (b - a) * (g'c * (f\ b - f\ a) - f'c * (g\ b - g\ a))$ **by** *simp*
}
moreover
{
have $?h\ b - ?h\ a =$
$((f\ b)*(g\ b) - (f\ a)*(g\ b) - (g\ b)*(f\ b) + (g\ a)*(f\ b)) -$
$((f\ b)*(g\ a) - (f\ a)*(g\ a) - (g\ b)*(f\ a) + (g\ a)*(f\ a))$
by (*simp add*: *mult-ac add-ac right-diff-distrib*)
hence $?h\ b - ?h\ a = 0$ **by** *auto*
}
ultimately have $(b - a) * (g'c * (f\ b - f\ a) - f'c * (g\ b - g\ a)) = 0$ **by** *auto*
with *alb* **have** $g'c * (f\ b - f\ a) - f'c * (g\ b - g\ a) = 0$ **by** *simp*
hence $g'c * (f\ b - f\ a) = f'c * (g\ b - g\ a)$ **by** *simp*
hence $(f\ b - f\ a) * g'c = (g\ b - g\ a) * f'c$ **by** (*simp add*: *mult-ac*)

with *g'cdef f'cdef cint* **show** *?thesis* **by** *auto*
qed

lemma *lemma-DERIV-subst*: [| *DERIV f x* :> *D*; *D* = *E* |] ==> *DERIV f x* :> *E*
by *auto*

end

15 NthRoot: Nth Roots of Real Numbers

theory *NthRoot*
imports *SEQ Parity Deriv*
begin

15.1 Existence of Nth Root

Existence follows from the Intermediate Value Theorem

lemma *realpow-pos-nth*:
assumes *n*: $0 < n$
assumes *a*: $0 < a$
shows $\exists r>0.\ r\ \hat{}\ n = (a::real)$
proof −
have $\exists r\geq 0.\ r \leq (max\ 1\ a) \wedge r\ \hat{}\ n = a$

```
  proof (rule IVT)
    show 0 ^ n ≤ a using n a by (simp add: power-0-left)
    show 0 ≤ max 1 a by simp
    from n have n1: 1 ≤ n by simp
    have a ≤ max 1 a ^ 1 by simp
    also have max 1 a ^ 1 ≤ max 1 a ^ n
      using n1 by (rule power-increasing, simp)
    finally show a ≤ max 1 a ^ n .
    show ∀ r. 0 ≤ r ∧ r ≤ max 1 a ⟶ isCont (λx. x ^ n) r
      by (simp add: isCont-power)
  qed
  then obtain r where r: 0 ≤ r ∧ r ^ n = a by fast
  with n a have r ≠ 0 by (auto simp add: power-0-left)
  with r have 0 < r ∧ r ^ n = a by simp
  thus ?thesis ..
qed
```

```
lemma realpow-pos-nth2: (0::real) < a ⟹ ∃ r>0. r ^ Suc n = a
by (blast intro: realpow-pos-nth)
```

Uniqueness of nth positive root

```
lemma realpow-pos-nth-unique:
  ⟦0 < n; 0 < a⟧ ⟹ ∃!r. 0 < r ∧ r ^ n = (a::real)
apply (auto intro!: realpow-pos-nth)
apply (rule-tac n=n in power-eq-imp-eq-base, simp-all)
done
```

15.2 Nth Root

We define roots of negative reals such that *root n* (− *x*) = − *root n x*. This allows us to omit side conditions from many theorems.

```
definition
  root :: [nat, real] ⇒ real where
  root n x = (if 0 < x then (THE u. 0 < u ∧ u ^ n = x) else
              if x < 0 then − (THE u. 0 < u ∧ u ^ n = − x) else 0)
```

```
lemma real-root-zero [simp]: root n 0 = 0
unfolding root-def by simp
```

```
lemma real-root-minus: 0 < n ⟹ root n (− x) = − root n x
unfolding root-def by simp
```

```
lemma real-root-gt-zero: ⟦0 < n; 0 < x⟧ ⟹ 0 < root n x
apply (simp add: root-def)
apply (drule (1) realpow-pos-nth-unique)
apply (erule theI' [THEN conjunct1])
done
```

lemma *real-root-pow-pos*:
$[\![0 < n;\ 0 < x]\!] \Longrightarrow root\ n\ x\ \hat{}\ n = x$
apply (*simp add*: *root-def*)
apply (*drule* (*1*) *realpow-pos-nth-unique*)
apply (*erule theI′* [*THEN conjunct2*])
done

lemma *real-root-pow-pos2* [*simp*]:
$[\![0 < n;\ 0 \le x]\!] \Longrightarrow root\ n\ x\ \hat{}\ n = x$
by (*auto simp add*: *order-le-less real-root-pow-pos*)

lemma *odd-pos*: $odd\ (n{::}nat) \Longrightarrow 0 < n$
by (*cases n*, *simp-all*)

lemma *odd-real-root-pow*: $odd\ n \Longrightarrow root\ n\ x\ \hat{}\ n = x$
apply (*rule-tac x=0* **and** *y=x* **in** *linorder-le-cases*)
apply (*erule* (*1*) *real-root-pow-pos2* [*OF odd-pos*])
apply (*subgoal-tac* $root\ n\ (-\ x)\ \hat{}\ n = -\ x$)
apply (*simp add*: *real-root-minus odd-pos*)
apply (*simp add*: *odd-pos*)
done

lemma *real-root-ge-zero*: $[\![0 < n;\ 0 \le x]\!] \Longrightarrow 0 \le root\ n\ x$
by (*auto simp add*: *order-le-less real-root-gt-zero*)

lemma *real-root-power-cancel*: $[\![0 < n;\ 0 \le x]\!] \Longrightarrow root\ n\ (x\ \hat{}\ n) = x$
apply (*subgoal-tac* $0 \le x\ \hat{}\ n$)
apply (*subgoal-tac* $0 \le root\ n\ (x\ \hat{}\ n)$)
apply (*subgoal-tac* $root\ n\ (x\ \hat{}\ n)\ \hat{}\ n = x\ \hat{}\ n$)
apply (*erule* (*3*) *power-eq-imp-eq-base*)
apply (*erule* (*1*) *real-root-pow-pos2*)
apply (*erule* (*1*) *real-root-ge-zero*)
apply (*erule zero-le-power*)
done

lemma *odd-real-root-power-cancel*: $odd\ n \Longrightarrow root\ n\ (x\ \hat{}\ n) = x$
apply (*rule-tac x=0* **and** *y=x* **in** *linorder-le-cases*)
apply (*erule* (*1*) *real-root-power-cancel* [*OF odd-pos*])
apply (*subgoal-tac* $root\ n\ ((-\ x)\ \hat{}\ n) = -\ x$)
apply (*simp add*: *real-root-minus odd-pos*)
apply (*erule real-root-power-cancel* [*OF odd-pos*], *simp*)
done

lemma *real-root-pos-unique*:
$[\![0 < n;\ 0 \le y;\ y\ \hat{}\ n = x]\!] \Longrightarrow root\ n\ x = y$
by (*erule subst*, *rule real-root-power-cancel*)

lemma *odd-real-root-unique*:

$[\![odd\ n;\ y\ \hat{}\ n = x]\!] \Longrightarrow root\ n\ x = y$
by (*erule subst*, *rule odd-real-root-power-cancel*)

lemma *real-root-one* [*simp*]: $0 < n \Longrightarrow root\ n\ 1 = 1$
by (*simp add*: *real-root-pos-unique*)

Root function is strictly monotonic, hence injective

lemma *real-root-less-mono-lemma*:
$[\![0 < n;\ 0 \leq x;\ x < y]\!] \Longrightarrow root\ n\ x < root\ n\ y$
apply (*subgoal-tac* $0 \leq y$)
apply (*subgoal-tac* $root\ n\ x\ \hat{}\ n < root\ n\ y\ \hat{}\ n$)
apply (*erule power-less-imp-less-base*)
apply (*erule* (*1*) *real-root-ge-zero*)
apply *simp*
apply *simp*
done

lemma *real-root-less-mono*: $[\![0 < n;\ x < y]\!] \Longrightarrow root\ n\ x < root\ n\ y$
apply (*cases* $0 \leq x$)
apply (*erule* (*2*) *real-root-less-mono-lemma*)
apply (*cases* $0 \leq y$)
apply (*rule-tac* $y=0$ **in** *order-less-le-trans*)
apply (*subgoal-tac* $0 < root\ n\ (-\ x)$)
apply (*simp add*: *real-root-minus*)
apply (*simp add*: *real-root-gt-zero*)
apply (*simp add*: *real-root-ge-zero*)
apply (*subgoal-tac* $root\ n\ (-\ y) < root\ n\ (-\ x)$)
apply (*simp add*: *real-root-minus*)
apply (*simp add*: *real-root-less-mono-lemma*)
done

lemma *real-root-le-mono*: $[\![0 < n;\ x \leq y]\!] \Longrightarrow root\ n\ x \leq root\ n\ y$
by (*auto simp add*: *order-le-less real-root-less-mono*)

lemma *real-root-less-iff* [*simp*]:
$0 < n \Longrightarrow (root\ n\ x < root\ n\ y) = (x < y)$
apply (*cases* $x < y$)
apply (*simp add*: *real-root-less-mono*)
apply (*simp add*: *linorder-not-less real-root-le-mono*)
done

lemma *real-root-le-iff* [*simp*]:
$0 < n \Longrightarrow (root\ n\ x \leq root\ n\ y) = (x \leq y)$
apply (*cases* $x \leq y$)
apply (*simp add*: *real-root-le-mono*)
apply (*simp add*: *linorder-not-le real-root-less-mono*)
done

lemma *real-root-eq-iff* [*simp*]:

$0 < n \Longrightarrow (root\ n\ x = root\ n\ y) = (x = y)$
by (*simp add*: *order-eq-iff*)

lemmas *real-root-gt-0-iff* [*simp*] = *real-root-less-iff* [**where** $x{=}0$, *simplified*]
lemmas *real-root-lt-0-iff* [*simp*] = *real-root-less-iff* [**where** $y{=}0$, *simplified*]
lemmas *real-root-ge-0-iff* [*simp*] = *real-root-le-iff* [**where** $x{=}0$, *simplified*]
lemmas *real-root-le-0-iff* [*simp*] = *real-root-le-iff* [**where** $y{=}0$, *simplified*]
lemmas *real-root-eq-0-iff* [*simp*] = *real-root-eq-iff* [**where** $y{=}0$, *simplified*]

lemma *real-root-gt-1-iff* [*simp*]: $0 < n \Longrightarrow (1 < root\ n\ y) = (1 < y)$
by (*insert real-root-less-iff* [**where** $x{=}1$], *simp*)

lemma *real-root-lt-1-iff* [*simp*]: $0 < n \Longrightarrow (root\ n\ x < 1) = (x < 1)$
by (*insert real-root-less-iff* [**where** $y{=}1$], *simp*)

lemma *real-root-ge-1-iff* [*simp*]: $0 < n \Longrightarrow (1 \le root\ n\ y) = (1 \le y)$
by (*insert real-root-le-iff* [**where** $x{=}1$], *simp*)

lemma *real-root-le-1-iff* [*simp*]: $0 < n \Longrightarrow (root\ n\ x \le 1) = (x \le 1)$
by (*insert real-root-le-iff* [**where** $y{=}1$], *simp*)

lemma *real-root-eq-1-iff* [*simp*]: $0 < n \Longrightarrow (root\ n\ x = 1) = (x = 1)$
by (*insert real-root-eq-iff* [**where** $y{=}1$], *simp*)

Roots of roots

lemma *real-root-Suc-0* [*simp*]: $root\ (Suc\ 0)\ x = x$
by (*simp add*: *odd-real-root-unique*)

lemma *real-root-pos-mult-exp*:
$[\![0 < m;\ 0 < n;\ 0 < x]\!] \Longrightarrow root\ (m * n)\ x = root\ m\ (root\ n\ x)$
by (*rule real-root-pos-unique*, *simp-all add*: *power-mult*)

lemma *real-root-mult-exp*:
$[\![0 < m;\ 0 < n]\!] \Longrightarrow root\ (m * n)\ x = root\ m\ (root\ n\ x)$
apply (*rule linorder-cases* [**where** $x{=}x$ **and** $y{=}0$])
apply (*subgoal-tac* $root\ (m * n)\ (-\ x) = root\ m\ (root\ n\ (-\ x))$)
apply (*simp add*: *real-root-minus*)
apply (*simp-all add*: *real-root-pos-mult-exp*)
done

lemma *real-root-commute*:
$[\![0 < m;\ 0 < n]\!] \Longrightarrow root\ m\ (root\ n\ x) = root\ n\ (root\ m\ x)$
by (*simp add*: *real-root-mult-exp* [*symmetric*] *mult-commute*)

Monotonicity in first argument

lemma *real-root-strict-decreasing*:
$[\![0 < n;\ n < N;\ 1 < x]\!] \Longrightarrow root\ N\ x < root\ n\ x$
apply (*subgoal-tac* $root\ n\ (root\ N\ x)\ \hat{}\ n < root\ N\ (root\ n\ x)\ \hat{}\ N$, *simp*)
apply (*simp add*: *real-root-commute power-strict-increasing*

del: *real-root-pow-pos2*)
done

lemma *real-root-strict-increasing*:
$[\![0 < n;\ n < N;\ 0 < x;\ x < 1]\!] \Longrightarrow root\ n\ x < root\ N\ x$
apply (*subgoal-tac root N* (*root n x*) ˆ *N* < *root n* (*root N x*) ˆ *n*, *simp*)
apply (*simp add*: *real-root-commute power-strict-decreasing*
del: *real-root-pow-pos2*)
done

lemma *real-root-decreasing*:
$[\![0 < n;\ n < N;\ 1 \le x]\!] \Longrightarrow root\ N\ x \le root\ n\ x$
by (*auto simp add*: *order-le-less real-root-strict-decreasing*)

lemma *real-root-increasing*:
$[\![0 < n;\ n < N;\ 0 \le x;\ x \le 1]\!] \Longrightarrow root\ n\ x \le root\ N\ x$
by (*auto simp add*: *order-le-less real-root-strict-increasing*)

Roots of multiplication and division

lemma *real-root-mult-lemma*:
$[\![0 < n;\ 0 \le x;\ 0 \le y]\!] \Longrightarrow root\ n\ (x * y) = root\ n\ x * root\ n\ y$
by (*simp add*: *real-root-pos-unique mult-nonneg-nonneg power-mult-distrib*)

lemma *real-root-inverse-lemma*:
$[\![0 < n;\ 0 \le x]\!] \Longrightarrow root\ n\ (inverse\ x) = inverse\ (root\ n\ x)$
by (*simp add*: *real-root-pos-unique power-inverse* [*symmetric*])

lemma *real-root-mult*:
  assumes *n*: $0 < n$
  shows $root\ n\ (x * y) = root\ n\ x * root\ n\ y$
proof (*rule linorder-le-cases*, *rule-tac* [!] *linorder-le-cases*)
  assume $0 \le x$ **and** $0 \le y$
  thus *?thesis* **by** (*rule real-root-mult-lemma* [*OF n*])
next
  assume $0 \le x$ **and** $y \le 0$
  hence $0 \le x$ **and** $0 \le -\ y$ **by** *simp-all*
  hence $root\ n\ (x * -\ y) = root\ n\ x * root\ n\ (-\ y)$
    by (*rule real-root-mult-lemma* [*OF n*])
  thus *?thesis* **by** (*simp add*: *real-root-minus* [*OF n*])
next
  assume $x \le 0$ **and** $0 \le y$
  hence $0 \le -\ x$ **and** $0 \le y$ **by** *simp-all*
  hence $root\ n\ (-\ x * y) = root\ n\ (-\ x) * root\ n\ y$
    by (*rule real-root-mult-lemma* [*OF n*])
  thus *?thesis* **by** (*simp add*: *real-root-minus* [*OF n*])
next
  assume $x \le 0$ **and** $y \le 0$
  hence $0 \le -\ x$ **and** $0 \le -\ y$ **by** *simp-all*
  hence $root\ n\ (-\ x * -\ y) = root\ n\ (-\ x) * root\ n\ (-\ y)$

by (*rule real-root-mult-lemma* [*OF n*])
thus *?thesis* **by** (*simp add*: *real-root-minus* [*OF n*])
qed

lemma *real-root-inverse*:
assumes *n*: $0 < n$
shows *root n* (*inverse x*) = *inverse* (*root n x*)
proof (*rule linorder-le-cases*)
assume $0 \leq x$
thus *?thesis* **by** (*rule real-root-inverse-lemma* [*OF n*])
next
assume $x \leq 0$
hence $0 \leq - x$ **by** *simp*
hence *root n* (*inverse* (− *x*)) = *inverse* (*root n* (− *x*))
by (*rule real-root-inverse-lemma* [*OF n*])
thus *?thesis* **by** (*simp add*: *real-root-minus* [*OF n*])
qed

lemma *real-root-divide*:
$0 < n \Longrightarrow$ *root n* (*x* / *y*) = *root n x* / *root n y*
by (*simp add*: *divide-inverse real-root-mult real-root-inverse*)

lemma *real-root-power*:
$0 < n \Longrightarrow$ *root n* (*x* ^ *k*) = *root n x* ^ *k*
by (*induct k*, *simp-all add*: *real-root-mult*)

lemma *real-root-abs*: $0 < n \Longrightarrow$ *root n* |*x*| = |*root n x*|
by (*simp add*: *abs-if real-root-minus*)

Continuity and derivatives

lemma *isCont-root-pos*:
assumes *n*: $0 < n$
assumes *x*: $0 < x$
shows *isCont* (*root n*) *x*
proof −
have *isCont* (*root n*) (*root n x* ^ *n*)
proof (*rule isCont-inverse-function* [**where** *f*=λ*a*. *a* ^ *n*])
show $0 <$ *root n x* **using** *n x* **by** *simp*
show ∀ *z*. |*z* − *root n x*| ≤ *root n x* ⟶ *root n* (*z* ^ *n*) = *z*
by (*simp add*: *abs-le-iff real-root-power-cancel n*)
show ∀ *z*. |*z* − *root n x*| ≤ *root n x* ⟶ *isCont* (λ*a*. *a* ^ *n*) *z*
by (*simp add*: *isCont-power*)
qed
thus *?thesis* **using** *n x* **by** *simp*
qed

lemma *isCont-root-neg*:
⟦$0 < n$; $x < 0$⟧ ⟹ *isCont* (*root n*) *x*
apply (*subgoal-tac isCont* (λ*x*. − *root n* (− *x*)) *x*)

```
apply (simp add: real-root-minus)
apply (rule isCont-o2 [OF isCont-minus [OF isCont-ident]])
apply (simp add: isCont-minus isCont-root-pos)
done

lemma isCont-root-zero:
  0 < n ⟹ isCont (root n) 0
unfolding isCont-def
apply (rule LIM-I)
apply (rule-tac x=r ^ n in exI, safe)
apply (simp add: zero-less-power)
apply (simp add: real-root-abs [symmetric])
apply (rule-tac n=n in power-less-imp-less-base, simp-all)
done

lemma isCont-real-root: 0 < n ⟹ isCont (root n) x
apply (rule-tac x=x and y=0 in linorder-cases)
apply (simp-all add: isCont-root-pos isCont-root-neg isCont-root-zero)
done

lemma DERIV-real-root:
  assumes n: 0 < n
  assumes x: 0 < x
  shows DERIV (root n) x :> inverse (real n ∗ root n x ^ (n − Suc 0))
proof (rule DERIV-inverse-function)
  show 0 < x using x .
  show x < x + 1 by simp
  show ∀ y. 0 < y ∧ y < x + 1 ⟶ root n y ^ n = y
    using n by simp
  show DERIV (λx. x ^ n) (root n x) :> real n ∗ root n x ^ (n − Suc 0)
    by (rule DERIV-pow)
  show real n ∗ root n x ^ (n − Suc 0) ≠ 0
    using n x by simp
  show isCont (root n) x
    using n by (rule isCont-real-root)
qed

lemma DERIV-odd-real-root:
  assumes n: odd n
  assumes x: x ≠ 0
  shows DERIV (root n) x :> inverse (real n ∗ root n x ^ (n − Suc 0))
proof (rule DERIV-inverse-function)
  show x − 1 < x by simp
  show x < x + 1 by simp
  show ∀ y. x − 1 < y ∧ y < x + 1 ⟶ root n y ^ n = y
    using n by (simp add: odd-real-root-pow)
  show DERIV (λx. x ^ n) (root n x) :> real n ∗ root n x ^ (n − Suc 0)
    by (rule DERIV-pow)
  show real n ∗ root n x ^ (n − Suc 0) ≠ 0
```

  using *odd-pos* [*OF n*] *x* **by** *simp*
 show *isCont* (*root n*) *x*
  using *odd-pos* [*OF n*] **by** (*rule isCont-real-root*)
qed

15.3 Square Root

definition
 sqrt :: *real* ⇒ *real* **where**
 sqrt = *root 2*

lemma *pos2*: $0 < (2::nat)$ **by** *simp*

lemma *real-sqrt-unique*: $[\![y^2 = x;\ 0 \le y]\!] \Longrightarrow sqrt\ x = y$
unfolding *sqrt-def* **by** (*rule real-root-pos-unique* [*OF pos2*])

lemma *real-sqrt-abs* [*simp*]: $sqrt\ (x^2) = |x|$
apply (*rule real-sqrt-unique*)
apply (*rule power2-abs*)
apply (*rule abs-ge-zero*)
done

lemma *real-sqrt-pow2* [*simp*]: $0 \le x \Longrightarrow (sqrt\ x)^2 = x$
unfolding *sqrt-def* **by** (*rule real-root-pow-pos2* [*OF pos2*])

lemma *real-sqrt-pow2-iff* [*simp*]: $((sqrt\ x)^2 = x) = (0 \le x)$
apply (*rule iffI*)
apply (*erule subst*)
apply (*rule zero-le-power2*)
apply (*erule real-sqrt-pow2*)
done

lemma *real-sqrt-zero* [*simp*]: *sqrt 0 = 0*
unfolding *sqrt-def* **by** (*rule real-root-zero*)

lemma *real-sqrt-one* [*simp*]: *sqrt 1 = 1*
unfolding *sqrt-def* **by** (*rule real-root-one* [*OF pos2*])

lemma *real-sqrt-minus*: $sqrt\ (-\ x) = -\ sqrt\ x$
unfolding *sqrt-def* **by** (*rule real-root-minus* [*OF pos2*])

lemma *real-sqrt-mult*: $sqrt\ (x * y) = sqrt\ x * sqrt\ y$
unfolding *sqrt-def* **by** (*rule real-root-mult* [*OF pos2*])

lemma *real-sqrt-inverse*: *sqrt* (*inverse x*) = *inverse* (*sqrt x*)
unfolding *sqrt-def* **by** (*rule real-root-inverse* [*OF pos2*])

lemma *real-sqrt-divide*: $sqrt\ (x\ /\ y) = sqrt\ x\ /\ sqrt\ y$
unfolding *sqrt-def* **by** (*rule real-root-divide* [*OF pos2*])

lemma *real-sqrt-power*: *sqrt* (*x* ˆ *k*) = *sqrt x* ˆ *k*
unfolding *sqrt-def* **by** (*rule real-root-power* [*OF pos2*])

lemma *real-sqrt-gt-zero*: *0* < *x* $\Longrightarrow$ *0* < *sqrt x*
unfolding *sqrt-def* **by** (*rule real-root-gt-zero* [*OF pos2*])

lemma *real-sqrt-ge-zero*: *0* $\leq$ *x* $\Longrightarrow$ *0* $\leq$ *sqrt x*
unfolding *sqrt-def* **by** (*rule real-root-ge-zero* [*OF pos2*])

lemma *real-sqrt-less-mono*: *x* < *y* $\Longrightarrow$ *sqrt x* < *sqrt y*
unfolding *sqrt-def* **by** (*rule real-root-less-mono* [*OF pos2*])

lemma *real-sqrt-le-mono*: *x* $\leq$ *y* $\Longrightarrow$ *sqrt x* $\leq$ *sqrt y*
unfolding *sqrt-def* **by** (*rule real-root-le-mono* [*OF pos2*])

lemma *real-sqrt-less-iff* [*simp*]: (*sqrt x* < *sqrt y*) = (*x* < *y*)
unfolding *sqrt-def* **by** (*rule real-root-less-iff* [*OF pos2*])

lemma *real-sqrt-le-iff* [*simp*]: (*sqrt x* $\leq$ *sqrt y*) = (*x* $\leq$ *y*)
unfolding *sqrt-def* **by** (*rule real-root-le-iff* [*OF pos2*])

lemma *real-sqrt-eq-iff* [*simp*]: (*sqrt x* = *sqrt y*) = (*x* = *y*)
unfolding *sqrt-def* **by** (*rule real-root-eq-iff* [*OF pos2*])

lemmas *real-sqrt-gt-0-iff* [*simp*] = *real-sqrt-less-iff* [**where** *x=0*, *simplified*]
lemmas *real-sqrt-lt-0-iff* [*simp*] = *real-sqrt-less-iff* [**where** *y=0*, *simplified*]
lemmas *real-sqrt-ge-0-iff* [*simp*] = *real-sqrt-le-iff* [**where** *x=0*, *simplified*]
lemmas *real-sqrt-le-0-iff* [*simp*] = *real-sqrt-le-iff* [**where** *y=0*, *simplified*]
lemmas *real-sqrt-eq-0-iff* [*simp*] = *real-sqrt-eq-iff* [**where** *y=0*, *simplified*]

lemmas *real-sqrt-gt-1-iff* [*simp*] = *real-sqrt-less-iff* [**where** *x=1*, *simplified*]
lemmas *real-sqrt-lt-1-iff* [*simp*] = *real-sqrt-less-iff* [**where** *y=1*, *simplified*]
lemmas *real-sqrt-ge-1-iff* [*simp*] = *real-sqrt-le-iff* [**where** *x=1*, *simplified*]
lemmas *real-sqrt-le-1-iff* [*simp*] = *real-sqrt-le-iff* [**where** *y=1*, *simplified*]
lemmas *real-sqrt-eq-1-iff* [*simp*] = *real-sqrt-eq-iff* [**where** *y=1*, *simplified*]

lemma *isCont-real-sqrt*: *isCont sqrt x*
unfolding *sqrt-def* **by** (*rule isCont-real-root* [*OF pos2*])

lemma *DERIV-real-sqrt*:
  0 < *x* $\Longrightarrow$ *DERIV sqrt x* :> *inverse* (*sqrt x*) / *2*
unfolding *sqrt-def* **by** (*rule DERIV-real-root* [*OF pos2*, *simplified*])

lemma *not-real-square-gt-zero* [*simp*]: (~ (*0*::*real*) < *x*∗*x*) = (*x* = *0*)
apply *auto*
apply (*cut-tac x* = *x* **and** *y* = *0* **in** *linorder-less-linear*)
apply (*simp add*: *zero-less-mult-iff*)
done

lemma *real-sqrt-abs2* [*simp*]: *sqrt*($x*x$) = $|x|$
apply (*subst power2-eq-square* [*symmetric*])
apply (*rule real-sqrt-abs*)
done

lemma *real-sqrt-pow2-gt-zero*: $0 < x$ ==> $0 <$ (*sqrt* x)^2
by *simp*

lemma *real-sqrt-not-eq-zero*: $0 < x$ ==> *sqrt* $x \neq 0$
by *simp*

lemma *real-inv-sqrt-pow2*: $0 < x$ ==> *inverse* (*sqrt*(x)) ^ 2 = *inverse* x
by (*simp add*: *power-inverse* [*symmetric*])

lemma *real-sqrt-eq-zero-cancel*: [| $0 \leq x$; *sqrt*(x) = 0|] ==> $x = 0$
by *simp*

lemma *real-sqrt-ge-one*: $1 \leq x$ ==> $1 \leq$ *sqrt* x
by *simp*

lemma *real-sqrt-two-gt-zero* [*simp*]: $0 <$ *sqrt* 2
by *simp*

lemma *real-sqrt-two-ge-zero* [*simp*]: $0 \leq$ *sqrt* 2
by *simp*

lemma *real-sqrt-two-gt-one* [*simp*]: $1 <$ *sqrt* 2
by *simp*

lemma *sqrt-divide-self-eq*:
  assumes *nneg*: $0 \leq x$
  shows *sqrt* x / x = *inverse* (*sqrt* x)
proof *cases*
  assume $x=0$ **thus** *?thesis* **by** *simp*
next
  assume *nz*: $x \neq 0$
  hence *pos*: $0<x$ **using** *nneg* **by** *arith*
  show *?thesis*
  proof (*rule right-inverse-eq* [*THEN iffD1*, *THEN sym*])
    show *sqrt* x / $x \neq 0$ **by** (*simp add*: *divide-inverse nneg nz*)
    show *inverse* (*sqrt* x) / (*sqrt* x / x) = 1
      by (*simp add*: *divide-inverse mult-assoc* [*symmetric*]
            power2-eq-square [*symmetric*] *real-inv-sqrt-pow2 pos nz*)
  qed
qed

lemma *real-divide-square-eq* [*simp*]: (((r::*real*) $*$ a) / ($r * r$)) = a / r
apply (*simp add*: *divide-inverse*)

apply (*case-tac* $r=0$)
apply (*auto simp add*: *mult-ac*)
done

lemma *lemma-real-divide-sqrt-less*: $0 < u ==> u \; / \; sqrt \; 2 < u$
by (*simp add*: *divide-less-eq mult-compare-simps*)

lemma *four-x-squared*:
  fixes *x*::*real*
  shows $4 * x^2 = (2 * x)^2$
by (*simp add*: *power2-eq-square*)

15.4 Square Root of Sum of Squares

lemma *real-sqrt-mult-self-sum-ge-zero* [*simp*]: $0 \le sqrt(x{*}x + y{*}y)$
by (*rule real-sqrt-ge-zero* [*OF sum-squares-ge-zero*])

lemma *real-sqrt-sum-squares-ge-zero* [*simp*]: $0 \le sqrt\ (x^2 + y^2)$
by *simp*

declare *real-sqrt-sum-squares-ge-zero* [*THEN abs-of-nonneg*, *simp*]

lemma *real-sqrt-sum-squares-mult-ge-zero* [*simp*]:
   $0 \le sqrt\ ((x^2 + y^2){*}(xa^2 + ya^2))$
by (*auto intro*!: *real-sqrt-ge-zero simp add*: *zero-le-mult-iff*)

lemma *real-sqrt-sum-squares-mult-squared-eq* [*simp*]:
   $sqrt\ ((x^2 + y^2) * (xa^2 + ya^2))\ \hat{}\ 2 = (x^2 + y^2) * (xa^2 + ya^2)$
by (*auto simp add*: *zero-le-mult-iff*)

lemma *real-sqrt-sum-squares-eq-cancel*: $sqrt\ (x^2 + y^2) = x \Longrightarrow y = 0$
by (*drule-tac* $f = \%x.\ x^2$ **in** *arg-cong*, *simp*)

lemma *real-sqrt-sum-squares-eq-cancel2*: $sqrt\ (x^2 + y^2) = y \Longrightarrow x = 0$
by (*drule-tac* $f = \%x.\ x^2$ **in** *arg-cong*, *simp*)

lemma *real-sqrt-sum-squares-ge1* [*simp*]: $x \le sqrt\ (x^2 + y^2)$
by (*rule power2-le-imp-le*, *simp-all*)

lemma *real-sqrt-sum-squares-ge2* [*simp*]: $y \le sqrt\ (x^2 + y^2)$
by (*rule power2-le-imp-le*, *simp-all*)

lemma *real-sqrt-ge-abs1* [*simp*]: $|x| \le sqrt\ (x^2 + y^2)$
by (*rule power2-le-imp-le*, *simp-all*)

lemma *real-sqrt-ge-abs2* [*simp*]: $|y| \le sqrt\ (x^2 + y^2)$
by (*rule power2-le-imp-le*, *simp-all*)

lemma *le-real-sqrt-sumsq* [*simp*]: $x \le sqrt\ (x * x + y * y)$

by (*simp add*: *power2-eq-square* [*symmetric*])

lemma *power2-sum*:
fixes $x\ y$ :: $'a$::{*number-ring*,*recpower*}
shows $(x + y)^2 = x^2 + y^2 + 2 * x * y$
by (*simp add*: *ring-distribs power2-eq-square*)

lemma *power2-diff*:
fixes $x\ y$ :: $'a$::{*number-ring*,*recpower*}
shows $(x - y)^2 = x^2 + y^2 - 2 * x * y$
by (*simp add*: *ring-distribs power2-eq-square*)

lemma *real-sqrt-sum-squares-triangle-ineq*:
$sqrt\ ((a + c)^2 + (b + d)^2) \leq sqrt\ (a^2 + b^2) + sqrt\ (c^2 + d^2)$
apply (*rule power2-le-imp-le*, *simp*)
apply (*simp add*: *power2-sum*)
apply (*simp only*: *mult-assoc right-distrib* [*symmetric*])
apply (*rule mult-left-mono*)
apply (*rule power2-le-imp-le*)
apply (*simp add*: *power2-sum power-mult-distrib*)
apply (*simp add*: *ring-distribs*)
apply (*subgoal-tac* $0 \leq b^2 * c^2 + a^2 * d^2 - 2 * (a * c) * (b * d)$, *simp*)
apply (*rule-tac* $b=(a * d - b * c)^2$ **in** *ord-le-eq-trans*)
apply (*rule zero-le-power2*)
apply (*simp add*: *power2-diff power-mult-distrib*)
apply (*simp add*: *mult-nonneg-nonneg*)
apply *simp*
apply (*simp add*: *add-increasing*)
done

lemma *real-sqrt-sum-squares-less*:
$[\![|x| < u\ /\ sqrt\ 2;\ |y| < u\ /\ sqrt\ 2]\!] \Longrightarrow sqrt\ (x^2 + y^2) < u$
apply (*rule power2-less-imp-less*, *simp*)
apply (*drule power-strict-mono* [*OF - abs-ge-zero pos2*])
apply (*drule power-strict-mono* [*OF - abs-ge-zero pos2*])
apply (*simp add*: *power-divide*)
apply (*drule order-le-less-trans* [*OF abs-ge-zero*])
apply (*simp add*: *zero-less-divide-iff*)
done

Needed for the infinitely close relation over the nonstandard complex numbers

lemma *lemma-sqrt-hcomplex-capprox*:
$[|\ 0 < u;\ x < u/2;\ y < u/2;\ 0 \leq x;\ 0 \leq y\ |] ==> sqrt\ (x^2 + y^2) < u$
apply (*rule-tac* $y = u/sqrt\ 2$ **in** *order-le-less-trans*)
apply (*erule-tac* [*2*] *lemma-real-divide-sqrt-less*)
apply (*rule power2-le-imp-le*)
apply (*auto simp add*: *real-0-le-divide-iff power-divide*)
apply (*rule-tac* $t = u^2$ **in** *real-sum-of-halves* [*THEN subst*])

apply (*rule add-mono*)
apply (*auto simp add*: *four-x-squared simp del*: *realpow-Suc intro*: *power-mono*)
done

Legacy theorem names:

lemmas *real-root-pos2* = *real-root-power-cancel*
lemmas *real-root-pos-pos* = *real-root-gt-zero* [*THEN order-less-imp-le*]
lemmas *real-root-pos-pos-le* = *real-root-ge-zero*
lemmas *real-sqrt-mult-distrib* = *real-sqrt-mult*
lemmas *real-sqrt-mult-distrib2* = *real-sqrt-mult*
lemmas *real-sqrt-eq-zero-cancel-iff* = *real-sqrt-eq-0-iff*

lemma *real-root-pos*: $0 < x \Longrightarrow root\ (Suc\ n)\ (x\ \hat{}\ (Suc\ n)) = x$
by (*rule real-root-power-cancel* [*OF zero-less-Suc order-less-imp-le*])

end

16 Fact: Factorial Function

theory *Fact*
imports ../*Real*/*Real*
begin

consts *fact* :: *nat* => *nat*
primrec
  fact-0: $fact\ 0 = 1$
  fact-Suc: $fact\ (Suc\ n) = (Suc\ n) * fact\ n$

lemma *fact-gt-zero* [*simp*]: $0 < fact\ n$
by (*induct n*) *auto*

lemma *fact-not-eq-zero* [*simp*]: $fact\ n \neq 0$
by *simp*

lemma *real-of-nat-fact-not-zero* [*simp*]: $real\ (fact\ n) \neq 0$
by *auto*

lemma *real-of-nat-fact-gt-zero* [*simp*]: $0 < real(fact\ n)$
by *auto*

lemma *real-of-nat-fact-ge-zero* [*simp*]: $0 \leq real(fact\ n)$
by *simp*

lemma *fact-ge-one* [*simp*]: $1 \leq fact\ n$
by (*induct n*) *auto*

lemma *fact-mono*: $m \leq n ==> fact\ m \leq fact\ n$
apply (*drule le-imp-less-or-eq*)
apply (*auto dest*!: *less-imp-Suc-add*)
apply (*induct-tac k*, *auto*)
done

Note that *fact 0* = *fact 1*

lemma *fact-less-mono*: [| $0 < m$; $m < n$ |] ==> $fact\ m < fact\ n$
apply (*drule-tac m* = *m* **in** *less-imp-Suc-add*, *auto*)
apply (*induct-tac k*, *auto*)
done

lemma *inv-real-of-nat-fact-gt-zero* [*simp*]: $0 < inverse\ (real\ (fact\ n))$
by (*auto simp add*: *positive-imp-inverse-positive*)

lemma *inv-real-of-nat-fact-ge-zero* [*simp*]: $0 \leq inverse\ (real\ (fact\ n))$
by (*auto intro*: *order-less-imp-le*)

lemma *fact-diff-Suc* [*rule-format*]:
$n < Suc\ m ==> fact\ (Suc\ m - n) = (Suc\ m - n) * fact\ (m - n)$
apply (*induct n arbitrary*: *m*)
apply *auto*
apply (*drule-tac x* = *m* − *1* **in** *meta-spec*, *auto*)
done

lemma *fact-num0* [*simp*]: *fact 0* = *1*
by *auto*

lemma *fact-num-eq-if*: $fact\ m = (if\ m{=}0\ then\ 1\ else\ m * fact\ (m - 1))$
by (*cases m*) *auto*

lemma *fact-add-num-eq-if*:
$fact\ (m + n) = (if\ m + n = 0\ then\ 1\ else\ (m + n) * fact\ (m + n - 1))$
by (*cases m* + *n*) *auto*

lemma *fact-add-num-eq-if2*:
$fact\ (m + n) = (if\ m = 0\ then\ fact\ n\ else\ (m + n) * fact\ ((m - 1) + n))$
by (*cases m*) *auto*

end

17 Series: Finite Summation and Infinite Series

theory *Series*
imports *SEQ*
begin

definition

sums :: $(nat \Rightarrow 'a{::}real\text{-}normed\text{-}vector) \Rightarrow 'a \Rightarrow bool$
(**infixr** *sums 80*) **where**
$f\ sums\ s = (\%n.\ setsum\ f\ \{0..{<}n\})\ {-}{-}{-}{-}{>}\ s$

definition
summable :: $(nat \Rightarrow 'a{::}real\text{-}normed\text{-}vector) \Rightarrow bool$ **where**
$summable\ f = (\exists\, s.\ f\ sums\ s)$

definition
suminf :: $(nat \Rightarrow 'a{::}real\text{-}normed\text{-}vector) \Rightarrow 'a$ **where**
$suminf\ f = (THE\ s.\ f\ sums\ s)$

syntax
$\text{-}suminf :: idt \Rightarrow 'a \Rightarrow 'a\ (\sum \text{-}.\ \text{-}\ [0,\ 10]\ 10)$
translations
$\sum i.\ b == CONST\ suminf\ (\%i.\ b)$

lemma *sumr-diff-mult-const*:
$setsum\ f\ \{0..{<}n\} - (real\ n{*}r) = setsum\ (\%i.\ f\ i - r)\ \{0..{<}n{::}nat\}$
by (*simp add*: *diff-minus setsum-addf real-of-nat-def*)

lemma *real-setsum-nat-ivl-bounded*:
$(!!p.\ p < n \Longrightarrow f(p) \le K)$
$\Longrightarrow setsum\ f\ \{0..{<}n{::}nat\} \le real\ n * K$
using *setsum-bounded*[**where** $A = \{0..{<}n\}$]
by (*auto simp*:*real-of-nat-def*)

lemma *sumr-minus-one-realpow-zero* [*simp*]:
$(\sum i{=}0..{<}2{*}n.\ (-1)\ \hat{}\ Suc\ i) = (0{::}real)$
by (*induct n*, *auto*)

lemma *sumr-one-lb-realpow-zero* [*simp*]:
$(\sum n{=}Suc\ 0..{<}n.\ f(n) * (0{::}real)\ \hat{}\ n) = 0$
by (*rule setsum-0'*, *simp*)

lemma *sumr-group*:
$(\sum m{=}0..{<}n{::}nat.\ setsum\ f\ \{m * k\ ..{<}\ m{*}k + k\}) = setsum\ f\ \{0\ ..{<}\ n * k\}$
apply (*subgoal-tac* $k = 0 \mid 0 < k$, *auto*)
apply (*induct n*)
apply (*simp-all add*: *setsum-add-nat-ivl add-commute*)
done

lemma *sumr-offset3*:
$setsum\ f\ \{0{::}nat..{<}n{+}k\} = (\sum m{=}0..{<}n.\ f\ (m{+}k)) + setsum\ f\ \{0..{<}k\}$
apply (*subst setsum-shift-bounds-nat-ivl* [*symmetric*])
apply (*simp add*: *setsum-add-nat-ivl add-commute*)

done

lemma *sumr-offset*:
  fixes f :: *nat* $\Rightarrow$ *'a*::*ab-group-add*
  shows $(\sum m{=}0..{<}n.\ f(m{+}k)) = setsum\ f\ \{0..{<}n{+}k\} - setsum\ f\ \{0..{<}k\}$
by (*simp add*: *sumr-offset3*)

lemma *sumr-offset2*:
$\forall f.\ (\sum m{=}0..{<}n{::}nat.\ f(m{+}k){::}real) = setsum\ f\ \{0..{<}n{+}k\} - setsum\ f\ \{0..{<}k\}$
by (*simp add*: *sumr-offset*)

lemma *sumr-offset4*:
  $\forall n\ f.\ setsum\ f\ \{0{::}nat..{<}n{+}k\} = (\sum m{=}0..{<}n.\ f\ (m{+}k){::}real) + setsum\ f\ \{0..{<}k\}$
by (*clarify*, *rule sumr-offset3*)

17.1 Infinite Sums, by the Properties of Limits

lemma *sums-summable*: *f sums l ==> summable f*
by (*simp add*: *sums-def summable-def*, *blast*)

lemma *summable-sums*: *summable f ==> f sums* (*suminf f*)
apply (*simp add*: *summable-def suminf-def sums-def*)
apply (*blast intro*: *theI LIMSEQ-unique*)
done

lemma *summable-sumr-LIMSEQ-suminf*:
    summable f ==> (%*n. setsum f* $\{0..{<}n\}$) *----->* (*suminf f*)
by (*rule summable-sums* [*unfolded sums-def*])

lemma *sums-unique*: *f sums s ==>* (*s = suminf f*)
apply (*frule sums-summable* [*THEN summable-sums*])
apply (*auto intro*!: *LIMSEQ-unique simp add*: *sums-def*)
done

lemma *sums-split-initial-segment*: *f sums s ==>*
  (%*n. f*(*n + k*)) *sums* (*s* − (*SUM i = 0..< k. f i*))
  apply (*unfold sums-def*)
  apply (*simp add*: *sumr-offset*)
  apply (*rule LIMSEQ-diff-const*)
  apply (*rule LIMSEQ-ignore-initial-segment*)
  apply *assumption*
done

lemma *summable-ignore-initial-segment*: *summable f ==>*
    summable (%*n. f*(*n + k*))
  apply (*unfold summable-def*)
  apply (*auto intro*: *sums-split-initial-segment*)

done

lemma *suminf-minus-initial-segment*: *summable f* $==>$
  suminf f $= s ==>$ *suminf* $(\%n.\ f(n + k)) = s - (SUM\ i = 0..< k.\ f\ i)$
 apply (*frule summable-ignore-initial-segment*)
 apply (*rule sums-unique* [*THEN sym*])
 apply (*frule summable-sums*)
 apply (*rule sums-split-initial-segment*)
 apply *auto*
done

lemma *suminf-split-initial-segment*: *summable f* $==>$
  suminf f $= (SUM\ i = 0..< k.\ f\ i) +$ *suminf* $(\%n.\ f(n + k))$
by (*auto simp add*: *suminf-minus-initial-segment*)

lemma *series-zero*:
  $(\forall m.\ n \le m --> f(m) = 0) ==> f$ *sums* (*setsum f* $\{0..<n\}$)
apply (*simp add*: *sums-def LIMSEQ-def diff-minus*[*symmetric*], *safe*)
apply (*rule-tac* $x = n$ **in** *exI*)
apply (*clarsimp simp add*:*setsum-diff* [*symmetric*] *cong*:*setsum-ivl-cong*)
done

lemma *sums-zero*: $(\lambda n.\ 0)$ *sums* 0
unfolding *sums-def* **by** (*simp add*: *LIMSEQ-const*)

lemma *summable-zero*: *summable* $(\lambda n.\ 0)$
by (*rule sums-zero* [*THEN sums-summable*])

lemma *suminf-zero*: *suminf* $(\lambda n.\ 0) = 0$
by (*rule sums-zero* [*THEN sums-unique*, *symmetric*])

lemma (**in** *bounded-linear*) *sums*:
 $(\lambda n.\ X\ n)$ *sums* $a \Longrightarrow (\lambda n.\ f\ (X\ n))$ *sums* $(f\ a)$
unfolding *sums-def* **by** (*drule LIMSEQ*, *simp only*: *setsum*)

lemma (**in** *bounded-linear*) *summable*:
 summable $(\lambda n.\ X\ n) \Longrightarrow$ *summable* $(\lambda n.\ f\ (X\ n))$
unfolding *summable-def* **by** (*auto intro*: *sums*)

lemma (**in** *bounded-linear*) *suminf*:
 summable $(\lambda n.\ X\ n) \Longrightarrow f\ (\sum n.\ X\ n) = (\sum n.\ f\ (X\ n))$
by (*intro sums-unique sums summable-sums*)

lemma *sums-mult*:
 fixes c :: *'a::real-normed-algebra*
 shows f *sums* $a \Longrightarrow (\lambda n.\ c * f\ n)$ *sums* $(c * a)$
by (*rule mult-right.sums*)

lemma *summable-mult*:

fixes c :: $'a$::*real-normed-algebra*
shows *summable f* $\Longrightarrow$ *summable* (%n. $c * f\ n$)
by (*rule mult-right.summable*)

lemma *suminf-mult*:
fixes c :: $'a$::*real-normed-algebra*
shows *summable f* $\Longrightarrow$ *suminf* (λn. $c * f\ n$) = c * *suminf f*
by (*rule mult-right.suminf* [*symmetric*])

lemma *sums-mult2*:
fixes c :: $'a$::*real-normed-algebra*
shows *f sums a* $\Longrightarrow$ (λn. $f\ n * c$) *sums* ($a * c$)
by (*rule mult-left.sums*)

lemma *summable-mult2*:
fixes c :: $'a$::*real-normed-algebra*
shows *summable f* $\Longrightarrow$ *summable* (λn. $f\ n * c$)
by (*rule mult-left.summable*)

lemma *suminf-mult2*:
fixes c :: $'a$::*real-normed-algebra*
shows *summable f* $\Longrightarrow$ *suminf f* $* c = (\sum n.\ f\ n * c)$
by (*rule mult-left.suminf*)

lemma *sums-divide*:
fixes c :: $'a$::*real-normed-field*
shows *f sums a* $\Longrightarrow$ (λn. $f\ n\ /\ c$) *sums* ($a\ /\ c$)
by (*rule divide.sums*)

lemma *summable-divide*:
fixes c :: $'a$::*real-normed-field*
shows *summable f* $\Longrightarrow$ *summable* (λn. $f\ n\ /\ c$)
by (*rule divide.summable*)

lemma *suminf-divide*:
fixes c :: $'a$::*real-normed-field*
shows *summable f* $\Longrightarrow$ *suminf* (λn. $f\ n\ /\ c$) = *suminf f* / c
by (*rule divide.suminf* [*symmetric*])

lemma *sums-add*: $[\![X$ *sums* a; Y *sums* $b]\!]$ $\Longrightarrow$ (λn. $X\ n + Y\ n$) *sums* ($a + b$)
unfolding *sums-def* **by** (*simp add*: *setsum-addf LIMSEQ-add*)

lemma *summable-add*: $[\![$*summable* X; *summable* $Y]\!]$ $\Longrightarrow$ *summable* (λn. $X\ n + Y$
n)
unfolding *summable-def* **by** (*auto intro*: *sums-add*)

lemma *suminf-add*:
$[\![$*summable* X; *summable* $Y]\!]$ $\Longrightarrow$ *suminf* X + *suminf* $Y = (\sum n.\ X\ n + Y\ n)$
by (*intro sums-unique sums-add summable-sums*)

lemma *sums-diff*: $[\![X\ sums\ a;\ Y\ sums\ b]\!] \Longrightarrow (\lambda n.\ X\ n - Y\ n)\ sums\ (a - b)$
unfolding *sums-def* **by** (*simp add*: *setsum-subtractf LIMSEQ-diff*)

lemma *summable-diff*: $[\![summable\ X;\ summable\ Y]\!] \Longrightarrow summable\ (\lambda n.\ X\ n - Y\ n)$
unfolding *summable-def* **by** (*auto intro*: *sums-diff*)

lemma *suminf-diff*:
$[\![summable\ X;\ summable\ Y]\!] \Longrightarrow suminf\ X - suminf\ Y = (\sum n.\ X\ n - Y\ n)$
by (*intro sums-unique sums-diff summable-sums*)

lemma *sums-minus*: $X\ sums\ a ==> (\lambda n.\ - X\ n)\ sums\ (- a)$
unfolding *sums-def* **by** (*simp add*: *setsum-negf LIMSEQ-minus*)

lemma *summable-minus*: $summable\ X \Longrightarrow summable\ (\lambda n.\ - X\ n)$
unfolding *summable-def* **by** (*auto intro*: *sums-minus*)

lemma *suminf-minus*: $summable\ X \Longrightarrow (\sum n.\ - X\ n) = - (\sum n.\ X\ n)$
by (*intro sums-unique* [*symmetric*] *sums-minus summable-sums*)

lemma *sums-group*:
[|*summable f*; *0* < *k* |] ==> (%*n*. *setsum f* {*n***k*..<*n***k*+*k*}) *sums* (*suminf f*)
apply (*drule summable-sums*)
apply (*simp only*: *sums-def sumr-group*)
apply (*unfold LIMSEQ-def*, *safe*)
apply (*drule-tac x*=*r* **in** *spec*, *safe*)
apply (*rule-tac x*=*no* **in** *exI*, *safe*)
apply (*drule-tac x*=*n***k* **in** *spec*)
apply (*erule mp*)
apply (*erule order-trans*)
apply *simp*
done

A summable series of positive terms has limit that is at least as great as any partial sum.

lemma *series-pos-le*:
fixes $f :: nat \Rightarrow real$
shows $[\![summable\ f;\ \forall m \geq n.\ 0 \leq f\ m]\!] \Longrightarrow setsum\ f\ \{0..<n\} \leq suminf\ f$
apply (*drule summable-sums*)
apply (*simp add*: *sums-def*)
apply (*cut-tac k* = *setsum f* {*0*..<*n*} **in** *LIMSEQ-const*)
apply (*erule LIMSEQ-le*, *blast*)
apply (*rule-tac x*=*n* **in** *exI*, *clarify*)
apply (*rule setsum-mono2*)
apply *auto*
done

lemma *series-pos-less*:

fixes f :: $nat \Rightarrow real$
shows $[\![summable\ f;\ \forall\, m{\geq}n.\ 0 < f\ m]\!] \Longrightarrow setsum\ f\ \{0..{<}n\} < suminf\ f$
apply (*rule-tac* $y{=}setsum\ f\ \{0..{<}Suc\ n\}$ **in** *order-less-le-trans*)
apply *simp*
apply (*erule series-pos-le*)
apply (*simp add*: *order-less-imp-le*)
done

lemma *suminf-gt-zero*:
fixes f :: $nat \Rightarrow real$
shows $[\![summable\ f;\ \forall\, n.\ 0 < f\ n]\!] \Longrightarrow 0 < suminf\ f$
by (*drule-tac* $n{=}0$ **in** *series-pos-less*, *simp-all*)

lemma *suminf-ge-zero*:
fixes f :: $nat \Rightarrow real$
shows $[\![summable\ f;\ \forall\, n.\ 0 \leq f\ n]\!] \Longrightarrow 0 \leq suminf\ f$
by (*drule-tac* $n{=}0$ **in** *series-pos-le*, *simp-all*)

lemma *sumr-pos-lt-pair*:
fixes f :: $nat \Rightarrow real$
shows $[\![summable\ f;$
$\forall\, d.\ 0 < f\ (k + (Suc(Suc\ 0) * d)) + f\ (k + ((Suc(Suc\ 0) * d) + 1))]\!]$
$\Longrightarrow setsum\ f\ \{0..{<}k\} < suminf\ f$
apply (*subst suminf-split-initial-segment* [**where** $k{=}k$])
apply *assumption*
apply *simp*
apply (*drule-tac* $k{=}k$ **in** *summable-ignore-initial-segment*)
apply (*drule-tac* $k{=}Suc\ (Suc\ 0)$ **in** *sums-group*, *simp*)
apply *simp*
apply (*frule sums-unique*)
apply (*drule sums-summable*)
apply *simp*
apply (*erule suminf-gt-zero*)
apply (*simp add*: *add-ac*)
done

Sum of a geometric progression.

lemmas *sumr-geometric* = *geometric-sum* [**where** $'a = real$]

lemma *geometric-sums*:
fixes x :: $'a$::{*real-normed-field*,*recpower*}
shows $norm\ x < 1 \Longrightarrow (\lambda n.\ x\ \hat{}\ n)\ sums\ (1\ /\ (1 - x))$
proof $-$
assume *less-1*: $norm\ x < 1$
hence *neq-1*: $x \neq 1$ **by** *auto*
hence *neq-0*: $x - 1 \neq 0$ **by** *simp*
from *less-1* **have** *lim-0*: $(\lambda n.\ x\ \hat{}\ n)$ ----> 0
by (*rule LIMSEQ-power-zero*)
hence $(\lambda n.\ x\ \hat{}\ n\ /\ (x - 1) - 1\ /\ (x - 1))$ ----> $0\ /\ (x - 1) - 1\ /\ (x -$

1)
  using *neq-0* **by** (*intro LIMSEQ-divide LIMSEQ-diff LIMSEQ-const*)
 hence ($\lambda n.$ $(x \,\hat{}\, n - 1) / (x - 1)$) $---->$ $1 / (1 - x)$
  by (*simp add*: *nonzero-minus-divide-right* [*OF neq-0*] *diff-divide-distrib*)
 thus ($\lambda n.$ $x \,\hat{}\, n$) *sums* ($1 / (1 - x)$)
  by (*simp add*: *sums-def geometric-sum neq-1*)
qed

lemma *summable-geometric*:
 fixes x :: $'a$::{*real-normed-field*,*recpower*}
 shows *norm* $x < 1 \Longrightarrow$ *summable* ($\lambda n.$ $x \,\hat{}\, n$)
by (*rule geometric-sums* [*THEN sums-summable*])

Cauchy-type criterion for convergence of series (c.f. Harrison)

lemma *summable-convergent-sumr-iff*:
summable f = *convergent* (%*n*. *setsum f* {*0*..<*n*})
by (*simp add*: *summable-def sums-def convergent-def*)

lemma *summable-LIMSEQ-zero*: *summable f* $\Longrightarrow$ f $---->$ *0*
apply (*drule summable-convergent-sumr-iff* [*THEN iffD1*])
apply (*drule convergent-Cauchy*)
apply (*simp only*: *Cauchy-def LIMSEQ-def*, *safe*)
apply (*drule-tac x=r* **in** *spec*, *safe*)
apply (*rule-tac x=M* **in** *exI*, *safe*)
apply (*drule-tac x=Suc n* **in** *spec*, *simp*)
apply (*drule-tac x=n* **in** *spec*, *simp*)
done

lemma *summable-Cauchy*:
  summable (f::*nat* $\Rightarrow$ $'a$::*banach*) =
  ($\forall\, e > 0.\ \exists N.\ \forall\, m \geq N.\ \forall\, n.$ *norm* (*setsum f* {*m*..<*n*}) $< e$)
apply (*simp only*: *summable-convergent-sumr-iff Cauchy-convergent-iff* [*symmetric*] *Cauchy-def*, *safe*)
apply (*drule spec*, *drule* (*1*) *mp*)
apply (*erule exE*, *rule-tac x=M* **in** *exI*, *clarify*)
apply (*rule-tac x=m* **and** *y=n* **in** *linorder-le-cases*)
apply (*frule* (*1*) *order-trans*)
apply (*drule-tac x=n* **in** *spec*, *drule* (*1*) *mp*)
apply (*drule-tac x=m* **in** *spec*, *drule* (*1*) *mp*)
apply (*simp add*: *setsum-diff* [*symmetric*])
apply *simp*
apply (*drule spec*, *drule* (*1*) *mp*)
apply (*erule exE*, *rule-tac x=N* **in** *exI*, *clarify*)
apply (*rule-tac x=m* **and** *y=n* **in** *linorder-le-cases*)
apply (*subst norm-minus-commute*)
apply (*simp add*: *setsum-diff* [*symmetric*])
apply (*simp add*: *setsum-diff* [*symmetric*])
done

Comparison test

lemma *norm-setsum*:
  fixes $f :: 'a \Rightarrow 'b::real\text{-}normed\text{-}vector$
  shows $norm\ (setsum\ f\ A) \le (\sum i{\in}A.\ norm\ (f\ i))$
apply (*case-tac finite A*)
apply (*erule finite-induct*)
apply *simp*
apply *simp*
apply (*erule order-trans* [*OF norm-triangle-ineq add-left-mono*])
apply *simp*
done

lemma *summable-comparison-test*:
  fixes $f :: nat \Rightarrow 'a::banach$
  shows $[\![\exists N.\ \forall n{\ge}N.\ norm\ (f\ n) \le g\ n;\ summable\ g]\!] \Longrightarrow summable\ f$
apply (*simp add*: *summable-Cauchy*, *safe*)
apply (*drule-tac* $x{=}e$ **in** *spec*, *safe*)
apply (*rule-tac* $x = N + Na$ **in** *exI*, *safe*)
apply (*rotate-tac 2*)
apply (*drule-tac* $x = m$ **in** *spec*)
apply (*auto*, *rotate-tac 2*, *drule-tac* $x = n$ **in** *spec*)
apply (*rule-tac* $y = \sum k{=}m..{<}n.\ norm\ (f\ k)$ **in** *order-le-less-trans*)
apply (*rule norm-setsum*)
apply (*rule-tac* $y = setsum\ g\ \{m..{<}n\}$ **in** *order-le-less-trans*)
apply (*auto intro*: *setsum-mono simp add*: *abs-less-iff*)
done

lemma *summable-norm-comparison-test*:
  fixes $f :: nat \Rightarrow 'a::banach$
  shows $[\![\exists N.\ \forall n{\ge}N.\ norm\ (f\ n) \le g\ n;\ summable\ g]\!]$
        $\Longrightarrow summable\ (\lambda n.\ norm\ (f\ n))$
apply (*rule summable-comparison-test*)
apply (*auto*)
done

lemma *summable-rabs-comparison-test*:
  fixes $f :: nat \Rightarrow real$
  shows $[\![\exists N.\ \forall n{\ge}N.\ |f\ n| \le g\ n;\ summable\ g]\!] \Longrightarrow summable\ (\lambda n.\ |f\ n|)$
apply (*rule summable-comparison-test*)
apply (*auto*)
done

Summability of geometric series for real algebras

lemma *complete-algebra-summable-geometric*:
  fixes $x :: 'a::\{real\text{-}normed\text{-}algebra\text{-}1, banach, recpower\}$
  shows $norm\ x < 1 \Longrightarrow summable\ (\lambda n.\ x\ \hat{}\ n)$
proof (*rule summable-comparison-test*)
  show $\exists N.\ \forall n{\ge}N.\ norm\ (x\ \hat{}\ n) \le norm\ x\ \hat{}\ n$
    by (*simp add*: *norm-power-ineq*)
  show $norm\ x < 1 \Longrightarrow summable\ (\lambda n.\ norm\ x\ \hat{}\ n)$

by (*simp add*: *summable-geometric*)
qed

Limit comparison property for series (c.f. jrh)

lemma *summable-le*:
fixes *f g* :: *nat* ⇒ *real*
shows $[\![\forall n.\ f\ n \le g\ n;\ summable\ f;\ summable\ g]\!] \Longrightarrow suminf\ f \le suminf\ g$
apply (*drule summable-sums*)+
apply (*simp only*: *sums-def*, *erule* (*1*) *LIMSEQ-le*)
apply (*rule exI*)
apply (*auto intro*!: *setsum-mono*)
done

lemma *summable-le2*:
fixes *f g* :: *nat* ⇒ *real*
shows $[\![\forall n.\ |f\ n| \le g\ n;\ summable\ g]\!] \Longrightarrow summable\ f \wedge suminf\ f \le suminf\ g$
apply (*subgoal-tac summable f*)
apply (*auto intro*!: *summable-le*)
apply (*simp add*: *abs-le-iff*)
apply (*rule-tac g=g* **in** *summable-comparison-test*, *simp-all*)
done

lemma *suminf-0-le*:
fixes *f*::*nat*⇒*real*
assumes *gt0*: $\forall n.\ 0 \le f\ n$ **and** *sm*: *summable f*
shows $0 \le suminf\ f$
proof −
let *?g* = (λ*n*. (*0*::*real*))
from *gt0* **have** $\forall n.\ ?g\ n \le f\ n$ **by** *simp*
moreover have *summable ?g* **by** (*rule summable-zero*)
moreover from *sm* **have** *summable f* **.**
ultimately have $suminf\ ?g \le suminf\ f$ **by** (*rule summable-le*)
then show $0 \le suminf\ f$ **by** (*simp add*: *suminf-zero*)
qed

Absolute convergence imples normal convergence

lemma *summable-norm-cancel*:
fixes *f* :: *nat* ⇒ *'a*::*banach*
shows *summable* (λ*n*. *norm* (*f n*)) ⟹ *summable f*
apply (*simp only*: *summable-Cauchy*, *safe*)
apply (*drule-tac x=e* **in** *spec*, *safe*)
apply (*rule-tac x=N* **in** *exI*, *safe*)
apply (*drule-tac x=m* **in** *spec*, *safe*)
apply (*rule order-le-less-trans* [*OF norm-setsum*])
apply (*rule order-le-less-trans* [*OF abs-ge-self*])
apply *simp*
done

lemma *summable-rabs-cancel*:
  fixes f :: $nat \Rightarrow real$
  shows $summable\ (\lambda n.\ |f\ n|) \Longrightarrow summable\ f$
by (*rule summable-norm-cancel*, *simp*)

Absolute convergence of series

lemma *summable-norm*:
  fixes f :: $nat \Rightarrow {}'a{::}banach$
  shows $summable\ (\lambda n.\ norm\ (f\ n)) \Longrightarrow norm\ (suminf\ f) \le (\sum n.\ norm\ (f\ n))$
by (*auto intro*: *LIMSEQ-le LIMSEQ-norm summable-norm-cancel*
  summable-sumr-LIMSEQ-suminf norm-setsum)

lemma *summable-rabs*:
  fixes f :: $nat \Rightarrow real$
  shows $summable\ (\lambda n.\ |f\ n|) \Longrightarrow |suminf\ f| \le (\sum n.\ |f\ n|)$
by (*fold real-norm-def*, *rule summable-norm*)

17.2 The Ratio Test

lemma *norm-ratiotest-lemma*:
  fixes $x\ y$ :: $'a{::}real\text{-}normed\text{-}vector$
  shows $[\![c \le 0;\ norm\ x \le c * norm\ y]\!] \Longrightarrow x = 0$
apply (*subgoal-tac* $norm\ x \le 0$, *simp*)
apply (*erule order-trans*)
apply (*simp add*: *mult-le-0-iff*)
done

lemma *rabs-ratiotest-lemma*: $[|\ c \le 0;\ abs\ x \le c * abs\ y\ |] ==> x = (0{::}real)$
by (*erule norm-ratiotest-lemma*, *simp*)

lemma *le-Suc-ex*: $(k{::}nat) \le l ==> (\exists n.\ l = k + n)$
apply (*drule le-imp-less-or-eq*)
apply (*auto dest*: *less-imp-Suc-add*)
done

lemma *le-Suc-ex-iff*: $((k{::}nat) \le l) = (\exists n.\ l = k + n)$
by (*auto simp add*: *le-Suc-ex*)

lemma *ratio-test-lemma2*:
  fixes f :: $nat \Rightarrow {}'a{::}banach$
  shows $[\![\forall n{\ge}N.\ norm\ (f\ (Suc\ n)) \le c * norm\ (f\ n)]\!] \Longrightarrow 0 < c \lor summable\ f$
apply (*simp* (*no-asm*) *add*: *linorder-not-le* [*symmetric*])
apply (*simp add*: *summable-Cauchy*)
apply (*safe*, *subgoal-tac* $\forall n.\ N < n\ {-}{-}{>}\ f\ (n) = 0$)
 prefer *2*
 apply *clarify*
 apply(*erule-tac* $x = n - 1$ **in** *allE*)
 apply (*simp add*:*diff-Suc split*:*nat.splits*)

```
 apply (blast intro: norm-ratiotest-lemma)
apply (rule-tac x = Suc N in exI, clarify)
apply(simp cong:setsum-ivl-cong)
done

lemma ratio-test:
  fixes f :: nat ⇒ 'a::banach
  shows ⟦c < 1; ∀ n≥N. norm (f (Suc n)) ≤ c ∗ norm (f n)⟧ ⟹ summable f
apply (frule ratio-test-lemma2, auto)
apply (rule-tac g = %n. (norm (f N) / (c ˆ N))∗c ˆ n
       in summable-comparison-test)
apply (rule-tac x = N in exI, safe)
apply (drule le-Suc-ex-iff [THEN iffD1])
apply (auto simp add: power-add field-power-not-zero)
apply (induct-tac na, auto)
apply (rule-tac y = c ∗ norm (f (N + n)) in order-trans)
apply (auto intro: mult-right-mono simp add: summable-def)
apply (simp add: mult-ac)
apply (rule-tac x = norm (f N) ∗ (1/ (1 − c)) / (c ˆ N) in exI)
apply (rule sums-divide)
apply (rule sums-mult)
apply (auto intro!: geometric-sums)
done
```

17.3 Cauchy Product Formula

```
lemma setsum-triangle-reindex:
  fixes n :: nat
  shows (∑(i,j)∈{(i,j). i+j < n}. f i j) = (∑k=0..<n. ∑i=0..k. f i (k − i))
proof −
  have (∑(i, j)∈{(i, j). i + j < n}. f i j) =
    (∑(k, i)∈(SIGMA k:{0..<n}. {0..k}). f i (k − i))
  proof (rule setsum-reindex-cong)
    show inj-on (λ(k,i). (i, k − i)) (SIGMA k:{0..<n}. {0..k})
      by (rule inj-on-inverseI [where g=λ(i,j). (i+j, i)], auto)
    show {(i,j). i + j < n} = (λ(k,i). (i, k − i)) ‘ (SIGMA k:{0..<n}. {0..k})
      by (safe, rule-tac x=(a+b,a) in image-eqI, auto)
    show ⋀a. (λ(k, i). f i (k − i)) a = split f ((λ(k, i). (i, k − i)) a)
      by clarify
  qed
  thus ?thesis by (simp add: setsum-Sigma)
qed

lemma Cauchy-product-sums:
  fixes a b :: nat ⇒ 'a::{real-normed-algebra,banach}
  assumes a: summable (λk. norm (a k))
  assumes b: summable (λk. norm (b k))
  shows (λk. ∑i=0..k. a i ∗ b (k − i)) sums ((∑k. a k) ∗ (∑k. b k))
proof −
```

let *?S1* = λ*n*::*nat*. {*0*..<*n*} × {*0*..<*n*}
let *?S2* = λ*n*::*nat*. {(*i*,*j*). *i* + *j* < *n*}
have *S1-mono*: ⋀*m n*. *m* ≤ *n* ⟹ *?S1 m* ⊆ *?S1 n* **by** *auto*
have *S2-le-S1*: ⋀*n*. *?S2 n* ⊆ *?S1 n* **by** *auto*
have *S1-le-S2*: ⋀*n*. *?S1* (*n div 2*) ⊆ *?S2 n* **by** *auto*
have *finite-S1*: ⋀*n*. *finite* (*?S1 n*) **by** *simp*
with *S2-le-S1* **have** *finite-S2*: ⋀*n*. *finite* (*?S2 n*) **by** (*rule finite-subset*)

let *?g* = λ(*i*,*j*). *a i* ∗ *b j*
let *?f* = λ(*i*,*j*). *norm* (*a i*) ∗ *norm* (*b j*)
have *f-nonneg*: ⋀*x*. *0* ≤ *?f x*
  by (*auto simp add*: *mult-nonneg-nonneg*)
hence *norm-setsum-f*: ⋀*A*. *norm* (*setsum ?f A*) = *setsum ?f A*
  unfolding *real-norm-def*
  by (*simp only*: *abs-of-nonneg setsum-nonneg* [*rule-format*])

have (λ*n*. (∑*k*=*0*..<*n*. *a k*) ∗ (∑*k*=*0*..<*n*. *b k*))
    −−−−> (∑*k*. *a k*) ∗ (∑*k*. *b k*)
  by (*intro LIMSEQ-mult summable-sumr-LIMSEQ-suminf*
    summable-norm-cancel [*OF a*] *summable-norm-cancel* [*OF b*])
hence *1*: (λ*n*. *setsum ?g* (*?S1 n*)) −−−−> (∑*k*. *a k*) ∗ (∑*k*. *b k*)
  by (*simp only*: *setsum-product setsum-Sigma* [*rule-format*]
    finite-atLeastLessThan)

have (λ*n*. (∑*k*=*0*..<*n*. *norm* (*a k*)) ∗ (∑*k*=*0*..<*n*. *norm* (*b k*)))
  −−−−> (∑*k*. *norm* (*a k*)) ∗ (∑*k*. *norm* (*b k*))
  using *a b* **by** (*intro LIMSEQ-mult summable-sumr-LIMSEQ-suminf*)
hence (λ*n*. *setsum ?f* (*?S1 n*)) −−−−> (∑*k*. *norm* (*a k*)) ∗ (∑*k*. *norm* (*b k*))
  by (*simp only*: *setsum-product setsum-Sigma* [*rule-format*]
    finite-atLeastLessThan)
hence *convergent* (λ*n*. *setsum ?f* (*?S1 n*))
  by (*rule convergentI*)
hence *Cauchy*: *Cauchy* (λ*n*. *setsum ?f* (*?S1 n*))
  by (*rule convergent-Cauchy*)
have *Zseq* (λ*n*. *setsum ?f* (*?S1 n* − *?S2 n*))
proof (*rule ZseqI*, *simp only*: *norm-setsum-f*)
  fix *r* :: *real*
  assume *r*: *0* < *r*
  from *CauchyD* [*OF Cauchy r*] **obtain** *N*
  where ∀*m*≥*N*. ∀*n*≥*N*. *norm* (*setsum ?f* (*?S1 m*) − *setsum ?f* (*?S1 n*)) < *r*
..
  hence ⋀*m n*. ⟦*N* ≤ *n*; *n* ≤ *m*⟧ ⟹ *norm* (*setsum ?f* (*?S1 m* − *?S1 n*)) < *r*
    by (*simp only*: *setsum-diff finite-S1 S1-mono*)
  hence *N*: ⋀*m n*. ⟦*N* ≤ *n*; *n* ≤ *m*⟧ ⟹ *setsum ?f* (*?S1 m* − *?S1 n*) < *r*
    by (*simp only*: *norm-setsum-f*)
  show ∃*N*. ∀*n*≥*N*. *setsum ?f* (*?S1 n* − *?S2 n*) < *r*
  proof (*intro exI allI impI*)
    fix *n* **assume** *2* ∗ *N* ≤ *n*
    hence *n*: *N* ≤ *n div 2* **by** *simp*

```
    have setsum ?f (?S1 n − ?S2 n) ≤ setsum ?f (?S1 n − ?S1 (n div 2))
      by (intro setsum-mono2 finite-Diff finite-S1 f-nonneg
              Diff-mono subset-refl S1-le-S2)
    also have ... < r
      using n div-le-dividend by (rule N)
    finally show setsum ?f (?S1 n − ?S2 n) < r .
  qed
 qed
 hence Zseq (λn. setsum ?g (?S1 n − ?S2 n))
   apply (rule Zseq-le [rule-format])
   apply (simp only: norm-setsum-f)
   apply (rule order-trans [OF norm-setsum setsum-mono])
   apply (auto simp add: norm-mult-ineq)
   done
 hence 2: (λn. setsum ?g (?S1 n) − setsum ?g (?S2 n)) −−−−> 0
   by (simp only: LIMSEQ-Zseq-iff setsum-diff finite-S1 S2-le-S1 diff-0-right)

 with 1 have (λn. setsum ?g (?S2 n)) −−−−> (∑ k. a k) ∗ (∑ k. b k)
   by (rule LIMSEQ-diff-approach-zero2)
 thus ?thesis by (simp only: sums-def setsum-triangle-reindex)
qed

lemma Cauchy-product:
 fixes a b :: nat ⇒ 'a::{real-normed-algebra,banach}
 assumes a: summable (λk. norm (a k))
 assumes b: summable (λk. norm (b k))
 shows (∑ k. a k) ∗ (∑ k. b k) = (∑ k. ∑ i=0..k. a i ∗ b (k − i))
using a b
by (rule Cauchy-product-sums [THEN sums-unique])

end
```

18 EvenOdd: Even and Odd Numbers: Compatibility file for Parity

```
theory EvenOdd
imports NthRoot
begin
```

18.1 General Lemmas About Division

```
lemma Suc-times-mod-eq: 1<k ==> Suc (k ∗ m) mod k = 1
apply (induct m)
apply (simp-all add: mod-Suc)
done

declare Suc-times-mod-eq [of number-of w, standard, simp]
```

```
lemma [simp]: n div k ≤ (Suc n) div k
by (simp add: div-le-mono)

lemma Suc-n-div-2-gt-zero [simp]: (0::nat) < n ==> 0 < (n + 1) div 2
by arith

lemma div-2-gt-zero [simp]: (1::nat) < n ==> 0 < n div 2
by arith

lemma mod-mult-self3 [simp]: (k*n + m) mod n = m mod (n::nat)
by (simp add: mult-ac add-ac)

lemma mod-mult-self4 [simp]: Suc (k*n + m) mod n = Suc m mod n
proof −
  have Suc (k * n + m) mod n = (k * n + Suc m) mod n by simp
  also have ... = Suc m mod n by (rule mod-mult-self3)
  finally show ?thesis .
qed

lemma mod-Suc-eq-Suc-mod: Suc m mod n = Suc (m mod n) mod n
apply (subst mod-Suc [of m])
apply (subst mod-Suc [of m mod n], simp)
done
```

18.2 More Even/Odd Results

```
lemma even-mult-two-ex: even(n) = (∃ m::nat. n = 2*m)
by (simp add: even-nat-equiv-def2 numeral-2-eq-2)

lemma odd-Suc-mult-two-ex: odd(n) = (∃ m. n = Suc (2*m))
by (simp add: odd-nat-equiv-def2 numeral-2-eq-2)

lemma even-add [simp]: even(m + n::nat) = (even m = even n)
by auto

lemma odd-add [simp]: odd(m + n::nat) = (odd m ≠ odd n)
by auto

lemma lemma-even-div2 [simp]: even (n::nat) ==> (n + 1) div 2 = n div 2
apply (simp add: numeral-2-eq-2)
apply (subst div-Suc)
apply (simp add: even-nat-mod-two-eq-zero)
done

lemma lemma-not-even-div2 [simp]: ~even n ==> (n + 1) div 2 = Suc (n div
2)
apply (simp add: numeral-2-eq-2)
apply (subst div-Suc)
```

apply (*simp add*: *odd-nat-mod-two-eq-one*)
done

lemma *even-num-iff*: *0* < *n* ==> *even n* = (~ *even*(*n* − *1* :: *nat*))
by (*case-tac n*, *auto*)

lemma *even-even-mod-4-iff*: *even* (*n*::*nat*) = *even* (*n mod 4*)
apply (*induct n*, *simp*)
apply (*subst mod-Suc*, *simp*)
done

lemma *lemma-odd-mod-4-div-2*: *n mod 4* = (*3*::*nat*) ==> *odd*((*n* − *1*) *div 2*)
apply (*rule-tac t* = *n* **and** *n1* = *4* **in** *mod-div-equality* [*THEN subst*])
apply (*simp add*: *even-num-iff*)
done

lemma *lemma-even-mod-4-div-2*: *n mod 4* = (*1*::*nat*) ==> *even* ((*n* − *1*) *div 2*)
by (*rule-tac t* = *n* **and** *n1* = *4* **in** *mod-div-equality* [*THEN subst*], *simp*)

end

19 Transcendental: Power Series, Transcendental Functions etc.

theory *Transcendental*
imports *NthRoot Fact Series EvenOdd Deriv*
begin

19.1 Properties of Power Series

lemma *lemma-realpow-diff*:
  fixes *y* :: *'a*::*recpower*
  shows $p \le n \Longrightarrow y$ ˆ (*Suc n* − *p*) = (*y* ˆ (*n* − *p*)) ∗ *y*
proof −
  assume $p \le n$
  hence *Suc n* − *p* = *Suc* (*n* − *p*) **by** (*rule Suc-diff-le*)
  thus *?thesis* **by** (*simp add*: *power-Suc power-commutes*)
qed

lemma *lemma-realpow-diff-sumr*:
  fixes *y* :: *'a*::{*recpower*,*comm-semiring-0*} **shows**
    (∑ *p*=*0*..<*Suc n*. (*x* ˆ *p*) ∗ *y* ˆ (*Suc n* − *p*)) =
    y ∗ (∑ *p*=*0*..<*Suc n*. (*x* ˆ *p*) ∗ *y* ˆ (*n* − *p*))
by (*auto simp add*: *setsum-right-distrib lemma-realpow-diff mult-ac*
  simp del: *setsum-op-ivl-Suc cong*: *strong-setsum-cong*)

lemma *lemma-realpow-diff-sumr2*:

fixes y :: $'a$::{*recpower*,*comm-ring*} **shows**
$x\ \hat{}\ (Suc\ n) - y\ \hat{}\ (Suc\ n) =$
$(x - y) * (\sum p{=}0..{<}Suc\ n.\ (x\ \hat{}\ p) * y\ \hat{}\ (n - p))$
apply (*induct n*, *simp add*: *power-Suc*)
apply (*simp add*: *power-Suc del*: *setsum-op-ivl-Suc*)
apply (*subst setsum-op-ivl-Suc*)
apply (*subst lemma-realpow-diff-sumr*)
apply (*simp add*: *right-distrib del*: *setsum-op-ivl-Suc*)
apply (*subst mult-left-commute* [**where** $a{=}x - y$])
apply (*erule subst*)
apply (*simp add*: *power-Suc ring-simps*)
done

lemma *lemma-realpow-rev-sumr*:
$(\sum p{=}0..{<}Suc\ n.\ (x\ \hat{}\ p) * (y\ \hat{}\ (n - p))) =$
$(\sum p{=}0..{<}Suc\ n.\ (x\ \hat{}\ (n - p)) * (y\ \hat{}\ p))$
apply (*rule setsum-reindex-cong* [**where** $f{=}\lambda i.\ n - i$])
apply (*rule inj-onI*, *simp*)
apply *auto*
apply (*rule-tac* $x{=}n - x$ **in** *image-eqI*, *simp*, *simp*)
done

Power series has a 'circle' of convergence, i.e. if it sums for x, then it sums absolutely for z with $|z| < |x|$.

lemma *powser-insidea*:
fixes $x\ z$ :: $'a$::{*real-normed-field*,*banach*,*recpower*}
assumes *1*: *summable* $(\lambda n.\ f\ n * x\ \hat{}\ n)$
assumes *2*: *norm* $z <$ *norm* x
shows *summable* $(\lambda n.$ *norm* $(f\ n * z\ \hat{}\ n))$
proof −
from *2* **have** *x-neq-0*: $x \neq 0$ **by** *clarsimp*
from *1* **have** $(\lambda n.\ f\ n * x\ \hat{}\ n)$ ----> 0
by (*rule summable-LIMSEQ-zero*)
hence *convergent* $(\lambda n.\ f\ n * x\ \hat{}\ n)$
by (*rule convergentI*)
hence *Cauchy* $(\lambda n.\ f\ n * x\ \hat{}\ n)$
by (*simp add*: *Cauchy-convergent-iff*)
hence *Bseq* $(\lambda n.\ f\ n * x\ \hat{}\ n)$
by (*rule Cauchy-Bseq*)
then obtain K **where** *3*: $0 < K$ **and** *4*: $\forall n.$ *norm* $(f\ n * x\ \hat{}\ n) \le K$
by (*simp add*: *Bseq-def*, *safe*)
have $\exists N.\ \forall n{\ge}N.$ *norm* (*norm* $(f\ n * z\ \hat{}\ n)) \le$
$K *$ *norm* $(z\ \hat{}\ n) *$ *inverse* (*norm* $(x\ \hat{}\ n)$)
proof (*intro exI allI impI*)
fix n::*nat* **assume** $0 \le n$
have *norm* (*norm* $(f\ n * z\ \hat{}\ n)) *$ *norm* $(x\ \hat{}\ n) =$
norm $(f\ n * x\ \hat{}\ n) *$ *norm* $(z\ \hat{}\ n)$
by (*simp add*: *norm-mult abs-mult*)
also have $\ldots \le K *$ *norm* $(z\ \hat{}\ n)$

by (*simp only*: *mult-right-mono 4 norm-ge-zero*)
also have ... = *K* ∗ *norm* (*z* ˆ *n*) ∗ (*inverse* (*norm* (*x* ˆ *n*)) ∗ *norm* (*x* ˆ *n*))
by (*simp add*: *x-neq-0*)
also have ... = *K* ∗ *norm* (*z* ˆ *n*) ∗ *inverse* (*norm* (*x* ˆ *n*)) ∗ *norm* (*x* ˆ *n*)
by (*simp only*: *mult-assoc*)
finally show *norm* (*norm* (*f n* ∗ *z* ˆ *n*)) ≤
K ∗ *norm* (*z* ˆ *n*) ∗ *inverse* (*norm* (*x* ˆ *n*))
by (*simp add*: *mult-le-cancel-right x-neq-0*)
qed
moreover have *summable* (λ*n*. *K* ∗ *norm* (*z* ˆ *n*) ∗ *inverse* (*norm* (*x* ˆ *n*)))
proof −
from *2* **have** *norm* (*norm* (*z* ∗ *inverse x*)) < *1*
using *x-neq-0*
by (*simp add*: *nonzero-norm-divide divide-inverse* [*symmetric*])
hence *summable* (λ*n*. *norm* (*z* ∗ *inverse x*) ˆ *n*)
by (*rule summable-geometric*)
hence *summable* (λ*n*. *K* ∗ *norm* (*z* ∗ *inverse x*) ˆ *n*)
by (*rule summable-mult*)
thus *summable* (λ*n*. *K* ∗ *norm* (*z* ˆ *n*) ∗ *inverse* (*norm* (*x* ˆ *n*)))
using *x-neq-0*
by (*simp add*: *norm-mult nonzero-norm-inverse power-mult-distrib*
power-inverse norm-power mult-assoc)
qed
ultimately show *summable* (λ*n*. *norm* (*f n* ∗ *z* ˆ *n*))
by (*rule summable-comparison-test*)
qed

lemma *powser-inside*:
fixes *f* :: *nat* ⇒ ′*a*::{*real-normed-field*,*banach*,*recpower*} **shows**
[| *summable* (%*n*. *f*(*n*) ∗ (*x* ˆ *n*)); *norm z* < *norm x* |]
==> *summable* (%*n*. *f*(*n*) ∗ (*z* ˆ *n*))
by (*rule powser-insidea* [*THEN summable-norm-cancel*])

19.2 Term-by-Term Differentiability of Power Series

definition
diffs :: (*nat* => ′*a*::*ring-1*) => *nat* => ′*a* **where**
diffs c = (%*n*. *of-nat* (*Suc n*) ∗ *c*(*Suc n*))

Lemma about distributing negation over it

lemma *diffs-minus*: *diffs* (%*n*. − *c n*) = (%*n*. − *diffs c n*)
by (*simp add*: *diffs-def*)

Show that we can shift the terms down one

lemma *lemma-diffs*:
($\sum$ *n*=*0*..<*n*. (*diffs c*)(*n*) ∗ (*x* ˆ *n*)) =
($\sum$ *n*=*0*..<*n*. *of-nat n* ∗ *c*(*n*) ∗ (*x* ˆ (*n* − *Suc 0*))) +
(*of-nat n* ∗ *c*(*n*) ∗ *x* ˆ (*n* − *Suc 0*))
apply (*induct n*)

apply (*auto simp add*: *mult-assoc add-assoc* [*symmetric*] *diffs-def*)
done

lemma *lemma-diffs2*:
$(\sum n{=}0..{<}n.\ \textit{of-nat}\ n * c(n) * (x \mathbin{\hat{}} (n - \textit{Suc}\ 0))) =$
$(\sum n{=}0..{<}n.\ (\textit{diffs}\ c)(n) * (x \mathbin{\hat{}} n)) -$
$(\textit{of-nat}\ n * c(n) * x \mathbin{\hat{}} (n - \textit{Suc}\ 0))$
by (*auto simp add*: *lemma-diffs*)

lemma *diffs-equiv*:
$\textit{summable}\ (\%n.\ (\textit{diffs}\ c)(n) * (x \mathbin{\hat{}} n)) ==>$
$(\%n.\ \textit{of-nat}\ n * c(n) * (x \mathbin{\hat{}} (n - \textit{Suc}\ 0)))\ \textit{sums}$
$(\sum n.\ (\textit{diffs}\ c)(n) * (x \mathbin{\hat{}} n))$
apply (*subgoal-tac* $(\%n.\ \textit{of-nat}\ n * c\ (n) * (x \mathbin{\hat{}} (n - \textit{Suc}\ 0)))$ $---->$ 0)
apply (*rule-tac* [*2*] *LIMSEQ-imp-Suc*)
apply (*drule summable-sums*)
apply (*auto simp add*: *sums-def*)
apply (*drule-tac* $X{=}(\lambda n.\ \sum n = 0..{<}n.\ \textit{diffs}\ c\ n * x \mathbin{\hat{}} n)$ **in** *LIMSEQ-diff*)
apply (*auto simp add*: *lemma-diffs2* [*symmetric*] *diffs-def* [*symmetric*])
apply (*simp add*: *diffs-def summable-LIMSEQ-zero*)
done

lemma *lemma-termdiff1*:
fixes $z :: 'a :: \{\textit{recpower},\textit{comm-ring}\}$ **shows**
$(\sum p{=}0..{<}m.\ (((z + h) \mathbin{\hat{}} (m - p)) * (z \mathbin{\hat{}} p)) - (z \mathbin{\hat{}} m)) =$
$(\sum p{=}0..{<}m.\ (z \mathbin{\hat{}} p) * (((z + h) \mathbin{\hat{}} (m - p)) - (z \mathbin{\hat{}} (m - p))))$
by (*auto simp add*: *right-distrib diff-minus power-add* [*symmetric*] *mult-ac cong*: *strong-setsum-cong*)

lemma *less-add-one*: $m < n ==> (\exists\, d.\ n = m + d + \textit{Suc}\ 0)$
by (*simp add*: *less-iff-Suc-add*)

lemma *sumdiff*: $a + b - (c + d) = a - c + b - (d{::}\textit{real})$
by *arith*

lemma *sumr-diff-mult-const2*:
$\textit{setsum}\ f\ \{0..{<}n\} - \textit{of-nat}\ n * (r{::}'a{::}\textit{ring-1}) = (\sum i = 0..{<}n.\ f\ i - r)$
by (*simp add*: *setsum-subtractf*)

lemma *lemma-termdiff2*:
fixes $h :: 'a :: \{\textit{recpower},\textit{field}\}$
assumes h: $h \neq 0$ **shows**
$((z + h) \mathbin{\hat{}} n - z \mathbin{\hat{}} n) / h - \textit{of-nat}\ n * z \mathbin{\hat{}} (n - \textit{Suc}\ 0) =$
$h * (\sum p{=}0..{<}\ n - \textit{Suc}\ 0.\ \sum q{=}0..{<}\ n - \textit{Suc}\ 0 - p.$
$(z + h) \mathbin{\hat{}} q * z \mathbin{\hat{}} (n - 2 - q))$ (**is** *?lhs* = *?rhs*)
apply (*subgoal-tac* $h * \textit{?lhs} = h * \textit{?rhs}$, *simp add*: h)
apply (*simp add*: *right-diff-distrib diff-divide-distrib* h)
apply (*simp add*: *mult-assoc* [*symmetric*])

apply (*cases n*, *simp*)
apply (*simp add*: *lemma-realpow-diff-sumr2 h*
right-diff-distrib [*symmetric*] *mult-assoc*
del: *realpow-Suc setsum-op-ivl-Suc of-nat-Suc*)
apply (*subst lemma-realpow-rev-sumr*)
apply (*subst sumr-diff-mult-const2*)
apply *simp*
apply (*simp only*: *lemma-termdiff1 setsum-right-distrib*)
apply (*rule setsum-cong* [*OF refl*])
apply (*simp add*: *diff-minus* [*symmetric*] *less-iff-Suc-add*)
apply (*clarify*)
apply (*simp add*: *setsum-right-distrib lemma-realpow-diff-sumr2 mult-ac*
del: *setsum-op-ivl-Suc realpow-Suc*)
apply (*subst mult-assoc* [*symmetric*], *subst power-add* [*symmetric*])
apply (*simp add*: *mult-ac*)
done

lemma *real-setsum-nat-ivl-bounded2*:
fixes K :: $'a$::*ordered-semidom*
assumes f: $\bigwedge p{::}nat.\ p < n \Longrightarrow f\ p \le K$
assumes K: $0 \le K$
shows *setsum* f $\{0..{<}n{-}k\} \le$ *of-nat* $n * K$
apply (*rule order-trans* [*OF setsum-mono*])
apply (*rule f*, *simp*)
apply (*simp add*: *mult-right-mono K*)
done

lemma *lemma-termdiff3*:
fixes $h\ z$ :: $'a$::{*real-normed-field*,*recpower*}
assumes *1*: $h \neq 0$
assumes *2*: *norm* $z \le K$
assumes *3*: *norm* $(z + h) \le K$
shows *norm* $(((z + h)\ \hat{}\ n - z\ \hat{}\ n) / h -$ *of-nat* $n * z\ \hat{}\ (n - Suc\ 0))$
$\le$ *of-nat* $n *$ *of-nat* $(n - Suc\ 0) * K\ \hat{}\ (n - 2) *$ *norm* h
proof −
have *norm* $(((z + h)\ \hat{}\ n - z\ \hat{}\ n) / h -$ *of-nat* $n * z\ \hat{}\ (n - Suc\ 0)) =$
norm $(\sum p = 0..{<}n - Suc\ 0.\ \sum q = 0..{<}n - Suc\ 0 - p.$
$(z + h)\ \hat{}\ q * z\ \hat{}\ (n - 2 - q)) *$ *norm* h
apply (*subst lemma-termdiff2* [*OF 1*])
apply (*subst norm-mult*)
apply (*rule mult-commute*)
done
also have ... $\le$ *of-nat* $n * ($*of-nat* $(n - Suc\ 0) * K\ \hat{}\ (n - 2)) *$ *norm* h
proof (*rule mult-right-mono* [*OF - norm-ge-zero*])
from *norm-ge-zero 2* **have** K: $0 \le K$ **by** (*rule order-trans*)
have *le-Kn*: $\bigwedge i\ j\ n.\ i + j = n \Longrightarrow$ *norm* $((z + h)\ \hat{}\ i * z\ \hat{}\ j) \le K\ \hat{}\ n$
apply (*erule subst*)
apply (*simp only*: *norm-mult norm-power power-add*)
apply (*intro mult-mono power-mono 2 3 norm-ge-zero zero-le-power K*)

```
      done
    show norm (∑ p = 0..<n − Suc 0. ∑ q = 0..<n − Suc 0 − p.
            (z + h) ^ q * z ^ (n − 2 − q))
          ≤ of-nat n * (of-nat (n − Suc 0) * K ^ (n − 2))
      apply (intro
         order-trans [OF norm-setsum]
         real-setsum-nat-ivl-bounded2
         mult-nonneg-nonneg
         zero-le-imp-of-nat
         zero-le-power K)
      apply (rule le-Kn, simp)
      done
  qed
  also have ... = of-nat n * of-nat (n − Suc 0) * K ^ (n − 2) * norm h
    by (simp only: mult-assoc)
  finally show ?thesis .
qed

lemma lemma-termdiff4:
  fixes f :: 'a::{real-normed-field,recpower} ⇒
             'b::real-normed-vector
  assumes k: 0 < (k::real)
  assumes le: ⋀h. ⟦h ≠ 0; norm h < k⟧ ⟹ norm (f h) ≤ K * norm h
  shows f −− 0 −−> 0
proof (simp add: LIM-def, safe)
  fix r::real assume r: 0 < r
  have zero-le-K: 0 ≤ K
    apply (cut-tac k)
    apply (cut-tac h=of-real (k/2) in le, simp)
    apply (simp del: of-real-divide)
    apply (drule order-trans [OF norm-ge-zero])
    apply (simp add: zero-le-mult-iff)
    done
  show ∃s. 0 < s ∧ (∀x. x ≠ 0 ∧ norm x < s ⟶ norm (f x) < r)
  proof (cases)
    assume K = 0
    with k r le have 0 < k ∧ (∀x. x ≠ 0 ∧ norm x < k ⟶ norm (f x) < r)
      by simp
    thus ∃s. 0 < s ∧ (∀x. x ≠ 0 ∧ norm x < s ⟶ norm (f x) < r) ..
  next
    assume K-neq-zero: K ≠ 0
    with zero-le-K have K: 0 < K by simp
    show ∃s. 0 < s ∧ (∀x. x ≠ 0 ∧ norm x < s ⟶ norm (f x) < r)
    proof (rule exI, safe)
      from k r K show 0 < min k (r * inverse K / 2)
        by (simp add: mult-pos-pos positive-imp-inverse-positive)
    next
      fix x::'a
      assume x1: x ≠ 0 and x2: norm x < min k (r * inverse K / 2)
```

from *x2* **have** *x3*: $norm\ x < k$ **and** *x4*: $norm\ x < r * inverse\ K\ /\ 2$
  by *simp-all*
from *x1 x3 le* **have** $norm\ (f\ x) \le K * norm\ x$ **by** *simp*
also from *x4 K* **have** $K * norm\ x < K * (r * inverse\ K\ /\ 2)$
  by (*rule mult-strict-left-mono*)
also have $\ldots = r\ /\ 2$
  using *K-neq-zero* **by** *simp*
also have $r\ /\ 2 < r$
  using *r* **by** *simp*
finally show $norm\ (f\ x) < r$ **.**
qed
qed
qed

lemma *lemma-termdiff5*:
  fixes $g :: 'a::\{recpower,real\text{-}normed\text{-}field\} \Rightarrow$
    $nat \Rightarrow 'b::banach$
  assumes *k*: $0 < (k::real)$
  assumes *f*: *summable f*
  assumes *le*: $\bigwedge h\ n.\ [\![h \ne 0;\ norm\ h < k]\!] \Longrightarrow norm\ (g\ h\ n) \le f\ n * norm\ h$
  shows $(\lambda h.\ suminf\ (g\ h))\ \text{--}\ 0\ \text{-->}\ 0$
proof (*rule lemma-termdiff4* [*OF k*])
  fix $h::'a$ **assume** $h \ne 0$ **and** $norm\ h < k$
  hence *A*: $\forall n.\ norm\ (g\ h\ n) \le f\ n * norm\ h$
    by (*simp add*: *le*)
  hence $\exists N.\ \forall n \ge N.\ norm\ (norm\ (g\ h\ n)) \le f\ n * norm\ h$
    by *simp*
  moreover from *f* **have** *B*: $summable\ (\lambda n.\ f\ n * norm\ h)$
    by (*rule summable-mult2*)
  ultimately have *C*: $summable\ (\lambda n.\ norm\ (g\ h\ n))$
    by (*rule summable-comparison-test*)
  hence $norm\ (suminf\ (g\ h)) \le (\sum n.\ norm\ (g\ h\ n))$
    by (*rule summable-norm*)
  also from *A C B* **have** $(\sum n.\ norm\ (g\ h\ n)) \le (\sum n.\ f\ n * norm\ h)$
    by (*rule summable-le*)
  also from *f* **have** $(\sum n.\ f\ n * norm\ h) = suminf\ f * norm\ h$
    by (*rule suminf-mult2* [*symmetric*])
  finally show $norm\ (suminf\ (g\ h)) \le suminf\ f * norm\ h$ **.**
qed

FIXME: Long proofs

lemma *termdiffs-aux*:
  fixes $x :: 'a::\{recpower,real\text{-}normed\text{-}field,banach\}$
  assumes *1*: $summable\ (\lambda n.\ diffs\ (diffs\ c)\ n * K\ \hat{}\ n)$
  assumes *2*: $norm\ x < norm\ K$
  shows $(\lambda h.\ \sum n.\ c\ n * (((x + h)\ \hat{}\ n - x\ \hat{}\ n)\ /\ h$
    $- of\text{-}nat\ n * x\ \hat{}\ (n - Suc\ 0)))\ \text{--}\ 0\ \text{-->}\ 0$
proof −
  from *dense* [*OF 2*]

```
  obtain r where r1: norm x < r and r2: r < norm K by fast
  from norm-ge-zero r1 have r: 0 < r
    by (rule order-le-less-trans)
  hence r-neq-0: r ≠ 0 by simp
  show ?thesis
  proof (rule lemma-termdiff5)
    show 0 < r − norm x using r1 by simp
  next
    from r r2 have norm (of-real r::'a) < norm K
      by simp
    with 1 have summable (λn. norm (diffs (diffs c) n ∗ (of-real r ˆ n)))
      by (rule powser-insidea)
    hence summable (λn. diffs (diffs (λn. norm (c n))) n ∗ r ˆ n)
      using r
      by (simp add: diffs-def norm-mult norm-power del: of-nat-Suc)
    hence summable (λn. of-nat n ∗ diffs (λn. norm (c n)) n ∗ r ˆ (n − Suc 0))
      by (rule diffs-equiv [THEN sums-summable])
    also have (λn. of-nat n ∗ diffs (λn. norm (c n)) n ∗ r ˆ (n − Suc 0))
      = (λn. diffs (%m. of-nat (m − Suc 0) ∗ norm (c m) ∗ inverse r) n ∗ (r ˆ
n))
      apply (rule ext)
      apply (simp add: diffs-def)
      apply (case-tac n, simp-all add: r-neq-0)
      done
    finally have summable
      (λn. of-nat n ∗ (of-nat (n − Suc 0) ∗ norm (c n) ∗ inverse r) ∗ r ˆ (n −
Suc 0))
      by (rule diffs-equiv [THEN sums-summable])
    also have
      (λn. of-nat n ∗ (of-nat (n − Suc 0) ∗ norm (c n) ∗ inverse r) ∗
           r ˆ (n − Suc 0)) =
       (λn. norm (c n) ∗ of-nat n ∗ of-nat (n − Suc 0) ∗ r ˆ (n − 2))
      apply (rule ext)
      apply (case-tac n, simp)
      apply (case-tac nat, simp)
      apply (simp add: r-neq-0)
      done
    finally show
      summable (λn. norm (c n) ∗ of-nat n ∗ of-nat (n − Suc 0) ∗ r ˆ (n − 2)) .
  next
    fix h::'a and n::nat
    assume h: h ≠ 0
    assume norm h < r − norm x
    hence norm x + norm h < r by simp
    with norm-triangle-ineq have xh: norm (x + h) < r
      by (rule order-le-less-trans)
    show norm (c n ∗ (((x + h) ˆ n − x ˆ n) / h − of-nat n ∗ x ˆ (n − Suc 0)))
         ≤ norm (c n) ∗ of-nat n ∗ of-nat (n − Suc 0) ∗ r ˆ (n − 2) ∗ norm h
      apply (simp only: norm-mult mult-assoc)
```

```
    apply (rule mult-left-mono [OF - norm-ge-zero])
    apply (simp (no-asm) add: mult-assoc [symmetric])
    apply (rule lemma-termdiff3)
    apply (rule h)
    apply (rule r1 [THEN order-less-imp-le])
    apply (rule xh [THEN order-less-imp-le])
    done
  qed
qed

lemma termdiffs:
  fixes K x :: 'a::{recpower,real-normed-field,banach}
  assumes 1: summable (λn. c n ∗ K ˆ n)
  assumes 2: summable (λn. (diffs c) n ∗ K ˆ n)
  assumes 3: summable (λn. (diffs (diffs c)) n ∗ K ˆ n)
  assumes 4: norm x < norm K
  shows DERIV (λx. ∑n. c n ∗ x ˆ n) x :> (∑n. (diffs c) n ∗ x ˆ n)
proof (simp add: deriv-def, rule LIM-zero-cancel)
  show (λh. (suminf (λn. c n ∗ (x + h) ˆ n) − suminf (λn. c n ∗ x ˆ n)) / h
          − suminf (λn. diffs c n ∗ x ˆ n)) −− 0 −−> 0
  proof (rule LIM-equal2)
    show 0 < norm K − norm x by (simp add: less-diff-eq 4)
  next
    fix h :: 'a
    assume h ≠ 0
    assume norm (h − 0) < norm K − norm x
    hence norm x + norm h < norm K by simp
    hence 5: norm (x + h) < norm K
      by (rule norm-triangle-ineq [THEN order-le-less-trans])
    have A: summable (λn. c n ∗ x ˆ n)
      by (rule powser-inside [OF 1 4])
    have B: summable (λn. c n ∗ (x + h) ˆ n)
      by (rule powser-inside [OF 1 5])
    have C: summable (λn. diffs c n ∗ x ˆ n)
      by (rule powser-inside [OF 2 4])
    show ((∑n. c n ∗ (x + h) ˆ n) − (∑n. c n ∗ x ˆ n)) / h
          − (∑n. diffs c n ∗ x ˆ n) =
        (∑n. c n ∗ (((x + h) ˆ n − x ˆ n) / h − of-nat n ∗ x ˆ (n − Suc 0)))
      apply (subst sums-unique [OF diffs-equiv [OF C]])
      apply (subst suminf-diff [OF B A])
      apply (subst suminf-divide [symmetric])
      apply (rule summable-diff [OF B A])
      apply (subst suminf-diff)
      apply (rule summable-divide)
      apply (rule summable-diff [OF B A])
      apply (rule sums-summable [OF diffs-equiv [OF C]])
      apply (rule-tac f=suminf in arg-cong)
      apply (rule ext)
      apply (simp add: ring-simps)
```

done
next
show $(\lambda h.\ \sum n.\ c\ n * (((x + h)\ \hat{}\ n - x\ \hat{}\ n)\ /\ h -$
$\quad \textit{of-nat}\ n * x\ \hat{}\ (n - \textit{Suc}\ 0)))\ {-}{-}\ 0\ {-}{-}{>}\ 0$
by $(\textit{rule termdiffs-aux}\ [\textit{OF}\ 3\ 4])$
qed
qed

19.3 Exponential Function

definition
$\textit{exp} :: {'a} \Rightarrow {'a}::\{\textit{recpower},\textit{real-normed-field},\textit{banach}\}$ **where**
$\textit{exp}\ x = (\sum n.\ x\ \hat{}\ n\ /_R\ \textit{real}\ (\textit{fact}\ n))$

definition
$\textit{sin} :: \textit{real} \Rightarrow \textit{real}$ **where**
$\textit{sin}\ x = (\sum n.\ (\textit{if}\ \textit{even}(n)\ \textit{then}\ 0\ \textit{else}$
$\quad (-1\ \hat{}\ ((n - \textit{Suc}\ 0)\ \textit{div}\ 2))/(\textit{real}\ (\textit{fact}\ n))) * x\ \hat{}\ n)$

definition
$\textit{cos} :: \textit{real} \Rightarrow \textit{real}$ **where**
$\textit{cos}\ x = (\sum n.\ (\textit{if}\ \textit{even}(n)\ \textit{then}\ (-1\ \hat{}\ (n\ \textit{div}\ 2))/(\textit{real}\ (\textit{fact}\ n))$
$\quad \textit{else}\ 0) * x\ \hat{}\ n)$

lemma *summable-exp-generic*:
fixes $x :: {'a}::\{\textit{real-normed-algebra-1},\textit{recpower},\textit{banach}\}$
defines *S-def*: $S \equiv \lambda n.\ x\ \hat{}\ n\ /_R\ \textit{real}\ (\textit{fact}\ n)$
shows *summable S*
proof $-$
have *S-Suc*: $\bigwedge n.\ S\ (\textit{Suc}\ n) = (x * S\ n)\ /_R\ \textit{real}\ (\textit{Suc}\ n)$
unfolding *S-def* **by** (*simp add*: *power-Suc del*: *mult-Suc*)
obtain $r :: \textit{real}$ **where** *r0*: $0 < r$ **and** *r1*: $r < 1$
using *dense* [*OF zero-less-one*] **by** *fast*
obtain $N :: \textit{nat}$ **where** N: $\textit{norm}\ x < \textit{real}\ N * r$
using *reals-Archimedean3* [*OF r0*] **by** *fast*
from *r1* **show** *?thesis*
proof (*rule ratio-test* [*rule-format*])
fix $n :: \textit{nat}$
assume n: $N \le n$
have $\textit{norm}\ x \le \textit{real}\ N * r$
using N **by** (*rule order-less-imp-le*)
also have $\textit{real}\ N * r \le \textit{real}\ (\textit{Suc}\ n) * r$
using *r0 n* **by** (*simp add*: *mult-right-mono*)
finally have $\textit{norm}\ x * \textit{norm}\ (S\ n) \le \textit{real}\ (\textit{Suc}\ n) * r * \textit{norm}\ (S\ n)$
using *norm-ge-zero* **by** (*rule mult-right-mono*)
hence $\textit{norm}\ (x * S\ n) \le \textit{real}\ (\textit{Suc}\ n) * r * \textit{norm}\ (S\ n)$
by (*rule order-trans* [*OF norm-mult-ineq*])
hence $\textit{norm}\ (x * S\ n)\ /\ \textit{real}\ (\textit{Suc}\ n) \le r * \textit{norm}\ (S\ n)$
by (*simp add*: *pos-divide-le-eq mult-ac*)

thus *norm* (*S* (*Suc n*)) $\leq$ *r* $*$ *norm* (*S n*)
by (*simp add*: *S-Suc norm-scaleR inverse-eq-divide*)
qed
qed

lemma *summable-norm-exp*:
fixes *x* :: *'a*::{*real-normed-algebra-1*,*recpower*,*banach*}
shows *summable* (λ*n*. *norm* (*x* ^ *n* $/_R$ *real* (*fact n*)))
proof (*rule summable-norm-comparison-test* [*OF exI*, *rule-format*])
show *summable* (λ*n*. *norm x* ^ *n* $/_R$ *real* (*fact n*))
by (*rule summable-exp-generic*)
next
fix *n* **show** *norm* (*x* ^ *n* $/_R$ *real* (*fact n*)) $\leq$ *norm x* ^ *n* $/_R$ *real* (*fact n*)
by (*simp add*: *norm-scaleR norm-power-ineq*)
qed

lemma *summable-exp*: *summable* (%*n*. *inverse* (*real* (*fact n*)) $*$ *x* ^ *n*)
by (*insert summable-exp-generic* [**where** *x*=*x*], *simp*)

lemma *summable-sin*:
summable (%*n*.
(*if even n then 0*
else $-$*1* ^ ((*n* $-$ *Suc 0*) *div 2*)/(*real* (*fact n*))) $*$
x ^ *n*)
apply (*rule-tac g* = (%*n*. *inverse* (*real* (*fact n*)) $*$ $|x|$ ^ *n*) **in** *summable-comparison-test*)
apply (*rule-tac* [*2*] *summable-exp*)
apply (*rule-tac x* = *0* **in** *exI*)
apply (*auto simp add*: *divide-inverse abs-mult power-abs* [*symmetric*] *zero-le-mult-iff*)
done

lemma *summable-cos*:
summable (%*n*.
(*if even n then*
$-$*1* ^ (*n div 2*)/(*real* (*fact n*)) *else 0*) $*$ *x* ^ *n*)
apply (*rule-tac g* = (%*n*. *inverse* (*real* (*fact n*)) $*$ $|x|$ ^ *n*) **in** *summable-comparison-test*)
apply (*rule-tac* [*2*] *summable-exp*)
apply (*rule-tac x* = *0* **in** *exI*)
apply (*auto simp add*: *divide-inverse abs-mult power-abs* [*symmetric*] *zero-le-mult-iff*)
done

lemma *lemma-STAR-sin*:
(*if even n then 0*
else $-$*1* ^ ((*n* $-$ *Suc 0*) *div 2*)/(*real* (*fact n*))) $*$ *0* ^ *n* = *0*
by (*induct n*, *auto*)

lemma *lemma-STAR-cos*:
0 < *n* $--$>
$-$*1* ^ (*n div 2*)/(*real* (*fact n*)) $*$ *0* ^ *n* = *0*
by (*induct n*, *auto*)

```
lemma lemma-STAR-cos1:
    0 < n -->
     (-1) ^ (n div 2)/(real (fact n)) * 0 ^ n = 0
by (induct n, auto)

lemma lemma-STAR-cos2:
  (∑ n=1..<n. if even n then -1 ^ (n div 2)/(real (fact n)) *  0 ^ n
                        else 0) = 0
apply (induct n)
apply (case-tac [2] n, auto)
done

lemma exp-converges: (λn. x ^ n /_R real (fact n)) sums exp x
unfolding exp-def by (rule summable-exp-generic [THEN summable-sums])

lemma sin-converges:
     (%n. (if even n then 0
          else -1 ^ ((n - Suc 0) div 2)/(real (fact n))) *
              x ^ n) sums sin(x)
unfolding sin-def by (rule summable-sin [THEN summable-sums])

lemma cos-converges:
     (%n. (if even n then
         -1 ^ (n div 2)/(real (fact n))
         else 0) * x ^ n) sums cos(x)
unfolding cos-def by (rule summable-cos [THEN summable-sums])
```

19.4 Formal Derivatives of Exp, Sin, and Cos Series

```
lemma exp-fdiffs:
     diffs (%n. inverse(real (fact n))) = (%n. inverse(real (fact n)))
by (simp add: diffs-def mult-assoc [symmetric] real-of-nat-def of-nat-mult
        del: mult-Suc of-nat-Suc)

lemma diffs-of-real: diffs (λn. of-real (f n)) = (λn. of-real (diffs f n))
by (simp add: diffs-def)

lemma sin-fdiffs:
     diffs(%n. if even n then 0
          else -1 ^ ((n - Suc 0) div 2)/(real (fact n)))
      = (%n. if even n then
              -1 ^ (n div 2)/(real (fact n))
            else 0)
by (auto intro!: ext
        simp add: diffs-def divide-inverse real-of-nat-def of-nat-mult
        simp del: mult-Suc of-nat-Suc)

lemma sin-fdiffs2:
```

diffs(%*n*. *if even n then 0*
else -1 ^ ((*n* − *Suc 0*) *div 2*)/(*real* (*fact n*))) *n*
= (*if even n then*
-1 ^ (*n div 2*)/(*real* (*fact n*))
else 0)
by (*simp only*: *sin-fdiffs*)

lemma *cos-fdiffs*:
diffs(%*n*. *if even n then*
-1 ^ (*n div 2*)/(*real* (*fact n*)) *else 0*)
= (%*n*. − (*if even n then 0*
else -1 ^ ((*n* − *Suc 0*)*div 2*)/(*real* (*fact n*))))
by (*auto intro*!: *ext*
simp add: *diffs-def divide-inverse odd-Suc-mult-two-ex real-of-nat-def of-nat-mult*
simp del: *mult-Suc of-nat-Suc*)

lemma *cos-fdiffs2*:
diffs(%*n*. *if even n then*
-1 ^ (*n div 2*)/(*real* (*fact n*)) *else 0*) *n*
= − (*if even n then 0*
else -1 ^ ((*n* − *Suc 0*)*div 2*)/(*real* (*fact n*)))
by (*simp only*: *cos-fdiffs*)

Now at last we can get the derivatives of exp, sin and cos

lemma *lemma-sin-minus*:
− *sin x* = ($\sum$ *n*. − ((*if even n then 0*
else -1 ^ ((*n* − *Suc 0*) *div 2*)/(*real* (*fact n*))) ∗ *x* ^ *n*))
by (*auto intro*!: *sums-unique sums-minus sin-converges*)

lemma *lemma-exp-ext*: *exp* = (λ*x*. $\sum$ *n*. *x* ^ *n* $/_R$ *real* (*fact n*))
by (*auto intro*!: *ext simp add*: *exp-def*)

lemma *DERIV-exp* [*simp*]: *DERIV exp x* :> *exp*(*x*)
apply (*simp add*: *exp-def*)
apply (*subst lemma-exp-ext*)
apply (*subgoal-tac DERIV* (λ*u*. $\sum$ *n*. *of-real* (*inverse* (*real* (*fact n*))) ∗ *u* ^ *n*) *x* :> ($\sum$ *n*. *diffs* (λ*n*. *of-real* (*inverse* (*real* (*fact n*)))) *n* ∗ *x* ^ *n*))
apply (*rule-tac* [*2*] *K* = *of-real* (*1* + *norm x*) **in** *termdiffs*)
apply (*simp-all only*: *diffs-of-real scaleR-conv-of-real exp-fdiffs*)
apply (*rule exp-converges* [*THEN sums-summable*, *unfolded scaleR-conv-of-real*])+
apply (*simp del*: *of-real-add*)
done

lemma *lemma-sin-ext*:
sin = (%*x*. $\sum$ *n*.
(*if even n then 0*
else -1 ^ ((*n* − *Suc 0*) *div 2*)/(*real* (*fact n*))) ∗

x ^ n)
by (*auto intro*!: *ext simp add*: *sin-def*)

lemma *lemma-cos-ext*:
cos = (%*x*. $\sum$ *n*.
(*if even n then* −*1* ^ (*n div 2*)/(*real* (*fact n*)) *else 0*) ∗
x ^ *n*)
by (*auto intro*!: *ext simp add*: *cos-def*)

lemma *DERIV-sin* [*simp*]: *DERIV sin x* :> *cos*(*x*)
apply (*simp add*: *cos-def*)
apply (*subst lemma-sin-ext*)
apply (*auto simp add*: *sin-fdiffs2* [*symmetric*])
apply (*rule-tac K* = *1* + |*x*| **in** *termdiffs*)
apply (*auto intro*: *sin-converges cos-converges sums-summable intro*!: *sums-minus* [*THEN sums-summable*] *simp add*: *cos-fdiffs sin-fdiffs*)
done

lemma *DERIV-cos* [*simp*]: *DERIV cos x* :> −*sin*(*x*)
apply (*subst lemma-cos-ext*)
apply (*auto simp add*: *lemma-sin-minus cos-fdiffs2* [*symmetric*] *minus-mult-left*)
apply (*rule-tac K* = *1* + |*x*| **in** *termdiffs*)
apply (*auto intro*: *sin-converges cos-converges sums-summable intro*!: *sums-minus* [*THEN sums-summable*] *simp add*: *cos-fdiffs sin-fdiffs diffs-minus*)
done

lemma *isCont-exp* [*simp*]: *isCont exp x*
by (*rule DERIV-exp* [*THEN DERIV-isCont*])

lemma *isCont-sin* [*simp*]: *isCont sin x*
by (*rule DERIV-sin* [*THEN DERIV-isCont*])

lemma *isCont-cos* [*simp*]: *isCont cos x*
by (*rule DERIV-cos* [*THEN DERIV-isCont*])

19.5 Properties of the Exponential Function

lemma *powser-zero*:
fixes *f* :: *nat* ⇒ '*a*::{*real-normed-algebra-1*,*recpower*}
shows ($\sum$ *n*. *f n* ∗ *0* ^ *n*) = *f 0*
proof −
have ($\sum$ *n* = *0*..<*1*. *f n* ∗ *0* ^ *n*) = ($\sum$ *n*. *f n* ∗ *0* ^ *n*)
by (*rule sums-unique* [*OF series-zero*], *simp add*: *power-0-left*)
thus *?thesis* **by** *simp*
qed

lemma *exp-zero* [*simp*]: *exp 0* = *1*
unfolding *exp-def* **by** (*simp add*: *scaleR-conv-of-real powser-zero*)

lemma *setsum-head2*:
$m \le n \Longrightarrow setsum\ f\ \{m..n\} = f\ m + setsum\ f\ \{Suc\ m..n\}$
by (*simp add*: *setsum-head atLeastSucAtMost-greaterThanAtMost*)

lemma *setsum-cl-ivl-Suc2*:
$(\sum i{=}m..Suc\ n.\ f\ i) = (if\ Suc\ n < m\ then\ 0\ else\ f\ m + (\sum i{=}m..n.\ f\ (Suc\ i)))$
by (*simp add*: *setsum-head2 setsum-shift-bounds-cl-Suc-ivl*
del: *setsum-cl-ivl-Suc*)

lemma *exp-series-add*:
fixes $x\ y$ :: *'a*::{*real-field*,*recpower*}
defines *S-def*: $S \equiv \lambda x\ n.\ x\ \hat{}\ n\ /_R\ real\ (fact\ n)$
shows $S\ (x + y)\ n = (\sum i{=}0..n.\ S\ x\ i * S\ y\ (n - i))$
proof (*induct n*)
case *0*
show *?case*
unfolding *S-def* **by** *simp*
next
case (*Suc n*)
have *S-Suc*: $\bigwedge x\ n.\ S\ x\ (Suc\ n) = (x * S\ x\ n)\ /_R\ real\ (Suc\ n)$
unfolding *S-def* **by** (*simp add*: *power-Suc del*: *mult-Suc*)
hence *times-S*: $\bigwedge x\ n.\ x * S\ x\ n = real\ (Suc\ n) *_R S\ x\ (Suc\ n)$
by *simp*

have $real\ (Suc\ n) *_R S\ (x + y)\ (Suc\ n) = (x + y) * S\ (x + y)\ n$
by (*simp only*: *times-S*)
also have $\ldots = (x + y) * (\sum i{=}0..n.\ S\ x\ i * S\ y\ (n{-}i))$
by (*simp only*: *Suc*)
also have $\ldots = x * (\sum i{=}0..n.\ S\ x\ i * S\ y\ (n{-}i))$
$+ y * (\sum i{=}0..n.\ S\ x\ i * S\ y\ (n{-}i))$
by (*rule left-distrib*)
also have $\ldots = (\sum i{=}0..n.\ (x * S\ x\ i) * S\ y\ (n{-}i))$
$+ (\sum i{=}0..n.\ S\ x\ i * (y * S\ y\ (n{-}i)))$
by (*simp only*: *setsum-right-distrib mult-ac*)
also have $\ldots = (\sum i{=}0..n.\ real\ (Suc\ i) *_R (S\ x\ (Suc\ i) * S\ y\ (n{-}i)))$
$+ (\sum i{=}0..n.\ real\ (Suc\ n{-}i) *_R (S\ x\ i * S\ y\ (Suc\ n{-}i)))$
by (*simp add*: *times-S Suc-diff-le*)
also have $(\sum i{=}0..n.\ real\ (Suc\ i) *_R (S\ x\ (Suc\ i) * S\ y\ (n{-}i))) =$
$(\sum i{=}0..Suc\ n.\ real\ i *_R (S\ x\ i * S\ y\ (Suc\ n{-}i)))$
by (*subst setsum-cl-ivl-Suc2*, *simp*)
also have $(\sum i{=}0..n.\ real\ (Suc\ n{-}i) *_R (S\ x\ i * S\ y\ (Suc\ n{-}i))) =$
$(\sum i{=}0..Suc\ n.\ real\ (Suc\ n{-}i) *_R (S\ x\ i * S\ y\ (Suc\ n{-}i)))$
by (*subst setsum-cl-ivl-Suc*, *simp*)
also have $(\sum i{=}0..Suc\ n.\ real\ i *_R (S\ x\ i * S\ y\ (Suc\ n{-}i))) +$
$(\sum i{=}0..Suc\ n.\ real\ (Suc\ n{-}i) *_R (S\ x\ i * S\ y\ (Suc\ n{-}i))) =$
$(\sum i{=}0..Suc\ n.\ real\ (Suc\ n) *_R (S\ x\ i * S\ y\ (Suc\ n{-}i)))$
by (*simp only*: *setsum-addf* [*symmetric*] *scaleR-left-distrib* [*symmetric*]
real-of-nat-add [*symmetric*], *simp*)
also have $\ldots = real\ (Suc\ n) *_R (\sum i{=}0..Suc\ n.\ S\ x\ i * S\ y\ (Suc\ n{-}i))$

  by (*simp only*: *scaleR-right.setsum*)
 finally show
  $S\ (x + y)\ (Suc\ n) = (\sum i{=}0..Suc\ n.\ S\ x\ i * S\ y\ (Suc\ n - i))$
  by (*simp add*: *scaleR-cancel-left del*: *setsum-cl-ivl-Suc*)
qed

lemma *exp-add*: $exp\ (x + y) = exp\ x * exp\ y$
unfolding *exp-def*
by (*simp only*: *Cauchy-product summable-norm-exp exp-series-add*)

lemma *exp-of-real*: $exp\ (of\text{-}real\ x) = of\text{-}real\ (exp\ x)$
unfolding *exp-def*
apply (*subst of-real.suminf*)
apply (*rule summable-exp-generic*)
apply (*simp add*: *scaleR-conv-of-real*)
done

lemma *exp-ge-add-one-self-aux*: $0 \le (x{::}real) ==> (1 + x) \le exp(x)$
apply (*drule order-le-imp-less-or-eq*, *auto*)
apply (*simp add*: *exp-def*)
apply (*rule real-le-trans*)
apply (*rule-tac* [*2*] $n = 2$ **and** $f = (\%n.\ inverse\ (real\ (fact\ n)) * x\ \hat{}\ n)$ **in** *series-pos-le*)
apply (*auto intro*: *summable-exp simp add*: *numeral-2-eq-2 zero-le-power zero-le-mult-iff*)
done

lemma *exp-gt-one* [*simp*]: $0 < (x{::}real) ==> 1 < exp\ x$
apply (*rule order-less-le-trans*)
apply (*rule-tac* [*2*] *exp-ge-add-one-self-aux*, *auto*)
done

lemma *DERIV-exp-add-const*: $DERIV\ (\%x.\ exp\ (x + y))\ x :> exp(x + y)$
proof −
 have $DERIV\ (exp \circ (\lambda x.\ x + y))\ x :> exp\ (x + y) * (1{+}0)$
  by (*fast intro*: *DERIV-chain DERIV-add DERIV-exp DERIV-ident DERIV-const*)

 thus *?thesis* **by** (*simp add*: *o-def*)
qed

lemma *DERIV-exp-minus* [*simp*]: $DERIV\ (\%x.\ exp\ (-x))\ x :> -\ exp(-x)$
proof −
 have $DERIV\ (exp \circ uminus)\ x :> exp\ (-\ x) * -\ 1$
  by (*fast intro*: *DERIV-chain DERIV-minus DERIV-exp DERIV-ident*)
 thus *?thesis* **by** (*simp add*: *o-def*)
qed

lemma *DERIV-exp-exp-zero* [*simp*]: $DERIV\ (\%x.\ exp\ (x + y) * exp\ (-\ x))\ x :>$
0
proof −

have *DERIV* $(\lambda x.\ exp\ (x + y) * exp\ (- x))\ x$
$:> exp\ (x + y) * exp\ (- x) + - exp\ (- x) * exp\ (x + y)$
by (*fast intro*: *DERIV-exp-add-const DERIV-exp-minus DERIV-mult*)
thus *?thesis* **by** (*simp add*: *mult-commute*)
qed

lemma *exp-add-mult-minus* [*simp*]: $exp(x + y)*exp(-x) = exp(y::real)$
proof $-$
have $\forall x.$ *DERIV* $(\%x.\ exp\ (x + y) * exp\ (- x))\ x :> 0$ **by** *simp*
hence $exp\ (x + y) * exp\ (- x) = exp\ (0 + y) * exp\ (- 0)$
by (*rule DERIV-isconst-all*)
thus *?thesis* **by** *simp*
qed

lemma *exp-mult-minus* [*simp*]: $exp\ x * exp(-x) = 1$
by (*simp add*: *exp-add* [*symmetric*])

lemma *exp-mult-minus2* [*simp*]: $exp(-x)*exp(x) = 1$
by (*simp add*: *mult-commute*)

lemma *exp-minus*: $exp(-x) = inverse(exp(x))$
by (*auto intro*: *inverse-unique* [*symmetric*])

Proof: because every exponential can be seen as a square.

lemma *exp-ge-zero* [*simp*]: $0 \le exp\ (x::real)$
apply (*rule-tac* $t = x$ **in** *real-sum-of-halves* [*THEN subst*])
apply (*subst exp-add*, *auto*)
done

lemma *exp-not-eq-zero* [*simp*]: $exp\ x \neq 0$
apply (*cut-tac* $x = x$ **in** *exp-mult-minus2*)
apply (*auto simp del*: *exp-mult-minus2*)
done

lemma *exp-gt-zero* [*simp*]: $0 < exp\ (x::real)$
by (*simp add*: *order-less-le*)

lemma *inv-exp-gt-zero* [*simp*]: $0 < inverse(exp\ x::real)$
by (*auto intro*: *positive-imp-inverse-positive*)

lemma *abs-exp-cancel* [*simp*]: $|exp\ x::real| = exp\ x$
by *auto*

lemma *exp-real-of-nat-mult*: $exp(real\ n * x) = exp(x)\ \hat{}\ n$
apply (*induct n*)
apply (*auto simp add*: *real-of-nat-Suc right-distrib exp-add mult-commute*)
done

lemma *exp-diff*: $exp(x - y) = exp(x)/(exp\ y)$
apply (*simp add*: *diff-minus divide-inverse*)
apply (*simp* (*no-asm*) *add*: *exp-add exp-minus*)
done

lemma *exp-less-mono*:
  fixes $x\ y$ :: *real*
  assumes *xy*: $x < y$ **shows** $exp\ x < exp\ y$
proof −
  from *xy* **have** $1 < exp\ (y + - x)$
    by (*rule real-less-sum-gt-zero* [*THEN exp-gt-one*])
  hence $exp\ x * inverse\ (exp\ x) < exp\ y * inverse\ (exp\ x)$
    by (*auto simp add*: *exp-add exp-minus*)
  thus *?thesis*
    by (*simp add*: *divide-inverse* [*symmetric*] *pos-less-divide-eq*
        del: *divide-self-if*)
qed

lemma *exp-less-cancel*: $exp\ (x::real) < exp\ y \implies x < y$
apply (*simp add*: *linorder-not-le* [*symmetric*])
apply (*auto simp add*: *order-le-less exp-less-mono*)
done

lemma *exp-less-cancel-iff* [*iff*]: $(exp(x::real) < exp(y)) = (x < y)$
by (*auto intro*: *exp-less-mono exp-less-cancel*)

lemma *exp-le-cancel-iff* [*iff*]: $(exp(x::real) \le exp(y)) = (x \le y)$
by (*auto simp add*: *linorder-not-less* [*symmetric*])

lemma *exp-inj-iff* [*iff*]: $(exp\ (x::real) = exp\ y) = (x = y)$
by (*simp add*: *order-eq-iff*)

lemma *lemma-exp-total*: $1 \le y \implies \exists x.\ 0 \le x\ \&\ x \le y - 1\ \&\ exp(x::real) = y$
apply (*rule IVT*)
apply (*auto intro*: *isCont-exp simp add*: *le-diff-eq*)
apply (*subgoal-tac* $1 + (y - 1) \le exp\ (y - 1)$)
apply *simp*
apply (*rule exp-ge-add-one-self-aux*, *simp*)
done

lemma *exp-total*: $0 < (y::real) \implies \exists x.\ exp\ x = y$
apply (*rule-tac* $x = 1$ **and** $y = y$ **in** *linorder-cases*)
apply (*drule order-less-imp-le* [*THEN lemma-exp-total*])
apply (*rule-tac* [*2*] $x = 0$ **in** *exI*)
apply (*frule-tac* [*3*] *real-inverse-gt-one*)
apply (*drule-tac* [*4*] *order-less-imp-le* [*THEN lemma-exp-total*], *auto*)
apply (*rule-tac* $x = -x$ **in** *exI*)
apply (*simp add*: *exp-minus*)

done

19.6 Properties of the Logarithmic Function

definition
ln :: *real* => *real* **where**
ln x = (*THE u*. *exp u* = *x*)

lemma *ln-exp* [*simp*]: *ln* (*exp x*) = *x*
by (*simp add*: *ln-def*)

lemma *exp-ln* [*simp*]: *0* < *x* ⟹ *exp* (*ln x*) = *x*
by (*auto dest*: *exp-total*)

lemma *exp-ln-iff* [*simp*]: (*exp* (*ln x*) = *x*) = (*0* < *x*)
apply (*auto dest*: *exp-total*)
apply (*erule subst*, *simp*)
done

lemma *ln-mult*: [| *0* < *x*; *0* < *y* |] ==> *ln*(*x* ∗ *y*) = *ln*(*x*) + *ln*(*y*)
apply (*rule exp-inj-iff* [*THEN iffD1*])
apply (*simp add*: *exp-add exp-ln mult-pos-pos*)
done

lemma *ln-inj-iff* [*simp*]: [| *0* < *x*; *0* < *y* |] ==> (*ln x* = *ln y*) = (*x* = *y*)
apply (*simp only*: *exp-ln-iff* [*symmetric*])
apply (*erule subst*)+
apply *simp*
done

lemma *ln-one*[*simp*]: *ln 1* = *0*
by (*rule exp-inj-iff* [*THEN iffD1*], *auto*)

lemma *ln-inverse*: *0* < *x* ==> *ln*(*inverse x*) = − *ln x*
apply (*rule-tac a1* = *ln x* **in** *add-left-cancel* [*THEN iffD1*])
apply (*auto simp add*: *positive-imp-inverse-positive ln-mult* [*symmetric*])
done

lemma *ln-div*:
[|*0* < *x*; *0* < *y*|] ==> *ln*(*x*/*y*) = *ln x* − *ln y*
apply (*simp add*: *divide-inverse*)
apply (*auto simp add*: *positive-imp-inverse-positive ln-mult ln-inverse*)
done

lemma *ln-less-cancel-iff* [*simp*]: [| *0* < *x*; *0* < *y*|] ==> (*ln x* < *ln y*) = (*x* < *y*)
apply (*simp only*: *exp-ln-iff* [*symmetric*])
apply (*erule subst*)+
apply *simp*
done

```
lemma ln-le-cancel-iff[simp]: [| 0 < x; 0 < y|] ==> (ln x ≤ ln y) = (x ≤ y)
by (auto simp add: linorder-not-less [symmetric])

lemma ln-realpow: 0 < x ==> ln(x ^ n) = real n * ln(x)
by (auto dest!: exp-total simp add: exp-real-of-nat-mult [symmetric])

lemma ln-add-one-self-le-self [simp]: 0 ≤ x ==> ln(1 + x) ≤ x
apply (rule ln-exp [THEN subst])
apply (rule ln-le-cancel-iff [THEN iffD2])
apply (auto simp add: exp-ge-add-one-self-aux)
done

lemma ln-less-self [simp]: 0 < x ==> ln x < x
apply (rule order-less-le-trans)
apply (rule-tac [2] ln-add-one-self-le-self)
apply (rule ln-less-cancel-iff [THEN iffD2], auto)
done

lemma ln-ge-zero [simp]:
  assumes x: 1 ≤ x shows 0 ≤ ln x
proof −
  have 0 < x using x by arith
  hence exp 0 ≤ exp (ln x)
    by (simp add: x)
  thus ?thesis by (simp only: exp-le-cancel-iff)
qed

lemma ln-ge-zero-imp-ge-one:
  assumes ln: 0 ≤ ln x
      and x:  0 < x
  shows 1 ≤ x
proof −
  from ln have ln 1 ≤ ln x by simp
  thus ?thesis by (simp add: x del: ln-one)
qed

lemma ln-ge-zero-iff [simp]: 0 < x ==> (0 ≤ ln x) = (1 ≤ x)
by (blast intro: ln-ge-zero ln-ge-zero-imp-ge-one)

lemma ln-less-zero-iff [simp]: 0 < x ==> (ln x < 0) = (x < 1)
by (insert ln-ge-zero-iff [of x], arith)

lemma ln-gt-zero:
  assumes x: 1 < x shows 0 < ln x
proof −
  have 0 < x using x by arith
  hence exp 0 < exp (ln x) by (simp add: x)
  thus ?thesis  by (simp only: exp-less-cancel-iff)
```

qed

lemma *ln-gt-zero-imp-gt-one*:
  assumes *ln*: *0 < ln x*
    and *x*: *0 < x*
  shows *1 < x*
proof −
  from *ln* **have** *ln 1 < ln x* **by** *simp*
  thus *?thesis* **by** (*simp add*: *x del*: *ln-one*)
qed

lemma *ln-gt-zero-iff* [*simp*]: *0 < x ==> (0 < ln x) = (1 < x)*
by (*blast intro*: *ln-gt-zero ln-gt-zero-imp-gt-one*)

lemma *ln-eq-zero-iff* [*simp*]: *0 < x ==> (ln x = 0) = (x = 1)*
by (*insert ln-less-zero-iff* [*of x*] *ln-gt-zero-iff* [*of x*], *arith*)

lemma *ln-less-zero*: [| *0 < x*; *x < 1* |] *==> ln x < 0*
by *simp*

lemma *exp-ln-eq*: *exp u = x ==> ln x = u*
by *auto*

lemma *isCont-ln*: *0 < x* $\Longrightarrow$ *isCont ln x*
apply (*subgoal-tac isCont ln* (*exp* (*ln x*)), *simp*)
apply (*rule isCont-inverse-function* [**where** *f=exp*], *simp-all*)
done

lemma *DERIV-ln*: *0 < x* $\Longrightarrow$ *DERIV ln x :> inverse x*
apply (*rule DERIV-inverse-function* [**where** *f=exp* **and** *a=0* **and** *b=x+1*])
apply (*erule lemma-DERIV-subst* [*OF DERIV-exp exp-ln*])
apply (*simp-all add*: *abs-if isCont-ln*)
done

19.7 Basic Properties of the Trigonometric Functions

lemma *sin-zero* [*simp*]: *sin 0 = 0*
unfolding *sin-def* **by** (*simp add*: *powser-zero*)

lemma *cos-zero* [*simp*]: *cos 0 = 1*
unfolding *cos-def* **by** (*simp add*: *powser-zero*)

lemma *DERIV-sin-sin-mult* [*simp*]:
    DERIV (%*x*. *sin*(*x*)∗*sin*(*x*)) *x* :> *cos*(*x*) ∗ *sin*(*x*) + *cos*(*x*) ∗ *sin*(*x*)
by (*rule DERIV-mult*, *auto*)

lemma *DERIV-sin-sin-mult2* [*simp*]:
    DERIV (%*x*. *sin*(*x*)∗*sin*(*x*)) *x* :> *2* ∗ *cos*(*x*) ∗ *sin*(*x*)
apply (*cut-tac x = x* **in** *DERIV-sin-sin-mult*)

apply (*auto simp add*: *mult-assoc*)
done

lemma *DERIV-sin-realpow2* [*simp*]:
 $DERIV\ (\%x.\ (sin\ x)^2)\ x :> cos(x) * sin(x) + cos(x) * sin(x)$
by (*auto simp add*: *numeral-2-eq-2 real-mult-assoc* [*symmetric*])

lemma *DERIV-sin-realpow2a* [*simp*]:
 $DERIV\ (\%x.\ (sin\ x)^2)\ x :> 2 * cos(x) * sin(x)$
by (*auto simp add*: *numeral-2-eq-2*)

lemma *DERIV-cos-cos-mult* [*simp*]:
 $DERIV\ (\%x.\ cos(x)*cos(x))\ x :> -sin(x) * cos(x) + -sin(x) * cos(x)$
by (*rule DERIV-mult*, *auto*)

lemma *DERIV-cos-cos-mult2* [*simp*]:
 $DERIV\ (\%x.\ cos(x)*cos(x))\ x :> -2 * cos(x) * sin(x)$
apply (*cut-tac* $x = x$ **in** *DERIV-cos-cos-mult*)
apply (*auto simp add*: *mult-ac*)
done

lemma *DERIV-cos-realpow2* [*simp*]:
 $DERIV\ (\%x.\ (cos\ x)^2)\ x :> -sin(x) * cos(x) + -sin(x) * cos(x)$
by (*auto simp add*: *numeral-2-eq-2 real-mult-assoc* [*symmetric*])

lemma *DERIV-cos-realpow2a* [*simp*]:
 $DERIV\ (\%x.\ (cos\ x)^2)\ x :> -2 * cos(x) * sin(x)$
by (*auto simp add*: *numeral-2-eq-2*)

lemma *lemma-DERIV-subst*: $[|\ DERIV\ f\ x :> D;\ D = E\ |] ==> DERIV\ f\ x :> E$
by *auto*

lemma *DERIV-cos-realpow2b*: $DERIV\ (\%x.\ (cos\ x)^2)\ x :> -(2 * cos(x) * sin(x))$
apply (*rule lemma-DERIV-subst*)
apply (*rule DERIV-cos-realpow2a*, *auto*)
done

lemma *DERIV-cos-cos-mult3* [*simp*]:
 $DERIV\ (\%x.\ cos(x)*cos(x))\ x :> -(2 * cos(x) * sin(x))$
apply (*rule lemma-DERIV-subst*)
apply (*rule DERIV-cos-cos-mult2*, *auto*)
done

lemma *DERIV-sin-circle-all*:
 $\forall x.\ DERIV\ (\%x.\ (sin\ x)^2 + (cos\ x)^2)\ x :>$
  $(2*cos(x)*sin(x) - 2*cos(x)*sin(x))$
apply (*simp only*: *diff-minus*, *safe*)

apply (*rule DERIV-add*)
apply (*auto simp add*: *numeral-2-eq-2*)
done

lemma *DERIV-sin-circle-all-zero* [*simp*]:
 $\forall x.$ *DERIV* (%*x*. $(sin\ x)^2 + (cos\ x)^2$) *x* :> *0*
by (*cut-tac DERIV-sin-circle-all*, *auto*)

lemma *sin-cos-squared-add* [*simp*]: $((sin\ x)^2) + ((cos\ x)^2) = 1$
apply (*cut-tac* $x = x$ **and** $y = 0$ **in** *DERIV-sin-circle-all-zero* [*THEN DERIV-isconst-all*])
apply (*auto simp add*: *numeral-2-eq-2*)
done

lemma *sin-cos-squared-add2* [*simp*]: $((cos\ x)^2) + ((sin\ x)^2) = 1$
apply (*subst add-commute*)
apply (*simp* (*no-asm*) *del*: *realpow-Suc*)
done

lemma *sin-cos-squared-add3* [*simp*]: $cos\ x * cos\ x + sin\ x * sin\ x = 1$
apply (*cut-tac* $x = x$ **in** *sin-cos-squared-add2*)
apply (*auto simp add*: *numeral-2-eq-2*)
done

lemma *sin-squared-eq*: $(sin\ x)^2 = 1 - (cos\ x)^2$
apply (*rule-tac* $a1 = (cos\ x)^2$ **in** *add-right-cancel* [*THEN iffD1*])
apply (*simp del*: *realpow-Suc*)
done

lemma *cos-squared-eq*: $(cos\ x)^2 = 1 - (sin\ x)^2$
apply (*rule-tac* $a1 = (sin\ x)^2$ **in** *add-right-cancel* [*THEN iffD1*])
apply (*simp del*: *realpow-Suc*)
done

lemma *real-gt-one-ge-zero-add-less*: [| $1 < x$; $0 \leq y$ |] ==> $1 < x + (y::real)$
by *arith*

lemma *abs-sin-le-one* [*simp*]: $|sin\ x| \leq 1$
by (*rule power2-le-imp-le*, *simp-all add*: *sin-squared-eq*)

lemma *sin-ge-minus-one* [*simp*]: $-1 \leq sin\ x$
apply (*insert abs-sin-le-one* [*of x*])
apply (*simp add*: *abs-le-iff del*: *abs-sin-le-one*)
done

lemma *sin-le-one* [*simp*]: $sin\ x \leq 1$
apply (*insert abs-sin-le-one* [*of x*])
apply (*simp add*: *abs-le-iff del*: *abs-sin-le-one*)
done

lemma *abs-cos-le-one* [*simp*]: $|cos\ x| \leq 1$
by (*rule power2-le-imp-le*, *simp-all add*: *cos-squared-eq*)

lemma *cos-ge-minus-one* [*simp*]: $-1 \leq cos\ x$
apply (*insert abs-cos-le-one* [*of x*])
apply (*simp add*: *abs-le-iff del*: *abs-cos-le-one*)
done

lemma *cos-le-one* [*simp*]: $cos\ x \leq 1$
apply (*insert abs-cos-le-one* [*of x*])
apply (*simp add*: *abs-le-iff del*: *abs-cos-le-one*)
done

lemma *DERIV-fun-pow*: *DERIV g x :> m ==>*
DERIV (%*x*. (*g x*) ^ *n*) *x* :> *real n* ∗ (*g x*) ^ (*n* − *1*) ∗ *m*
apply (*rule lemma-DERIV-subst*)
apply (*rule-tac f* = (%*x*. *x* ^ *n*) **in** *DERIV-chain2*)
apply (*rule DERIV-pow*, *auto*)
done

lemma *DERIV-fun-exp*:
DERIV g x :> m ==> DERIV (%*x*. *exp*(*g x*)) *x* :> *exp*(*g x*) ∗ *m*
apply (*rule lemma-DERIV-subst*)
apply (*rule-tac f* = *exp* **in** *DERIV-chain2*)
apply (*rule DERIV-exp*, *auto*)
done

lemma *DERIV-fun-sin*:
DERIV g x :> m ==> DERIV (%*x*. *sin*(*g x*)) *x* :> *cos*(*g x*) ∗ *m*
apply (*rule lemma-DERIV-subst*)
apply (*rule-tac f* = *sin* **in** *DERIV-chain2*)
apply (*rule DERIV-sin*, *auto*)
done

lemma *DERIV-fun-cos*:
DERIV g x :> m ==> DERIV (%*x*. *cos*(*g x*)) *x* :> −*sin*(*g x*) ∗ *m*
apply (*rule lemma-DERIV-subst*)
apply (*rule-tac f* = *cos* **in** *DERIV-chain2*)
apply (*rule DERIV-cos*, *auto*)
done

lemmas *DERIV-intros* = *DERIV-ident DERIV-const DERIV-cos DERIV-cmult*
DERIV-sin DERIV-exp DERIV-inverse DERIV-pow
DERIV-add DERIV-diff DERIV-mult DERIV-minus
DERIV-inverse-fun DERIV-quotient DERIV-fun-pow
DERIV-fun-exp DERIV-fun-sin DERIV-fun-cos

lemma *lemma-DERIV-sin-cos-add*:

$\forall x.$
$DERIV\ (\%x.\ (sin\ (x + y) - (sin\ x * cos\ y + cos\ x * sin\ y))\ \hat{}\ 2 +$
$(cos\ (x + y) - (cos\ x * cos\ y - sin\ x * sin\ y))\ \hat{}\ 2)\ x :> 0$
apply (*safe*, *rule lemma-DERIV-subst*)
apply (*best intro*!: *DERIV-intros intro*: *DERIV-chain2*)
— replaces the old *DERIV-tac*
apply (*auto simp add*: *diff-minus left-distrib right-distrib mult-ac add-ac*)
done

lemma *sin-cos-add* [*simp*]:
$(sin\ (x + y) - (sin\ x * cos\ y + cos\ x * sin\ y))\ \hat{}\ 2 +$
$(cos\ (x + y) - (cos\ x * cos\ y - sin\ x * sin\ y))\ \hat{}\ 2 = 0$
apply (*cut-tac* $y = 0$ **and** $x = x$ **and** $y7 = y$
in *lemma-DERIV-sin-cos-add* [*THEN DERIV-isconst-all*])
apply (*auto simp add*: *numeral-2-eq-2*)
done

lemma *sin-add*: $sin\ (x + y) = sin\ x * cos\ y + cos\ x * sin\ y$
apply (*cut-tac* $x = x$ **and** $y = y$ **in** *sin-cos-add*)
apply (*simp del*: *sin-cos-add*)
done

lemma *cos-add*: $cos\ (x + y) = cos\ x * cos\ y - sin\ x * sin\ y$
apply (*cut-tac* $x = x$ **and** $y = y$ **in** *sin-cos-add*)
apply (*simp del*: *sin-cos-add*)
done

lemma *lemma-DERIV-sin-cos-minus*:
$\forall x.\ DERIV\ (\%x.\ (sin(-x) + (sin\ x))\ \hat{}\ 2 + (cos(-x) - (cos\ x))\ \hat{}\ 2)\ x :> 0$
apply (*safe*, *rule lemma-DERIV-subst*)
apply (*best intro*!: *DERIV-intros intro*: *DERIV-chain2*)
apply (*auto simp add*: *diff-minus left-distrib right-distrib mult-ac add-ac*)
done

lemma *sin-cos-minus* [*simp*]:
$(sin(-x) + (sin\ x))\ \hat{}\ 2 + (cos(-x) - (cos\ x))\ \hat{}\ 2 = 0$
apply (*cut-tac* $y = 0$ **and** $x = x$
in *lemma-DERIV-sin-cos-minus* [*THEN DERIV-isconst-all*])
apply *simp*
done

lemma *sin-minus* [*simp*]: $sin\ (-x) = -sin(x)$
apply (*cut-tac* $x = x$ **in** *sin-cos-minus*)
apply (*simp del*: *sin-cos-minus*)
done

lemma *cos-minus* [*simp*]: $cos\ (-x) = cos(x)$
apply (*cut-tac* $x = x$ **in** *sin-cos-minus*)
apply (*simp del*: *sin-cos-minus*)

done

lemma *sin-diff*: $sin\ (x - y) = sin\ x * cos\ y - cos\ x * sin\ y$
by (*simp add*: *diff-minus sin-add*)

lemma *sin-diff2*: $sin\ (x - y) = cos\ y * sin\ x - sin\ y * cos\ x$
by (*simp add*: *sin-diff mult-commute*)

lemma *cos-diff*: $cos\ (x - y) = cos\ x * cos\ y + sin\ x * sin\ y$
by (*simp add*: *diff-minus cos-add*)

lemma *cos-diff2*: $cos\ (x - y) = cos\ y * cos\ x + sin\ y * sin\ x$
by (*simp add*: *cos-diff mult-commute*)

lemma *sin-double* [*simp*]: $sin(2 * x) = 2*\ sin\ x * cos\ x$
by (*cut-tac* $x = x$ **and** $y = x$ **in** *sin-add*, *auto*)

lemma *cos-double*: $cos(2*\ x) = ((cos\ x)^2) - ((sin\ x)^2)$
apply (*cut-tac* $x = x$ **and** $y = x$ **in** *cos-add*)
apply (*simp add*: *power2-eq-square*)
done

19.8 The Constant Pi

definition
  pi :: *real* **where**
  $pi = 2 * (THE\ x.\ 0 \le (x::real)\ \&\ x \le 2\ \&\ cos\ x = 0)$

Show that there's a least positive x with $cos\ x = 0$; hence define pi.

lemma *sin-paired*:
    $(\%n.\ -1\ \hat{}\ n\ /(real\ (fact\ (2 * n + 1))) * x\ \hat{}\ (2 * n + 1))$
     sums *sin x*
proof −
  have $(\lambda n. \sum k = n * 2..<n * 2 + 2.$
        $(if\ even\ k\ then\ 0$
         $else\ -1\ \hat{}\ ((k - Suc\ 0)\ div\ 2)\ /\ real\ (fact\ k)) *$
        $x\ \hat{}\ k)$
      sums sin x
    unfolding *sin-def*
    by (*rule sin-converges* [*THEN sums-summable*, *THEN sums-group*], *simp*)
  thus *?thesis* **by** (*simp add*: *mult-ac*)
qed

lemma *sin-gt-zero*: $[|0 < x;\ x < 2\ |] ==> 0 < sin\ x$
apply (*subgoal-tac*
    $(\lambda n. \sum k = n * 2..<n * 2 + 2.$
        $-1\ \hat{}\ k\ /\ real\ (fact\ (2 * k + 1)) * x\ \hat{}\ (2 * k + 1))$
   $sums\ (\sum n.\ -1\ \hat{}\ n\ /\ real\ (fact\ (2 * n + 1)) * x\ \hat{}\ (2 * n + 1)))$

```
 prefer 2
 apply (rule sin-paired [THEN sums-summable, THEN sums-group], simp)
apply (rotate-tac 2)
apply (drule sin-paired [THEN sums-unique, THEN ssubst])
apply (auto simp del: fact-Suc realpow-Suc)
apply (frule sums-unique)
apply (auto simp del: fact-Suc realpow-Suc)
apply (rule-tac n1 = 0 in series-pos-less [THEN [2] order-le-less-trans])
apply (auto simp del: fact-Suc realpow-Suc)
apply (erule sums-summable)
apply (case-tac m=0)
apply (simp (no-asm-simp))
apply (subgoal-tac 6 * (x * (x * x) / real (Suc (Suc (Suc (Suc (Suc (Suc 0))))))))
< 6 * x)
apply (simp only: mult-less-cancel-left, simp)
apply (simp (no-asm-simp) add: numeral-2-eq-2 [symmetric] mult-assoc [symmetric])
apply (subgoal-tac x*x < 2*3, simp)
apply (rule mult-strict-mono)
apply (auto simp add: real-0-less-add-iff real-of-nat-Suc simp del: fact-Suc)
apply (subst fact-Suc)
apply (subst fact-Suc)
apply (subst fact-Suc)
apply (subst fact-Suc)
apply (subst real-of-nat-mult)
apply (subst real-of-nat-mult)
apply (subst real-of-nat-mult)
apply (subst real-of-nat-mult)
apply (simp (no-asm) add: divide-inverse del: fact-Suc)
apply (auto simp add: mult-assoc [symmetric] simp del: fact-Suc)
apply (rule-tac c=real (Suc (Suc (4*m))) in mult-less-imp-less-right)
apply (auto simp add: mult-assoc simp del: fact-Suc)
apply (rule-tac c=real (Suc (Suc (Suc (4*m)))) in mult-less-imp-less-right)
apply (auto simp add: mult-assoc mult-less-cancel-left simp del: fact-Suc)
apply (subgoal-tac x * (x * x ^ (4*m)) = (x ^ (4*m)) * (x * x))
apply (erule ssubst)+
apply (auto simp del: fact-Suc)
apply (subgoal-tac 0 < x ^ (4 * m) )
 prefer 2 apply (simp only: zero-less-power)
apply (simp (no-asm-simp) add: mult-less-cancel-left)
apply (rule mult-strict-mono)
apply (simp-all (no-asm-simp))
done

lemma sin-gt-zero1: [|0 < x; x < 2 |] ==> 0 < sin x
by (auto intro: sin-gt-zero)

lemma cos-double-less-one: [| 0 < x; x < 2 |] ==> cos (2 * x) < 1
apply (cut-tac x = x in sin-gt-zero1)
apply (auto simp add: cos-squared-eq cos-double)
```

done

lemma *cos-paired*:
  $(\%n. -1 \ \hat{}\ n \ /(real\ (fact\ (2 * n))) * x \ \hat{}\ (2 * n))$ *sums cos x*
proof −
  have $(\lambda n. \sum k = n * 2..<n * 2 + 2.$
      $(if\ even\ k\ then\ -1 \ \hat{}\ (k\ div\ 2) \ / \ real\ (fact\ k)\ else\ 0) *$
      $x \ \hat{}\ k)$
    sums cos x
   unfolding *cos-def*
   by (*rule cos-converges* [*THEN sums-summable*, *THEN sums-group*], *simp*)
  thus *?thesis* **by** (*simp add*: *mult-ac*)
qed

declare *zero-less-power* [*simp*]

lemma *fact-lemma*: $real\ (n::nat) * 4 = real\ (4 * n)$
by *simp*

lemma *cos-two-less-zero* [*simp*]: $cos\ (2) < 0$
apply (*cut-tac* $x = 2$ **in** *cos-paired*)
apply (*drule sums-minus*)
apply (*rule neg-less-iff-less* [*THEN iffD1*])
apply (*frule sums-unique*, *auto*)
apply (*rule-tac* $y =$
$\sum n=0..< Suc(Suc(Suc\ 0)). - (-1 \ \hat{}\ n \ / \ (real(fact\ (2*n))) * 2 \ \hat{}\ (2*n))$
  in *order-less-trans*)
apply (*simp* (*no-asm*) *add*: *fact-num-eq-if realpow-num-eq-if del*: *fact-Suc realpow-Suc*)
apply (*simp* (*no-asm*) *add*: *mult-assoc del*: *setsum-op-ivl-Suc*)
apply (*rule sumr-pos-lt-pair*)
apply (*erule sums-summable*, *safe*)
apply (*simp* (*no-asm*) *add*: *divide-inverse real-0-less-add-iff mult-assoc* [*symmetric*]

   del: *fact-Suc*)
apply (*rule real-mult-inverse-cancel2*)
apply (*rule real-of-nat-fact-gt-zero*)+
apply (*simp* (*no-asm*) *add*: *mult-assoc* [*symmetric*] *del*: *fact-Suc*)
apply (*subst fact-lemma*)
apply (*subst fact-Suc* [*of* $Suc\ (Suc\ (Suc\ (Suc\ (Suc\ (Suc\ (Suc\ (4 * d)))))))$])
apply (*simp only*: *real-of-nat-mult*)
apply (*rule mult-strict-mono*, *force*)
 apply (*rule-tac* [*3*] *real-of-nat-fact-ge-zero*)
 prefer *2* **apply** *force*
apply (*rule real-of-nat-less-iff* [*THEN iffD2*])
apply (*rule fact-less-mono*, *auto*)
done

lemmas *cos-two-neq-zero* [*simp*] = *cos-two-less-zero* [*THEN less-imp-neq*]
lemmas *cos-two-le-zero* [*simp*] = *cos-two-less-zero* [*THEN order-less-imp-le*]

lemma *cos-is-zero*: *EX*! *x*. $0 \leq x$ & $x \leq 2$ & *cos* $x = 0$
apply (*subgoal-tac* $\exists x$. $0 \leq x$ & $x \leq 2$ & *cos* $x = 0$)
apply (*rule-tac* [*2*] *IVT2*)
apply (*auto intro*: *DERIV-isCont DERIV-cos*)
apply (*cut-tac* $x = xa$ **and** $y = y$ **in** *linorder-less-linear*)
apply (*rule ccontr*)
apply (*subgoal-tac* ($\forall x$. *cos differentiable* x) & ($\forall x$. *isCont cos* x))
apply (*auto intro*: *DERIV-cos DERIV-isCont simp add*: *differentiable-def*)
apply (*drule-tac* f = *cos* **in** *Rolle*)
apply (*drule-tac* [*5*] f = *cos* **in** *Rolle*)
apply (*auto dest*!: *DERIV-cos* [*THEN DERIV-unique*] *simp add*: *differentiable-def*)
apply (*drule-tac* $y1 = xa$ **in** *order-le-less-trans* [*THEN sin-gt-zero*])
apply (*assumption*, *rule-tac* $y{=}y$ **in** *order-less-le-trans*, *simp-all*)
apply (*drule-tac* $y1 = y$ **in** *order-le-less-trans* [*THEN sin-gt-zero*], *assumption*, *simp-all*)
done

lemma *pi-half*: *pi*/*2* = (*THE* *x*. $0 \leq x$ & $x \leq 2$ & *cos* $x = 0$)
by (*simp add*: *pi-def*)

lemma *cos-pi-half* [*simp*]: *cos* (*pi* / *2*) = *0*
by (*simp add*: *pi-half cos-is-zero* [*THEN theI'*])

lemma *pi-half-gt-zero* [*simp*]: *0* < *pi* / *2*
apply (*rule order-le-neq-trans*)
apply (*simp add*: *pi-half cos-is-zero* [*THEN theI'*])
apply (*rule notI*, *drule arg-cong* [**where** *f*=*cos*], *simp*)
done

lemmas *pi-half-neq-zero* [*simp*] = *pi-half-gt-zero* [*THEN less-imp-neq*, *symmetric*]
lemmas *pi-half-ge-zero* [*simp*] = *pi-half-gt-zero* [*THEN order-less-imp-le*]

lemma *pi-half-less-two* [*simp*]: *pi* / *2* < *2*
apply (*rule order-le-neq-trans*)
apply (*simp add*: *pi-half cos-is-zero* [*THEN theI'*])
apply (*rule notI*, *drule arg-cong* [**where** *f*=*cos*], *simp*)
done

lemmas *pi-half-neq-two* [*simp*] = *pi-half-less-two* [*THEN less-imp-neq*]
lemmas *pi-half-le-two* [*simp*] = *pi-half-less-two* [*THEN order-less-imp-le*]

lemma *pi-gt-zero* [*simp*]: *0* < *pi*
by (*insert pi-half-gt-zero*, *simp*)

lemma *pi-ge-zero* [*simp*]: *0* ≤ *pi*
by (*rule pi-gt-zero* [*THEN order-less-imp-le*])

lemma *pi-neq-zero* [*simp*]: *pi* ≠ *0*

by (*rule pi-gt-zero* [*THEN less-imp-neq*, *THEN not-sym*])

lemma *pi-not-less-zero* [*simp*]: $\neg$ *pi* < *0*
by (*simp add*: *linorder-not-less*)

lemma *minus-pi-half-less-zero* [*simp*]: $-$(*pi*/*2*) < *0*
by *auto*

lemma *sin-pi-half* [*simp*]: *sin*(*pi*/*2*) = *1*
apply (*cut-tac x* = *pi*/*2* **in** *sin-cos-squared-add2*)
apply (*cut-tac sin-gt-zero* [*OF pi-half-gt-zero pi-half-less-two*])
apply (*simp add*: *power2-eq-square*)
done

lemma *cos-pi* [*simp*]: *cos pi* = $-$*1*
by (*cut-tac x* = *pi*/*2* **and** *y* = *pi*/*2* **in** *cos-add*, *simp*)

lemma *sin-pi* [*simp*]: *sin pi* = *0*
by (*cut-tac x* = *pi*/*2* **and** *y* = *pi*/*2* **in** *sin-add*, *simp*)

lemma *sin-cos-eq*: *sin x* = *cos* (*pi*/*2* $-$ *x*)
by (*simp add*: *diff-minus cos-add*)
declare *sin-cos-eq* [*symmetric*, *simp*]

lemma *minus-sin-cos-eq*: $-$*sin x* = *cos* (*x* + *pi*/*2*)
by (*simp add*: *cos-add*)
declare *minus-sin-cos-eq* [*symmetric*, *simp*]

lemma *cos-sin-eq*: *cos x* = *sin* (*pi*/*2* $-$ *x*)
by (*simp add*: *diff-minus sin-add*)
declare *cos-sin-eq* [*symmetric*, *simp*]

lemma *sin-periodic-pi* [*simp*]: *sin* (*x* + *pi*) = $-$ *sin x*
by (*simp add*: *sin-add*)

lemma *sin-periodic-pi2* [*simp*]: *sin* (*pi* + *x*) = $-$ *sin x*
by (*simp add*: *sin-add*)

lemma *cos-periodic-pi* [*simp*]: *cos* (*x* + *pi*) = $-$ *cos x*
by (*simp add*: *cos-add*)

lemma *sin-periodic* [*simp*]: *sin* (*x* + *2***pi*) = *sin x*
by (*simp add*: *sin-add cos-double*)

lemma *cos-periodic* [*simp*]: *cos* (*x* + *2***pi*) = *cos x*
by (*simp add*: *cos-add cos-double*)

lemma *cos-npi* [*simp*]: *cos* (*real n* * *pi*) = $-$*1* ^ *n*
apply (*induct n*)

apply (*auto simp add*: *real-of-nat-Suc left-distrib*)
done

lemma *cos-npi2* [*simp*]: *cos* (*pi* ∗ *real n*) = −*1* ˆ *n*
proof −
  have *cos* (*pi* ∗ *real n*) = *cos* (*real n* ∗ *pi*) **by** (*simp only*: *mult-commute*)
  also have ... = −*1* ˆ *n* **by** (*rule cos-npi*)
  finally show *?thesis* **.**
qed

lemma *sin-npi* [*simp*]: *sin* (*real* (*n*::*nat*) ∗ *pi*) = *0*
apply (*induct n*)
apply (*auto simp add*: *real-of-nat-Suc left-distrib*)
done

lemma *sin-npi2* [*simp*]: *sin* (*pi* ∗ *real* (*n*::*nat*)) = *0*
by (*simp add*: *mult-commute* [*of pi*])

lemma *cos-two-pi* [*simp*]: *cos* (*2* ∗ *pi*) = *1*
by (*simp add*: *cos-double*)

lemma *sin-two-pi* [*simp*]: *sin* (*2* ∗ *pi*) = *0*
by *simp*

lemma *sin-gt-zero2*: [| *0* < *x*; *x* < *pi*/*2* |] ==> *0* < *sin x*
apply (*rule sin-gt-zero*, *assumption*)
apply (*rule order-less-trans*, *assumption*)
apply (*rule pi-half-less-two*)
done

lemma *sin-less-zero*:
  assumes *lb*: − *pi*/*2* < *x* **and** *x* < *0* **shows** *sin x* < *0*
proof −
  have *0* < *sin* (− *x*) **using** *prems* **by** (*simp only*: *sin-gt-zero2*)
  thus *?thesis* **by** *simp*
qed

lemma *pi-less-4*: *pi* < *4*
by (*cut-tac pi-half-less-two*, *auto*)

lemma *cos-gt-zero*: [| *0* < *x*; *x* < *pi*/*2* |] ==> *0* < *cos x*
apply (*cut-tac pi-less-4*)
apply (*cut-tac f* = *cos* **and** *a* = *0* **and** *b* = *x* **and** *y* = *0* **in** *IVT2-objl*, *safe*, *simp-all*)
apply (*cut-tac cos-is-zero*, *safe*)
apply (*rename-tac y z*)
apply (*drule-tac x* = *y* **in** *spec*)
apply (*drule-tac x* = *pi*/*2* **in** *spec*, *simp*)
done

lemma *cos-gt-zero-pi*: $[\mid -(pi/2) < x;\ x < pi/2 \mid] ==> 0 < cos\ x$
apply (*rule-tac* $x = x$ **and** $y = 0$ **in** *linorder-cases*)
apply (*rule cos-minus* [*THEN subst*])
apply (*rule cos-gt-zero*)
apply (*auto intro*: *cos-gt-zero*)
done

lemma *cos-ge-zero*: $[\mid -(pi/2) \le x;\ x \le pi/2 \mid] ==> 0 \le cos\ x$
apply (*auto simp add*: *order-le-less cos-gt-zero-pi*)
apply (*subgoal-tac* $x = pi/2$, *auto*)
done

lemma *sin-gt-zero-pi*: $[\mid 0 < x;\ x < pi \mid] ==> 0 < sin\ x$
apply (*subst sin-cos-eq*)
apply (*rotate-tac 1*)
apply (*drule real-sum-of-halves* [*THEN ssubst*])
apply (*auto intro*!: *cos-gt-zero-pi simp del*: *sin-cos-eq* [*symmetric*])
done

lemma *sin-ge-zero*: $[\mid 0 \le x;\ x \le pi \mid] ==> 0 \le sin\ x$
by (*auto simp add*: *order-le-less sin-gt-zero-pi*)

lemma *cos-total*: $[\mid -1 \le y;\ y \le 1 \mid] ==> EX!\ x.\ 0 \le x\ \&\ x \le pi\ \&\ (cos\ x = y)$
apply (*subgoal-tac* $\exists x.\ 0 \le x\ \&\ x \le pi\ \&\ cos\ x = y$)
apply (*rule-tac* [2] *IVT2*)
apply (*auto intro*: *order-less-imp-le DERIV-isCont DERIV-cos*)
apply (*cut-tac* $x = xa$ **and** $y = y$ **in** *linorder-less-linear*)
apply (*rule ccontr*, *auto*)
apply (*drule-tac* $f = cos$ **in** *Rolle*)
apply (*drule-tac* [5] $f = cos$ **in** *Rolle*)
apply (*auto intro*: *order-less-imp-le DERIV-isCont DERIV-cos*
  dest!: *DERIV-cos* [*THEN DERIV-unique*]
  simp add: *differentiable-def*)
apply (*auto dest*: *sin-gt-zero-pi* [*OF order-le-less-trans order-less-le-trans*])
done

lemma *sin-total*:
  $[\mid -1 \le y;\ y \le 1 \mid] ==> EX!\ x.\ -(pi/2) \le x\ \&\ x \le pi/2\ \&\ (sin\ x = y)$
apply (*rule ccontr*)
apply (*subgoal-tac* $\forall x.\ (-\ (pi/2) \le x\ \&\ x \le pi/2\ \&\ (sin\ x = y)) = (0 \le (x + pi/2)\ \&\ (x + pi/2) \le pi\ \&\ (cos\ (x + pi/2) = -y))$)
apply (*erule contrapos-np*)
apply (*simp del*: *minus-sin-cos-eq* [*symmetric*])
apply (*cut-tac* $y{=}{-}y$ **in** *cos-total*, *simp*) **apply** *simp*
apply (*erule ex1E*)
apply (*rule-tac* $a = x - (pi/2)$ **in** *ex1I*)
apply (*simp* (*no-asm*) *add*: *add-assoc*)

```
apply (rotate-tac 3)
apply (drule-tac x = xa + pi/2 in spec, safe, simp-all)
done

lemma reals-Archimedean4:
     [| 0 < y; 0 ≤ x |] ==> ∃ n. real n ∗ y ≤ x & x < real (Suc n) ∗ y
apply (auto dest!: reals-Archimedean3)
apply (drule-tac x = x in spec, clarify)
apply (subgoal-tac x < real(LEAST m::nat. x < real m ∗ y) ∗ y)
 prefer 2 apply (erule LeastI)
apply (case-tac LEAST m::nat. x < real m ∗ y, simp)
apply (subgoal-tac ~ x < real nat ∗ y)
 prefer 2 apply (rule not-less-Least, simp, force)
done


lemma cos-zero-lemma:
     [| 0 ≤ x; cos x = 0 |] ==>
      ∃ n::nat. ~even n & x = real n ∗ (pi/2)
apply (drule pi-gt-zero [THEN reals-Archimedean4], safe)
apply (subgoal-tac 0 ≤ x − real n ∗ pi &
                    (x − real n ∗ pi) ≤ pi & (cos (x − real n ∗ pi) = 0) )
apply (auto simp add: compare-rls)
  prefer 3 apply (simp add: cos-diff)
 prefer 2 apply (simp add: real-of-nat-Suc left-distrib)
apply (simp add: cos-diff)
apply (subgoal-tac EX! x. 0 ≤ x & x ≤ pi & cos x = 0)
apply (rule-tac [2] cos-total, safe)
apply (drule-tac x = x − real n ∗ pi in spec)
apply (drule-tac x = pi/2 in spec)
apply (simp add: cos-diff)
apply (rule-tac x = Suc (2 ∗ n) in exI)
apply (simp add: real-of-nat-Suc left-distrib, auto)
done

lemma sin-zero-lemma:
     [| 0 ≤ x; sin x = 0 |] ==>
      ∃ n::nat. even n & x = real n ∗ (pi/2)
apply (subgoal-tac ∃ n::nat. ~ even n & x + pi/2 = real n ∗ (pi/2) )
 apply (clarify, rule-tac x = n − 1 in exI)
 apply (force simp add: odd-Suc-mult-two-ex real-of-nat-Suc left-distrib)
apply (rule cos-zero-lemma)
apply (simp-all add: add-increasing)
done


lemma cos-zero-iff:
     (cos x = 0) =
      ((∃ n::nat. ~even n & (x = real n ∗ (pi/2))) |
```

(∃ *n::nat*. ~*even n* & (*x* = −(*real n* ∗ (*pi/2*)))))
apply (*rule iffI*)
apply (*cut-tac linorder-linear* [*of 0 x*], *safe*)
apply (*drule cos-zero-lemma*, *assumption*+)
apply (*cut-tac x*=−*x* **in** *cos-zero-lemma*, *simp*, *simp*)
apply (*force simp add*: *minus-equation-iff* [*of x*])
apply (*auto simp only*: *odd-Suc-mult-two-ex real-of-nat-Suc left-distrib*)
apply (*auto simp add*: *cos-add*)
done

lemma *sin-zero-iff*:
(*sin x* = *0*) =
((∃ *n::nat*. *even n* & (*x* = *real n* ∗ (*pi/2*))) |
(∃ *n::nat*. *even n* & (*x* = −(*real n* ∗ (*pi/2*)))))
apply (*rule iffI*)
apply (*cut-tac linorder-linear* [*of 0 x*], *safe*)
apply (*drule sin-zero-lemma*, *assumption*+)
apply (*cut-tac x*=−*x* **in** *sin-zero-lemma*, *simp*, *simp*, *safe*)
apply (*force simp add*: *minus-equation-iff* [*of x*])
apply (*auto simp add*: *even-mult-two-ex*)
done

19.9 Tangent

definition
tan :: *real* => *real* **where**
tan x = (*sin x*)/(*cos x*)

lemma *tan-zero* [*simp*]: *tan 0* = *0*
by (*simp add*: *tan-def*)

lemma *tan-pi* [*simp*]: *tan pi* = *0*
by (*simp add*: *tan-def*)

lemma *tan-npi* [*simp*]: *tan* (*real* (*n::nat*) ∗ *pi*) = *0*
by (*simp add*: *tan-def*)

lemma *tan-minus* [*simp*]: *tan* (−*x*) = − *tan x*
by (*simp add*: *tan-def minus-mult-left*)

lemma *tan-periodic* [*simp*]: *tan* (*x* + *2*∗*pi*) = *tan x*
by (*simp add*: *tan-def*)

lemma *lemma-tan-add1*:
[| *cos x* ≠ *0*; *cos y* ≠ *0* |]
==> *1* − *tan*(*x*)∗*tan*(*y*) = *cos* (*x* + *y*)/(*cos x* ∗ *cos y*)
apply (*simp add*: *tan-def divide-inverse*)
apply (*auto simp del*: *inverse-mult-distrib*

simp add: *inverse-mult-distrib* [*symmetric*] *mult-ac*)
apply (*rule-tac c1* = *cos x* ∗ *cos y* **in** *real-mult-right-cancel* [*THEN subst*])
apply (*auto simp del*: *inverse-mult-distrib*
simp add: *mult-assoc left-diff-distrib cos-add*)
done

lemma *add-tan-eq*:
[| *cos x* ≠ *0*; *cos y* ≠ *0* |]
==> *tan x* + *tan y* = *sin*(*x* + *y*)/(*cos x* ∗ *cos y*)
apply (*simp add*: *tan-def*)
apply (*rule-tac c1* = *cos x* ∗ *cos y* **in** *real-mult-right-cancel* [*THEN subst*])
apply (*auto simp add*: *mult-assoc left-distrib*)
apply (*simp add*: *sin-add*)
done

lemma *tan-add*:
[| *cos x* ≠ *0*; *cos y* ≠ *0*; *cos* (*x* + *y*) ≠ *0* |]
==> *tan*(*x* + *y*) = (*tan*(*x*) + *tan*(*y*))/(*1* − *tan*(*x*) ∗ *tan*(*y*))
apply (*simp* (*no-asm-simp*) *add*: *add-tan-eq lemma-tan-add1*)
apply (*simp add*: *tan-def*)
done

lemma *tan-double*:
[| *cos x* ≠ *0*; *cos* (*2* ∗ *x*) ≠ *0* |]
==> *tan* (*2* ∗ *x*) = (*2* ∗ *tan x*)/(*1* − (*tan*(*x*) ^ *2*))
apply (*insert tan-add* [*of x x*])
apply (*simp add*: *mult-2* [*symmetric*])
apply (*auto simp add*: *numeral-2-eq-2*)
done

lemma *tan-gt-zero*: [| *0* < *x*; *x* < *pi*/*2* |] ==> *0* < *tan x*
by (*simp add*: *tan-def zero-less-divide-iff sin-gt-zero2 cos-gt-zero-pi*)

lemma *tan-less-zero*:
assumes *lb*: − *pi*/*2* < *x* **and** *x* < *0* **shows** *tan x* < *0*
proof −
have *0* < *tan* (− *x*) **using** *prems* **by** (*simp only*: *tan-gt-zero*)
thus *?thesis* **by** *simp*
qed

lemma *lemma-DERIV-tan*:
cos x ≠ *0* ==> *DERIV* (%*x*. *sin*(*x*)/*cos*(*x*)) *x* :> *inverse*((*cos x*)^2)
apply (*rule lemma-DERIV-subst*)
apply (*best intro*!: *DERIV-intros intro*: *DERIV-chain2*)
apply (*auto simp add*: *divide-inverse numeral-2-eq-2*)
done

lemma *DERIV-tan* [*simp*]: *cos x* ≠ *0* ==> *DERIV tan x* :> *inverse*((*cos x*)^2)
by (*auto dest*: *lemma-DERIV-tan simp add*: *tan-def* [*symmetric*])

lemma *isCont-tan* [*simp*]: *cos x* $\neq$ *0* ==> *isCont tan x*
by (*rule DERIV-tan* [*THEN DERIV-isCont*])

lemma *LIM-cos-div-sin* [*simp*]: (%*x*. *cos*(*x*)/*sin*(*x*)) −− *pi*/*2* −−> *0*
apply (*subgoal-tac* (λ*x*. *cos x* ∗ *inverse* (*sin x*)) −− *pi* ∗ *inverse 2* −−> *0*∗*1*)
apply (*simp add*: *divide-inverse* [*symmetric*])
apply (*rule LIM-mult*)
apply (*rule-tac* [*2*] *inverse-1* [*THEN subst*])
apply (*rule-tac* [*2*] *LIM-inverse*)
apply (*simp-all add*: *divide-inverse* [*symmetric*])
apply (*simp-all only*: *isCont-def* [*symmetric*] *cos-pi-half* [*symmetric*] *sin-pi-half* [*symmetric*])
apply (*blast intro*!: *DERIV-isCont DERIV-sin DERIV-cos*)+
done

lemma *lemma-tan-total*: *0* < *y* ==> ∃ *x*. *0* < *x* & *x* < *pi*/*2* & *y* < *tan x*
apply (*cut-tac LIM-cos-div-sin*)
apply (*simp only*: *LIM-def*)
apply (*drule-tac x* = *inverse y* **in** *spec*, *safe*, *force*)
apply (*drule-tac ?d1.0* = *s* **in** *pi-half-gt-zero* [*THEN* [*2*] *real-lbound-gt-zero*], *safe*)
apply (*rule-tac x* = (*pi*/*2*) − *e* **in** *exI*)
apply (*simp* (*no-asm-simp*))
apply (*drule-tac x* = (*pi*/*2*) − *e* **in** *spec*)
apply (*auto simp add*: *tan-def*)
apply (*rule inverse-less-iff-less* [*THEN iffD1*])
apply (*auto simp add*: *divide-inverse*)
apply (*rule real-mult-order*)
apply (*subgoal-tac* [*3*] *0* < *sin e* & *0* < *cos e*)
apply (*auto intro*: *cos-gt-zero sin-gt-zero2 simp add*: *mult-commute*)
done

lemma *tan-total-pos*: *0* ≤ *y* ==> ∃ *x*. *0* ≤ *x* & *x* < *pi*/*2* & *tan x* = *y*
apply (*frule order-le-imp-less-or-eq*, *safe*)
 prefer *2* **apply** *force*
apply (*drule lemma-tan-total*, *safe*)
apply (*cut-tac f* = *tan* **and** *a* = *0* **and** *b* = *x* **and** *y* = *y* **in** *IVT-objl*)
apply (*auto intro*!: *DERIV-tan* [*THEN DERIV-isCont*])
apply (*drule-tac y* = *xa* **in** *order-le-imp-less-or-eq*)
apply (*auto dest*: *cos-gt-zero*)
done

lemma *lemma-tan-total1*: ∃ *x*. −(*pi*/*2*) < *x* & *x* < (*pi*/*2*) & *tan x* = *y*
apply (*cut-tac linorder-linear* [*of 0 y*], *safe*)
apply (*drule tan-total-pos*)
apply (*cut-tac* [*2*] *y*=−*y* **in** *tan-total-pos*, *safe*)
apply (*rule-tac* [*3*] *x* = −*x* **in** *exI*)
apply (*auto intro*!: *exI*)
done

lemma *tan-total*: *EX*! x. $-(pi/2) < x$ & $x < (pi/2)$ & *tan* $x = y$
apply (*cut-tac* $y = y$ **in** *lemma-tan-total1*, *auto*)
apply (*cut-tac* $x = xa$ **and** $y = y$ **in** *linorder-less-linear*, *auto*)
apply (*subgoal-tac* [*2*] $\exists z$. $y < z$ & $z < xa$ & *DERIV tan* z :> *0*)
apply (*subgoal-tac* $\exists z$. $xa < z$ & $z < y$ & *DERIV tan* z :> *0*)
apply (*rule-tac* [*4*] *Rolle*)
apply (*rule-tac* [*2*] *Rolle*)
apply (*auto intro*!: *DERIV-tan DERIV-isCont exI*
  simp add: *differentiable-def*)

Now, simulate TRYALL

apply (*rule-tac* [!] *DERIV-tan asm-rl*)
apply (*auto dest*!: *DERIV-unique* [*OF* - *DERIV-tan*]
  simp add: *cos-gt-zero-pi* [*THEN less-imp-neq*, *THEN not-sym*])
done

19.10 Inverse Trigonometric Functions

definition
arcsin :: *real* => *real* **where**
arcsin y = (*THE* x. $-(pi/2) \leq x$ & $x \leq pi/2$ & *sin* $x = y$)

definition
arccos :: *real* => *real* **where**
arccos y = (*THE* x. $0 \leq x$ & $x \leq pi$ & *cos* $x = y$)

definition
arctan :: *real* => *real* **where**
arctan y = (*THE* x. $-(pi/2) < x$ & $x < pi/2$ & *tan* $x = y$)

lemma *arcsin*:
[| $-1 \leq y$; $y \leq 1$ |]
==> $-(pi/2) \leq$ *arcsin* y &
arcsin $y \leq pi/2$ & *sin*(*arcsin* y) = y
unfolding *arcsin-def* **by** (*rule theI'* [*OF sin-total*])

lemma *arcsin-pi*:
[| $-1 \leq y$; $y \leq 1$ |]
==> $-(pi/2) \leq$ *arcsin* y & *arcsin* $y \leq pi$ & *sin*(*arcsin* y) = y
apply (*drule* (*1*) *arcsin*)
apply (*force intro*: *order-trans*)
done

lemma *sin-arcsin* [*simp*]: [| $-1 \leq y$; $y \leq 1$ |] ==> *sin*(*arcsin* y) = y
by (*blast dest*: *arcsin*)

lemma *arcsin-bounded*:
[| $-1 \leq y$; $y \leq 1$ |] ==> $-(pi/2) \leq$ *arcsin* y & *arcsin* $y \leq pi/2$
by (*blast dest*: *arcsin*)

lemma *arcsin-lbound*: [| $-1 \le y$; $y \le 1$ |] ==> $-(pi/2) \le arcsin\ y$
by (*blast dest*: *arcsin*)

lemma *arcsin-ubound*: [| $-1 \le y$; $y \le 1$ |] ==> $arcsin\ y \le pi/2$
by (*blast dest*: *arcsin*)

lemma *arcsin-lt-bounded*:
  [| $-1 < y$; $y < 1$ |] ==> $-(pi/2) < arcsin\ y$ & $arcsin\ y < pi/2$
apply (*frule order-less-imp-le*)
apply (*frule-tac* $y = y$ **in** *order-less-imp-le*)
apply (*frule arcsin-bounded*)
apply (*safe*, *simp*)
apply (*drule-tac* $y = arcsin\ y$ **in** *order-le-imp-less-or-eq*)
apply (*drule-tac* [2] $y = pi/2$ **in** *order-le-imp-less-or-eq*, *safe*)
apply (*drule-tac* [!] $f = sin$ **in** *arg-cong*, *auto*)
done

lemma *arcsin-sin*: [|$-(pi/2) \le x$; $x \le pi/2$ |] ==> $arcsin(sin\ x) = x$
apply (*unfold arcsin-def*)
apply (*rule the1-equality*)
apply (*rule sin-total*, *auto*)
done

lemma *arccos*:
  [| $-1 \le y$; $y \le 1$ |]
   ==> $0 \le arccos\ y$ & $arccos\ y \le pi$ & $cos(arccos\ y) = y$
unfolding *arccos-def* **by** (*rule theI'* [*OF cos-total*])

lemma *cos-arccos* [*simp*]: [| $-1 \le y$; $y \le 1$ |] ==> $cos(arccos\ y) = y$
by (*blast dest*: *arccos*)

lemma *arccos-bounded*: [| $-1 \le y$; $y \le 1$ |] ==> $0 \le arccos\ y$ & $arccos\ y \le pi$
by (*blast dest*: *arccos*)

lemma *arccos-lbound*: [| $-1 \le y$; $y \le 1$ |] ==> $0 \le arccos\ y$
by (*blast dest*: *arccos*)

lemma *arccos-ubound*: [| $-1 \le y$; $y \le 1$ |] ==> $arccos\ y \le pi$
by (*blast dest*: *arccos*)

lemma *arccos-lt-bounded*:
  [| $-1 < y$; $y < 1$ |]
   ==> $0 < arccos\ y$ & $arccos\ y < pi$
apply (*frule order-less-imp-le*)
apply (*frule-tac* $y = y$ **in** *order-less-imp-le*)
apply (*frule arccos-bounded*, *auto*)
apply (*drule-tac* $y = arccos\ y$ **in** *order-le-imp-less-or-eq*)
apply (*drule-tac* [2] $y = pi$ **in** *order-le-imp-less-or-eq*, *auto*)

apply (*drule-tac* [!] $f = cos$ **in** *arg-cong*, *auto*)
done

lemma *arccos-cos*: $[|0 \leq x;\ x \leq pi\ |] ==> arccos(cos\ x) = x$
apply (*simp add*: *arccos-def*)
apply (*auto intro*!: *the1-equality cos-total*)
done

lemma *arccos-cos2*: $[|x \leq 0;\ -pi \leq x\ |] ==> arccos(cos\ x) = -x$
apply (*simp add*: *arccos-def*)
apply (*auto intro*!: *the1-equality cos-total*)
done

lemma *cos-arcsin*: $[\![-1 \leq x;\ x \leq 1]\!] \Longrightarrow cos\ (arcsin\ x) = sqrt\ (1 - x^2)$
apply (*subgoal-tac* $x^2 \leq 1$)
apply (*rule power2-eq-imp-eq*)
apply (*simp add*: *cos-squared-eq*)
apply (*rule cos-ge-zero*)
apply (*erule* (*1*) *arcsin-lbound*)
apply (*erule* (*1*) *arcsin-ubound*)
apply *simp*
apply (*subgoal-tac* $|x|^2 \leq 1^2$, *simp*)
apply (*rule power-mono*, *simp*, *simp*)
done

lemma *sin-arccos*: $[\![-1 \leq x;\ x \leq 1]\!] \Longrightarrow sin\ (arccos\ x) = sqrt\ (1 - x^2)$
apply (*subgoal-tac* $x^2 \leq 1$)
apply (*rule power2-eq-imp-eq*)
apply (*simp add*: *sin-squared-eq*)
apply (*rule sin-ge-zero*)
apply (*erule* (*1*) *arccos-lbound*)
apply (*erule* (*1*) *arccos-ubound*)
apply *simp*
apply (*subgoal-tac* $|x|^2 \leq 1^2$, *simp*)
apply (*rule power-mono*, *simp*, *simp*)
done

lemma *arctan* [*simp*]:
   $-\ (pi/2) < arctan\ y\ \ \&\ arctan\ y < pi/2\ \&\ tan\ (arctan\ y) = y$
unfolding *arctan-def* **by** (*rule theI'* [*OF tan-total*])

lemma *tan-arctan*: $tan(arctan\ y) = y$
by *auto*

lemma *arctan-bounded*: $-\ (pi/2) < arctan\ y\ \ \&\ arctan\ y < pi/2$
by (*auto simp only*: *arctan*)

lemma *arctan-lbound*: $-\ (pi/2) < arctan\ y$
by *auto*

lemma *arctan-ubound*: *arctan* $y < pi/2$
by (*auto simp only*: *arctan*)

lemma *arctan-tan*:
  $[|-(pi/2) < x; x < pi/2 |] ==> arctan(tan\ x) = x$
apply (*unfold arctan-def*)
apply (*rule the1-equality*)
apply (*rule tan-total*, *auto*)
done

lemma *arctan-zero-zero* [*simp*]: *arctan* $0 = 0$
by (*insert arctan-tan* [*of 0*], *simp*)

lemma *cos-arctan-not-zero* [*simp*]: $cos(arctan\ x) \neq 0$
apply (*auto simp add*: *cos-zero-iff*)
apply (*case-tac n*)
apply (*case-tac* [*3*] *n*)
apply (*cut-tac* [*2*] $y = x$ **in** *arctan-ubound*)
apply (*cut-tac* [*4*] $y = x$ **in** *arctan-lbound*)
apply (*auto simp add*: *real-of-nat-Suc left-distrib mult-less-0-iff*)
done

lemma *tan-sec*: $cos\ x \neq 0 ==> 1 + tan(x)\ \hat{}\ 2 = inverse(cos\ x)\ \hat{}\ 2$
apply (*rule power-inverse* [*THEN subst*])
apply (*rule-tac* $c1 = (cos\ x)^2$ **in** *real-mult-right-cancel* [*THEN iffD1*])
apply (*auto dest*: *field-power-not-zero*
  simp add: *power-mult-distrib left-distrib power-divide tan-def*
    mult-assoc power-inverse [*symmetric*]
  simp del: *realpow-Suc*)
done

lemma *isCont-inverse-function2*:
 fixes *f g* :: *real* ⇒ *real* **shows**
 $[\![a < x; x < b;$
  $\forall z.\ a \leq z \wedge z \leq b \longrightarrow g\ (f\ z) = z;$
  $\forall z.\ a \leq z \wedge z \leq b \longrightarrow isCont\ f\ z]\!]$
 $\Longrightarrow isCont\ g\ (f\ x)$
apply (*rule isCont-inverse-function*
  [**where** *f*=*f* **and** d=*min* $(x - a)\ (b - x)$])
apply (*simp-all add*: *abs-le-iff*)
done

lemma *isCont-arcsin*: $[\![-1 < x; x < 1]\!] \Longrightarrow isCont\ arcsin\ x$
apply (*subgoal-tac isCont arcsin* (*sin* (*arcsin x*)), *simp*)
apply (*rule isCont-inverse-function2* [**where** *f*=*sin*])
apply (*erule* (*1*) *arcsin-lt-bounded* [*THEN conjunct1*])
apply (*erule* (*1*) *arcsin-lt-bounded* [*THEN conjunct2*])
apply (*fast intro*: *arcsin-sin*, *simp*)

done

lemma *isCont-arccos*: $[\![-1 < x; x < 1]\!] \Longrightarrow isCont\ arccos\ x$
apply (*subgoal-tac isCont arccos* (*cos* (*arccos x*)), *simp*)
apply (*rule isCont-inverse-function2* [**where** *f*=*cos*])
apply (*erule* (*1*) *arccos-lt-bounded* [*THEN conjunct1*])
apply (*erule* (*1*) *arccos-lt-bounded* [*THEN conjunct2*])
apply (*fast intro*: *arccos-cos*, *simp*)
done

lemma *isCont-arctan*: *isCont arctan x*
apply (*rule arctan-lbound* [*of x*, *THEN dense*, *THEN exE*], *clarify*)
apply (*rule arctan-ubound* [*of x*, *THEN dense*, *THEN exE*], *clarify*)
apply (*subgoal-tac isCont arctan* (*tan* (*arctan x*)), *simp*)
apply (*erule* (*1*) *isCont-inverse-function2* [**where** *f*=*tan*])
apply (*clarify*, *rule arctan-tan*)
apply (*erule* (*1*) *order-less-le-trans*)
apply (*erule* (*1*) *order-le-less-trans*)
apply (*clarify*, *rule isCont-tan*)
apply (*rule less-imp-neq* [*symmetric*])
apply (*rule cos-gt-zero-pi*)
apply (*erule* (*1*) *order-less-le-trans*)
apply (*erule* (*1*) *order-le-less-trans*)
done

lemma *DERIV-arcsin*:
$[\![-1 < x; x < 1]\!] \Longrightarrow DERIV\ arcsin\ x :> inverse\ (sqrt\ (1 - x^2))$
apply (*rule DERIV-inverse-function* [**where** *f*=*sin* **and** *a*=-1 **and** *b*=*1*])
apply (*rule lemma-DERIV-subst* [*OF DERIV-sin*])
apply (*simp add*: *cos-arcsin*)
apply (*subgoal-tac* $|x|^2 < 1^2$, *simp*)
apply (*rule power-strict-mono*, *simp*, *simp*, *simp*)
apply *assumption*
apply *assumption*
apply *simp*
apply (*erule* (*1*) *isCont-arcsin*)
done

lemma *DERIV-arccos*:
$[\![-1 < x; x < 1]\!] \Longrightarrow DERIV\ arccos\ x :> inverse\ (-\ sqrt\ (1 - x^2))$
apply (*rule DERIV-inverse-function* [**where** *f*=*cos* **and** *a*=-1 **and** *b*=*1*])
apply (*rule lemma-DERIV-subst* [*OF DERIV-cos*])
apply (*simp add*: *sin-arccos*)
apply (*subgoal-tac* $|x|^2 < 1^2$, *simp*)
apply (*rule power-strict-mono*, *simp*, *simp*, *simp*)
apply *assumption*
apply *assumption*
apply *simp*
apply (*erule* (*1*) *isCont-arccos*)

done

lemma *DERIV-arctan*: *DERIV arctan x :> inverse* ($1 + x^2$)
apply (*rule DERIV-inverse-function* [**where** f=*tan* **and** a=$x - 1$ **and** b=$x + 1$])
apply (*rule lemma-DERIV-subst* [*OF DERIV-tan*])
apply (*rule cos-arctan-not-zero*)
apply (*simp add*: *power-inverse tan-sec* [*symmetric*])
apply (*subgoal-tac* $0 < 1 + x^2$, *simp*)
apply (*simp add*: *add-pos-nonneg*)
apply (*simp*, *simp*, *simp*, *rule isCont-arctan*)
done

19.11 More Theorems about Sin and Cos

lemma *cos-45*: *cos* (*pi* / *4*) = *sqrt 2* / *2*
proof −
  let *?c* = *cos* (*pi* / *4*) **and** *?s* = *sin* (*pi* / *4*)
  have *nonneg*: $0 \leq$ *?c*
    by (*rule cos-ge-zero*, *rule order-trans* [**where** y=*0*], *simp-all*)
  have *0* = *cos* (*pi* / *4* + *pi* / *4*)
    by *simp*
  also have *cos* (*pi* / *4* + *pi* / *4*) = $?c^2 - ?s^2$
    by (*simp only*: *cos-add power2-eq-square*)
  also have ... = $2 * ?c^2 - 1$
    by (*simp add*: *sin-squared-eq*)
  finally have $?c^2$ = $(sqrt\ 2\ /\ 2)^2$
    by (*simp add*: *power-divide*)
  thus *?thesis*
    using *nonneg* **by** (*rule power2-eq-imp-eq*) *simp*
qed

lemma *cos-30*: *cos* (*pi* / *6*) = *sqrt 3* / *2*
proof −
  let *?c* = *cos* (*pi* / *6*) **and** *?s* = *sin* (*pi* / *6*)
  have *pos-c*: $0 <$ *?c*
    by (*rule cos-gt-zero*, *simp*, *simp*)
  have *0* = *cos* (*pi* / *6* + *pi* / *6* + *pi* / *6*)
    by *simp*
  also have ... = $(?c * ?c - ?s * ?s) * ?c - (?s * ?c + ?c * ?s) * ?s$
    by (*simp only*: *cos-add sin-add*)
  also have ... = $?c * (?c^2 - 3 * ?s^2)$
    by (*simp add*: *ring-simps power2-eq-square*)
  finally have $?c^2$ = $(sqrt\ 3\ /\ 2)^2$
    using *pos-c* **by** (*simp add*: *sin-squared-eq power-divide*)
  thus *?thesis*
    using *pos-c* [*THEN order-less-imp-le*]
    by (*rule power2-eq-imp-eq*) *simp*
qed

```
lemma sin-45: sin (pi / 4) = sqrt 2 / 2
proof –
  have sin (pi / 4) = cos (pi / 2 − pi / 4) by (rule sin-cos-eq)
  also have pi / 2 − pi / 4 = pi / 4 by simp
  also have cos (pi / 4) = sqrt 2 / 2 by (rule cos-45)
  finally show ?thesis .
qed

lemma sin-60: sin (pi / 3) = sqrt 3 / 2
proof –
  have sin (pi / 3) = cos (pi / 2 − pi / 3) by (rule sin-cos-eq)
  also have pi / 2 − pi / 3 = pi / 6 by simp
  also have cos (pi / 6) = sqrt 3 / 2 by (rule cos-30)
  finally show ?thesis .
qed

lemma cos-60: cos (pi / 3) = 1 / 2
apply (rule power2-eq-imp-eq)
apply (simp add: cos-squared-eq sin-60 power-divide)
apply (rule cos-ge-zero, rule order-trans [where y=0], simp-all)
done

lemma sin-30: sin (pi / 6) = 1 / 2
proof –
  have sin (pi / 6) = cos (pi / 2 − pi / 6) by (rule sin-cos-eq)
  also have pi / 2 − pi / 6 = pi / 3 by simp
  also have cos (pi / 3) = 1 / 2 by (rule cos-60)
  finally show ?thesis .
qed

lemma tan-30: tan (pi / 6) = 1 / sqrt 3
unfolding tan-def by (simp add: sin-30 cos-30)

lemma tan-45: tan (pi / 4) = 1
unfolding tan-def by (simp add: sin-45 cos-45)

lemma tan-60: tan (pi / 3) = sqrt 3
unfolding tan-def by (simp add: sin-60 cos-60)
```

NEEDED??

```
lemma [simp]:
    sin (x + 1 / 2 ∗ real (Suc m) ∗ pi) =
     cos (x + 1 / 2 ∗ real  (m) ∗ pi)
by (simp only: cos-add sin-add real-of-nat-Suc left-distrib right-distrib, auto)
```

NEEDED??

```
lemma [simp]:
    sin (x + real (Suc m) ∗ pi / 2) =
```

```
     cos (x + real (m) * pi / 2)
by (simp only: cos-add sin-add real-of-nat-Suc add-divide-distrib left-distrib, auto)

lemma DERIV-sin-add [simp]: DERIV (%x. sin (x + k)) xa :> cos (xa + k)
apply (rule lemma-DERIV-subst)
apply (rule-tac f = sin and g = %x. x + k in DERIV-chain2)
apply (best intro!: DERIV-intros intro: DERIV-chain2)+
apply (simp (no-asm))
done

lemma sin-cos-npi [simp]: sin (real (Suc (2 * n)) * pi / 2) = (-1) ^ n
proof -
  have sin ((real n + 1/2) * pi) = cos (real n * pi)
    by (auto simp add: right-distrib sin-add left-distrib mult-ac)
  thus ?thesis
    by (simp add: real-of-nat-Suc left-distrib add-divide-distrib
              mult-commute [of pi])
qed

lemma cos-2npi [simp]: cos (2 * real (n::nat) * pi) = 1
by (simp add: cos-double mult-assoc power-add [symmetric] numeral-2-eq-2)

lemma cos-3over2-pi [simp]: cos (3 / 2 * pi) = 0
apply (subgoal-tac cos (pi + pi/2) = 0, simp)
apply (subst cos-add, simp)
done

lemma sin-2npi [simp]: sin (2 * real (n::nat) * pi) = 0
by (auto simp add: mult-assoc)

lemma sin-3over2-pi [simp]: sin (3 / 2 * pi) = - 1
apply (subgoal-tac sin (pi + pi/2) = - 1, simp)
apply (subst sin-add, simp)
done


lemma [simp]:
     cos(x + 1 / 2 * real(Suc m) * pi) = -sin (x + 1 / 2 * real m * pi)
apply (simp only: cos-add sin-add real-of-nat-Suc right-distrib left-distrib minus-mult-right,
auto)
done


lemma [simp]: cos (x + real(Suc m) * pi / 2) = -sin (x + real m * pi / 2)
by (simp only: cos-add sin-add real-of-nat-Suc left-distrib add-divide-distrib, auto)

lemma cos-pi-eq-zero [simp]: cos (pi * real (Suc (2 * m)) / 2) = 0
by (simp only: cos-add sin-add real-of-nat-Suc left-distrib right-distrib add-divide-distrib,
auto)
```

lemma *DERIV-cos-add* [*simp*]: *DERIV* (%*x*. *cos* (*x* + *k*)) *xa* :> − *sin* (*xa* + *k*)
apply (*rule lemma-DERIV-subst*)
apply (*rule-tac f* = *cos* **and** *g* = %*x*. *x* + *k* **in** *DERIV-chain2*)
apply (*best intro*!: *DERIV-intros intro*: *DERIV-chain2*)+
apply (*simp* (*no-asm*))
done

lemma *sin-zero-abs-cos-one*: *sin x* = *0* ==> |*cos x*| = *1*
by (*auto simp add*: *sin-zero-iff even-mult-two-ex*)

lemma *exp-eq-one-iff* [*simp*]: (*exp* (*x*::*real*) = *1*) = (*x* = *0*)
apply *auto*
apply (*drule-tac f* = *ln* **in** *arg-cong*, *auto*)
done

lemma *cos-one-sin-zero*: *cos x* = *1* ==> *sin x* = *0*
by (*cut-tac x* = *x* **in** *sin-cos-squared-add3*, *auto*)

19.12 Existence of Polar Coordinates

lemma *cos-x-y-le-one*: |*x* / *sqrt* (x^2 + y^2)| ≤ *1*
apply (*rule power2-le-imp-le* [*OF* - *zero-le-one*])
apply (*simp add*: *abs-divide power-divide divide-le-eq not-sum-power2-lt-zero*)
done

lemma *cos-arccos-abs*: |*y*| ≤ *1* ⟹ *cos* (*arccos y*) = *y*
by (*simp add*: *abs-le-iff*)

lemma *sin-arccos-abs*: |*y*| ≤ *1* ⟹ *sin* (*arccos y*) = *sqrt* (*1* − y^2)
by (*simp add*: *sin-arccos abs-le-iff*)

lemmas *cos-arccos-lemma1* = *cos-arccos-abs* [*OF cos-x-y-le-one*]

lemmas *sin-arccos-lemma1* = *sin-arccos-abs* [*OF cos-x-y-le-one*]

lemma *polar-ex1*:
  0 < *y* ==> ∃ *r a*. *x* = *r* ∗ *cos a* & *y* = *r* ∗ *sin a*
apply (*rule-tac x* = *sqrt* (x^2 + y^2) **in** *exI*)
apply (*rule-tac x* = *arccos* (*x* / *sqrt* (x^2 + y^2)) **in** *exI*)
apply (*simp add*: *cos-arccos-lemma1*)
apply (*simp add*: *sin-arccos-lemma1*)
apply (*simp add*: *power-divide*)
apply (*simp add*: *real-sqrt-mult* [*symmetric*])
apply (*simp add*: *right-diff-distrib*)
done

lemma *polar-ex2*:
  y < *0* ==> ∃ *r a*. *x* = *r* ∗ *cos a* & *y* = *r* ∗ *sin a*

apply (*insert polar-ex1* [**where** $x{=}x$ **and** $y{=}{-}y$], *simp*, *clarify*)
apply (*rule-tac* $x = r$ **in** *exI*)
apply (*rule-tac* $x = -a$ **in** *exI*, *simp*)
done

lemma *polar-Ex*: $\exists\, r\ a.\ x = r * cos\ a\ \&\ y = r * sin\ a$
apply (*rule-tac* $x{=}0$ **and** $y{=}y$ **in** *linorder-cases*)
apply (*erule polar-ex1*)
apply (*rule-tac* $x{=}x$ **in** *exI*, *rule-tac* $x{=}0$ **in** *exI*, *simp*)
apply (*erule polar-ex2*)
done

19.13 Theorems about Limits

lemma *isCont-inv-fun*:
  fixes $f\ g :: real \Rightarrow real$
  shows $[|\ 0 < d;\ \forall\, z.\ |z - x| \le d\ {-}{-}{>}\ g(f(z)) = z;$
    $\forall\, z.\ |z - x| \le d\ {-}{-}{>}\ isCont\ f\ z\ |]$
   $={=}{>}\ isCont\ g\ (f\ x)$
by (*rule isCont-inverse-function*)

lemma *isCont-inv-fun-inv*:
  fixes $f\ g :: real \Rightarrow real$
  shows $[|\ 0 < d;$
    $\forall\, z.\ |z - x| \le d\ {-}{-}{>}\ g(f(z)) = z;$
    $\forall\, z.\ |z - x| \le d\ {-}{-}{>}\ isCont\ f\ z\ |]$
   $={=}{>}\ \exists\, e.\ 0 < e\ \&$
     $(\forall\, y.\ 0 < |y - f(x)|\ \&\ |y - f(x)| < e\ {-}{-}{>}\ f(g(y)) = y)$
apply (*drule isCont-inj-range*)
prefer *2* **apply** (*assumption*, *assumption*, *auto*)
apply (*rule-tac* $x = e$ **in** *exI*, *auto*)
apply (*rotate-tac 2*)
apply (*drule-tac* $x = y$ **in** *spec*, *auto*)
done

Bartle/Sherbert: Introduction to Real Analysis, Theorem 4.2.9, p. 110

lemma *LIM-fun-gt-zero*:
   $[|\ f\ {-}{-}\ c\ {-}{-}{>}\ (l{::}real);\ 0 < l\ |]$
    $={=}{>}\ \exists\, r.\ 0 < r\ \&\ (\forall\, x{::}real.\ x \ne c\ \&\ |c - x| < r\ {-}{-}{>}\ 0 < f\ x)$
apply (*auto simp add*: *LIM-def*)
apply (*drule-tac* $x = l/2$ **in** *spec*, *safe*, *force*)
apply (*rule-tac* $x = s$ **in** *exI*)
apply (*auto simp only*: *abs-less-iff*)
done

lemma *LIM-fun-less-zero*:
   $[|\ f\ {-}{-}\ c\ {-}{-}{>}\ (l{::}real);\ l < 0\ |]$
   $={=}{>}\ \exists\, r.\ 0 < r\ \&\ (\forall\, x{::}real.\ x \ne c\ \&\ |c - x| < r\ {-}{-}{>}\ f\ x < 0)$
apply (*auto simp add*: *LIM-def*)

apply (*drule-tac* $x = -l/2$ **in** *spec*, *safe*, *force*)
apply (*rule-tac* $x = s$ **in** *exI*)
apply (*auto simp only*: *abs-less-iff*)
done

lemma *LIM-fun-not-zero*:
[| f −− c −−> (l::*real*); $l \neq 0$ |]
==> $\exists\, r.\ 0 < r$ & ($\forall\, x$::*real*. $x \neq c$ & $|c - x| < r$ −−> $f\, x \neq 0$)
apply (*cut-tac* $x = l$ **and** $y = 0$ **in** *linorder-less-linear*, *auto*)
apply (*drule LIM-fun-less-zero*)
apply (*drule-tac* [3] *LIM-fun-gt-zero*)
apply *force*+
done

end

20 Complex: Complex Numbers: Rectangular and Polar Representations

theory *Complex*
imports ../*Hyperreal*/*Transcendental*
begin

datatype *complex* = *Complex real real*

consts *Re* :: *complex* ⇒ *real*
primrec *Re*: *Re* (*Complex x y*) = *x*

consts *Im* :: *complex* ⇒ *real*
primrec *Im*: *Im* (*Complex x y*) = *y*

lemma *complex-surj* [*simp*]: *Complex* (*Re z*) (*Im z*) = *z*
by (*induct z*) *simp*

lemma *complex-equality* [*intro?*]: ⟦*Re x* = *Re y*; *Im x* = *Im y*⟧ ⟹ *x* = *y*
by (*induct x*, *induct y*) *simp*

lemma *expand-complex-eq*: (*x* = *y*) = (*Re x* = *Re y* ∧ *Im x* = *Im y*)
by (*induct x*, *induct y*) *simp*

lemmas *complex-Re-Im-cancel-iff* = *expand-complex-eq*

20.1 Addition and Subtraction

instance *complex* :: *zero*
complex-zero-def:

  0 ≡ *Complex 0 0* **..**

instance *complex* :: *plus*
  complex-add-def:
    x + *y* ≡ *Complex* (*Re x* + *Re y*) (*Im x* + *Im y*) **..**

instance *complex* :: *minus*
  complex-minus-def:
    − *x* ≡ *Complex* (− *Re x*) (− *Im x*)
  complex-diff-def:
    x − *y* ≡ *x* + − *y* **..**

lemma *Complex-eq-0* [*simp*]: (*Complex a b* = *0*) = (*a* = *0* ∧ *b* = *0*)
by (*simp add*: *complex-zero-def*)

lemma *complex-Re-zero* [*simp*]: *Re 0* = *0*
by (*simp add*: *complex-zero-def*)

lemma *complex-Im-zero* [*simp*]: *Im 0* = *0*
by (*simp add*: *complex-zero-def*)

lemma *complex-add* [*simp*]:
  Complex a b + *Complex c d* = *Complex* (*a* + *c*) (*b* + *d*)
by (*simp add*: *complex-add-def*)

lemma *complex-Re-add* [*simp*]: *Re* (*x* + *y*) = *Re x* + *Re y*
by (*simp add*: *complex-add-def*)

lemma *complex-Im-add* [*simp*]: *Im* (*x* + *y*) = *Im x* + *Im y*
by (*simp add*: *complex-add-def*)

lemma *complex-minus* [*simp*]: − (*Complex a b*) = *Complex* (− *a*) (− *b*)
by (*simp add*: *complex-minus-def*)

lemma *complex-Re-minus* [*simp*]: *Re* (− *x*) = − *Re x*
by (*simp add*: *complex-minus-def*)

lemma *complex-Im-minus* [*simp*]: *Im* (− *x*) = − *Im x*
by (*simp add*: *complex-minus-def*)

lemma *complex-diff* [*simp*]:
  Complex a b − *Complex c d* = *Complex* (*a* − *c*) (*b* − *d*)
by (*simp add*: *complex-diff-def*)

lemma *complex-Re-diff* [*simp*]: *Re* (*x* − *y*) = *Re x* − *Re y*
by (*simp add*: *complex-diff-def*)

lemma *complex-Im-diff* [*simp*]: *Im* (*x* − *y*) = *Im x* − *Im y*
by (*simp add*: *complex-diff-def*)

instance *complex* :: *ab-group-add*
proof
  fix x y z :: *complex*
  show $(x + y) + z = x + (y + z)$
    by (*simp add*: *expand-complex-eq add-assoc*)
  show $x + y = y + x$
    by (*simp add*: *expand-complex-eq add-commute*)
  show $0 + x = x$
    by (*simp add*: *expand-complex-eq*)
  show $- x + x = 0$
    by (*simp add*: *expand-complex-eq*)
  show $x - y = x + - y$
    by (*simp add*: *expand-complex-eq*)
qed

20.2 Multiplication and Division

instance *complex* :: *one*
  complex-one-def:
    $1 \equiv Complex\ 1\ 0$ **..**

instance *complex* :: *times*
  complex-mult-def:
    $x * y \equiv Complex\ (Re\ x * Re\ y - Im\ x * Im\ y)\ (Re\ x * Im\ y + Im\ x * Re\ y)$
..

instance *complex* :: *inverse*
  complex-inverse-def:
    $inverse\ x \equiv$
     $Complex\ (Re\ x\ /\ ((Re\ x)^2 + (Im\ x)^2))\ (- Im\ x\ /\ ((Re\ x)^2 + (Im\ x)^2))$
  complex-divide-def:
    $x\ /\ y \equiv x * inverse\ y$ **..**

lemma *Complex-eq-1* [*simp*]: $(Complex\ a\ b = 1) = (a = 1 \wedge b = 0)$
by (*simp add*: *complex-one-def*)

lemma *complex-Re-one* [*simp*]: $Re\ 1 = 1$
by (*simp add*: *complex-one-def*)

lemma *complex-Im-one* [*simp*]: $Im\ 1 = 0$
by (*simp add*: *complex-one-def*)

lemma *complex-mult* [*simp*]:
  $Complex\ a\ b * Complex\ c\ d = Complex\ (a * c - b * d)\ (a * d + b * c)$
by (*simp add*: *complex-mult-def*)

lemma *complex-Re-mult* [*simp*]: $Re\ (x * y) = Re\ x * Re\ y - Im\ x * Im\ y$
by (*simp add*: *complex-mult-def*)

lemma *complex-Im-mult* [*simp*]: *Im* $(x * y) = Re\ x * Im\ y + Im\ x * Re\ y$
by (*simp add*: *complex-mult-def*)

lemma *complex-inverse* [*simp*]:
inverse (*Complex a b*) = *Complex* $(a / (a^2 + b^2)) (- b / (a^2 + b^2))$
by (*simp add*: *complex-inverse-def*)

lemma *complex-Re-inverse*:
Re (*inverse x*) = $Re\ x / ((Re\ x)^2 + (Im\ x)^2)$
by (*simp add*: *complex-inverse-def*)

lemma *complex-Im-inverse*:
Im (*inverse x*) = $- Im\ x / ((Re\ x)^2 + (Im\ x)^2)$
by (*simp add*: *complex-inverse-def*)

instance *complex* :: *field*
proof
  fix *x y z* :: *complex*
  show $(x * y) * z = x * (y * z)$
    by (*simp add*: *expand-complex-eq ring-simps*)
  show $x * y = y * x$
    by (*simp add*: *expand-complex-eq mult-commute add-commute*)
  show $1 * x = x$
    by (*simp add*: *expand-complex-eq*)
  show $0 \neq (1::complex)$
    by (*simp add*: *expand-complex-eq*)
  show $(x + y) * z = x * z + y * z$
    by (*simp add*: *expand-complex-eq ring-simps*)
  show $x / y = x * inverse\ y$
    by (*simp only*: *complex-divide-def*)
  show $x \neq 0 \Longrightarrow inverse\ x * x = 1$
    by (*induct x*, *simp add*: *power2-eq-square add-divide-distrib* [*symmetric*])
qed

instance *complex* :: *division-by-zero*
proof
  show *inverse 0* = (*0*::*complex*)
    by (*simp add*: *complex-inverse-def*)
qed

20.3 Exponentiation

instance *complex* :: *power* **..**

primrec
    complexpow-0: $z \hat{}\ 0 = 1$
    complexpow-Suc: $z \hat{}\ (Suc\ n) = (z::complex) * (z \hat{}\ n)$

instance *complex* :: *recpower*
proof
fix x :: *complex* **and** n :: *nat*
show x ^ $0 = 1$ **by** *simp*
show x ^ *Suc* $n = x * x$ ^ n **by** *simp*
qed

20.4 Numerals and Arithmetic

instance *complex* :: *number*
complex-number-of-def:
number-of $w \equiv$ *of-int* w **..**

instance *complex* :: *number-ring*
by (*intro-classes*, *simp only*: *complex-number-of-def*)

lemma *complex-Re-of-nat* [*simp*]: *Re* (*of-nat* n) = *of-nat* n
by (*induct* n) *simp-all*

lemma *complex-Im-of-nat* [*simp*]: *Im* (*of-nat* n) = 0
by (*induct* n) *simp-all*

lemma *complex-Re-of-int* [*simp*]: *Re* (*of-int* z) = *of-int* z
by (*cases* z *rule*: *int-diff-cases*) *simp*

lemma *complex-Im-of-int* [*simp*]: *Im* (*of-int* z) = 0
by (*cases* z *rule*: *int-diff-cases*) *simp*

lemma *complex-Re-number-of* [*simp*]: *Re* (*number-of* v) = *number-of* v
unfolding *number-ring-class.axioms* **by** (*rule complex-Re-of-int*)

lemma *complex-Im-number-of* [*simp*]: *Im* (*number-of* v) = 0
unfolding *number-ring-class.axioms* **by** (*rule complex-Im-of-int*)

lemma *Complex-eq-number-of* [*simp*]:
(*Complex* a b = *number-of* w) = (a = *number-of* $w \wedge b = 0$)
by (*simp add*: *expand-complex-eq*)

20.5 Scalar Multiplication

instance *complex* :: *scaleR*
complex-scaleR-def:
scaleR r $x \equiv$ *Complex* (r * *Re* x) (r * *Im* x) **..**

lemma *complex-scaleR* [*simp*]:
scaleR r (*Complex* a b) = *Complex* ($r * a$) ($r * b$)
unfolding *complex-scaleR-def* **by** *simp*

lemma *complex-Re-scaleR* [*simp*]: *Re* (*scaleR* r x) = r * *Re* x
unfolding *complex-scaleR-def* **by** *simp*

lemma *complex-Im-scaleR* [*simp*]: *Im* (*scaleR r x*) = *r* ∗ *Im x*
unfolding *complex-scaleR-def* **by** *simp*

instance *complex* :: *real-field*
proof
  fix *a b* :: *real* **and** *x y* :: *complex*
  show *scaleR a* (*x* + *y*) = *scaleR a x* + *scaleR a y*
    by (*simp add*: *expand-complex-eq right-distrib*)
  show *scaleR* (*a* + *b*) *x* = *scaleR a x* + *scaleR b x*
    by (*simp add*: *expand-complex-eq left-distrib*)
  show *scaleR a* (*scaleR b x*) = *scaleR* (*a* ∗ *b*) *x*
    by (*simp add*: *expand-complex-eq mult-assoc*)
  show *scaleR 1 x* = *x*
    by (*simp add*: *expand-complex-eq*)
  show *scaleR a x* ∗ *y* = *scaleR a* (*x* ∗ *y*)
    by (*simp add*: *expand-complex-eq ring-simps*)
  show *x* ∗ *scaleR a y* = *scaleR a* (*x* ∗ *y*)
    by (*simp add*: *expand-complex-eq ring-simps*)
qed

20.6 Properties of Embedding from Reals

abbreviation
  complex-of-real :: *real* ⇒ *complex* **where**
    complex-of-real ≡ *of-real*

lemma *complex-of-real-def*: *complex-of-real r* = *Complex r 0*
by (*simp add*: *of-real-def complex-scaleR-def*)

lemma *Re-complex-of-real* [*simp*]: *Re* (*complex-of-real z*) = *z*
by (*simp add*: *complex-of-real-def*)

lemma *Im-complex-of-real* [*simp*]: *Im* (*complex-of-real z*) = *0*
by (*simp add*: *complex-of-real-def*)

lemma *Complex-add-complex-of-real* [*simp*]:
    Complex x y + *complex-of-real r* = *Complex* (*x*+*r*) *y*
by (*simp add*: *complex-of-real-def*)

lemma *complex-of-real-add-Complex* [*simp*]:
    complex-of-real r + *Complex x y* = *Complex* (*r*+*x*) *y*
by (*simp add*: *complex-of-real-def*)

lemma *Complex-mult-complex-of-real*:
    Complex x y ∗ *complex-of-real r* = *Complex* (*x*∗*r*) (*y*∗*r*)
by (*simp add*: *complex-of-real-def*)

lemma *complex-of-real-mult-Complex*:

complex-of-real r $*$ *Complex x y* = *Complex* (*r*$*$*x*) (*r*$*$*y*)
by (*simp add*: *complex-of-real-def*)

20.7 Vector Norm

instance *complex* :: *norm*
complex-norm-def:
norm z ≡ *sqrt* ((*Re z*)^2 + (*Im z*)^2) **..**

abbreviation
cmod :: *complex* ⇒ *real* **where**
cmod ≡ *norm*

instance *complex* :: *sgn*
complex-sgn-def: *sgn x* == *x* $/_R$ *cmod x* **..**

lemmas *cmod-def* = *complex-norm-def*

lemma *complex-norm* [*simp*]: *cmod* (*Complex x y*) = *sqrt* (x^2 + y^2)
by (*simp add*: *complex-norm-def*)

instance *complex* :: *real-normed-field*
proof
fix *r* :: *real* **and** *x y* :: *complex*
show *0* ≤ *norm x*
by (*induct x*) *simp*
show (*norm x* = *0*) = (*x* = *0*)
by (*induct x*) *simp*
show *norm* (*x* + *y*) ≤ *norm x* + *norm y*
by (*induct x*, *induct y*)
(*simp add*: *real-sqrt-sum-squares-triangle-ineq*)
show *norm* (*scaleR r x*) = |*r*| $*$ *norm x*
by (*induct x*)
(*simp add*: *power-mult-distrib right-distrib* [*symmetric*] *real-sqrt-mult*)
show *norm* (*x* $*$ *y*) = *norm x* $*$ *norm y*
by (*induct x*, *induct y*)
(*simp add*: *real-sqrt-mult* [*symmetric*] *power2-eq-square ring-simps*)
show *sgn x* = *x* $/_R$ *cmod x* **by**(*simp add*: *complex-sgn-def*)
qed

lemma *cmod-unit-one* [*simp*]: *cmod* (*Complex* (*cos a*) (*sin a*)) = *1*
by *simp*

lemma *cmod-complex-polar* [*simp*]:
cmod (*complex-of-real r* $*$ *Complex* (*cos a*) (*sin a*)) = *abs r*
by (*simp add*: *norm-mult*)

lemma *complex-Re-le-cmod*: *Re x* ≤ *cmod x*
unfolding *complex-norm-def*

by (*rule real-sqrt-sum-squares-ge1*)

lemma *complex-mod-minus-le-complex-mod* [*simp*]: $-$ *cmod x* $\leq$ *cmod x*
by (*rule order-trans* [*OF - norm-ge-zero*], *simp*)

lemma *complex-mod-triangle-ineq2* [*simp*]: *cmod*(*b* + *a*) $-$ *cmod b* $\leq$ *cmod a*
by (*rule ord-le-eq-trans* [*OF norm-triangle-ineq2*], *simp*)

lemmas *real-sum-squared-expand* = *power2-sum* [**where** *'a=real*]

20.8 Completeness of the Complexes

interpretation *Re*: *bounded-linear* [*Re*]
apply (*unfold-locales*, *simp*, *simp*)
apply (*rule-tac x=1* **in** *exI*)
apply (*simp add*: *complex-norm-def*)
done

interpretation *Im*: *bounded-linear* [*Im*]
apply (*unfold-locales*, *simp*, *simp*)
apply (*rule-tac x=1* **in** *exI*)
apply (*simp add*: *complex-norm-def*)
done

lemma *LIMSEQ-Complex*:
⟦*X* −−−−> *a*; *Y* −−−−> *b*⟧ ⟹ (λ*n*. *Complex* (*X n*) (*Y n*)) −−−−> *Complex a b*
apply (*rule LIMSEQ-I*)
apply (*subgoal-tac 0* < *r* / *sqrt 2*)
apply (*drule-tac r=r* / *sqrt 2* **in** *LIMSEQ-D*, *safe*)
apply (*drule-tac r=r* / *sqrt 2* **in** *LIMSEQ-D*, *safe*)
apply (*rename-tac M N*, *rule-tac x=max M N* **in** *exI*, *safe*)
apply (*simp add*: *real-sqrt-sum-squares-less*)
apply (*simp add*: *divide-pos-pos*)
done

instance *complex* :: *banach*
proof
  fix *X* :: *nat* ⇒ *complex*
  assume *X*: *Cauchy X*
  from *Re.Cauchy* [*OF X*] **have** *1*: (λ*n*. *Re* (*X n*)) −−−−> *lim* (λ*n*. *Re* (*X n*))
    by (*simp add*: *Cauchy-convergent-iff convergent-LIMSEQ-iff*)
  from *Im.Cauchy* [*OF X*] **have** *2*: (λ*n*. *Im* (*X n*)) −−−−> *lim* (λ*n*. *Im* (*X n*))
    by (*simp add*: *Cauchy-convergent-iff convergent-LIMSEQ-iff*)
  have *X* −−−−> *Complex* (*lim* (λ*n*. *Re* (*X n*))) (*lim* (λ*n*. *Im* (*X n*)))
    using *LIMSEQ-Complex* [*OF 1 2*] **by** *simp*
  thus *convergent X*
    by (*rule convergentI*)
qed

20.9 The Complex Number i

definition
ii :: *complex* (i) **where**
i-def: $ii \equiv Complex\ 0\ 1$

lemma *complex-Re-i* [*simp*]: $Re\ ii = 0$
by (*simp add*: *i-def*)

lemma *complex-Im-i* [*simp*]: $Im\ ii = 1$
by (*simp add*: *i-def*)

lemma *Complex-eq-i* [*simp*]: $(Complex\ x\ y = ii) = (x = 0 \wedge y = 1)$
by (*simp add*: *i-def*)

lemma *complex-i-not-zero* [*simp*]: $ii \neq 0$
by (*simp add*: *expand-complex-eq*)

lemma *complex-i-not-one* [*simp*]: $ii \neq 1$
by (*simp add*: *expand-complex-eq*)

lemma *complex-i-not-number-of* [*simp*]: $ii \neq number\text{-}of\ w$
by (*simp add*: *expand-complex-eq*)

lemma *i-mult-Complex* [*simp*]: $ii * Complex\ a\ b = Complex\ (-\ b)\ a$
by (*simp add*: *expand-complex-eq*)

lemma *Complex-mult-i* [*simp*]: $Complex\ a\ b * ii = Complex\ (-\ b)\ a$
by (*simp add*: *expand-complex-eq*)

lemma *i-complex-of-real* [*simp*]: $ii * complex\text{-}of\text{-}real\ r = Complex\ 0\ r$
by (*simp add*: *i-def complex-of-real-def*)

lemma *complex-of-real-i* [*simp*]: $complex\text{-}of\text{-}real\ r * ii = Complex\ 0\ r$
by (*simp add*: *i-def complex-of-real-def*)

lemma *i-squared* [*simp*]: $ii * ii = -1$
by (*simp add*: *i-def*)

lemma *power2-i* [*simp*]: $ii^2 = -1$
by (*simp add*: *power2-eq-square*)

lemma *inverse-i* [*simp*]: $inverse\ ii = -\ ii$
by (*rule inverse-unique*, *simp*)

20.10 Complex Conjugation

definition
cnj :: $complex \Rightarrow complex$ **where**
$cnj\ z = Complex\ (Re\ z)\ (-\ Im\ z)$

lemma *complex-cnj* [*simp*]: *cnj* (*Complex a b*) = *Complex a* (− *b*)
by (*simp add*: *cnj-def*)

lemma *complex-Re-cnj* [*simp*]: *Re* (*cnj x*) = *Re x*
by (*simp add*: *cnj-def*)

lemma *complex-Im-cnj* [*simp*]: *Im* (*cnj x*) = − *Im x*
by (*simp add*: *cnj-def*)

lemma *complex-cnj-cancel-iff* [*simp*]: (*cnj x* = *cnj y*) = (*x* = *y*)
by (*simp add*: *expand-complex-eq*)

lemma *complex-cnj-cnj* [*simp*]: *cnj* (*cnj z*) = *z*
by (*simp add*: *cnj-def*)

lemma *complex-cnj-zero* [*simp*]: *cnj 0* = *0*
by (*simp add*: *expand-complex-eq*)

lemma *complex-cnj-zero-iff* [*iff*]: (*cnj z* = *0*) = (*z* = *0*)
by (*simp add*: *expand-complex-eq*)

lemma *complex-cnj-add*: *cnj* (*x* + *y*) = *cnj x* + *cnj y*
by (*simp add*: *expand-complex-eq*)

lemma *complex-cnj-diff*: *cnj* (*x* − *y*) = *cnj x* − *cnj y*
by (*simp add*: *expand-complex-eq*)

lemma *complex-cnj-minus*: *cnj* (− *x*) = − *cnj x*
by (*simp add*: *expand-complex-eq*)

lemma *complex-cnj-one* [*simp*]: *cnj 1* = *1*
by (*simp add*: *expand-complex-eq*)

lemma *complex-cnj-mult*: *cnj* (*x* ∗ *y*) = *cnj x* ∗ *cnj y*
by (*simp add*: *expand-complex-eq*)

lemma *complex-cnj-inverse*: *cnj* (*inverse x*) = *inverse* (*cnj x*)
by (*simp add*: *complex-inverse-def*)

lemma *complex-cnj-divide*: *cnj* (*x* / *y*) = *cnj x* / *cnj y*
by (*simp add*: *complex-divide-def complex-cnj-mult complex-cnj-inverse*)

lemma *complex-cnj-power*: *cnj* (*x* ˆ *n*) = *cnj x* ˆ *n*
by (*induct n*, *simp-all add*: *complex-cnj-mult*)

lemma *complex-cnj-of-nat* [*simp*]: *cnj* (*of-nat n*) = *of-nat n*
by (*simp add*: *expand-complex-eq*)

lemma *complex-cnj-of-int* [*simp*]: *cnj* (*of-int z*) = *of-int z*
by (*simp add*: *expand-complex-eq*)

lemma *complex-cnj-number-of* [*simp*]: *cnj* (*number-of w*) = *number-of w*
by (*simp add*: *expand-complex-eq*)

lemma *complex-cnj-scaleR*: *cnj* (*scaleR r x*) = *scaleR r* (*cnj x*)
by (*simp add*: *expand-complex-eq*)

lemma *complex-mod-cnj* [*simp*]: *cmod* (*cnj z*) = *cmod z*
by (*simp add*: *complex-norm-def*)

lemma *complex-cnj-complex-of-real* [*simp*]: *cnj* (*of-real x*) = *of-real x*
by (*simp add*: *expand-complex-eq*)

lemma *complex-cnj-i* [*simp*]: *cnj ii* = − *ii*
by (*simp add*: *expand-complex-eq*)

lemma *complex-add-cnj*: *z* + *cnj z* = *complex-of-real* (*2* ∗ *Re z*)
by (*simp add*: *expand-complex-eq*)

lemma *complex-diff-cnj*: *z* − *cnj z* = *complex-of-real* (*2* ∗ *Im z*) ∗ *ii*
by (*simp add*: *expand-complex-eq*)

lemma *complex-mult-cnj*: *z* ∗ *cnj z* = *complex-of-real* $((Re\ z)^2 + (Im\ z)^2)$
by (*simp add*: *expand-complex-eq power2-eq-square*)

lemma *complex-mod-mult-cnj*: *cmod* (*z* ∗ *cnj z*) = $(cmod\ z)^2$
by (*simp add*: *norm-mult power2-eq-square*)

interpretation *cnj*: *bounded-linear* [*cnj*]
apply (*unfold-locales*)
apply (*rule complex-cnj-add*)
apply (*rule complex-cnj-scaleR*)
apply (*rule-tac x=1* **in** *exI*, *simp*)
done

20.11 The Functions *sgn* and *arg*

———— Argand ————-

definition
arg :: *complex* => *real* **where**
arg z = (*SOME a*. *Re*(*sgn z*) = *cos a* & *Im*(*sgn z*) = *sin a* & −*pi* < *a* & *a* ≤ *pi*)

lemma *sgn-eq*: *sgn z* = *z* / *complex-of-real* (*cmod z*)
by (*simp add*: *complex-sgn-def divide-inverse scaleR-conv-of-real mult-commute*)

lemma *i-mult-eq*: *ii* ∗ *ii* = *complex-of-real* (−*1*)

by (*simp add*: *i-def complex-of-real-def*)

lemma *i-mult-eq2* [*simp*]: $ii * ii = -(1{::}complex)$
by (*simp add*: *i-def complex-one-def*)

lemma *complex-eq-cancel-iff2* [*simp*]:
$(Complex\ x\ y = complex\text{-}of\text{-}real\ xa) = (x = xa\ \&\ y = 0)$
by (*simp add*: *complex-of-real-def*)

lemma *Re-sgn* [*simp*]: $Re(sgn\ z) = Re(z)/cmod\ z$
by (*simp add*: *complex-sgn-def divide-inverse*)

lemma *Im-sgn* [*simp*]: $Im(sgn\ z) = Im(z)/cmod\ z$
by (*simp add*: *complex-sgn-def divide-inverse*)

lemma *complex-inverse-complex-split*:
$inverse(complex\text{-}of\text{-}real\ x + ii * complex\text{-}of\text{-}real\ y) =$
$complex\text{-}of\text{-}real(x/(x\ \hat{}\ 2 + y\ \hat{}\ 2)) -$
$ii * complex\text{-}of\text{-}real(y/(x\ \hat{}\ 2 + y\ \hat{}\ 2))$
by (*simp add*: *complex-of-real-def i-def diff-minus divide-inverse*)

lemma *cos-arg-i-mult-zero-pos*:
$0 < y \mathrel{==>} cos\ (arg(Complex\ 0\ y)) = 0$
apply (*simp add*: *arg-def abs-if*)
apply (*rule-tac* $a = pi/2$ **in** *someI2*, *auto*)
apply (*rule order-less-trans* [*of - 0*], *auto*)
done

lemma *cos-arg-i-mult-zero-neg*:
$y < 0 \mathrel{==>} cos\ (arg(Complex\ 0\ y)) = 0$
apply (*simp add*: *arg-def abs-if*)
apply (*rule-tac* $a = -\ pi/2$ **in** *someI2*, *auto*)
apply (*rule order-trans* [*of - 0*], *auto*)
done

lemma *cos-arg-i-mult-zero* [*simp*]:
$y \neq 0 \mathrel{==>} cos\ (arg(Complex\ 0\ y)) = 0$
by (*auto simp add*: *linorder-neq-iff cos-arg-i-mult-zero-pos cos-arg-i-mult-zero-neg*)

20.12 Finally! Polar Form for Complex Numbers

definition

cis :: *real* => *complex* **where**
cis a = *Complex* (*cos a*) (*sin a*)

definition

rcis :: [*real*, *real*] => *complex* **where**
rcis r a = *complex-of-real r* ∗ *cis a*

definition

expi :: *complex* => *complex* **where**
expi z = *complex-of-real*(*exp* (*Re z*)) ∗ *cis* (*Im z*)

lemma *complex-split-polar*:
∃ *r a*. *z* = *complex-of-real r* ∗ (*Complex* (*cos a*) (*sin a*))
apply (*induct z*)
apply (*auto simp add*: *polar-Ex complex-of-real-mult-Complex*)
done

lemma *rcis-Ex*: ∃ *r a*. *z* = *rcis r a*
apply (*induct z*)
apply (*simp add*: *rcis-def cis-def polar-Ex complex-of-real-mult-Complex*)
done

lemma *Re-rcis* [*simp*]: *Re*(*rcis r a*) = *r* ∗ *cos a*
by (*simp add*: *rcis-def cis-def*)

lemma *Im-rcis* [*simp*]: *Im*(*rcis r a*) = *r* ∗ *sin a*
by (*simp add*: *rcis-def cis-def*)

lemma *sin-cos-squared-add2-mult*: $(r * cos\ a)^2 + (r * sin\ a)^2 = r^2$
proof −
have $(r * cos\ a)^2 + (r * sin\ a)^2 = r^2 * ((cos\ a)^2 + (sin\ a)^2)$
by (*simp only*: *power-mult-distrib right-distrib*)
thus *?thesis* **by** *simp*
qed

lemma *complex-mod-rcis* [*simp*]: *cmod*(*rcis r a*) = *abs r*
by (*simp add*: *rcis-def cis-def sin-cos-squared-add2-mult*)

lemma *complex-Re-cnj* [*simp*]: *Re*(*cnj z*) = *Re z*
by (*induct z*, *simp add*: *complex-cnj*)

lemma *complex-Im-cnj* [*simp*]: *Im*(*cnj z*) = − *Im z*
by (*induct z*, *simp add*: *complex-cnj*)

lemma *complex-mod-sqrt-Re-mult-cnj*: *cmod z* = *sqrt* (*Re* (*z* ∗ *cnj z*))
by (*simp add*: *cmod-def power2-eq-square*)

lemma *complex-In-mult-cnj-zero* [*simp*]: *Im* (*z* ∗ *cnj z*) = *0*
by *simp*

lemma *cis-rcis-eq*: *cis a* = *rcis 1 a*
by (*simp add*: *rcis-def*)

lemma *rcis-mult*: *rcis r1 a* ∗ *rcis r2 b* = *rcis* (*r1*∗*r2*) (*a* + *b*)
by (*simp add*: *rcis-def cis-def cos-add sin-add right-distrib right-diff-distrib complex-of-real-def*)

lemma *cis-mult*: *cis a* ∗ *cis b* = *cis* (*a* + *b*)
by (*simp add*: *cis-rcis-eq rcis-mult*)

lemma *cis-zero* [*simp*]: *cis 0* = *1*
by (*simp add*: *cis-def complex-one-def*)

lemma *rcis-zero-mod* [*simp*]: *rcis 0 a* = *0*
by (*simp add*: *rcis-def*)

lemma *rcis-zero-arg* [*simp*]: *rcis r 0* = *complex-of-real r*
by (*simp add*: *rcis-def*)

lemma *complex-of-real-minus-one*:
complex-of-real (−(*1*::*real*)) = −(*1*::*complex*)
by (*simp add*: *complex-of-real-def complex-one-def*)

lemma *complex-i-mult-minus* [*simp*]: *ii* ∗ (*ii* ∗ *x*) = − *x*
by (*simp add*: *mult-assoc* [*symmetric*])

lemma *cis-real-of-nat-Suc-mult*:
cis (*real* (*Suc n*) ∗ *a*) = *cis a* ∗ *cis* (*real n* ∗ *a*)
by (*simp add*: *cis-def real-of-nat-Suc left-distrib cos-add sin-add right-distrib*)

lemma *DeMoivre*: (*cis a*) ˆ *n* = *cis* (*real n* ∗ *a*)
apply (*induct-tac n*)
apply (*auto simp add*: *cis-real-of-nat-Suc-mult*)
done

lemma *DeMoivre2*: (*rcis r a*) ˆ *n* = *rcis* (*r* ˆ *n*) (*real n* ∗ *a*)
by (*simp add*: *rcis-def power-mult-distrib DeMoivre*)

lemma *cis-inverse* [*simp*]: *inverse*(*cis a*) = *cis* (−*a*)
by (*simp add*: *cis-def complex-inverse-complex-split diff-minus*)

lemma *rcis-inverse*: $inverse(rcis\ r\ a) = rcis\ (1/r)\ (-a)$
by (*simp add*: *divide-inverse rcis-def*)

lemma *cis-divide*: $cis\ a\ /\ cis\ b = cis\ (a - b)$
by (*simp add*: *complex-divide-def cis-mult real-diff-def*)

lemma *rcis-divide*: $rcis\ r1\ a\ /\ rcis\ r2\ b = rcis\ (r1/r2)\ (a - b)$
apply (*simp add*: *complex-divide-def*)
apply (*case-tac* $r2=0$, *simp*)
apply (*simp add*: *rcis-inverse rcis-mult real-diff-def*)
done

lemma *Re-cis* [*simp*]: $Re(cis\ a) = cos\ a$
by (*simp add*: *cis-def*)

lemma *Im-cis* [*simp*]: $Im(cis\ a) = sin\ a$
by (*simp add*: *cis-def*)

lemma *cos-n-Re-cis-pow-n*: $cos\ (real\ n * a) = Re(cis\ a\ \hat{}\ n)$
by (*auto simp add*: *DeMoivre*)

lemma *sin-n-Im-cis-pow-n*: $sin\ (real\ n * a) = Im(cis\ a\ \hat{}\ n)$
by (*auto simp add*: *DeMoivre*)

lemma *expi-add*: $expi(a + b) = expi(a) * expi(b)$
by (*simp add*: *expi-def exp-add cis-mult* [*symmetric*] *mult-ac*)

lemma *expi-zero* [*simp*]: $expi\ (0::complex) = 1$
by (*simp add*: *expi-def*)

lemma *complex-expi-Ex*: $\exists\, a\ r.\ z = complex\text{-}of\text{-}real\ r * expi\ a$
apply (*insert rcis-Ex* [*of z*])
apply (*auto simp add*: *expi-def rcis-def mult-assoc* [*symmetric*])
apply (*rule-tac* $x = ii * complex\text{-}of\text{-}real\ a$ **in** *exI*, *auto*)
done

lemma *expi-two-pi-i* [*simp*]: $expi((2::complex) * complex\text{-}of\text{-}real\ pi * ii) = 1$
by (*simp add*: *expi-def cis-def*)

end

21 Zorn: Zorn's Lemma

theory *Zorn*
imports *Main*

begin

The lemma and section numbers refer to an unpublished article [**?**].

definition
  chain :: *'a set set => 'a set set set* **where**
  chain S $= \{F.\ F \subseteq S\ \&\ (\forall x \in F.\ \forall y \in F.\ x \subseteq y \mid y \subseteq x)\}$

definition
  super :: *['a set set,'a set set] => 'a set set set* **where**
  super S c $= \{d.\ d \in$ *chain S* $\&\ c \subset d\}$

definition
  maxchain :: *'a set set => 'a set set set* **where**
  maxchain S $= \{c.\ c \in$ *chain S* $\&$ *super S c* $= \{\}\}$

definition
  succ :: *['a set set,'a set set] => 'a set set* **where**
  succ S c =
    (*if c* $\notin$ *chain S* $\mid c \in$ *maxchain S*
    then c else SOME c'. c' $\in$ *super S c*)

inductive-set
  TFin :: *'a set set => 'a set set set*
  for *S* :: *'a set set*
  where
    succI: $x \in$ *TFin S* ==> *succ S x* $\in$ *TFin S*
  | *Pow-UnionI*: $Y \in$ *Pow*(*TFin S*) ==> *Union*(Y) $\in$ *TFin S*
  monos *Pow-mono*

21.1 Mathematical Preamble

lemma *Union-lemma0*:
    $(\forall x \in C.\ x \subseteq A \mid B \subseteq x)$ ==> *Union*(C) $\subseteq A \mid B \subseteq$ *Union*(C)
  by *blast*

This is theorem *increasingD2* of ZF/Zorn.thy

lemma *Abrial-axiom1*: $x \subseteq$ *succ S x*
  apply (*unfold succ-def*)
  apply (*rule split-if* [*THEN iffD2*])
  apply (*auto simp add*: *super-def maxchain-def psubset-def*)
  apply (*rule contrapos-np*, *assumption*)
  apply (*rule someI2*, *blast+*)
  done

lemmas *TFin-UnionI* = *TFin.Pow-UnionI* [*OF PowI*]

lemma *TFin-induct*:
        [| $n \in$ *TFin S*;
          !!x. [| $x \in$ *TFin S*; $P(x)$ |] ==> P(*succ S x*);

```
        !!Y. [| Y ⊆ TFin S; Ball Y P |] ==> P(Union Y) |]
      ==> P(n)
  apply (induct set: TFin)
   apply blast+
  done
```

```
lemma succ-trans: x ⊆ y ==> x ⊆ succ S y
  apply (erule subset-trans)
  apply (rule Abrial-axiom1)
  done
```

Lemma 1 of section 3.1

```
lemma TFin-linear-lemma1:
     [| n ∈ TFin S;  m ∈ TFin S;
        ∀x ∈ TFin S. x ⊆ m --> x = m | succ S x ⊆ m
      |] ==> n ⊆ m | succ S m ⊆ n
  apply (erule TFin-induct)
   apply (erule-tac [2] Union-lemma0)
  apply (blast del: subsetI intro: succ-trans)
  done
```

Lemma 2 of section 3.2

```
lemma TFin-linear-lemma2:
     m ∈ TFin S ==> ∀n ∈ TFin S. n ⊆ m --> n=m | succ S n ⊆ m
  apply (erule TFin-induct)
   apply (rule impI [THEN ballI])
```

case split using *TFin-linear-lemma1*

```
   apply (rule-tac n1 = n and m1 = x in TFin-linear-lemma1 [THEN disjE],
     assumption+)
    apply (drule-tac x = n in bspec, assumption)
    apply (blast del: subsetI intro: succ-trans, blast)
```

second induction step

```
  apply (rule impI [THEN ballI])
  apply (rule Union-lemma0 [THEN disjE])
    apply (rule-tac [3] disjI2)
    prefer 2 apply blast
   apply (rule ballI)
   apply (rule-tac n1 = n and m1 = x in TFin-linear-lemma1 [THEN disjE],
     assumption+, auto)
  apply (blast intro!: Abrial-axiom1 [THEN subsetD])
  done
```

Re-ordering the premises of Lemma 2

```
lemma TFin-subsetD:
     [| n ⊆ m;  m ∈ TFin S;  n ∈ TFin S |] ==> n=m | succ S n ⊆ m
  by (rule TFin-linear-lemma2 [rule-format])
```

Consequences from section 3.3 – Property 3.2, the ordering is total

lemma *TFin-subset-linear*: $[| \ m \in TFin\ S; \ \ n \in TFin\ S|] ==> n \subseteq m \mid m \subseteq n$
  apply (*rule disjE*)
    apply (*rule TFin-linear-lemma1* [*OF - -TFin-linear-lemma2*])
      apply (*assumption+*, *erule disjI2*)
  apply (*blast del*: *subsetI*
    intro: *subsetI Abrial-axiom1* [*THEN subset-trans*])
  done

Lemma 3 of section 3.3

lemma *eq-succ-upper*: $[| \ n \in TFin\ S; \ \ m \in TFin\ S; \ \ m = succ\ S\ m \ |] ==> n \subseteq m$
  apply (*erule TFin-induct*)
   apply (*drule TFin-subsetD*)
    apply (*assumption+*, *force*, *blast*)
  done

Property 3.3 of section 3.3

lemma *equal-succ-Union*: $m \in TFin\ S ==> (m = succ\ S\ m) = (m = Union(TFin\ S))$
  apply (*rule iffI*)
   apply (*rule Union-upper* [*THEN equalityI*])
    apply *assumption*
   apply (*rule eq-succ-upper* [*THEN Union-least*], *assumption+*)
  apply (*erule ssubst*)
  apply (*rule Abrial-axiom1* [*THEN equalityI*])
  apply (*blast del*: *subsetI intro*: *subsetI TFin-UnionI TFin.succI*)
  done

21.2 Hausdorff's Theorem: Every Set Contains a Maximal Chain.

NB: We assume the partial ordering is ⊆, the subset relation!

lemma *empty-set-mem-chain*: $(\{\} :: 'a\ set\ set) \in chain\ S$
  by (*unfold chain-def*) *auto*

lemma *super-subset-chain*: $super\ S\ c \subseteq chain\ S$
  by (*unfold super-def*) *blast*

lemma *maxchain-subset-chain*: $maxchain\ S \subseteq chain\ S$
  by (*unfold maxchain-def*) *blast*

lemma *mem-super-Ex*: $c \in chain\ S - maxchain\ S ==> ?\ d.\ d \in super\ S\ c$
  by (*unfold super-def maxchain-def*) *auto*

lemma *select-super*:
    $c \in chain\ S - maxchain\ S ==> (\epsilon\, c'.\ c': super\ S\ c): super\ S\ c$
  apply (*erule mem-super-Ex* [*THEN exE*])

apply (*rule someI2*, *auto*)
done

lemma *select-not-equals*:
$c \in chain\ S - maxchain\ S ==> (\epsilon\, c'.\ c'\colon super\ S\ c) \neq c$
apply (*rule notI*)
apply (*drule select-super*)
apply (*simp add*: *super-def psubset-def*)
done

lemma *succI3*: $c \in chain\ S - maxchain\ S ==> succ\ S\ c = (\epsilon\, c'.\ c'\colon super\ S\ c)$
by (*unfold succ-def*) (*blast intro*!: *if-not-P*)

lemma *succ-not-equals*: $c \in chain\ S - maxchain\ S ==> succ\ S\ c \neq c$
apply (*frule succI3*)
apply (*simp* (*no-asm-simp*))
apply (*rule select-not-equals*, *assumption*)
done

lemma *TFin-chain-lemma4*: $c \in TFin\ S ==> (c :: 'a\ set\ set)\colon chain\ S$
apply (*erule TFin-induct*)
apply (*simp add*: *succ-def select-super* [*THEN super-subset-chain*[*THEN subsetD*]])
apply (*unfold chain-def*)
apply (*rule CollectI*, *safe*)
apply (*drule bspec*, *assumption*)
apply (*rule-tac* [2] *m1* = *Xa* **and** *n1* = *X* **in** *TFin-subset-linear* [*THEN disjE*], *blast*+)
done

theorem *Hausdorff*: $\exists\, c.\ (c :: 'a\ set\ set)\colon maxchain\ S$
apply (*rule-tac x* = *Union* (*TFin S*) **in** *exI*)
apply (*rule classical*)
apply (*subgoal-tac succ S* (*Union* (*TFin S*)) = *Union* (*TFin S*))
prefer *2*
apply (*blast intro*!: *TFin-UnionI equal-succ-Union* [*THEN iffD2*, *symmetric*])
apply (*cut-tac subset-refl* [*THEN TFin-UnionI*, *THEN TFin-chain-lemma4*])
apply (*drule DiffI* [*THEN succ-not-equals*], *blast*+)
done

21.3 Zorn's Lemma: If All Chains Have Upper Bounds Then There Is a Maximal Element

lemma *chain-extend*:
$[|\ c \in chain\ S;\ z \in S;$
$\forall\, x \in c.\ x \subseteq (z{::}\ 'a\ set)\ |] ==> \{z\}\ Un\ c \in chain\ S$
by (*unfold chain-def*) *blast*

lemma *chain-Union-upper*: $[|\ c \in chain\ S;\ x \in c\ |] ==> x \subseteq Union(c)$

```
  by (unfold chain-def) auto

lemma chain-ball-Union-upper: c ∈ chain S ==> ∀x ∈ c. x ⊆ Union(c)
  by (unfold chain-def) auto

lemma maxchain-Zorn:
     [| c ∈ maxchain S; u ∈ S; Union(c) ⊆ u |] ==> Union(c) = u
  apply (rule ccontr)
  apply (simp add: maxchain-def)
  apply (erule conjE)
  apply (subgoal-tac ({u} Un c) ∈ super S c)
   apply simp
  apply (unfold super-def psubset-def)
  apply (blast intro: chain-extend dest: chain-Union-upper)
  done

theorem Zorn-Lemma:
     ∀c ∈ chain S. Union(c): S ==> ∃y ∈ S. ∀z ∈ S. y ⊆ z --> y = z
  apply (cut-tac Hausdorff maxchain-subset-chain)
  apply (erule exE)
  apply (drule subsetD, assumption)
  apply (drule bspec, assumption)
  apply (rule-tac x = Union(c) in bexI)
   apply (rule ballI, rule impI)
   apply (blast dest!: maxchain-Zorn, assumption)
  done
```

21.4 Alternative version of Zorn's Lemma

```
lemma Zorn-Lemma2:
  ∀c ∈ chain S. ∃y ∈ S. ∀x ∈ c. x ⊆ y
    ==> ∃y ∈ S. ∀x ∈ S. (y :: 'a set) ⊆ x --> y = x
  apply (cut-tac Hausdorff maxchain-subset-chain)
  apply (erule exE)
  apply (drule subsetD, assumption)
  apply (drule bspec, assumption, erule bexE)
  apply (rule-tac x = y in bexI)
   prefer 2 apply assumption
  apply clarify
  apply (rule ccontr)
  apply (frule-tac z = x in chain-extend)
    apply (assumption, blast)
  apply (unfold maxchain-def super-def psubset-def)
  apply (blast elim!: equalityCE)
  done
```

Various other lemmas

```
lemma chainD: [| c ∈ chain S; x ∈ c; y ∈ c |] ==> x ⊆ y | y ⊆ x
  by (unfold chain-def) blast
```

lemma *chainD2*: $!!(c :: 'a\ set\ set).\ c \in chain\ S ==> c \subseteq S$
by (*unfold chain-def*) *blast*

end

22 Filter: Filters and Ultrafilters

theory *Filter*
imports *Zorn Infinite-Set*
begin

22.1 Definitions and basic properties

22.1.1 Filters

locale *filter* =
fixes $F :: 'a\ set\ set$
assumes *UNIV* [*iff*]: $UNIV \in F$
assumes *empty* [*iff*]: $\{\} \notin F$
assumes *Int*: $[\![u \in F;\ v \in F]\!] \Longrightarrow u \cap v \in F$
assumes *subset*: $[\![u \in F;\ u \subseteq v]\!] \Longrightarrow v \in F$

lemma (**in** *filter*) *memD*: $A \in F \Longrightarrow -\ A \notin F$
proof
assume $A \in F$ **and** $-\ A \in F$
hence $A \cap (-\ A) \in F$ **by** (*rule Int*)
thus *False* **by** *simp*
qed

lemma (**in** *filter*) *not-memI*: $-\ A \in F \Longrightarrow A \notin F$
by (*drule memD*, *simp*)

lemma (**in** *filter*) *Int-iff*: $(x \cap y \in F) = (x \in F \wedge y \in F)$
by (*auto elim*: *subset intro*: *Int*)

22.1.2 Ultrafilters

locale *ultrafilter* = *filter* +
assumes *ultra*: $A \in F \vee -\ A \in F$

lemma (**in** *ultrafilter*) *memI*: $-\ A \notin F \Longrightarrow A \in F$
by (*cut-tac ultra* [*of A*], *simp*)

lemma (**in** *ultrafilter*) *not-memD*: $A \notin F \Longrightarrow -\ A \in F$
by (*rule memI*, *simp*)

lemma (**in** *ultrafilter*) *not-mem-iff*: $(A \notin F) = (-\ A \in F)$

by (*rule iffI* [*OF not-memD not-memI*])

lemma (**in** *ultrafilter*) *Compl-iff*: $(- A \in F) = (A \notin F)$
by (*rule iffI* [*OF not-memI not-memD*])

lemma (**in** *ultrafilter*) *Un-iff*: $(x \cup y \in F) = (x \in F \vee y \in F)$
 apply (*rule iffI*)
  apply (*erule contrapos-pp*)
  apply (*simp add*: *Int-iff not-mem-iff*)
 apply (*auto elim*: *subset*)
done

22.1.3 Free Ultrafilters

locale *freeultrafilter* = *ultrafilter* +
  assumes *infinite*: $A \in F \Longrightarrow$ *infinite* A

lemma (**in** *freeultrafilter*) *finite*: *finite* $A \Longrightarrow A \notin F$
by (*erule contrapos-pn*, *erule infinite*)

lemma (**in** *freeultrafilter*) *singleton*: $\{x\} \notin F$
by (*rule finite*, *simp*)

lemma (**in** *freeultrafilter*) *insert-iff* [*simp*]: (*insert* x $A \in F) = (A \in F)$
apply (*subst insert-is-Un*)
apply (*subst Un-iff*)
apply (*simp add*: *singleton*)
done

lemma (**in** *freeultrafilter*) *filter*: *filter* F **by** *unfold-locales*

lemma (**in** *freeultrafilter*) *ultrafilter*: *ultrafilter* F
  by *unfold-locales*

22.2 Collect properties

lemma (**in** *filter*) *Collect-ex*:
  $(\{n.\ \exists x.\ P\ n\ x\} \in F) = (\exists X.\ \{n.\ P\ n\ (X\ n)\} \in F)$
proof
  assume $\{n.\ \exists x.\ P\ n\ x\} \in F$
  hence $\{n.\ P\ n\ (SOME\ x.\ P\ n\ x)\} \in F$
    by (*auto elim*: *someI subset*)
  thus $\exists X.\ \{n.\ P\ n\ (X\ n)\} \in F$ **by** *fast*
next
  show $\exists X.\ \{n.\ P\ n\ (X\ n)\} \in F \Longrightarrow \{n.\ \exists x.\ P\ n\ x\} \in F$
    by (*auto elim*: *subset*)
qed

lemma (**in** *filter*) *Collect-conj*:
  $(\{n.\ P\ n \wedge Q\ n\} \in F) = (\{n.\ P\ n\} \in F \wedge \{n.\ Q\ n\} \in F)$

by (*subst Collect-conj-eq*, *rule Int-iff*)

lemma (**in** *ultrafilter*) *Collect-not*:
$(\{n.\ \neg\ P\ n\} \in F) = (\{n.\ P\ n\} \notin F)$
by (*subst Collect-neg-eq*, *rule Compl-iff*)

lemma (**in** *ultrafilter*) *Collect-disj*:
$(\{n.\ P\ n \lor Q\ n\} \in F) = (\{n.\ P\ n\} \in F \lor \{n.\ Q\ n\} \in F)$
by (*subst Collect-disj-eq*, *rule Un-iff*)

lemma (**in** *ultrafilter*) *Collect-all*:
$(\{n.\ \forall x.\ P\ n\ x\} \in F) = (\forall X.\ \{n.\ P\ n\ (X\ n)\} \in F)$
apply (*rule Not-eq-iff* [*THEN iffD1*])
apply (*simp add*: *Collect-not* [*symmetric*])
apply (*rule Collect-ex*)
done

22.3 Maximal filter = Ultrafilter

A filter F is an ultrafilter iff it is a maximal filter, i.e. whenever G is a filter and $F \subseteq G$ then $F = G$

Lemmas that shows existence of an extension to what was assumed to be a maximal filter. Will be used to derive contradiction in proof of property of ultrafilter.

lemma *extend-lemma1*: $UNIV \in F \Longrightarrow A \in \{X.\ \exists f{\in}F.\ A \cap f \subseteq X\}$
by *blast*

lemma *extend-lemma2*: $F \subseteq \{X.\ \exists f{\in}F.\ A \cap f \subseteq X\}$
by *blast*

lemma (**in** *filter*) *extend-filter*:
assumes *A*: $-\ A \notin F$
shows *filter* $\{X.\ \exists f{\in}F.\ A \cap f \subseteq X\}$ (**is** *filter ?X*)
proof (*rule filter.intro*)
  show $UNIV \in ?X$ **by** *blast*
next
  show $\{\} \notin ?X$
  proof (*clarify*)
    fix *f* **assume** *f*: $f \in F$ **and** *Af*: $A \cap f \subseteq \{\}$
    from *Af* **have** *fA*: $f \subseteq -\ A$ **by** *blast*
    from *f fA* **have** $-\ A \in F$ **by** (*rule subset*)
    with *A* **show** *False* **by** *simp*
  qed
next
  fix *u* **and** *v*
  assume *u*: $u \in ?X$ **and** *v*: $v \in ?X$
  from *u* **obtain** *f* **where** *f*: $f \in F$ **and** *Af*: $A \cap f \subseteq u$ **by** *blast*
  from *v* **obtain** *g* **where** *g*: $g \in F$ **and** *Ag*: $A \cap g \subseteq v$ **by** *blast*

```
  from f g have fg: f ∩ g ∈ F by (rule Int)
  from Af Ag have Afg: A ∩ (f ∩ g) ⊆ u ∩ v by blast
  from fg Afg show u ∩ v ∈ ?X by blast
next
  fix u and v
  assume uv: u ⊆ v and u: u ∈ ?X
  from u obtain f where f: f ∈ F and Afu: A ∩ f ⊆ u by blast
  from Afu uv have Afv: A ∩ f ⊆ v by blast
  from f Afv have ∃f∈F. A ∩ f ⊆ v by blast
  thus v ∈ ?X by simp
qed

lemma (in filter) max-filter-ultrafilter:
assumes max: ⋀G. ⟦filter G; F ⊆ G⟧ ⟹ F = G
shows ultrafilter-axioms F
proof (rule ultrafilter-axioms.intro)
  fix A show A ∈ F ∨ − A ∈ F
  proof (rule disjCI)
    let ?X = {X. ∃f∈F. A ∩ f ⊆ X}
    assume AF: − A ∉ F
    from AF have X: filter ?X by (rule extend-filter)
    from UNIV have AX: A ∈ ?X by (rule extend-lemma1)
    have FX: F ⊆ ?X by (rule extend-lemma2)
    from X FX have F = ?X by (rule max)
    with AX show A ∈ F by simp
  qed
qed

lemma (in ultrafilter) max-filter:
assumes G: filter G and sub: F ⊆ G shows F = G
proof
  show F ⊆ G using sub .
  show G ⊆ F
  proof
    fix A assume A: A ∈ G
    from G A have − A ∉ G by (rule filter.memD)
    with sub have B: − A ∉ F by blast
    thus A ∈ F by (rule memI)
  qed
qed
```

22.4 Ultrafilter Theorem

A locale makes proof of ultrafilter Theorem more modular

```
locale (open) UFT =
  fixes  frechet :: 'a set set
  and     superfrechet :: 'a set set set

  assumes infinite-UNIV: infinite (UNIV :: 'a set)
```

defines *frechet-def*: *frechet* $\equiv$ {*A*. *finite* ($-$ *A*)}
and *superfrechet-def*: *superfrechet* $\equiv$ {*G*. *filter G* $\wedge$ *frechet* $\subseteq$ *G*}

lemma (**in** *UFT*) *superfrechetI*:
$[\![$*filter G*; *frechet* $\subseteq$ *G*$]\!] \Longrightarrow$ *G* $\in$ *superfrechet*
by (*simp add*: *superfrechet-def*)

lemma (**in** *UFT*) *superfrechetD1*:
G $\in$ *superfrechet* $\Longrightarrow$ *filter G*
by (*simp add*: *superfrechet-def*)

lemma (**in** *UFT*) *superfrechetD2*:
G $\in$ *superfrechet* $\Longrightarrow$ *frechet* $\subseteq$ *G*
by (*simp add*: *superfrechet-def*)

A few properties of free filters

lemma *filter-cofinite*:
assumes *inf*: *infinite* (*UNIV* :: *'a set*)
shows *filter* {*A*:: *'a set*. *finite* ($-$ *A*)} (**is** *filter ?F*)
proof (*rule filter.intro*)
show *UNIV* $\in$ *?F* **by** *simp*
next
show {} $\notin$ *?F* **using** *inf* **by** *simp*
next
fix *u v* **assume** *u* $\in$ *?F* **and** *v* $\in$ *?F*
thus *u* $\cap$ *v* $\in$ *?F* **by** *simp*
next
fix *u v* **assume** *uv*: *u* $\subseteq$ *v* **and** *u*: *u* $\in$ *?F*
from *uv* **have** *vu*: $-$ *v* $\subseteq$ $-$ *u* **by** *simp*
from *u* **show** *v* $\in$ *?F*
by (*simp add*: *finite-subset* [*OF vu*])
qed

We prove: 1. Existence of maximal filter i.e. ultrafilter; 2. Freeness property i.e ultrafilter is free. Use a locale to prove various lemmas and then export main result: The ultrafilter Theorem

lemma (**in** *UFT*) *filter-frechet*: *filter frechet*
by (*unfold frechet-def*, *rule filter-cofinite* [*OF infinite-UNIV*])

lemma (**in** *UFT*) *frechet-in-superfrechet*: *frechet* $\in$ *superfrechet*
by (*rule superfrechetI* [*OF filter-frechet subset-refl*])

lemma (**in** *UFT*) *lemma-mem-chain-filter*:
$[\![$*c* $\in$ *chain superfrechet*; *x* $\in$ *c*$]\!] \Longrightarrow$ *filter x*
by (*unfold chain-def superfrechet-def*, *blast*)

22.4.1 Unions of chains of superfrechets

In this section we prove that superfrechet is closed with respect to unions of non-empty chains. We must show 1) Union of a chain is a filter, 2) Union of a chain contains frechet.

Number 2 is trivial, but 1 requires us to prove all the filter rules.

lemma (**in** *UFT*) *Union-chain-UNIV*:
$[\![c \in \textit{chain superfrechet};\ c \neq \{\}]\!] \Longrightarrow \textit{UNIV} \in \bigcup c$
proof −
  assume *1*: $c \in$ *chain superfrechet* **and** *2*: $c \neq \{\}$
  from *2* **obtain** x **where** *3*: $x \in c$ **by** *blast*
  from *1 3* **have** *filter* x **by** (*rule lemma-mem-chain-filter*)
  hence $\textit{UNIV} \in x$ **by** (*rule filter.UNIV*)
  with *3* **show** $\textit{UNIV} \in \bigcup c$ **by** *blast*
qed

lemma (**in** *UFT*) *Union-chain-empty*:
$c \in \textit{chain superfrechet} \Longrightarrow \{\} \notin \bigcup c$
proof
  assume *1*: $c \in$ *chain superfrechet* **and** *2*: $\{\} \in \bigcup c$
  from *2* **obtain** x **where** *3*: $x \in c$ **and** *4*: $\{\} \in x$ **..**
  from *1 3* **have** *filter* x **by** (*rule lemma-mem-chain-filter*)
  hence $\{\} \notin x$ **by** (*rule filter.empty*)
  with *4* **show** *False* **by** *simp*
qed

lemma (**in** *UFT*) *Union-chain-Int*:
$[\![c \in \textit{chain superfrechet};\ u \in \bigcup c;\ v \in \bigcup c]\!] \Longrightarrow u \cap v \in \bigcup c$
proof −
  assume *c*: $c \in$ *chain superfrechet*
  assume $u \in \bigcup c$
    then obtain x **where** *ux*: $u \in x$ **and** *xc*: $x \in c$ **..**
  assume $v \in \bigcup c$
    then obtain y **where** *vy*: $v \in y$ **and** *yc*: $y \in c$ **..**
  from *c xc yc* **have** $x \subseteq y \lor y \subseteq x$ **by** (*rule chainD*)
  with *xc yc* **have** *xyc*: $x \cup y \in c$
    by (*auto simp add*: *Un-absorb1 Un-absorb2*)
  with *c* **have** *fxy*: *filter* $(x \cup y)$ **by** (*rule lemma-mem-chain-filter*)
  from *ux* **have** *uxy*: $u \in x \cup y$ **by** *simp*
  from *vy* **have** *vxy*: $v \in x \cup y$ **by** *simp*
  from *fxy uxy vxy* **have** $u \cap v \in x \cup y$ **by** (*rule filter.Int*)
  with *xyc* **show** $u \cap v \in \bigcup c$ **..**
qed

lemma (**in** *UFT*) *Union-chain-subset*:
$[\![c \in \textit{chain superfrechet};\ u \in \bigcup c;\ u \subseteq v]\!] \Longrightarrow v \in \bigcup c$
proof −
  assume *c*: $c \in$ *chain superfrechet*

and u: $u \in \bigcup c$ **and** uv: $u \subseteq v$
from u **obtain** x **where** ux: $u \in x$ **and** xc: $x \in c$ **..**
from c xc **have** fx: *filter* x **by** (*rule lemma-mem-chain-filter*)
from fx ux uv **have** vx: $v \in x$ **by** (*rule filter.subset*)
with xc **show** $v \in \bigcup c$ **..**
qed

lemma (**in** *UFT*) *Union-chain-filter*:
assumes *chain*: $c \in$ *chain superfrechet* **and** *nonempty*: $c \neq \{\}$
shows *filter* $(\bigcup c)$
proof (*rule filter.intro*)
show *UNIV* $\in \bigcup c$ **using** *chain nonempty* **by** (*rule Union-chain-UNIV*)
next
show $\{\} \notin \bigcup c$ **using** *chain* **by** (*rule Union-chain-empty*)
next
fix u v **assume** $u \in \bigcup c$ **and** $v \in \bigcup c$
with *chain* **show** $u \cap v \in \bigcup c$ **by** (*rule Union-chain-Int*)
next
fix u v **assume** $u \in \bigcup c$ **and** $u \subseteq v$
with *chain* **show** $v \in \bigcup c$ **by** (*rule Union-chain-subset*)
qed

lemma (**in** *UFT*) *lemma-mem-chain-frechet-subset*:
$[\![c \in$ *chain superfrechet*; $x \in c]\!] \Longrightarrow$ *frechet* $\subseteq x$
by (*unfold superfrechet-def chain-def*, *blast*)

lemma (**in** *UFT*) *Union-chain-superfrechet*:
$[\![c \neq \{\}; c \in$ *chain superfrechet*$]\!] \Longrightarrow \bigcup c \in$ *superfrechet*
proof (*rule superfrechetI*)
assume *1*: $c \in$ *chain superfrechet* **and** *2*: $c \neq \{\}$
thus *filter* $(\bigcup c)$ **by** (*rule Union-chain-filter*)
from *2* **obtain** x **where** *3*: $x \in c$ **by** *blast*
from *1 3* **have** *frechet* $\subseteq x$ **by** (*rule lemma-mem-chain-frechet-subset*)
also from *3* **have** $x \subseteq \bigcup c$ **by** *blast*
finally show *frechet* $\subseteq \bigcup c$ **.**
qed

22.4.2 Existence of free ultrafilter

lemma (**in** *UFT*) *max-cofinite-filter-Ex*:
$\exists\, U \in$*superfrechet*. $\forall\, G \in$*superfrechet*. $U \subseteq G \longrightarrow U = G$
proof (*rule Zorn-Lemma2* [*rule-format*])
fix c **assume** c: $c \in$ *chain superfrechet*
show $\exists\, U \in$*superfrechet*. $\forall\, G \in c$. $G \subseteq U$ (**is** *?U*)
proof (*cases*)
assume $c = \{\}$
with *frechet-in-superfrechet* **show** *?U* **by** *blast*
next
assume A: $c \neq \{\}$

```
    from A c have ⋃c ∈ superfrechet
      by (rule Union-chain-superfrechet)
    thus ?U by blast
  qed
qed

lemma (in UFT) mem-superfrechet-all-infinite:
  ⟦U ∈ superfrechet; A ∈ U⟧ ⟹ infinite A
proof
  assume U: U ∈ superfrechet and A: A ∈ U and fin: finite A
  from U have fil: filter U and fre: frechet ⊆ U
    by (simp-all add: superfrechet-def)
  from fin have − A ∈ frechet by (simp add: frechet-def)
  with fre have cA: − A ∈ U by (rule subsetD)
  from fil A cA have A ∩ − A ∈ U by (rule filter.Int)
  with fil show False by (simp add: filter.empty)
qed
```

There exists a free ultrafilter on any infinite set

```
lemma (in UFT) freeultrafilter-ex:
  ∃U::'a set set. freeultrafilter U
proof −
  from max-cofinite-filter-Ex obtain U
    where U: U ∈ superfrechet
      and max [rule-format]: ∀G∈superfrechet. U ⊆ G ⟶ U = G ..
  from U have fil: filter U by (rule superfrechetD1)
  from U have fre: frechet ⊆ U by (rule superfrechetD2)
  have ultra: ultrafilter-axioms U
  proof (rule filter.max-filter-ultrafilter [OF fil])
    fix G assume G: filter G and UG: U ⊆ G
    from fre UG have frechet ⊆ G by simp
    with G have G ∈ superfrechet by (rule superfrechetI)
    from this UG show U = G by (rule max)
  qed
  have free: freeultrafilter-axioms U
  proof (rule freeultrafilter-axioms.intro)
    fix A assume A ∈ U
    with U show infinite A by (rule mem-superfrechet-all-infinite)
  qed
  from fil ultra free have freeultrafilter U
    by (rule freeultrafilter.intro [OF ultrafilter.intro])

  thus ?thesis ..
qed

lemmas freeultrafilter-Ex = UFT.freeultrafilter-ex

hide (open) const filter
```

end

23 StarDef: Construction of Star Types Using Ultrafilters

theory *StarDef*
imports *Filter*
uses (*transfer.ML*)
begin

23.1 A Free Ultrafilter over the Naturals

definition
  FreeUltrafilterNat :: *nat set set* ($\mathcal{U}$) **where**
  $\mathcal{U}$ = (*SOME U. freeultrafilter U*)

lemma *freeultrafilter-FreeUltrafilterNat*: *freeultrafilter* $\mathcal{U}$
apply (*unfold FreeUltrafilterNat-def*)
apply (*rule someI-ex*)
apply (*rule freeultrafilter-Ex*)
apply (*rule nat-infinite*)
done

interpretation *FreeUltrafilterNat*: *freeultrafilter* [*FreeUltrafilterNat*]
by (*rule freeultrafilter-FreeUltrafilterNat*)

This rule takes the place of the old ultra tactic

lemma *ultra*:
  $[\![\{n.\ P\ n\} \in \mathcal{U}; \{n.\ P\ n \longrightarrow Q\ n\} \in \mathcal{U}]\!] \Longrightarrow \{n.\ Q\ n\} \in \mathcal{U}$
by (*simp add*: *Collect-imp-eq*
    FreeUltrafilterNat.Un-iff FreeUltrafilterNat.Compl-iff)

23.2 Definition of *star* type constructor

definition
  starrel :: $((nat \Rightarrow 'a) \times (nat \Rightarrow 'a))$ *set* **where**
  $starrel = \{(X,Y).\ \{n.\ X\ n = Y\ n\} \in \mathcal{U}\}$

typedef *'a star* = (*UNIV* :: $(nat \Rightarrow 'a)$ *set*) // *starrel*
by (*auto intro*: *quotientI*)

definition
  star-n :: $(nat \Rightarrow 'a) \Rightarrow 'a$ *star* **where**
  star-n X = *Abs-star* (*starrel* `` $\{X\}$)

theorem *star-cases* [*case-names star-n*, *cases type*: *star*]:
  $(\bigwedge X.\ x = star\text{-}n\ X \Longrightarrow P) \Longrightarrow P$

by (*cases* x, *unfold star-n-def star-def*, *erule quotientE*, *fast*)

lemma *all-star-eq*: $(\forall x.\ P\ x) = (\forall X.\ P\ (star\text{-}n\ X))$
by (*auto*, *rule-tac* $x{=}x$ **in** *star-cases*, *simp*)

lemma *ex-star-eq*: $(\exists x.\ P\ x) = (\exists X.\ P\ (star\text{-}n\ X))$
by (*auto*, *rule-tac* $x{=}x$ **in** *star-cases*, *auto*)

Proving that *starrel* is an equivalence relation

lemma *starrel-iff* [*iff*]: $((X,Y) \in starrel) = (\{n.\ X\ n = Y\ n\} \in \mathcal{U})$
by (*simp add*: *starrel-def*)

lemma *equiv-starrel*: *equiv UNIV starrel*
proof (*rule equiv.intro*)
  show *reflexive starrel* **by** (*simp add*: *refl-def*)
  show *sym starrel* **by** (*simp add*: *sym-def eq-commute*)
  show *trans starrel* **by** (*auto intro*: *transI elim*!: *ultra*)
qed

lemmas *equiv-starrel-iff* =
  eq-equiv-class-iff [*OF equiv-starrel UNIV-I UNIV-I*]

lemma *starrel-in-star*: $starrel``\{x\} \in star$
by (*simp add*: *star-def quotientI*)

lemma *star-n-eq-iff*: $(star\text{-}n\ X = star\text{-}n\ Y) = (\{n.\ X\ n = Y\ n\} \in \mathcal{U})$
by (*simp add*: *star-n-def Abs-star-inject starrel-in-star equiv-starrel-iff*)

23.3 Transfer principle

This introduction rule starts each transfer proof.

lemma *transfer-start*:
  $P \equiv \{n.\ Q\} \in \mathcal{U} \Longrightarrow Trueprop\ P \equiv Trueprop\ Q$
by (*subgoal-tac* $P \equiv Q$, *simp*, *simp add*: *atomize-eq*)

Initialize transfer tactic.

use *transfer.ML*
setup *Transfer.setup*

Transfer introduction rules.

lemma *transfer-ex* [*transfer-intro*]:
  $[\![\bigwedge X.\ p\ (star\text{-}n\ X) \equiv \{n.\ P\ n\ (X\ n)\} \in \mathcal{U}]\!]$
    $\Longrightarrow \exists x{::}'a\ star.\ p\ x \equiv \{n.\ \exists x.\ P\ n\ x\} \in \mathcal{U}$
by (*simp only*: *ex-star-eq FreeUltrafilterNat.Collect-ex*)

lemma *transfer-all* [*transfer-intro*]:
  $[\![\bigwedge X.\ p\ (star\text{-}n\ X) \equiv \{n.\ P\ n\ (X\ n)\} \in \mathcal{U}]\!]$
    $\Longrightarrow \forall x{::}'a\ star.\ p\ x \equiv \{n.\ \forall x.\ P\ n\ x\} \in \mathcal{U}$

by (*simp only*: *all-star-eq FreeUltrafilterNat.Collect-all*)

lemma *transfer-not* [*transfer-intro*]:
$[\![p \equiv \{n.\ P\ n\} \in \mathcal{U}]\!] \Longrightarrow \neg\ p \equiv \{n.\ \neg\ P\ n\} \in \mathcal{U}$
by (*simp only*: *FreeUltrafilterNat.Collect-not*)

lemma *transfer-conj* [*transfer-intro*]:
$[\![p \equiv \{n.\ P\ n\} \in \mathcal{U};\ q \equiv \{n.\ Q\ n\} \in \mathcal{U}]\!]$
$\Longrightarrow p \wedge q \equiv \{n.\ P\ n \wedge Q\ n\} \in \mathcal{U}$
by (*simp only*: *FreeUltrafilterNat.Collect-conj*)

lemma *transfer-disj* [*transfer-intro*]:
$[\![p \equiv \{n.\ P\ n\} \in \mathcal{U};\ q \equiv \{n.\ Q\ n\} \in \mathcal{U}]\!]$
$\Longrightarrow p \vee q \equiv \{n.\ P\ n \vee Q\ n\} \in \mathcal{U}$
by (*simp only*: *FreeUltrafilterNat.Collect-disj*)

lemma *transfer-imp* [*transfer-intro*]:
$[\![p \equiv \{n.\ P\ n\} \in \mathcal{U};\ q \equiv \{n.\ Q\ n\} \in \mathcal{U}]\!]$
$\Longrightarrow p \longrightarrow q \equiv \{n.\ P\ n \longrightarrow Q\ n\} \in \mathcal{U}$
by (*simp only*: *imp-conv-disj transfer-disj transfer-not*)

lemma *transfer-iff* [*transfer-intro*]:
$[\![p \equiv \{n.\ P\ n\} \in \mathcal{U};\ q \equiv \{n.\ Q\ n\} \in \mathcal{U}]\!]$
$\Longrightarrow p = q \equiv \{n.\ P\ n = Q\ n\} \in \mathcal{U}$
by (*simp only*: *iff-conv-conj-imp transfer-conj transfer-imp*)

lemma *transfer-if-bool* [*transfer-intro*]:
$[\![p \equiv \{n.\ P\ n\} \in \mathcal{U};\ x \equiv \{n.\ X\ n\} \in \mathcal{U};\ y \equiv \{n.\ Y\ n\} \in \mathcal{U}]\!]$
$\Longrightarrow (\mathit{if}\ p\ \mathit{then}\ x\ \mathit{else}\ y) \equiv \{n.\ \mathit{if}\ P\ n\ \mathit{then}\ X\ n\ \mathit{else}\ Y\ n\} \in \mathcal{U}$
by (*simp only*: *if-bool-eq-conj transfer-conj transfer-imp transfer-not*)

lemma *transfer-eq* [*transfer-intro*]:
$[\![x \equiv \mathit{star\text{-}n}\ X;\ y \equiv \mathit{star\text{-}n}\ Y]\!] \Longrightarrow x = y \equiv \{n.\ X\ n = Y\ n\} \in \mathcal{U}$
by (*simp only*: *star-n-eq-iff*)

lemma *transfer-if* [*transfer-intro*]:
$[\![p \equiv \{n.\ P\ n\} \in \mathcal{U};\ x \equiv \mathit{star\text{-}n}\ X;\ y \equiv \mathit{star\text{-}n}\ Y]\!]$
$\Longrightarrow (\mathit{if}\ p\ \mathit{then}\ x\ \mathit{else}\ y) \equiv \mathit{star\text{-}n}\ (\lambda n.\ \mathit{if}\ P\ n\ \mathit{then}\ X\ n\ \mathit{else}\ Y\ n)$
apply (*rule eq-reflection*)
apply (*auto simp add*: *star-n-eq-iff transfer-not elim*!: *ultra*)
done

lemma *transfer-fun-eq* [*transfer-intro*]:
$[\![\bigwedge X.\ f\ (\mathit{star\text{-}n}\ X) = g\ (\mathit{star\text{-}n}\ X)$
$\equiv \{n.\ F\ n\ (X\ n) = G\ n\ (X\ n)\} \in \mathcal{U}]\!]$
$\Longrightarrow f = g \equiv \{n.\ F\ n = G\ n\} \in \mathcal{U}$
by (*simp only*: *expand-fun-eq transfer-all*)

lemma *transfer-star-n* [*transfer-intro*]: $\mathit{star\text{-}n}\ X \equiv \mathit{star\text{-}n}\ (\lambda n.\ X\ n)$

by (*rule reflexive*)

lemma *transfer-bool* [*transfer-intro*]: $p \equiv \{n.\ p\} \in \mathcal{U}$
by (*simp add*: *atomize-eq*)

23.4 Standard elements

definition
star-of :: *'a* ⇒ *'a star* **where**
star-of x == *star-n* ($\lambda n.\ x$)

definition
Standard :: *'a star set* **where**
Standard = *range star-of*

Transfer tactic should remove occurrences of *star-of*

setup ⟪ *Transfer.add-const StarDef.star-of* ⟫

declare *star-of-def* [*transfer-intro*]

lemma *star-of-inject*: (*star-of* x = *star-of* y) = (x = y)
by (*transfer*, *rule refl*)

lemma *Standard-star-of* [*simp*]: *star-of* x ∈ *Standard*
by (*simp add*: *Standard-def*)

23.5 Internal functions

definition
Ifun :: (*'a* ⇒ *'b*) *star* ⇒ *'a star* ⇒ *'b star* (- ⋆ - [*300*,*301*] *300*) **where**
Ifun $f \equiv \lambda x.$ *Abs-star*
($\bigcup F \in$ *Rep-star* $f.$ $\bigcup X \in$ *Rep-star* $x.$ *starrel*''$\{\lambda n.\ F\ n\ (X\ n)\}$)

lemma *Ifun-congruent2*:
congruent2 starrel starrel ($\lambda F\ X.$ *starrel*''$\{\lambda n.\ F\ n\ (X\ n)\}$)
by (*auto simp add*: *congruent2-def equiv-starrel-iff elim*!: *ultra*)

lemma *Ifun-star-n*: *star-n* $F \star$ *star-n* X = *star-n* ($\lambda n.\ F\ n\ (X\ n)$)
by (*simp add*: *Ifun-def star-n-def Abs-star-inverse starrel-in-star*
UN-equiv-class2 [*OF equiv-starrel equiv-starrel Ifun-congruent2*])

Transfer tactic should remove occurrences of *Ifun*

setup ⟪ *Transfer.add-const StarDef.Ifun* ⟫

lemma *transfer-Ifun* [*transfer-intro*]:
⟦$f \equiv$ *star-n* F; $x \equiv$ *star-n* X⟧ ⟹ $f \star x \equiv$ *star-n* ($\lambda n.\ F\ n\ (X\ n)$)
by (*simp only*: *Ifun-star-n*)

lemma *Ifun-star-of* [*simp*]: *star-of* $f \star$ *star-of* x = *star-of* ($f\ x$)

by (*transfer*, *rule refl*)

lemma *Standard-Ifun* [*simp*]:
$[\![f \in \mathit{Standard}; x \in \mathit{Standard}]\!] \Longrightarrow f \star x \in \mathit{Standard}$
by (*auto simp add*: *Standard-def*)

Nonstandard extensions of functions

definition
starfun :: $('a \Rightarrow 'b) \Rightarrow ('a\ star \Rightarrow 'b\ star)$ ($*f*$ - [*80*] *80*) **where**
starfun $f == \lambda x.$ *star-of* $f \star x$

definition
starfun2 :: $('a \Rightarrow 'b \Rightarrow 'c) \Rightarrow ('a\ star \Rightarrow 'b\ star \Rightarrow 'c\ star)$
($*f2*$ - [*80*] *80*) **where**
starfun2 $f == \lambda x\ y.$ *star-of* $f \star x \star y$

declare *starfun-def* [*transfer-unfold*]
declare *starfun2-def* [*transfer-unfold*]

lemma *starfun-star-n*: $(*f* f)$ (*star-n* X) = *star-n* $(\lambda n.\ f\ (X\ n))$
by (*simp only*: *starfun-def star-of-def Ifun-star-n*)

lemma *starfun2-star-n*:
$(*f2* f)$ (*star-n* X) (*star-n* Y) = *star-n* $(\lambda n.\ f\ (X\ n)\ (Y\ n))$
by (*simp only*: *starfun2-def star-of-def Ifun-star-n*)

lemma *starfun-star-of* [*simp*]: $(*f* f)$ (*star-of* x) = *star-of* $(f\ x)$
by (*transfer*, *rule refl*)

lemma *starfun2-star-of* [*simp*]: $(*f2* f)$ (*star-of* x) = $*f*\ f\ x$
by (*transfer*, *rule refl*)

lemma *Standard-starfun* [*simp*]: $x \in \mathit{Standard} \Longrightarrow$ *starfun* $f\ x \in \mathit{Standard}$
by (*simp add*: *starfun-def*)

lemma *Standard-starfun2* [*simp*]:
$[\![x \in \mathit{Standard}; y \in \mathit{Standard}]\!] \Longrightarrow$ *starfun2* $f\ x\ y \in \mathit{Standard}$
by (*simp add*: *starfun2-def*)

lemma *Standard-starfun-iff*:
assumes *inj*: $\bigwedge x\ y.\ f\ x = f\ y \Longrightarrow x = y$
shows (*starfun* $f\ x \in \mathit{Standard}$) = $(x \in \mathit{Standard})$
proof
assume $x \in \mathit{Standard}$
thus *starfun* $f\ x \in \mathit{Standard}$ **by** *simp*
next
have *inj'*: $\bigwedge x\ y.$ *starfun* $f\ x$ = *starfun* $f\ y \Longrightarrow x = y$
using *inj* **by** *transfer*
assume *starfun* $f\ x \in \mathit{Standard}$

then obtain b **where** b: *starfun* $f\ x = $ *star-of* b
  unfolding *Standard-def* **..**
hence $\exists\, x.$ *starfun* $f\ x = $ *star-of* b **..**
hence $\exists\, a.\ f\ a = b$ **by** *transfer*
then obtain a **where** $f\ a = b$ **..**
hence *starfun* f (*star-of* a) = *star-of* b **by** *transfer*
with b **have** *starfun* $f\ x = $ *starfun* f (*star-of* a) **by** *simp*
hence $x = $ *star-of* a **by** (*rule inj'*)
thus $x \in$ *Standard*
  unfolding *Standard-def* **by** *auto*
qed

lemma *Standard-starfun2-iff*:
  assumes *inj*: $\bigwedge a\ b\ a'\ b'.\ f\ a\ b = f\ a'\ b' \Longrightarrow a = a' \wedge b = b'$
  shows (*starfun2* $f\ x\ y \in$ *Standard*) = ($x \in$ *Standard* $\wedge\ y \in$ *Standard*)
proof
  assume $x \in$ *Standard* $\wedge\ y \in$ *Standard*
  thus *starfun2* $f\ x\ y \in$ *Standard* **by** *simp*
next
  have *inj'*: $\bigwedge x\ y\ z\ w.$ *starfun2* $f\ x\ y = $ *starfun2* $f\ z\ w \Longrightarrow x = z \wedge y = w$
    using *inj* **by** *transfer*
  assume *starfun2* $f\ x\ y \in$ *Standard*
  then obtain c **where** c: *starfun2* $f\ x\ y = $ *star-of* c
    unfolding *Standard-def* **..**
  hence $\exists\, x\ y.$ *starfun2* $f\ x\ y = $ *star-of* c **by** *auto*
  hence $\exists\, a\ b.\ f\ a\ b = c$ **by** *transfer*
  then obtain $a\ b$ **where** $f\ a\ b = c$ **by** *auto*
  hence *starfun2* f (*star-of* a) (*star-of* b) = *star-of* c
    by *transfer*
  with c **have** *starfun2* $f\ x\ y = $ *starfun2* f (*star-of* a) (*star-of* b)
    by *simp*
  hence $x = $ *star-of* $a\ \wedge\ y = $ *star-of* b
    by (*rule inj'*)
  thus $x \in$ *Standard* $\wedge\ y \in$ *Standard*
    unfolding *Standard-def* **by** *auto*
qed

23.6 Internal predicates

definition
  unstar :: *bool star* $\Rightarrow$ *bool* **where**
  unstar $b = (b = $ *star-of True*)

lemma *unstar-star-n*: *unstar* (*star-n* P) = ($\{n.\ P\ n\} \in \mathcal{U}$)
by (*simp add*: *unstar-def star-of-def star-n-eq-iff*)

lemma *unstar-star-of* [*simp*]: *unstar* (*star-of* p) = p
by (*simp add*: *unstar-def star-of-inject*)

Transfer tactic should remove occurrences of *unstar*

setup ⟪ *Transfer.add-const StarDef.unstar* ⟫

lemma *transfer-unstar* [*transfer-intro*]:
$p \equiv \textit{star-n}\ P \Longrightarrow \textit{unstar}\ p \equiv \{n.\ P\ n\} \in \mathcal{U}$
by (*simp only*: *unstar-star-n*)

definition
$\textit{starP} :: (\prime a \Rightarrow \textit{bool}) \Rightarrow \prime a\ \textit{star} \Rightarrow \textit{bool}\ \ (*p* \text{ - } [80]\ 80)$ **where**
$*p*\ P = (\lambda x.\ \textit{unstar}\ (\textit{star-of}\ P \star x))$

definition
$\textit{starP2} :: (\prime a \Rightarrow \prime b \Rightarrow \textit{bool}) \Rightarrow \prime a\ \textit{star} \Rightarrow \prime b\ \textit{star} \Rightarrow \textit{bool}\ \ (*p2* \text{ - } [80]\ 80)$ **where**
$*p2*\ P = (\lambda x\ y.\ \textit{unstar}\ (\textit{star-of}\ P \star x \star y))$

declare *starP-def* [*transfer-unfold*]
declare *starP2-def* [*transfer-unfold*]

lemma *starP-star-n*: $(\ *p*\ P)\ (\textit{star-n}\ X) = (\{n.\ P\ (X\ n)\} \in \mathcal{U})$
by (*simp only*: *starP-def star-of-def Ifun-star-n unstar-star-n*)

lemma *starP2-star-n*:
$(\ *p2*\ P)\ (\textit{star-n}\ X)\ (\textit{star-n}\ Y) = (\{n.\ P\ (X\ n)\ (Y\ n)\} \in \mathcal{U})$
by (*simp only*: *starP2-def star-of-def Ifun-star-n unstar-star-n*)

lemma *starP-star-of* [*simp*]: $(\ *p*\ P)\ (\textit{star-of}\ x) = P\ x$
by (*transfer*, *rule refl*)

lemma *starP2-star-of* [*simp*]: $(\ *p2*\ P)\ (\textit{star-of}\ x) = *p*\ P\ x$
by (*transfer*, *rule refl*)

23.7 Internal sets

definition
$\textit{Iset} :: \prime a\ \textit{set star} \Rightarrow \prime a\ \textit{star set}$ **where**
$\textit{Iset}\ A = \{x.\ (\ *p2*\ \textit{op} \in)\ x\ A\}$

lemma *Iset-star-n*:
$(\textit{star-n}\ X \in \textit{Iset}\ (\textit{star-n}\ A)) = (\{n.\ X\ n \in A\ n\} \in \mathcal{U})$
by (*simp add*: *Iset-def starP2-star-n*)

Transfer tactic should remove occurrences of *Iset*

setup ⟪ *Transfer.add-const StarDef.Iset* ⟫

lemma *transfer-mem* [*transfer-intro*]:
$[\![x \equiv \textit{star-n}\ X;\ a \equiv \textit{Iset}\ (\textit{star-n}\ A)]\!]$
$\Longrightarrow x \in a \equiv \{n.\ X\ n \in A\ n\} \in \mathcal{U}$
by (*simp only*: *Iset-star-n*)

lemma *transfer-Collect* [*transfer-intro*]:

$[\![\bigwedge X.\ p\ (\textit{star-n}\ X) \equiv \{n.\ P\ n\ (X\ n)\} \in \mathcal{U}]\!]$
$\Longrightarrow \textit{Collect}\ p \equiv \textit{Iset}\ (\textit{star-n}\ (\lambda n.\ \textit{Collect}\ (P\ n)))$
by (*simp add*: *atomize-eq expand-set-eq all-star-eq Iset-star-n*)

lemma *transfer-set-eq* [*transfer-intro*]:
$[\![a \equiv \textit{Iset}\ (\textit{star-n}\ A);\ b \equiv \textit{Iset}\ (\textit{star-n}\ B)]\!]$
$\Longrightarrow a = b \equiv \{n.\ A\ n = B\ n\} \in \mathcal{U}$
by (*simp only*: *expand-set-eq transfer-all transfer-iff transfer-mem*)

lemma *transfer-ball* [*transfer-intro*]:
$[\![a \equiv \textit{Iset}\ (\textit{star-n}\ A);\ \bigwedge X.\ p\ (\textit{star-n}\ X) \equiv \{n.\ P\ n\ (X\ n)\} \in \mathcal{U}]\!]$
$\Longrightarrow \forall x{\in}a.\ p\ x \equiv \{n.\ \forall x{\in}A\ n.\ P\ n\ x\} \in \mathcal{U}$
by (*simp only*: *Ball-def transfer-all transfer-imp transfer-mem*)

lemma *transfer-bex* [*transfer-intro*]:
$[\![a \equiv \textit{Iset}\ (\textit{star-n}\ A);\ \bigwedge X.\ p\ (\textit{star-n}\ X) \equiv \{n.\ P\ n\ (X\ n)\} \in \mathcal{U}]\!]$
$\Longrightarrow \exists x{\in}a.\ p\ x \equiv \{n.\ \exists x{\in}A\ n.\ P\ n\ x\} \in \mathcal{U}$
by (*simp only*: *Bex-def transfer-ex transfer-conj transfer-mem*)

lemma *transfer-Iset* [*transfer-intro*]:
$[\![a \equiv \textit{star-n}\ A]\!] \Longrightarrow \textit{Iset}\ a \equiv \textit{Iset}\ (\textit{star-n}\ (\lambda n.\ A\ n))$
by *simp*

Nonstandard extensions of sets.

definition
starset :: $'a\ set \Rightarrow\ 'a\ star\ set$ ($*s*$ - [*80*] *80*) **where**
starset A = *Iset* (*star-of* A)

declare *starset-def* [*transfer-unfold*]

lemma *starset-mem*: (*star-of* $x \in *s*\ A) = (x \in A)$
by (*transfer*, *rule refl*)

lemma *starset-UNIV*: $*s*$ (*UNIV*::$'a$ *set*) = (*UNIV*::$'a$ *star set*)
by (*transfer UNIV-def*, *rule refl*)

lemma *starset-empty*: $*s*\ \{\} = \{\}$
by (*transfer empty-def*, *rule refl*)

lemma *starset-insert*: $*s*$ (*insert* x A) = *insert* (*star-of* x) ($*s*\ A$)
by (*transfer insert-def Un-def*, *rule refl*)

lemma *starset-Un*: $*s*\ (A \cup B) = *s*\ A \cup *s*\ B$
by (*transfer Un-def*, *rule refl*)

lemma *starset-Int*: $*s*\ (A \cap B) = *s*\ A \cap *s*\ B$
by (*transfer Int-def*, *rule refl*)

lemma *starset-Compl*: $*s*\ {-A} = -(\ *s*\ A)$

by (*transfer Compl-def*, *rule refl*)

lemma *starset-diff*: $*s*\ (A - B) = *s*\ A - *s*\ B$
by (*transfer set-diff-def*, *rule refl*)

lemma *starset-image*: $*s*\ (f\ `\ A) = (\ *f*\ f)\ `\ (\ *s*\ A)$
by (*transfer image-def*, *rule refl*)

lemma *starset-vimage*: $*s*\ (f\ -`\ A) = (\ *f*\ f)\ -`\ (\ *s*\ A)$
by (*transfer vimage-def*, *rule refl*)

lemma *starset-subset*: $(\ *s*\ A \subseteq *s*\ B) = (A \subseteq B)$
by (*transfer subset-def*, *rule refl*)

lemma *starset-eq*: $(\ *s*\ A = *s*\ B) = (A = B)$
by (*transfer*, *rule refl*)

lemmas *starset-simps* [*simp*] =
starset-mem *starset-UNIV*
starset-empty *starset-insert*
starset-Un *starset-Int*
starset-Compl *starset-diff*
starset-image *starset-vimage*
starset-subset *starset-eq*

end

24 StarClasses: Class Instances

theory *StarClasses*
imports *StarDef*
begin

24.1 Syntactic classes

instance *star* :: (*zero*) *zero*
star-zero-def: $0 \equiv$ *star-of* 0 **..**

instance *star* :: (*one*) *one*
star-one-def: $1 \equiv$ *star-of* 1 **..**

instance *star* :: (*plus*) *plus*
star-add-def: $(op\ +) \equiv *f2*\ (op\ +)$ **..**

instance *star* :: (*times*) *times*
star-mult-def: $(op\ *) \equiv *f2*\ (op\ *)$ **..**

instance *star* :: (*minus*) *minus*

star-minus-def: *uminus* $\equiv$ **f* uminus*
star-diff-def: $(op\ -) \equiv *f2* (op\ -)$ **..**

instance *star* :: (*abs*) *abs*
star-abs-def: *abs* $\equiv$ **f* abs* **..**

instance *star* :: (*sgn*) *sgn*
star-sgn-def: *sgn* $\equiv$ **f* sgn* **..**

instance *star* :: (*inverse*) *inverse*
star-divide-def: $(op\ /) \equiv *f2* (op\ /)$
star-inverse-def: *inverse* $\equiv$ **f* inverse* **..**

instance *star* :: (*number*) *number*
star-number-def: *number-of b* $\equiv$ *star-of* (*number-of b*) **..**

instance *star* :: (*Divides.div*) *Divides.div*
star-div-def: (*op div*) $\equiv$ **f2** (*op div*)
star-mod-def: (*op mod*) $\equiv$ **f2** (*op mod*) **..**

instance *star* :: (*power*) *power*
star-power-def: $(op\ \hat{}) \equiv \lambda x\ n.\ (*f* (\lambda x.\ x\ \hat{}\ n))\ x$ **..**

instance *star* :: (*ord*) *ord*
star-le-def: $(op \leq) \equiv *p2* (op \leq)$
star-less-def: $(op <) \equiv *p2* (op <)$ **..**

lemmas *star-class-defs* [*transfer-unfold*] =
star-zero-def *star-one-def* *star-number-def*
star-add-def *star-diff-def* *star-minus-def*
star-mult-def *star-divide-def* *star-inverse-def*
star-le-def *star-less-def* *star-abs-def* *star-sgn-def*
star-div-def *star-mod-def* *star-power-def*

Class operations preserve standard elements

lemma *Standard-zero*: $0 \in$ *Standard*
by (*simp add*: *star-zero-def*)

lemma *Standard-one*: $1 \in$ *Standard*
by (*simp add*: *star-one-def*)

lemma *Standard-number-of*: *number-of b* $\in$ *Standard*
by (*simp add*: *star-number-def*)

lemma *Standard-add*: $[\![x \in Standard;\ y \in Standard]\!] \Longrightarrow x + y \in Standard$
by (*simp add*: *star-add-def*)

lemma *Standard-diff*: $[\![x \in Standard;\ y \in Standard]\!] \Longrightarrow x - y \in Standard$
by (*simp add*: *star-diff-def*)

lemma *Standard-minus*: $x \in Standard \Longrightarrow -\ x \in Standard$
by (*simp add*: *star-minus-def*)

lemma *Standard-mult*: $[\![x \in Standard;\ y \in Standard]\!] \Longrightarrow x * y \in Standard$
by (*simp add*: *star-mult-def*)

lemma *Standard-divide*: $[\![x \in Standard;\ y \in Standard]\!] \Longrightarrow x\ /\ y \in Standard$
by (*simp add*: *star-divide-def*)

lemma *Standard-inverse*: $x \in Standard \Longrightarrow inverse\ x \in Standard$
by (*simp add*: *star-inverse-def*)

lemma *Standard-abs*: $x \in Standard \Longrightarrow abs\ x \in Standard$
by (*simp add*: *star-abs-def*)

lemma *Standard-div*: $[\![x \in Standard;\ y \in Standard]\!] \Longrightarrow x\ div\ y \in Standard$
by (*simp add*: *star-div-def*)

lemma *Standard-mod*: $[\![x \in Standard;\ y \in Standard]\!] \Longrightarrow x\ mod\ y \in Standard$
by (*simp add*: *star-mod-def*)

lemma *Standard-power*: $x \in Standard \Longrightarrow x\ \hat{}\ n \in Standard$
by (*simp add*: *star-power-def*)

lemmas *Standard-simps* [*simp*] =
Standard-zero *Standard-one* *Standard-number-of*
Standard-add *Standard-diff* *Standard-minus*
Standard-mult *Standard-divide* *Standard-inverse*
Standard-abs *Standard-div* *Standard-mod*
Standard-power

star-of preserves class operations

lemma *star-of-add*: *star-of* $(x + y) =$ *star-of* $x +$ *star-of* y
by *transfer* (*rule refl*)

lemma *star-of-diff*: *star-of* $(x - y) =$ *star-of* $x -$ *star-of* y
by *transfer* (*rule refl*)

lemma *star-of-minus*: *star-of* $(-x) = -$ *star-of* x
by *transfer* (*rule refl*)

lemma *star-of-mult*: *star-of* $(x * y) =$ *star-of* $x *$ *star-of* y
by *transfer* (*rule refl*)

lemma *star-of-divide*: *star-of* $(x\ /\ y) =$ *star-of* $x\ /$ *star-of* y
by *transfer* (*rule refl*)

lemma *star-of-inverse*: *star-of* (*inverse* x) = *inverse* (*star-of* x)

by *transfer* (*rule refl*)

lemma *star-of-div*: *star-of* (*x div y*) = *star-of x div star-of y*
by *transfer* (*rule refl*)

lemma *star-of-mod*: *star-of* (*x mod y*) = *star-of x mod star-of y*
by *transfer* (*rule refl*)

lemma *star-of-power*: *star-of* (*x* ˆ *n*) = *star-of x* ˆ *n*
by *transfer* (*rule refl*)

lemma *star-of-abs*: *star-of* (*abs x*) = *abs* (*star-of x*)
by *transfer* (*rule refl*)

star-of preserves numerals

lemma *star-of-zero*: *star-of 0* = *0*
by *transfer* (*rule refl*)

lemma *star-of-one*: *star-of 1* = *1*
by *transfer* (*rule refl*)

lemma *star-of-number-of*: *star-of* (*number-of x*) = *number-of x*
by *transfer* (*rule refl*)

star-of preserves orderings

lemma *star-of-less*: (*star-of x* < *star-of y*) = (*x* < *y*)
by *transfer* (*rule refl*)

lemma *star-of-le*: (*star-of x* ≤ *star-of y*) = (*x* ≤ *y*)
by *transfer* (*rule refl*)

lemma *star-of-eq*: (*star-of x* = *star-of y*) = (*x* = *y*)
by *transfer* (*rule refl*)

As above, for 0

lemmas *star-of-0-less* = *star-of-less* [*of 0*, *simplified star-of-zero*]
lemmas *star-of-0-le* = *star-of-le* [*of 0*, *simplified star-of-zero*]
lemmas *star-of-0-eq* = *star-of-eq* [*of 0*, *simplified star-of-zero*]

lemmas *star-of-less-0* = *star-of-less* [*of - 0*, *simplified star-of-zero*]
lemmas *star-of-le-0* = *star-of-le* [*of - 0*, *simplified star-of-zero*]
lemmas *star-of-eq-0* = *star-of-eq* [*of - 0*, *simplified star-of-zero*]

As above, for 1

lemmas *star-of-1-less* = *star-of-less* [*of 1*, *simplified star-of-one*]
lemmas *star-of-1-le* = *star-of-le* [*of 1*, *simplified star-of-one*]
lemmas *star-of-1-eq* = *star-of-eq* [*of 1*, *simplified star-of-one*]

lemmas *star-of-less-1* = *star-of-less* [*of* - *1*, *simplified star-of-one*]
lemmas *star-of-le-1* = *star-of-le* [*of* - *1*, *simplified star-of-one*]
lemmas *star-of-eq-1* = *star-of-eq* [*of* - *1*, *simplified star-of-one*]

As above, for numerals

lemmas *star-of-number-less* =
star-of-less [*of number-of w*, *standard*, *simplified star-of-number-of*]
lemmas *star-of-number-le* =
star-of-le [*of number-of w*, *standard*, *simplified star-of-number-of*]
lemmas *star-of-number-eq* =
star-of-eq [*of number-of w*, *standard*, *simplified star-of-number-of*]

lemmas *star-of-less-number* =
star-of-less [*of* - *number-of w*, *standard*, *simplified star-of-number-of*]
lemmas *star-of-le-number* =
star-of-le [*of* - *number-of w*, *standard*, *simplified star-of-number-of*]
lemmas *star-of-eq-number* =
star-of-eq [*of* - *number-of w*, *standard*, *simplified star-of-number-of*]

lemmas *star-of-simps* [*simp*] =
star-of-add star-of-diff star-of-minus
star-of-mult star-of-divide star-of-inverse
star-of-div star-of-mod
star-of-power star-of-abs
star-of-zero star-of-one star-of-number-of
star-of-less star-of-le star-of-eq
star-of-0-less star-of-0-le star-of-0-eq
star-of-less-0 star-of-le-0 star-of-eq-0
star-of-1-less star-of-1-le star-of-1-eq
star-of-less-1 star-of-le-1 star-of-eq-1
star-of-number-less star-of-number-le star-of-number-eq
star-of-less-number star-of-le-number star-of-eq-number

24.2 Ordering and lattice classes

instance *star* :: (*order*) *order*
apply (*intro-classes*)
apply (*transfer*, *rule order-less-le*)
apply (*transfer*, *rule order-refl*)
apply (*transfer*, *erule* (*1*) *order-trans*)
apply (*transfer*, *erule* (*1*) *order-antisym*)
done

instance *star* :: (*lower-semilattice*) *lower-semilattice*
star-inf-def [*transfer-unfold*]: *inf* ≡ **f2** *inf*
by *default* (*transfer star-inf-def*, *auto*)+

instance *star* :: (*upper-semilattice*) *upper-semilattice*
star-sup-def [*transfer-unfold*]: *sup* ≡ **f2** *sup*

by *default* (*transfer star-sup-def*, *auto*)+

instance *star* :: (*lattice*) *lattice* **..**

instance *star* :: (*distrib-lattice*) *distrib-lattice*
by *default* (*transfer*, *auto simp add*: *sup-inf-distrib1*)

lemma *Standard-inf* [*simp*]:
$[\![x \in \mathit{Standard};\ y \in \mathit{Standard}]\!] \Longrightarrow \mathit{inf}\ x\ y \in \mathit{Standard}$
by (*simp add*: *star-inf-def*)

lemma *Standard-sup* [*simp*]:
$[\![x \in \mathit{Standard};\ y \in \mathit{Standard}]\!] \Longrightarrow \mathit{sup}\ x\ y \in \mathit{Standard}$
by (*simp add*: *star-sup-def*)

lemma *star-of-inf* [*simp*]: *star-of* (*inf x y*) = *inf* (*star-of x*) (*star-of y*)
by *transfer* (*rule refl*)

lemma *star-of-sup* [*simp*]: *star-of* (*sup x y*) = *sup* (*star-of x*) (*star-of y*)
by *transfer* (*rule refl*)

instance *star* :: (*linorder*) *linorder*
by (*intro-classes*, *transfer*, *rule linorder-linear*)

lemma *star-max-def* [*transfer-unfold*]: *max* = **f2* max*
apply (*rule ext*, *rule ext*)
apply (*unfold max-def*, *transfer*, *fold max-def*)
apply (*rule refl*)
done

lemma *star-min-def* [*transfer-unfold*]: *min* = **f2* min*
apply (*rule ext*, *rule ext*)
apply (*unfold min-def*, *transfer*, *fold min-def*)
apply (*rule refl*)
done

lemma *Standard-max* [*simp*]:
$[\![x \in \mathit{Standard};\ y \in \mathit{Standard}]\!] \Longrightarrow \mathit{max}\ x\ y \in \mathit{Standard}$
by (*simp add*: *star-max-def*)

lemma *Standard-min* [*simp*]:
$[\![x \in \mathit{Standard};\ y \in \mathit{Standard}]\!] \Longrightarrow \mathit{min}\ x\ y \in \mathit{Standard}$
by (*simp add*: *star-min-def*)

lemma *star-of-max* [*simp*]: *star-of* (*max x y*) = *max* (*star-of x*) (*star-of y*)
by *transfer* (*rule refl*)

lemma *star-of-min* [*simp*]: *star-of* (*min x y*) = *min* (*star-of x*) (*star-of y*)
by *transfer* (*rule refl*)

24.3 Ordered group classes

instance *star* :: (*semigroup-add*) *semigroup-add*
by (*intro-classes*, *transfer*, *rule add-assoc*)

instance *star* :: (*ab-semigroup-add*) *ab-semigroup-add*
by (*intro-classes*, *transfer*, *rule add-commute*)

instance *star* :: (*semigroup-mult*) *semigroup-mult*
by (*intro-classes*, *transfer*, *rule mult-assoc*)

instance *star* :: (*ab-semigroup-mult*) *ab-semigroup-mult*
by (*intro-classes*, *transfer*, *rule mult-commute*)

instance *star* :: (*comm-monoid-add*) *comm-monoid-add*
by (*intro-classes*, *transfer*, *rule comm-monoid-add-class.zero-plus.add-0*)

instance *star* :: (*monoid-mult*) *monoid-mult*
apply (*intro-classes*)
apply (*transfer*, *rule mult-1-left*)
apply (*transfer*, *rule mult-1-right*)
done

instance *star* :: (*comm-monoid-mult*) *comm-monoid-mult*
by (*intro-classes*, *transfer*, *rule mult-1*)

instance *star* :: (*cancel-semigroup-add*) *cancel-semigroup-add*
apply (*intro-classes*)
apply (*transfer*, *erule add-left-imp-eq*)
apply (*transfer*, *erule add-right-imp-eq*)
done

instance *star* :: (*cancel-ab-semigroup-add*) *cancel-ab-semigroup-add*
by (*intro-classes*, *transfer*, *rule add-imp-eq*)

instance *star* :: (*ab-group-add*) *ab-group-add*
apply (*intro-classes*)
apply (*transfer*, *rule left-minus*)
apply (*transfer*, *rule diff-minus*)
done

instance *star* :: (*pordered-ab-semigroup-add*) *pordered-ab-semigroup-add*
by (*intro-classes*, *transfer*, *rule add-left-mono*)

instance *star* :: (*pordered-cancel-ab-semigroup-add*) *pordered-cancel-ab-semigroup-add*
..

instance *star* :: (*pordered-ab-semigroup-add-imp-le*) *pordered-ab-semigroup-add-imp-le*
by (*intro-classes*, *transfer*, *rule add-le-imp-le-left*)

instance *star* :: (*pordered-comm-monoid-add*) *pordered-comm-monoid-add* **..**
instance *star* :: (*pordered-ab-group-add*) *pordered-ab-group-add* **..**

instance *star* :: (*pordered-ab-group-add-abs*) *pordered-ab-group-add-abs*
  by *intro-classes* (*transfer*,
    simp add: *abs-ge-self abs-leI abs-triangle-ineq*)+

instance *star* :: (*ordered-cancel-ab-semigroup-add*) *ordered-cancel-ab-semigroup-add*
..
instance *star* :: (*lordered-ab-group-add-meet*) *lordered-ab-group-add-meet* **..**
instance *star* :: (*lordered-ab-group-add-meet*) *lordered-ab-group-add-meet* **..**
instance *star* :: (*lordered-ab-group-add*) *lordered-ab-group-add* **..**

instance *star* :: (*lordered-ab-group-add-abs*) *lordered-ab-group-add-abs*
by (*intro-classes*, *transfer*, *rule abs-lattice*)

24.4 Ring and field classes

instance *star* :: (*semiring*) *semiring*
apply (*intro-classes*)
apply (*transfer*, *rule left-distrib*)
apply (*transfer*, *rule right-distrib*)
done

instance *star* :: (*semiring-0*) *semiring-0*
by *intro-classes* (*transfer*, *simp*)+

instance *star* :: (*semiring-0-cancel*) *semiring-0-cancel* **..**

instance *star* :: (*comm-semiring*) *comm-semiring*
by (*intro-classes*, *transfer*, *rule left-distrib*)

instance *star* :: (*comm-semiring-0*) *comm-semiring-0* **..**
instance *star* :: (*comm-semiring-0-cancel*) *comm-semiring-0-cancel* **..**

instance *star* :: (*zero-neq-one*) *zero-neq-one*
by (*intro-classes*, *transfer*, *rule zero-neq-one*)

instance *star* :: (*semiring-1*) *semiring-1* **..**
instance *star* :: (*comm-semiring-1*) *comm-semiring-1* **..**

instance *star* :: (*no-zero-divisors*) *no-zero-divisors*
by (*intro-classes*, *transfer*, *rule no-zero-divisors*)

instance *star* :: (*semiring-1-cancel*) *semiring-1-cancel* **..**
instance *star* :: (*comm-semiring-1-cancel*) *comm-semiring-1-cancel* **..**
instance *star* :: (*ring*) *ring* **..**
instance *star* :: (*comm-ring*) *comm-ring* **..**
instance *star* :: (*ring-1*) *ring-1* **..**

instance *star* :: (*comm-ring-1*) *comm-ring-1* **..**
instance *star* :: (*ring-no-zero-divisors*) *ring-no-zero-divisors* **..**
instance *star* :: (*ring-1-no-zero-divisors*) *ring-1-no-zero-divisors* **..**
instance *star* :: (*idom*) *idom* **..**

instance *star* :: (*division-ring*) *division-ring*
apply (*intro-classes*)
apply (*transfer*, *erule left-inverse*)
apply (*transfer*, *erule right-inverse*)
done

instance *star* :: (*field*) *field*
apply (*intro-classes*)
apply (*transfer*, *erule left-inverse*)
apply (*transfer*, *rule divide-inverse*)
done

instance *star* :: (*division-by-zero*) *division-by-zero*
by (*intro-classes*, *transfer*, *rule inverse-zero*)

instance *star* :: (*pordered-semiring*) *pordered-semiring*
apply (*intro-classes*)
apply (*transfer*, *erule* (*1*) *mult-left-mono*)
apply (*transfer*, *erule* (*1*) *mult-right-mono*)
done

instance *star* :: (*pordered-cancel-semiring*) *pordered-cancel-semiring* **..**

instance *star* :: (*ordered-semiring-strict*) *ordered-semiring-strict*
apply (*intro-classes*)
apply (*transfer*, *erule* (*1*) *mult-strict-left-mono*)
apply (*transfer*, *erule* (*1*) *mult-strict-right-mono*)
done

instance *star* :: (*pordered-comm-semiring*) *pordered-comm-semiring*
by (*intro-classes*, *transfer*, *rule mult-mono1-class.times-zero-less-eq-less.mult-mono1*)

instance *star* :: (*pordered-cancel-comm-semiring*) *pordered-cancel-comm-semiring*
..

instance *star* :: (*ordered-comm-semiring-strict*) *ordered-comm-semiring-strict*
by (*intro-classes*, *transfer*, *rule ordered-comm-semiring-strict-class.plus-times-zero-less-eq-less.mult-strict-mon*

instance *star* :: (*pordered-ring*) *pordered-ring* **..**
instance *star* :: (*pordered-ring-abs*) *pordered-ring-abs*
  by *intro-classes* (*transfer*, *rule abs-eq-mult*)
instance *star* :: (*lordered-ring*) *lordered-ring* **..**

instance *star* :: (*abs-if*) *abs-if*

by (*intro-classes*, *transfer*, *rule abs-if*)

instance *star* :: (*sgn-if*) *sgn-if*
by (*intro-classes*, *transfer*, *rule sgn-if*)

instance *star* :: (*ordered-ring-strict*) *ordered-ring-strict* **..**
instance *star* :: (*pordered-comm-ring*) *pordered-comm-ring* **..**

instance *star* :: (*ordered-semidom*) *ordered-semidom*
by (*intro-classes*, *transfer*, *rule zero-less-one*)

instance *star* :: (*ordered-idom*) *ordered-idom* **..**
instance *star* :: (*ordered-field*) *ordered-field* **..**

24.5 Power classes

Proving the class axiom *power-Suc* for type *'a star* is a little tricky, because it quantifies over values of type *nat*. The transfer principle does not handle quantification over non-star types in general, but we can work around this by fixing an arbitrary *nat* value, and then applying the transfer principle.

instance *star* :: (*recpower*) *recpower*
proof
  show $\bigwedge a::'a\ star.\ a\ \hat{}\ 0 = 1$
    by *transfer* (*rule power-0*)
next
  fix *n* **show** $\bigwedge a::'a\ star.\ a\ \hat{}\ Suc\ n = a * a\ \hat{}\ n$
    by *transfer* (*rule power-Suc*)
qed

24.6 Number classes

lemma *star-of-nat-def* [*transfer-unfold*]: *of-nat n* = *star-of* (*of-nat n*)
by (*induct n*, *simp-all*)

lemma *Standard-of-nat* [*simp*]: *of-nat n* ∈ *Standard*
by (*simp add*: *star-of-nat-def*)

lemma *star-of-of-nat* [*simp*]: *star-of* (*of-nat n*) = *of-nat n*
by *transfer* (*rule refl*)

lemma *star-of-int-def* [*transfer-unfold*]: *of-int z* = *star-of* (*of-int z*)
by (*rule-tac z=z* **in** *int-diff-cases*, *simp*)

lemma *Standard-of-int* [*simp*]: *of-int z* ∈ *Standard*
by (*simp add*: *star-of-int-def*)

lemma *star-of-of-int* [*simp*]: *star-of* (*of-int z*) = *of-int z*
by *transfer* (*rule refl*)

instance *star* :: (*semiring-char-0*) *semiring-char-0*
by *intro-classes* (*simp only*: *star-of-nat-def star-of-eq of-nat-eq-iff*)

instance *star* :: (*ring-char-0*) *ring-char-0* **..**

instance *star* :: (*number-ring*) *number-ring*
by (*intro-classes*, *simp only*: *star-number-def star-of-int-def number-of-eq*)

24.7 Finite class

lemma *starset-finite*: *finite* $A \Longrightarrow *s*\ A = star\text{-}of\ `\ A$
by (*erule finite-induct*, *simp-all*)

instance *star* :: (*finite*) *finite*
apply (*intro-classes*)
apply (*subst starset-UNIV* [*symmetric*])
apply (*subst starset-finite* [*OF finite*])
apply (*rule finite-imageI* [*OF finite*])
done

end

25 HyperNat: Hypernatural numbers

theory *HyperNat*
imports *StarClasses*
begin

types *hypnat* = *nat star*

abbreviation
hypnat-of-nat :: *nat* => *nat star* **where**
hypnat-of-nat == *star-of*

definition
hSuc :: *hypnat* => *hypnat* **where**
hSuc-def [*transfer-unfold*]: *hSuc* = $*f*$ *Suc*

25.1 Properties Transferred from Naturals

lemma *hSuc-not-zero* [*iff*]: $\bigwedge m.\ hSuc\ m \neq 0$
by *transfer* (*rule Suc-not-Zero*)

lemma *zero-not-hSuc* [*iff*]: $\bigwedge m.\ 0 \neq hSuc\ m$
by *transfer* (*rule Zero-not-Suc*)

lemma *hSuc-hSuc-eq* [*iff*]: $\bigwedge m\ n.\ (hSuc\ m = hSuc\ n) = (m = n)$

by *transfer* (*rule Suc-Suc-eq*)

lemma *zero-less-hSuc* [*iff*]: $\bigwedge n.\ 0 < hSuc\ n$
by *transfer* (*rule zero-less-Suc*)

lemma *hypnat-minus-zero* [*simp*]: $!!z.\ z - z = (0::hypnat)$
by *transfer* (*rule diff-self-eq-0*)

lemma *hypnat-diff-0-eq-0* [*simp*]: $!!n.\ (0::hypnat) - n = 0$
by *transfer* (*rule diff-0-eq-0*)

lemma *hypnat-add-is-0* [*iff*]: $!!m\ n.\ (m+n = (0::hypnat)) = (m=0\ \&\ n=0)$
by *transfer* (*rule add-is-0*)

lemma *hypnat-diff-diff-left*: $!!i\ j\ k.\ (i::hypnat) - j - k = i - (j+k)$
by *transfer* (*rule diff-diff-left*)

lemma *hypnat-diff-commute*: $!!i\ j\ k.\ (i::hypnat) - j - k = i{-}k{-}j$
by *transfer* (*rule diff-commute*)

lemma *hypnat-diff-add-inverse* [*simp*]: $!!m\ n.\ ((n::hypnat) + m) - n = m$
by *transfer* (*rule diff-add-inverse*)

lemma *hypnat-diff-add-inverse2* [*simp*]: $!!m\ n.\ ((m::hypnat) + n) - n = m$
by *transfer* (*rule diff-add-inverse2*)

lemma *hypnat-diff-cancel* [*simp*]: $!!k\ m\ n.\ ((k::hypnat) + m) - (k+n) = m - n$
by *transfer* (*rule diff-cancel*)

lemma *hypnat-diff-cancel2* [*simp*]: $!!k\ m\ n.\ ((m::hypnat) + k) - (n+k) = m - n$
by *transfer* (*rule diff-cancel2*)

lemma *hypnat-diff-add-0* [*simp*]: $!!m\ n.\ (n::hypnat) - (n+m) = (0::hypnat)$
by *transfer* (*rule diff-add-0*)

lemma *hypnat-diff-mult-distrib*: $!!k\ m\ n.\ ((m::hypnat) - n) * k = (m * k) - (n * k)$
by *transfer* (*rule diff-mult-distrib*)

lemma *hypnat-diff-mult-distrib2*: $!!k\ m\ n.\ (k::hypnat) * (m - n) = (k * m) - (k * n)$
by *transfer* (*rule diff-mult-distrib2*)

lemma *hypnat-le-zero-cancel* [*iff*]: $!!n.\ (n \le (0::hypnat)) = (n = 0)$
by *transfer* (*rule le-0-eq*)

lemma *hypnat-mult-is-0* [*simp*]: $!!m\ n.\ (m*n = (0::hypnat)) = (m=0 \mid n=0)$
by *transfer* (*rule mult-is-0*)

lemma *hypnat-diff-is-0-eq* [*simp*]: !!m n. ((m::*hypnat*) − n = *0*) = (m ≤ n)
by *transfer* (*rule diff-is-0-eq*)

lemma *hypnat-not-less0* [*iff*]: !!n. ~ n < (*0*::*hypnat*)
by *transfer* (*rule not-less0*)

lemma *hypnat-less-one* [*iff*]:
  !!n. (n < (*1*::*hypnat*)) = (n=*0*)
by *transfer* (*rule less-one*)

lemma *hypnat-add-diff-inverse*: !!m n. ~ m<n ==> n+(m−n) = (m::*hypnat*)
by *transfer* (*rule add-diff-inverse*)

lemma *hypnat-le-add-diff-inverse* [*simp*]: !!m n. n ≤ m ==> n+(m−n) = (m::*hypnat*)
by *transfer* (*rule le-add-diff-inverse*)

lemma *hypnat-le-add-diff-inverse2* [*simp*]: !!m n. n≤m ==> (m−n)+n = (m::*hypnat*)
by *transfer* (*rule le-add-diff-inverse2*)

declare *hypnat-le-add-diff-inverse2* [*OF order-less-imp-le*]

lemma *hypnat-le0* [*iff*]: !!n. (*0*::*hypnat*) ≤ n
by *transfer* (*rule le0*)

lemma *hypnat-le-add1* [*simp*]: !!x n. (x::*hypnat*) ≤ x + n
by *transfer* (*rule le-add1*)

lemma *hypnat-add-self-le* [*simp*]: !!x n. (x::*hypnat*) ≤ n + x
by *transfer* (*rule le-add2*)

lemma *hypnat-add-one-self-less* [*simp*]: (x::*hypnat*) < x + (*1*::*hypnat*)
by (*insert add-strict-left-mono* [*OF zero-less-one*], *auto*)

lemma *hypnat-neq0-conv* [*iff*]: !!n. (n ≠ *0*) = (*0* < (n::*hypnat*))
by *transfer* (*rule neq0-conv*)

lemma *hypnat-gt-zero-iff*: ((*0*::*hypnat*) < n) = ((*1*::*hypnat*) ≤ n)
by (*auto simp add*: *linorder-not-less* [*symmetric*])

lemma *hypnat-gt-zero-iff2*: (*0* < n) = (∃ m. n = m + (*1*::*hypnat*))
apply *safe*
 apply (*rule-tac* x = n − (*1*::*hypnat*) **in** *exI*)
 apply (*simp add*: *hypnat-gt-zero-iff*)
apply (*insert add-le-less-mono* [*OF - zero-less-one*, *of 0*], *auto*)
done

lemma *hypnat-add-self-not-less*: ~ (x + y < (x::*hypnat*))
by (*simp add*: *linorder-not-le* [*symmetric*] *add-commute* [*of x*])

lemma *hypnat-diff-split*:
P(*a* − *b*::*hypnat*) = ((*a*<*b* −−> *P 0*) & (*ALL d*. *a* = *b* + *d* −−> *P d*))
— elimination of − on *hypnat*
proof (*cases a*<*b rule*: *case-split*)
case *True*
thus *?thesis*
by (*auto simp add*: *hypnat-add-self-not-less order-less-imp-le*
hypnat-diff-is-0-eq [*THEN iffD2*])
next
case *False*
thus *?thesis*
by (*auto simp add*: *linorder-not-less dest*: *order-le-less-trans*)
qed

25.2 Properties of the set of embedded natural numbers

lemma *of-nat-eq-star-of* [*simp*]: *of-nat* = *star-of*
proof
fix *n* :: *nat*
show *of-nat n* = *star-of n* **by** *transfer simp*
qed

lemma *Nats-eq-Standard*: (*Nats* :: *nat star set*) = *Standard*
by (*auto simp add*: *Nats-def Standard-def*)

lemma *hypnat-of-nat-mem-Nats* [*simp*]: *hypnat-of-nat n* ∈ *Nats*
by (*simp add*: *Nats-eq-Standard*)

lemma *hypnat-of-nat-one* [*simp*]: *hypnat-of-nat* (*Suc 0*) = (*1*::*hypnat*)
by *transfer simp*

lemma *hypnat-of-nat-Suc* [*simp*]:
hypnat-of-nat (*Suc n*) = *hypnat-of-nat n* + (*1*::*hypnat*)
by *transfer simp*

lemma *of-nat-eq-add* [*rule-format*]:
∀ *d*::*hypnat*. *of-nat m* = *of-nat n* + *d* −−> *d* ∈ *range of-nat*
apply (*induct n*)
apply (*auto simp add*: *add-assoc*)
apply (*case-tac x*)
apply (*auto simp add*: *add-commute* [*of 1*])
done

lemma *Nats-diff* [*simp*]: [|*a* ∈ *Nats*; *b* ∈ *Nats*|] ==> (*a*−*b* :: *hypnat*) ∈ *Nats*
by (*simp add*: *Nats-eq-Standard*)

25.3 Infinite Hypernatural Numbers – *HNatInfinite*

definition

HNatInfinite :: *hypnat set* **where**
HNatInfinite = $\{n.\ n \notin Nats\}$

lemma *Nats-not-HNatInfinite-iff*: $(x \in Nats) = (x \notin HNatInfinite)$
by (*simp add*: *HNatInfinite-def*)

lemma *HNatInfinite-not-Nats-iff*: $(x \in HNatInfinite) = (x \notin Nats)$
by (*simp add*: *HNatInfinite-def*)

lemma *star-of-neq-HNatInfinite*: $N \in HNatInfinite \Longrightarrow$ *star-of* $n \neq N$
by (*auto simp add*: *HNatInfinite-def Nats-eq-Standard*)

lemma *star-of-Suc-lessI*:
$\bigwedge N.$ ⟦*star-of* $n < N$; *star-of* (*Suc* n) $\neq N$⟧ $\Longrightarrow$ *star-of* (*Suc* n) $< N$
by *transfer* (*rule Suc-lessI*)

lemma *star-of-less-HNatInfinite*:
assumes N: $N \in HNatInfinite$
shows *star-of* $n < N$
proof (*induct n*)
case *0*
from N **have** *star-of* $0 \neq N$ **by** (*rule star-of-neq-HNatInfinite*)
thus *star-of* $0 < N$ **by** *simp*
next
case (*Suc n*)
from N **have** *star-of* (*Suc* n) $\neq N$ **by** (*rule star-of-neq-HNatInfinite*)
with *Suc* **show** *star-of* (*Suc* n) $< N$ **by** (*rule star-of-Suc-lessI*)
qed

lemma *star-of-le-HNatInfinite*: $N \in HNatInfinite \Longrightarrow$ *star-of* $n \leq N$
by (*rule star-of-less-HNatInfinite* [*THEN order-less-imp-le*])

25.3.1 Closure Rules

lemma *Nats-less-HNatInfinite*: ⟦$x \in Nats$; $y \in HNatInfinite$⟧ $\Longrightarrow x < y$
by (*auto simp add*: *Nats-def star-of-less-HNatInfinite*)

lemma *Nats-le-HNatInfinite*: ⟦$x \in Nats$; $y \in HNatInfinite$⟧ $\Longrightarrow x \leq y$
by (*rule Nats-less-HNatInfinite* [*THEN order-less-imp-le*])

lemma *zero-less-HNatInfinite*: $x \in HNatInfinite \Longrightarrow 0 < x$
by (*simp add*: *Nats-less-HNatInfinite*)

lemma *one-less-HNatInfinite*: $x \in HNatInfinite \Longrightarrow 1 < x$
by (*simp add*: *Nats-less-HNatInfinite*)

lemma *one-le-HNatInfinite*: $x \in HNatInfinite \Longrightarrow 1 \leq x$
by (*simp add*: *Nats-le-HNatInfinite*)

lemma *zero-not-mem-HNatInfinite* [*simp*]: $0 \notin HNatInfinite$
by (*simp add*: *HNatInfinite-def*)

lemma *Nats-downward-closed*:
$[\![x \in Nats; (y::hypnat) \leq x]\!] \Longrightarrow y \in Nats$
apply (*simp only*: *linorder-not-less* [*symmetric*])
apply (*erule contrapos-np*)
apply (*drule HNatInfinite-not-Nats-iff* [*THEN iffD2*])
apply (*erule* (*1*) *Nats-less-HNatInfinite*)
done

lemma *HNatInfinite-upward-closed*:
$[\![x \in HNatInfinite; x \leq y]\!] \Longrightarrow y \in HNatInfinite$
apply (*simp only*: *HNatInfinite-not-Nats-iff*)
apply (*erule contrapos-nn*)
apply (*erule* (*1*) *Nats-downward-closed*)
done

lemma *HNatInfinite-add*: $x \in HNatInfinite \Longrightarrow x + y \in HNatInfinite$
apply (*erule HNatInfinite-upward-closed*)
apply (*rule hypnat-le-add1*)
done

lemma *HNatInfinite-add-one*: $x \in HNatInfinite \Longrightarrow x + 1 \in HNatInfinite$
by (*rule HNatInfinite-add*)

lemma *HNatInfinite-diff*:
$[\![x \in HNatInfinite; y \in Nats]\!] \Longrightarrow x - y \in HNatInfinite$
apply (*frule* (*1*) *Nats-le-HNatInfinite*)
apply (*simp only*: *HNatInfinite-not-Nats-iff*)
apply (*erule contrapos-nn*)
apply (*drule* (*1*) *Nats-add*, *simp*)
done

lemma *HNatInfinite-is-Suc*: $x \in HNatInfinite$ ==> $\exists y.\ x = y + (1::hypnat)$
apply (*rule-tac* $x = x - (1::hypnat)$ **in** *exI*)
apply (*simp add*: *Nats-le-HNatInfinite*)
done

25.4 Existence of an infinite hypernatural number

definition

whn :: *hypnat* **where**
hypnat-omega-def: *whn* = *star-n* (%*n*::*nat*. *n*)

lemma *hypnat-of-nat-neq-whn*: *hypnat-of-nat* $n \neq$ *whn*
by (*simp add*: *hypnat-omega-def star-of-def star-n-eq-iff*)

lemma *whn-neq-hypnat-of-nat*: $whn \neq hypnat\text{-}of\text{-}nat\ n$
by (*simp add*: *hypnat-omega-def star-of-def star-n-eq-iff*)

lemma *whn-not-Nats* [*simp*]: $whn \notin Nats$
by (*simp add*: *Nats-def image-def whn-neq-hypnat-of-nat*)

lemma *HNatInfinite-whn* [*simp*]: $whn \in HNatInfinite$
by (*simp add*: *HNatInfinite-def*)

lemma *lemma-unbounded-set* [*simp*]: $\{n::nat.\ m < n\} \in FreeUltrafilterNat$
apply (*insert finite-atMost* [*of m*])
apply (*simp add*: *atMost-def*)
apply (*drule FreeUltrafilterNat.finite*)
apply (*drule FreeUltrafilterNat.not-memD*)
apply (*simp add*: *Collect-neg-eq* [*symmetric*] *linorder-not-le*)
done

lemma *Compl-Collect-le*: $- \{n::nat.\ N \leq n\} = \{n.\ n < N\}$
by (*simp add*: *Collect-neg-eq* [*symmetric*] *linorder-not-le*)

lemma *hypnat-of-nat-eq*:
$hypnat\text{-}of\text{-}nat\ m\ = star\text{-}n\ (\%n::nat.\ m)$
by (*simp add*: *star-of-def*)

lemma *SHNat-eq*: $Nats = \{n.\ \exists N.\ n = hypnat\text{-}of\text{-}nat\ N\}$
by (*simp add*: *Nats-def image-def*)

lemma *Nats-less-whn*: $n \in Nats \Longrightarrow n < whn$
by (*simp add*: *Nats-less-HNatInfinite*)

lemma *Nats-le-whn*: $n \in Nats \Longrightarrow n \leq whn$
by (*simp add*: *Nats-le-HNatInfinite*)

lemma *hypnat-of-nat-less-whn* [*simp*]: $hypnat\text{-}of\text{-}nat\ n < whn$
by (*simp add*: *Nats-less-whn*)

lemma *hypnat-of-nat-le-whn* [*simp*]: $hypnat\text{-}of\text{-}nat\ n \leq whn$
by (*simp add*: *Nats-le-whn*)

lemma *hypnat-zero-less-hypnat-omega* [*simp*]: $0 < whn$
by (*simp add*: *Nats-less-whn*)

lemma *hypnat-one-less-hypnat-omega* [*simp*]: $1 < whn$
by (*simp add*: *Nats-less-whn*)

25.4.1 Alternative characterization of the set of infinite hypernaturals

$HNatInfinite = \{N.\ \forall n{\in}\mathbb{N}.\ n < N\}$

lemma *HNatInfinite-FreeUltrafilterNat-lemma*:
  $\forall N{::}nat.\ \{n.\ f\ n \neq N\} \in FreeUltrafilterNat$
  $==> \{n.\ N < f\ n\} \in FreeUltrafilterNat$
apply (*induct-tac N*)
apply (*drule-tac x = 0* **in** *spec*, *simp*)
apply (*drule-tac x = Suc n* **in** *spec*)
apply (*elim ultra*, *auto*)
done

lemma *HNatInfinite-iff*: $HNatInfinite = \{N.\ \forall n \in Nats.\ n < N\}$
apply (*safe intro*!: *Nats-less-HNatInfinite*)
apply (*auto simp add*: *HNatInfinite-def*)
done

25.4.2 Alternative Characterization of *HNatInfinite* using Free Ultrafilter

lemma *HNatInfinite-FreeUltrafilterNat*:
  $star\text{-}n\ X \in HNatInfinite ==> \forall u.\ \{n.\ u < X\ n\}:\ FreeUltrafilterNat$
apply (*auto simp add*: *HNatInfinite-iff SHNat-eq*)
apply (*drule-tac x=star-of u* **in** *spec*, *simp*)
apply (*simp add*: *star-of-def star-less-def starP2-star-n*)
done

lemma *FreeUltrafilterNat-HNatInfinite*:
  $\forall u.\ \{n.\ u < X\ n\}:\ FreeUltrafilterNat ==> star\text{-}n\ X \in HNatInfinite$
by (*auto simp add*: *star-less-def starP2-star-n HNatInfinite-iff SHNat-eq hypnat-of-nat-eq*)

lemma *HNatInfinite-FreeUltrafilterNat-iff*:
  $(star\text{-}n\ X \in HNatInfinite) = (\forall u.\ \{n.\ u < X\ n\}:\ FreeUltrafilterNat)$
by (*rule iffI* [*OF HNatInfinite-FreeUltrafilterNat*
  FreeUltrafilterNat-HNatInfinite])

25.5 Embedding of the Hypernaturals into other types

definition
  of-hypnat :: *hypnat* ⇒ *'a::semiring-1-cancel star* **where**
  of-hypnat-def [*transfer-unfold*]: $of\text{-}hypnat = {*}f{*}\ of\text{-}nat$

lemma *of-hypnat-0* [*simp*]: $of\text{-}hypnat\ 0 = 0$
by *transfer* (*rule of-nat-0*)

lemma *of-hypnat-1* [*simp*]: $of\text{-}hypnat\ 1 = 1$
by *transfer* (*rule of-nat-1*)

lemma *of-hypnat-hSuc*: $\bigwedge m.\ of\text{-}hypnat\ (hSuc\ m) = 1 + of\text{-}hypnat\ m$
by *transfer* (*rule of-nat-Suc*)

lemma *of-hypnat-add* [*simp*]:
  $\bigwedge m\ n.\ of\text{-}hypnat\ (m + n) = of\text{-}hypnat\ m + of\text{-}hypnat\ n$

by *transfer* (*rule of-nat-add*)

lemma *of-hypnat-mult* [*simp*]:
$\bigwedge m\ n.\ \textit{of-hypnat}\ (m * n) = \textit{of-hypnat}\ m * \textit{of-hypnat}\ n$
by *transfer* (*rule of-nat-mult*)

lemma *of-hypnat-less-iff* [*simp*]:
$\bigwedge m\ n.\ (\textit{of-hypnat}\ m < (\textit{of-hypnat}\ n{::}'a{::}\textit{ordered-semidom star})) = (m < n)$
by *transfer* (*rule of-nat-less-iff*)

lemma *of-hypnat-0-less-iff* [*simp*]:
$\bigwedge n.\ (0 < (\textit{of-hypnat}\ n{::}'a{::}\textit{ordered-semidom star})) = (0 < n)$
by *transfer* (*rule of-nat-0-less-iff*)

lemma *of-hypnat-less-0-iff* [*simp*]:
$\bigwedge m.\ \neg\ (\textit{of-hypnat}\ m{::}'a{::}\textit{ordered-semidom star}) < 0$
by *transfer* (*rule of-nat-less-0-iff*)

lemma *of-hypnat-le-iff* [*simp*]:
$\bigwedge m\ n.\ (\textit{of-hypnat}\ m \le (\textit{of-hypnat}\ n{::}'a{::}\textit{ordered-semidom star})) = (m \le n)$
by *transfer* (*rule of-nat-le-iff*)

lemma *of-hypnat-0-le-iff* [*simp*]:
$\bigwedge n.\ 0 \le (\textit{of-hypnat}\ n{::}'a{::}\textit{ordered-semidom star})$
by *transfer* (*rule of-nat-0-le-iff*)

lemma *of-hypnat-le-0-iff* [*simp*]:
$\bigwedge m.\ ((\textit{of-hypnat}\ m{::}'a{::}\textit{ordered-semidom star}) \le 0) = (m = 0)$
by *transfer* (*rule of-nat-le-0-iff*)

lemma *of-hypnat-eq-iff* [*simp*]:
$\bigwedge m\ n.\ (\textit{of-hypnat}\ m = (\textit{of-hypnat}\ n{::}'a{::}\textit{ordered-semidom star})) = (m = n)$
by *transfer* (*rule of-nat-eq-iff*)

lemma *of-hypnat-eq-0-iff* [*simp*]:
$\bigwedge m.\ ((\textit{of-hypnat}\ m{::}'a{::}\textit{ordered-semidom star}) = 0) = (m = 0)$
by *transfer* (*rule of-nat-eq-0-iff*)

lemma *HNatInfinite-of-hypnat-gt-zero*:
$N \in \textit{HNatInfinite} \Longrightarrow (0{::}'a{::}\textit{ordered-semidom star}) < \textit{of-hypnat}\ N$
by (*rule ccontr*, *simp add*: *linorder-not-less*)

end

26 HyperDef: Construction of Hyperreals Using Ultrafilters

theory *HyperDef*
imports *HyperNat ../Real/Real*
uses (*hypreal-arith.ML*)
begin

types *hypreal* = *real star*

abbreviation
hypreal-of-real :: *real* => *real star* **where**
hypreal-of-real == *star-of*

abbreviation
hypreal-of-hypnat :: *hypnat* ⇒ *hypreal* **where**
hypreal-of-hypnat ≡ *of-hypnat*

definition
omega :: *hypreal* **where**
— an infinite number = [<*1*,*2*,*3*,...>]
omega = *star-n* (λ*n*. *real* (*Suc n*))

definition
epsilon :: *hypreal* **where**
— an infinitesimal number = [<*1*,*1*/*2*,*1*/*3*,...>]
epsilon = *star-n* (λ*n*. *inverse* (*real* (*Suc n*)))

notation (*xsymbols*)
omega (ω) **and**
epsilon (ε)

notation (*HTML* **output**)
omega (ω) **and**
epsilon (ε)

26.1 Real vector class instances

instance *star* :: (*scaleR*) *scaleR* ..

defs (**overloaded**)
star-scaleR-def [*transfer-unfold*]: *scaleR r* ≡ **f** (*scaleR r*)

lemma *Standard-scaleR* [*simp*]: *x* ∈ *Standard* ⟹ *scaleR r x* ∈ *Standard*
by (*simp add*: *star-scaleR-def*)

lemma *star-of-scaleR* [*simp*]: *star-of* (*scaleR r x*) = *scaleR r* (*star-of x*)
by *transfer* (*rule refl*)

instance *star* :: (*real-vector*) *real-vector*
proof
  fix *a b* :: *real*
  show $\bigwedge x\ y{::}'a\ star.\ scaleR\ a\ (x + y) = scaleR\ a\ x + scaleR\ a\ y$
    by *transfer* (*rule scaleR-right-distrib*)
  show $\bigwedge x{::}'a\ star.\ scaleR\ (a + b)\ x = scaleR\ a\ x + scaleR\ b\ x$
    by *transfer* (*rule scaleR-left-distrib*)
  show $\bigwedge x{::}'a\ star.\ scaleR\ a\ (scaleR\ b\ x) = scaleR\ (a * b)\ x$
    by *transfer* (*rule scaleR-scaleR*)
  show $\bigwedge x{::}'a\ star.\ scaleR\ 1\ x = x$
    by *transfer* (*rule scaleR-one*)
qed

instance *star* :: (*real-algebra*) *real-algebra*
proof
  fix *a* :: *real*
  show $\bigwedge x\ y{::}'a\ star.\ scaleR\ a\ x * y = scaleR\ a\ (x * y)$
    by *transfer* (*rule mult-scaleR-left*)
  show $\bigwedge x\ y{::}'a\ star.\ x * scaleR\ a\ y = scaleR\ a\ (x * y)$
    by *transfer* (*rule mult-scaleR-right*)
qed

instance *star* :: (*real-algebra-1*) *real-algebra-1* **..**

instance *star* :: (*real-div-algebra*) *real-div-algebra* **..**

instance *star* :: (*real-field*) *real-field* **..**

lemma *star-of-real-def* [*transfer-unfold*]: *of-real r* = *star-of* (*of-real r*)
by (*unfold of-real-def*, *transfer*, *rule refl*)

lemma *Standard-of-real* [*simp*]: *of-real r* ∈ *Standard*
by (*simp add*: *star-of-real-def*)

lemma *star-of-of-real* [*simp*]: *star-of* (*of-real r*) = *of-real r*
by *transfer* (*rule refl*)

lemma *of-real-eq-star-of* [*simp*]: *of-real* = *star-of*
proof
  fix *r* :: *real*
  show *of-real r* = *star-of r*
    by *transfer simp*
qed

lemma *Reals-eq-Standard*: (*Reals* :: *hypreal set*) = *Standard*
by (*simp add*: *Reals-def Standard-def*)

26.2 Injection from *hypreal*

definition
of-hypreal :: *hypreal* ⇒ *'a*::*real-algebra-1 star* **where**
of-hypreal = ∗*f*∗ *of-real*

declare *of-hypreal-def* [*transfer-unfold*]

lemma *Standard-of-hypreal* [*simp*]:
$r \in Standard \Longrightarrow$ *of-hypreal* $r \in Standard$
by (*simp add*: *of-hypreal-def*)

lemma *of-hypreal-0* [*simp*]: *of-hypreal 0* = *0*
by *transfer* (*rule of-real-0*)

lemma *of-hypreal-1* [*simp*]: *of-hypreal 1* = *1*
by *transfer* (*rule of-real-1*)

lemma *of-hypreal-add* [*simp*]:
$\bigwedge x\ y.$ *of-hypreal* $(x + y) =$ *of-hypreal* $x +$ *of-hypreal* y
by *transfer* (*rule of-real-add*)

lemma *of-hypreal-minus* [*simp*]: $\bigwedge x.$ *of-hypreal* $(- x) = -$ *of-hypreal* x
by *transfer* (*rule of-real-minus*)

lemma *of-hypreal-diff* [*simp*]:
$\bigwedge x\ y.$ *of-hypreal* $(x - y) =$ *of-hypreal* $x -$ *of-hypreal* y
by *transfer* (*rule of-real-diff*)

lemma *of-hypreal-mult* [*simp*]:
$\bigwedge x\ y.$ *of-hypreal* $(x * y) =$ *of-hypreal* $x *$ *of-hypreal* y
by *transfer* (*rule of-real-mult*)

lemma *of-hypreal-inverse* [*simp*]:
$\bigwedge x.$ *of-hypreal* (*inverse* x) =
inverse (*of-hypreal* x :: *'a*::{*real-div-algebra*,*division-by-zero*} *star*)
by *transfer* (*rule of-real-inverse*)

lemma *of-hypreal-divide* [*simp*]:
$\bigwedge x\ y.$ *of-hypreal* $(x\ /\ y)$ =
(*of-hypreal* x / *of-hypreal* y :: *'a*::{*real-field*,*division-by-zero*} *star*)
by *transfer* (*rule of-real-divide*)

lemma *of-hypreal-eq-iff* [*simp*]:
$\bigwedge x\ y.$ (*of-hypreal* x = *of-hypreal* y) = $(x = y)$
by *transfer* (*rule of-real-eq-iff*)

lemma *of-hypreal-eq-0-iff* [*simp*]:
$\bigwedge x.$ (*of-hypreal* x = *0*) = (x = *0*)
by *transfer* (*rule of-real-eq-0-iff*)

26.3 Properties of *starrel*

lemma *lemma-starrel-refl* [*simp*]: $x \in starrel `` \{x\}$
by (*simp add*: *starrel-def*)

lemma *starrel-in-hypreal* [*simp*]: $starrel``\{x\}:star$
by (*simp add*: *star-def starrel-def quotient-def*, *blast*)

declare *Abs-star-inject* [*simp*] *Abs-star-inverse* [*simp*]
declare *equiv-starrel* [*THEN eq-equiv-class-iff*, *simp*]

26.4 *hypreal-of-real*: the Injection from *real* to *hypreal*

lemma *inj-star-of*: *inj star-of*
by (*rule inj-onI*, *simp*)

lemma *mem-Rep-star-iff*: $(X \in \mathit{Rep\text{-}star}\ x) = (x = \mathit{star\text{-}n}\ X)$
by (*cases x*, *simp add*: *star-n-def*)

lemma *Rep-star-star-n-iff* [*simp*]:
$(X \in \mathit{Rep\text{-}star}\ (\mathit{star\text{-}n}\ Y)) = (\{n.\ Y\ n = X\ n\} \in \mathcal{U})$
by (*simp add*: *star-n-def*)

lemma *Rep-star-star-n*: $X \in \mathit{Rep\text{-}star}\ (\mathit{star\text{-}n}\ X)$
by *simp*

26.5 Properties of *star-n*

lemma *star-n-add*:
$\mathit{star\text{-}n}\ X + \mathit{star\text{-}n}\ Y = \mathit{star\text{-}n}\ (\%n.\ X\ n + Y\ n)$
by (*simp only*: *star-add-def starfun2-star-n*)

lemma *star-n-minus*:
$- \mathit{star\text{-}n}\ X = \mathit{star\text{-}n}\ (\%n.\ -(X\ n))$
by (*simp only*: *star-minus-def starfun-star-n*)

lemma *star-n-diff*:
$\mathit{star\text{-}n}\ X - \mathit{star\text{-}n}\ Y = \mathit{star\text{-}n}\ (\%n.\ X\ n - Y\ n)$
by (*simp only*: *star-diff-def starfun2-star-n*)

lemma *star-n-mult*:
$\mathit{star\text{-}n}\ X * \mathit{star\text{-}n}\ Y = \mathit{star\text{-}n}\ (\%n.\ X\ n * Y\ n)$
by (*simp only*: *star-mult-def starfun2-star-n*)

lemma *star-n-inverse*:
$\mathit{inverse}\ (\mathit{star\text{-}n}\ X) = \mathit{star\text{-}n}\ (\%n.\ \mathit{inverse}(X\ n))$
by (*simp only*: *star-inverse-def starfun-star-n*)

lemma *star-n-le*:
$\mathit{star\text{-}n}\ X \le \mathit{star\text{-}n}\ Y =$

({n. X n ≤ Y n} ∈ *FreeUltrafilterNat*)
by (*simp only*: *star-le-def starP2-star-n*)

lemma *star-n-less*:
star-n X < *star-n Y* = ({n. X n < Y n} ∈ *FreeUltrafilterNat*)
by (*simp only*: *star-less-def starP2-star-n*)

lemma *star-n-zero-num*: *0* = *star-n* (%n. *0*)
by (*simp only*: *star-zero-def star-of-def*)

lemma *star-n-one-num*: *1* = *star-n* (%n. *1*)
by (*simp only*: *star-one-def star-of-def*)

lemma *star-n-abs*:
abs (*star-n X*) = *star-n* (%n. *abs* (X n))
by (*simp only*: *star-abs-def starfun-star-n*)

26.6 Misc Others

lemma *hypreal-not-refl2*: !!(x::*hypreal*). x < y ==> x ≠ y
by (*auto*)

lemma *hypreal-eq-minus-iff*: ((x::*hypreal*) = y) = (x + − y = *0*)
by *auto*

lemma *hypreal-mult-left-cancel*: (c::*hypreal*) ≠ *0* ==> (c∗a=c∗b) = (a=b)
by *auto*

lemma *hypreal-mult-right-cancel*: (c::*hypreal*) ≠ *0* ==> (a∗c=b∗c) = (a=b)
by *auto*

lemma *hypreal-omega-gt-zero* [*simp*]: *0* < *omega*
by (*simp add*: *omega-def star-n-zero-num star-n-less*)

26.7 Existence of Infinite Hyperreal Number

Existence of infinite number not corresponding to any real number. Use assumption that member $\mathcal{U}$ is not finite.

A few lemmas first

lemma *lemma-omega-empty-singleton-disj*: {n::*nat*. x = *real* n} = {} |
(∃ y. {n::*nat*. x = *real* n} = {y})
by *force*

lemma *lemma-finite-omega-set*: *finite* {n::*nat*. x = *real* n}
by (*cut-tac* x = x **in** *lemma-omega-empty-singleton-disj*, *auto*)

lemma *not-ex-hypreal-of-real-eq-omega*:
~ (∃ x. *hypreal-of-real* x = *omega*)

apply (*simp add*: *omega-def*)
apply (*simp add*: *star-of-def star-n-eq-iff*)
apply (*auto simp add*: *real-of-nat-Suc diff-eq-eq* [*symmetric*]
lemma-finite-omega-set [*THEN FreeUltrafilterNat.finite*])
done

lemma *hypreal-of-real-not-eq-omega*: *hypreal-of-real* $x \neq$ *omega*
by (*insert not-ex-hypreal-of-real-eq-omega*, *auto*)

Existence of infinitesimal number also not corresponding to any real number

lemma *lemma-epsilon-empty-singleton-disj*:
$\{n::nat.\ x = inverse(real(Suc\ n))\} = \{\}$ |
$(\exists\, y.\ \{n::nat.\ x = inverse(real(Suc\ n))\} = \{y\})$
by *auto*

lemma *lemma-finite-epsilon-set*: *finite* $\{n.\ x = inverse(real(Suc\ n))\}$
by (*cut-tac* $x = x$ **in** *lemma-epsilon-empty-singleton-disj*, *auto*)

lemma *not-ex-hypreal-of-real-eq-epsilon*: ~ ($\exists\, x.$ *hypreal-of-real* x = *epsilon*)
by (*auto simp add*: *epsilon-def star-of-def star-n-eq-iff*
lemma-finite-epsilon-set [*THEN FreeUltrafilterNat.finite*])

lemma *hypreal-of-real-not-eq-epsilon*: *hypreal-of-real* $x \neq$ *epsilon*
by (*insert not-ex-hypreal-of-real-eq-epsilon*, *auto*)

lemma *hypreal-epsilon-not-zero*: *epsilon* $\neq$ *0*
by (*simp add*: *epsilon-def star-zero-def star-of-def star-n-eq-iff*
del: *star-of-zero*)

lemma *hypreal-epsilon-inverse-omega*: *epsilon* = *inverse*(*omega*)
by (*simp add*: *epsilon-def omega-def star-n-inverse*)

lemma *hypreal-epsilon-gt-zero*: *0* < *epsilon*
by (*simp add*: *hypreal-epsilon-inverse-omega*)

26.8 Absolute Value Function for the Hyperreals

lemma *hrabs-add-less*:
[| *abs* $x < r$; *abs* $y < s$ |] ==> *abs*$(x+y) < r + (s::hypreal)$
by (*simp add*: *abs-if split*: *split-if-asm*)

lemma *hrabs-less-gt-zero*: *abs* $x < r$ ==> $(0::hypreal) < r$
by (*blast intro*!: *order-le-less-trans abs-ge-zero*)

lemma *hrabs-disj*: *abs* $x = (x::'a::abs\text{-}if)$ | *abs* $x = -x$
by (*simp add*: *abs-if*)

lemma *hrabs-add-lemma-disj*: $(y::hypreal) + - x + (y + - z) = abs\ (x + - z)$
==> $y = z$ | $x = y$

by (*simp add*: *abs-if split add*: *split-if-asm*)

26.9 Embedding the Naturals into the Hyperreals

abbreviation
  hypreal-of-nat :: *nat* => *hypreal* **where**
  hypreal-of-nat == *of-nat*

lemma *SNat-eq*: *Nats* = $\{n.\ \exists\, N.\ n = $ *hypreal-of-nat N*$\}$
by (*simp add*: *Nats-def image-def*)

lemma *hypreal-of-nat-eq*:
    hypreal-of-nat (*n*::*nat*) = *hypreal-of-real* (*real n*)
by (*simp add*: *real-of-nat-def*)

lemma *hypreal-of-nat*:
    hypreal-of-nat m = *star-n* (%*n*. *real m*)
apply (*fold star-of-def*)
apply (*simp add*: *real-of-nat-def*)
done

use *hypreal-arith.ML*
declaration ⟪ *K hypreal-arith-setup* ⟫

26.10 Exponentials on the Hyperreals

lemma *hpowr-0* [*simp*]: *r* ˆ *0* = (*1*::*hypreal*)
by (*rule power-0*)

lemma *hpowr-Suc* [*simp*]: *r* ˆ (*Suc n*) = (*r*::*hypreal*) ∗ (*r* ˆ *n*)
by (*rule power-Suc*)

lemma *hrealpow-two*: (*r*::*hypreal*) ˆ *Suc* (*Suc 0*) = *r* ∗ *r*
by *simp*

lemma *hrealpow-two-le* [*simp*]: (*0*::*hypreal*) ≤ *r* ˆ *Suc* (*Suc 0*)
by (*auto simp add*: *zero-le-mult-iff*)

lemma *hrealpow-two-le-add-order* [*simp*]:
    (*0*::*hypreal*) ≤ *u* ˆ *Suc* (*Suc 0*) + *v* ˆ *Suc* (*Suc 0*)
by (*simp only*: *hrealpow-two-le add-nonneg-nonneg*)

lemma *hrealpow-two-le-add-order2* [*simp*]:

(*0*::*hypreal*) ≤ *u* ^ *Suc* (*Suc* *0*) + *v* ^ *Suc* (*Suc* *0*) + *w* ^ *Suc* (*Suc* *0*)
by (*simp only*: *hrealpow-two-le add-nonneg-nonneg*)

lemma *hypreal-add-nonneg-eq-0-iff*:
[| *0* ≤ *x*; *0* ≤ *y* |] ==> (*x*+*y* = *0*) = (*x* = *0* & *y* = (*0*::*hypreal*))
by *arith*

FIXME: DELETE THESE

lemma *hypreal-three-squares-add-zero-iff*:
(*x*∗*x* + *y*∗*y* + *z*∗*z* = *0*) = (*x* = *0* & *y* = *0* & *z* = (*0*::*hypreal*))
apply (*simp only*: *zero-le-square add-nonneg-nonneg hypreal-add-nonneg-eq-0-iff*, *auto*)
done

lemma *hrealpow-three-squares-add-zero-iff* [*simp*]:
(*x* ^ *Suc* (*Suc* *0*) + *y* ^ *Suc* (*Suc* *0*) + *z* ^ *Suc* (*Suc* *0*) = (*0*::*hypreal*)) =
(*x* = *0* & *y* = *0* & *z* = *0*)
by (*simp only*: *hypreal-three-squares-add-zero-iff hrealpow-two*)

lemma *hrabs-hrealpow-two* [*simp*]:
abs(*x* ^ *Suc* (*Suc* *0*)) = (*x*::*hypreal*) ^ *Suc* (*Suc* *0*)
by (*simp add*: *abs-mult*)

lemma *two-hrealpow-ge-one* [*simp*]: (*1*::*hypreal*) ≤ *2* ^ *n*
by (*insert power-increasing* [*of 0 n 2*::*hypreal*], *simp*)

lemma *two-hrealpow-gt* [*simp*]: *hypreal-of-nat n* < *2* ^ *n*
apply (*induct-tac n*)
apply (*auto simp add*: *left-distrib*)
apply (*cut-tac n* = *n* **in** *two-hrealpow-ge-one*, *arith*)
done

lemma *hrealpow*:
star-n X ^ *m* = *star-n* (%*n*. (*X n*::*real*) ^ *m*)
apply (*induct-tac m*)
apply (*auto simp add*: *star-n-one-num star-n-mult power-0*)
done

lemma *hrealpow-sum-square-expand*:
(*x* + (*y*::*hypreal*)) ^ *Suc* (*Suc* *0*) =
x ^ *Suc* (*Suc* *0*) + *y* ^ *Suc* (*Suc* *0*) + (*hypreal-of-nat* (*Suc* (*Suc* *0*)))∗*x*∗*y*
by (*simp add*: *right-distrib left-distrib*)

lemma *power-hypreal-of-real-number-of*:
(*number-of v* :: *hypreal*) ^ *n* = *hypreal-of-real* ((*number-of v*) ^ *n*)
by *simp*
declare *power-hypreal-of-real-number-of* [*of - number-of w*, *standard*, *simp*]

26.11 Powers with Hypernatural Exponents

definition

pow :: [′*a*::*power star*, *nat star*] ⇒ ′*a star* (**infixr** *pow 80*) **where**
hyperpow-def [*transfer-unfold*]:
R pow N = (∗*f2*∗ *op* ^) *R N*

lemma *Standard-hyperpow* [*simp*]:
⟦*r* ∈ *Standard*; *n* ∈ *Standard*⟧ ⟹ *r pow n* ∈ *Standard*
unfolding *hyperpow-def* **by** *simp*

lemma *hyperpow*: *star-n X pow star-n Y* = *star-n* (%*n*. *X n* ^ *Y n*)
by (*simp add*: *hyperpow-def starfun2-star-n*)

lemma *hyperpow-zero* [*simp*]:
⋀*n*. (*0*::′*a*::{*recpower*,*semiring-0*} *star*) *pow* (*n* + (*1*::*hypnat*)) = *0*
by *transfer simp*

lemma *hyperpow-not-zero*:
⋀*r n*. *r* ≠ (*0*::′*a*::{*recpower*,*field*} *star*) ==> *r pow n* ≠ *0*
by *transfer* (*rule field-power-not-zero*)

lemma *hyperpow-inverse*:
⋀*r n*. *r* ≠ (*0*::′*a*::{*recpower*,*division-by-zero*,*field*} *star*)
⟹ *inverse* (*r pow n*) = (*inverse r*) *pow n*
by *transfer* (*rule power-inverse*)

lemma *hyperpow-hrabs*:
⋀*r n*. *abs* (*r*::′*a*::{*recpower*,*ordered-idom*} *star*) *pow n* = *abs* (*r pow n*)
by *transfer* (*rule power-abs* [*symmetric*])

lemma *hyperpow-add*:
⋀*r n m*. (*r*::′*a*::*recpower star*) *pow* (*n* + *m*) = (*r pow n*) ∗ (*r pow m*)
by *transfer* (*rule power-add*)

lemma *hyperpow-one* [*simp*]:
⋀*r*. (*r*::′*a*::*recpower star*) *pow* (*1*::*hypnat*) = *r*
by *transfer* (*rule power-one-right*)

lemma *hyperpow-two*:
⋀*r*. (*r*::′*a*::*recpower star*) *pow* ((*1*::*hypnat*) + (*1*::*hypnat*)) = *r* ∗ *r*
by *transfer* (*simp add*: *power-Suc*)

lemma *hyperpow-gt-zero*:
⋀*r n*. (*0*::′*a*::{*recpower*,*ordered-semidom*} *star*) < *r* ⟹ *0* < *r pow n*
by *transfer* (*rule zero-less-power*)

lemma *hyperpow-ge-zero*:
⋀*r n*. (*0*::′*a*::{*recpower*,*ordered-semidom*} *star*) ≤ *r* ⟹ *0* ≤ *r pow n*

by *transfer* (*rule zero-le-power*)

lemma *hyperpow-le*:
$\bigwedge x\ y\ n.\ [\![(0::'a::\{recpower,ordered\text{-}semidom\}\ star) < x;\ x \le y]\!]$
$\Longrightarrow x\ pow\ n \le y\ pow\ n$
by *transfer* (*rule power-mono* [*OF - order-less-imp-le*])

lemma *hyperpow-eq-one* [*simp*]:
$\bigwedge n.\ 1\ pow\ n = (1::'a::recpower\ star)$
by *transfer* (*rule power-one*)

lemma *hrabs-hyperpow-minus-one* [*simp*]:
$\bigwedge n.\ abs(-1\ pow\ n) = (1::'a::\{number\text{-}ring,recpower,ordered\text{-}idom\}\ star)$
by *transfer* (*rule abs-power-minus-one*)

lemma *hyperpow-mult*:
$\bigwedge r\ s\ n.\ (r * s::'a::\{comm\text{-}monoid\text{-}mult,recpower\}\ star)\ pow\ n$
$= (r\ pow\ n) * (s\ pow\ n)$
by *transfer* (*rule power-mult-distrib*)

lemma *hyperpow-two-le* [*simp*]:
$(0::'a::\{recpower,ordered\text{-}ring\text{-}strict\}\ star) \le r\ pow\ (1 + 1)$
by (*auto simp add*: *hyperpow-two zero-le-mult-iff*)

lemma *hrabs-hyperpow-two* [*simp*]:
$abs(x\ pow\ (1 + 1)) =$
$(x::'a::\{recpower,ordered\text{-}ring\text{-}strict\}\ star)\ pow\ (1 + 1)$
by (*simp only*: *abs-of-nonneg hyperpow-two-le*)

lemma *hyperpow-two-hrabs* [*simp*]:
$abs(x::'a::\{recpower,ordered\text{-}idom\}\ star)\ pow\ (1 + 1) = x\ pow\ (1 + 1)$
by (*simp add*: *hyperpow-hrabs*)

The precondition could be weakened to $(0::'a) \le x$

lemma *hypreal-mult-less-mono*:
$[|\ u<v;\ \ x<y;\ \ (0::hypreal) < v;\ \ 0 < x\ |] ==> u*x < v*\ y$
by (*simp add*: *Ring-and-Field.mult-strict-mono order-less-imp-le*)

lemma *hyperpow-two-gt-one*:
$\bigwedge r::'a::\{recpower,ordered\text{-}semidom\}\ star.\ 1 < r \Longrightarrow 1 < r\ pow\ (1 + 1)$
by *transfer* (*simp add*: *power-gt1*)

lemma *hyperpow-two-ge-one*:
$\bigwedge r::'a::\{recpower,ordered\text{-}semidom\}\ star.\ 1 \le r \Longrightarrow 1 \le r\ pow\ (1 + 1)$
by *transfer* (*simp add*: *one-le-power*)

lemma *two-hyperpow-ge-one* [*simp*]: $(1::hypreal) \le 2\ pow\ n$
apply (*rule-tac* $y = 1\ pow\ n$ **in** *order-trans*)
apply (*rule-tac* [*2*] *hyperpow-le*, *auto*)

done

lemma *hyperpow-minus-one2* [*simp*]:
  !!*n*. -1 *pow* ((*1* + *1*)∗*n*) = (*1*::*hypreal*)
by *transfer* (*subst power-mult*, *simp*)

lemma *hyperpow-less-le*:
  !!*r n N*. [|(*0*::*hypreal*) ≤ *r*; *r* ≤ *1*; *n* < *N*|] ==> *r pow N* ≤ *r pow n*
by *transfer* (*rule power-decreasing* [*OF order-less-imp-le*])

lemma *hyperpow-SHNat-le*:
  [| *0* ≤ *r*; *r* ≤ (*1*::*hypreal*); *N* ∈ *HNatInfinite* |]
  ==> *ALL n*: *Nats*. *r pow N* ≤ *r pow n*
by (*auto intro*!: *hyperpow-less-le simp add*: *HNatInfinite-iff*)

lemma *hyperpow-realpow*:
  (*hypreal-of-real r*) *pow* (*hypnat-of-nat n*) = *hypreal-of-real* (*r* ˆ *n*)
by *transfer* (*rule refl*)

lemma *hyperpow-SReal* [*simp*]:
  (*hypreal-of-real r*) *pow* (*hypnat-of-nat n*) ∈ *Reals*
by (*simp add*: *Reals-eq-Standard*)

lemma *hyperpow-zero-HNatInfinite* [*simp*]:
  N ∈ *HNatInfinite* ==> (*0*::*hypreal*) *pow N* = *0*
by (*drule HNatInfinite-is-Suc*, *auto*)

lemma *hyperpow-le-le*:
  [| (*0*::*hypreal*) ≤ *r*; *r* ≤ *1*; *n* ≤ *N* |] ==> *r pow N* ≤ *r pow n*
apply (*drule order-le-less* [*of n*, *THEN iffD1*])
apply (*auto intro*: *hyperpow-less-le*)
done

lemma *hyperpow-Suc-le-self2*:
  [| (*0*::*hypreal*) ≤ *r*; *r* < *1* |] ==> *r pow* (*n* + (*1*::*hypnat*)) ≤ *r*
apply (*drule-tac n* = (*1*::*hypnat*) **in** *hyperpow-le-le*)
apply *auto*
done

lemma *hyperpow-hypnat-of-nat*: ⋀*x*. *x pow hypnat-of-nat n* = *x* ˆ *n*
by *transfer* (*rule refl*)

lemma *of-hypreal-hyperpow*:
  ⋀*x n*. *of-hypreal* (*x pow n*) =
  (*of-hypreal x*::′*a*::{*real-algebra-1*,*recpower*} *star*) *pow n*
by *transfer* (*rule of-real-power*)

end

27 NSA: Infinite Numbers, Infinitesimals, Infinitely Close Relation

theory *NSA*
imports *HyperDef ../Real/RComplete*
begin

definition
hnorm :: $'a$::*norm star* $\Rightarrow$ *real star* **where**
hnorm = $*f*$ *norm*

definition
Infinitesimal :: ($'a$::*real-normed-vector*) *star set* **where**
Infinitesimal = $\{x.\ \forall r \in Reals.\ 0 < r --> hnorm\ x < r\}$

definition
HFinite :: ($'a$::*real-normed-vector*) *star set* **where**
HFinite = $\{x.\ \exists r \in Reals.\ hnorm\ x < r\}$

definition
HInfinite :: ($'a$::*real-normed-vector*) *star set* **where**
HInfinite = $\{x.\ \forall r \in Reals.\ r < hnorm\ x\}$

definition
approx :: [$'a$::*real-normed-vector star*, $'a$ *star*] => *bool* (**infixl** @= *50*) **where**
— the 'infinitely close' relation
(x @= y) = (($x - y$) $\in$ *Infinitesimal*)

definition
st :: *hypreal* => *hypreal* **where**
— the standard part of a hyperreal
st = (%x. @r. $x \in$ *HFinite* & $r \in$ *Reals* & r @= x)

definition
monad :: $'a$::*real-normed-vector star* => $'a$ *star set* **where**
monad x = $\{y.\ x$ @= $y\}$

definition
galaxy :: $'a$::*real-normed-vector star* => $'a$ *star set* **where**
galaxy x = $\{y.\ (x + -y) \in HFinite\}$

notation (*xsymbols*)
approx (**infixl** $\approx$ *50*)

notation (*HTML* **output**)
approx (**infixl** $\approx$ *50*)

lemma *SReal-def*: *Reals* == $\{x.\ \exists r.\ x = hypreal\text{-}of\text{-}real\ r\}$
by (*simp add*: *Reals-def image-def*)

27.1 Nonstandard Extension of the Norm Function

definition
scaleHR :: *real star* ⇒ *'a star* ⇒ *'a::real-normed-vector star* **where**
scaleHR = *starfun2 scaleR*

declare *hnorm-def* [*transfer-unfold*]
declare *scaleHR-def* [*transfer-unfold*]

lemma *Standard-hnorm* [*simp*]: $x \in Standard \Longrightarrow hnorm\ x \in Standard$
by (*simp add*: *hnorm-def*)

lemma *star-of-norm* [*simp*]: *star-of* (*norm x*) = *hnorm* (*star-of x*)
by *transfer* (*rule refl*)

lemma *hnorm-ge-zero* [*simp*]:
$\bigwedge x$::*'a::real-normed-vector star*. $0 \le hnorm\ x$
by *transfer* (*rule norm-ge-zero*)

lemma *hnorm-eq-zero* [*simp*]:
$\bigwedge x$::*'a::real-normed-vector star*. $(hnorm\ x = 0) = (x = 0)$
by *transfer* (*rule norm-eq-zero*)

lemma *hnorm-triangle-ineq*:
$\bigwedge x\ y$::*'a::real-normed-vector star*. $hnorm\ (x + y) \le hnorm\ x + hnorm\ y$
by *transfer* (*rule norm-triangle-ineq*)

lemma *hnorm-triangle-ineq3*:
$\bigwedge x\ y$::*'a::real-normed-vector star*. $|hnorm\ x - hnorm\ y| \le hnorm\ (x - y)$
by *transfer* (*rule norm-triangle-ineq3*)

lemma *hnorm-scaleR*:
$\bigwedge x$::*'a::real-normed-vector star*.
$hnorm\ (a *_R x) = |star\text{-}of\ a| * hnorm\ x$
by *transfer* (*rule norm-scaleR*)

lemma *hnorm-scaleHR*:
$\bigwedge a$ (x::*'a::real-normed-vector star*).
$hnorm\ (scaleHR\ a\ x) = |a| * hnorm\ x$
by *transfer* (*rule norm-scaleR*)

lemma *hnorm-mult-ineq*:
$\bigwedge x\ y$::*'a::real-normed-algebra star*. $hnorm\ (x * y) \le hnorm\ x * hnorm\ y$
by *transfer* (*rule norm-mult-ineq*)

lemma *hnorm-mult*:
$\bigwedge x\ y$::*'a::real-normed-div-algebra star*. $hnorm\ (x * y) = hnorm\ x * hnorm\ y$
by *transfer* (*rule norm-mult*)

lemma *hnorm-hyperpow*:

$\bigwedge(x::'a::\{real\text{-}normed\text{-}div\text{-}algebra,recpower\}\ star)\ n.$
$hnorm\ (x\ pow\ n) = hnorm\ x\ pow\ n$
by *transfer* (*rule norm-power*)

lemma *hnorm-one* [*simp*]:
$hnorm\ (1::'a::real\text{-}normed\text{-}div\text{-}algebra\ star) = 1$
by *transfer* (*rule norm-one*)

lemma *hnorm-zero* [*simp*]:
$hnorm\ (0::'a::real\text{-}normed\text{-}vector\ star) = 0$
by *transfer* (*rule norm-zero*)

lemma *zero-less-hnorm-iff* [*simp*]:
$\bigwedge x::'a::real\text{-}normed\text{-}vector\ star.\ (0 < hnorm\ x) = (x \neq 0)$
by *transfer* (*rule zero-less-norm-iff*)

lemma *hnorm-minus-cancel* [*simp*]:
$\bigwedge x::'a::real\text{-}normed\text{-}vector\ star.\ hnorm\ (-\ x) = hnorm\ x$
by *transfer* (*rule norm-minus-cancel*)

lemma *hnorm-minus-commute*:
$\bigwedge a\ b::'a::real\text{-}normed\text{-}vector\ star.\ hnorm\ (a - b) = hnorm\ (b - a)$
by *transfer* (*rule norm-minus-commute*)

lemma *hnorm-triangle-ineq2*:
$\bigwedge a\ b::'a::real\text{-}normed\text{-}vector\ star.\ hnorm\ a - hnorm\ b \leq hnorm\ (a - b)$
by *transfer* (*rule norm-triangle-ineq2*)

lemma *hnorm-triangle-ineq4*:
$\bigwedge a\ b::'a::real\text{-}normed\text{-}vector\ star.\ hnorm\ (a - b) \leq hnorm\ a + hnorm\ b$
by *transfer* (*rule norm-triangle-ineq4*)

lemma *abs-hnorm-cancel* [*simp*]:
$\bigwedge a::'a::real\text{-}normed\text{-}vector\ star.\ |hnorm\ a| = hnorm\ a$
by *transfer* (*rule abs-norm-cancel*)

lemma *hnorm-of-hypreal* [*simp*]:
$\bigwedge r.\ hnorm\ (of\text{-}hypreal\ r::'a::real\text{-}normed\text{-}algebra\text{-}1\ star) = |r|$
by *transfer* (*rule norm-of-real*)

lemma *nonzero-hnorm-inverse*:
$\bigwedge a::'a::real\text{-}normed\text{-}div\text{-}algebra\ star.$
$a \neq 0 \Longrightarrow hnorm\ (inverse\ a) = inverse\ (hnorm\ a)$
by *transfer* (*rule nonzero-norm-inverse*)

lemma *hnorm-inverse*:
$\bigwedge a::'a::\{real\text{-}normed\text{-}div\text{-}algebra,division\text{-}by\text{-}zero\}\ star.$
$hnorm\ (inverse\ a) = inverse\ (hnorm\ a)$
by *transfer* (*rule norm-inverse*)

lemma *hnorm-divide*:
$\bigwedge a\ b::'a::\{real\text{-}normed\text{-}field,division\text{-}by\text{-}zero\}\ star.$
$hnorm\ (a\ /\ b) = hnorm\ a\ /\ hnorm\ b$
by *transfer* (*rule norm-divide*)

lemma *hypreal-hnorm-def* [*simp*]:
$\bigwedge r::hypreal.\ hnorm\ r \equiv |r|$
by *transfer* (*rule real-norm-def*)

lemma *hnorm-add-less*:
$\bigwedge(x::'a::real\text{-}normed\text{-}vector\ star)\ y\ r\ s.$
$[\![hnorm\ x < r;\ hnorm\ y < s]\!] \Longrightarrow hnorm\ (x + y) < r + s$
by *transfer* (*rule norm-add-less*)

lemma *hnorm-mult-less*:
$\bigwedge(x::'a::real\text{-}normed\text{-}algebra\ star)\ y\ r\ s.$
$[\![hnorm\ x < r;\ hnorm\ y < s]\!] \Longrightarrow hnorm\ (x * y) < r * s$
by *transfer* (*rule norm-mult-less*)

lemma *hnorm-scaleHR-less*:
$[\![|x| < r;\ hnorm\ y < s]\!] \Longrightarrow hnorm\ (scaleHR\ x\ y) < r * s$
apply (*simp only*: *hnorm-scaleHR*)
apply (*simp add*: *mult-strict-mono'*)
done

27.2 Closure Laws for the Standard Reals

lemma *Reals-minus-iff* [*simp*]: $(-x \in Reals) = (x \in Reals)$
apply *auto*
apply (*drule Reals-minus*, *auto*)
done

lemma *Reals-add-cancel*: $[\![x + y \in Reals;\ y \in Reals]\!] \Longrightarrow x \in Reals$
by (*drule* (*1*) *Reals-diff*, *simp*)

lemma *SReal-hrabs*: $(x::hypreal) \in Reals ==> abs\ x \in Reals$
by (*simp add*: *Reals-eq-Standard*)

lemma *SReal-hypreal-of-real* [*simp*]: $hypreal\text{-}of\text{-}real\ x \in Reals$
by (*simp add*: *Reals-eq-Standard*)

lemma *SReal-divide-number-of*: $r \in Reals ==> r/(number\text{-}of\ w::hypreal) \in Reals$
by *simp*

epsilon is not in Reals because it is an infinitesimal

lemma *SReal-epsilon-not-mem*: $epsilon \notin Reals$
apply (*simp add*: *SReal-def*)
apply (*auto simp add*: *hypreal-of-real-not-eq-epsilon* [*THEN not-sym*])

done

lemma *SReal-omega-not-mem*: *omega* $\notin$ *Reals*
apply (*simp add*: *SReal-def*)
apply (*auto simp add*: *hypreal-of-real-not-eq-omega* [*THEN not-sym*])
done

lemma *SReal-UNIV-real*: $\{x.$ *hypreal-of-real* $x \in$ *Reals*$\}$ = (*UNIV*::*real set*)
by *simp*

lemma *SReal-iff*: ($x \in$ *Reals*) = ($\exists y.$ x = *hypreal-of-real* y)
by (*simp add*: *SReal-def*)

lemma *hypreal-of-real-image*: *hypreal-of-real* '(*UNIV*::*real set*) = *Reals*
by (*simp add*: *Reals-eq-Standard Standard-def*)

lemma *inv-hypreal-of-real-image*: *inv hypreal-of-real* ' *Reals* = *UNIV*
apply (*auto simp add*: *SReal-def*)
apply (*rule inj-star-of* [*THEN inv-f-f*, *THEN subst*], *blast*)
done

lemma *SReal-hypreal-of-real-image*:
  [| $\exists x.$ x: P; $P \subseteq$ *Reals* |] ==> $\exists Q.$ P = *hypreal-of-real* ' Q
by (*simp add*: *SReal-def image-def*, *blast*)

lemma *SReal-dense*:
  [| (x::*hypreal*) $\in$ *Reals*; $y \in$ *Reals*; $x<y$ |] ==> $\exists r \in$ *Reals*. $x<r$ & $r<y$
apply (*auto simp add*: *SReal-def*)
apply (*drule dense*, *auto*)
done

Completeness of Reals, but both lemmas are unused.

lemma *SReal-sup-lemma*:
  $P \subseteq$ *Reals* ==> (($\exists x \in P.$ $y < x$) =
  ($\exists X.$ *hypreal-of-real* $X \in P$ & $y <$ *hypreal-of-real* X))
by (*blast dest*!: *SReal-iff* [*THEN iffD1*])

lemma *SReal-sup-lemma2*:
  [| $P \subseteq$ *Reals*; $\exists x.$ $x \in P$; $\exists y \in$ *Reals*. $\forall x \in P.$ $x < y$ |]
  ==> ($\exists X.$ $X \in \{w.$ *hypreal-of-real* $w \in P\}$) &
    ($\exists Y.$ $\forall X \in \{w.$ *hypreal-of-real* $w \in P\}.$ $X < Y$)
apply (*rule conjI*)
apply (*fast dest*!: *SReal-iff* [*THEN iffD1*])
apply (*auto*, *frule subsetD*, *assumption*)
apply (*drule SReal-iff* [*THEN iffD1*])
apply (*auto*, *rule-tac* x = *ya* **in** *exI*, *auto*)
done

27.3 Set of Finite Elements is a Subring of the Extended Reals

lemma *HFinite-add*: [|$x \in$ *HFinite*; $y \in$ *HFinite*|] ==> $(x+y) \in$ *HFinite*
apply (*simp add*: *HFinite-def*)
apply (*blast intro*!: *Reals-add hnorm-add-less*)
done

lemma *HFinite-mult*:
  fixes x y :: $'a$::*real-normed-algebra star*
  shows [|$x \in$ *HFinite*; $y \in$ *HFinite*|] ==> $x*y \in$ *HFinite*
apply (*simp add*: *HFinite-def*)
apply (*blast intro*!: *Reals-mult hnorm-mult-less*)
done

lemma *HFinite-scaleHR*:
  [|$x \in$ *HFinite*; $y \in$ *HFinite*|] ==> *scaleHR* x $y \in$ *HFinite*
apply (*simp add*: *HFinite-def*)
apply (*blast intro*!: *Reals-mult hnorm-scaleHR-less*)
done

lemma *HFinite-minus-iff*: $(-x \in$ *HFinite*$) = (x \in$ *HFinite*$)$
by (*simp add*: *HFinite-def*)

lemma *HFinite-star-of* [*simp*]: *star-of* $x \in$ *HFinite*
apply (*simp add*: *HFinite-def*)
apply (*rule-tac* x=*star-of* (*norm* x) + *1* **in** *bexI*)
apply (*transfer*, *simp*)
apply (*blast intro*: *Reals-add SReal-hypreal-of-real Reals-1*)
done

lemma *SReal-subset-HFinite*: (*Reals*::*hypreal set*) $\subseteq$ *HFinite*
by (*auto simp add*: *SReal-def*)

lemma *HFiniteD*: $x \in$ *HFinite* ==> $\exists\, t \in$ *Reals*. *hnorm* $x < t$
by (*simp add*: *HFinite-def*)

lemma *HFinite-hrabs-iff* [*iff*]: (*abs* (x::*hypreal*) $\in$ *HFinite*) = ($x \in$ *HFinite*)
by (*simp add*: *HFinite-def*)

lemma *HFinite-hnorm-iff* [*iff*]:
  (*hnorm* (x::*hypreal*) $\in$ *HFinite*) = ($x \in$ *HFinite*)
by (*simp add*: *HFinite-def*)

lemma *HFinite-number-of* [*simp*]: *number-of* $w \in$ *HFinite*
unfolding *star-number-def* **by** (*rule HFinite-star-of*)

lemma *HFinite-0* [*simp*]: *0* $\in$ *HFinite*

unfolding *star-zero-def* **by** (*rule HFinite-star-of*)

lemma *HFinite-1* [*simp*]: *1* $\in$ *HFinite*
unfolding *star-one-def* **by** (*rule HFinite-star-of*)

lemma *hrealpow-HFinite*:
  fixes *x* :: *'a*::{*real-normed-algebra*,*recpower*} *star*
  shows *x* $\in$ *HFinite* ==> *x* ^ *n* $\in$ *HFinite*
apply (*induct-tac n*)
apply (*auto simp add*: *power-Suc intro*: *HFinite-mult*)
done

lemma *HFinite-bounded*:
  [|(*x*::*hypreal*) $\in$ *HFinite*; *y* $\leq$ *x*; *0* $\leq$ *y* |] ==> *y* $\in$ *HFinite*
apply (*case-tac x* $\leq$ *0*)
apply (*drule-tac y* = *x* **in** *order-trans*)
apply (*drule-tac* [*2*] *order-antisym*)
apply (*auto simp add*: *linorder-not-le*)
apply (*auto intro*: *order-le-less-trans simp add*: *abs-if HFinite-def*)
done

27.4 Set of Infinitesimals is a Subring of the Hyperreals

lemma *InfinitesimalI*:
  ($\bigwedge$*r*. $[\![$*r* $\in$ $\mathbb{R}$; *0* < *r*$]\!]$ $\Longrightarrow$ *hnorm x* < *r*) $\Longrightarrow$ *x* $\in$ *Infinitesimal*
by (*simp add*: *Infinitesimal-def*)

lemma *InfinitesimalD*:
    x $\in$ *Infinitesimal* ==> $\forall$ *r* $\in$ *Reals*. *0* < *r* --> *hnorm x* < *r*
by (*simp add*: *Infinitesimal-def*)

lemma *InfinitesimalI2*:
  ($\bigwedge$*r*. *0* < *r* $\Longrightarrow$ *hnorm x* < *star-of r*) $\Longrightarrow$ *x* $\in$ *Infinitesimal*
by (*auto simp add*: *Infinitesimal-def SReal-def*)

lemma *InfinitesimalD2*:
  $[\![$*x* $\in$ *Infinitesimal*; *0* < *r*$]\!]$ $\Longrightarrow$ *hnorm x* < *star-of r*
by (*auto simp add*: *Infinitesimal-def SReal-def*)

lemma *Infinitesimal-zero* [*iff*]: *0* $\in$ *Infinitesimal*
by (*simp add*: *Infinitesimal-def*)

lemma *hypreal-sum-of-halves*: *x*/(*2*::*hypreal*) + *x*/(*2*::*hypreal*) = *x*
by *auto*

lemma *Infinitesimal-add*:
    [| *x* $\in$ *Infinitesimal*; *y* $\in$ *Infinitesimal* |] ==> (*x*+*y*) $\in$ *Infinitesimal*
apply (*rule InfinitesimalI*)
apply (*rule hypreal-sum-of-halves* [*THEN subst*])

```
apply (drule half-gt-zero)
apply (blast intro: hnorm-add-less SReal-divide-number-of dest: InfinitesimalD)
done

lemma Infinitesimal-minus-iff [simp]: (−x:Infinitesimal) = (x:Infinitesimal)
by (simp add: Infinitesimal-def)

lemma Infinitesimal-hnorm-iff:
  (hnorm x ∈ Infinitesimal) = (x ∈ Infinitesimal)
by (simp add: Infinitesimal-def)

lemma Infinitesimal-hrabs-iff [iff]:
  (abs (x::hypreal) ∈ Infinitesimal) = (x ∈ Infinitesimal)
by (simp add: abs-if)

lemma Infinitesimal-of-hypreal-iff [simp]:
  ((of-hypreal x::'a::real-normed-algebra-1 star) ∈ Infinitesimal) =
   (x ∈ Infinitesimal)
by (subst Infinitesimal-hnorm-iff [symmetric], simp)

lemma Infinitesimal-diff:
     [| x ∈ Infinitesimal;  y ∈ Infinitesimal |] ==> x−y ∈ Infinitesimal
by (simp add: diff-def Infinitesimal-add)

lemma Infinitesimal-mult:
  fixes x y :: 'a::real-normed-algebra star
  shows [|x ∈ Infinitesimal; y ∈ Infinitesimal|] ==> (x ∗ y) ∈ Infinitesimal
apply (rule InfinitesimalI)
apply (subgoal-tac hnorm (x ∗ y) < 1 ∗ r, simp only: mult-1)
apply (rule hnorm-mult-less)
apply (simp-all add: InfinitesimalD)
done

lemma Infinitesimal-HFinite-mult:
  fixes x y :: 'a::real-normed-algebra star
  shows [| x ∈ Infinitesimal; y ∈ HFinite |] ==> (x ∗ y) ∈ Infinitesimal
apply (rule InfinitesimalI)
apply (drule HFiniteD, clarify)
apply (subgoal-tac 0 < t)
apply (subgoal-tac hnorm (x ∗ y) < (r / t) ∗ t, simp)
apply (subgoal-tac 0 < r / t)
apply (rule hnorm-mult-less)
apply (simp add: InfinitesimalD Reals-divide)
apply assumption
apply (simp only: divide-pos-pos)
apply (erule order-le-less-trans [OF hnorm-ge-zero])
done

lemma Infinitesimal-HFinite-scaleHR:
```

[| *x* ∈ *Infinitesimal*; *y* ∈ *HFinite* |] ==> *scaleHR x y* ∈ *Infinitesimal*
apply (*rule InfinitesimalI*)
apply (*drule HFiniteD*, *clarify*)
apply (*drule InfinitesimalD*)
apply (*simp add*: *hnorm-scaleHR*)
apply (*subgoal-tac 0* < *t*)
apply (*subgoal-tac* |*x*| * *hnorm y* < (*r* / *t*) * *t*, *simp*)
apply (*subgoal-tac 0* < *r* / *t*)
apply (*rule mult-strict-mono'*, *simp-all*)
apply (*simp only*: *divide-pos-pos*)
apply (*erule order-le-less-trans* [*OF hnorm-ge-zero*])
done

lemma *Infinitesimal-HFinite-mult2*:
fixes *x y* :: *'a::real-normed-algebra star*
shows [| *x* ∈ *Infinitesimal*; *y* ∈ *HFinite* |] ==> (*y* * *x*) ∈ *Infinitesimal*
apply (*rule InfinitesimalI*)
apply (*drule HFiniteD*, *clarify*)
apply (*subgoal-tac 0* < *t*)
apply (*subgoal-tac hnorm* (*y* * *x*) < *t* * (*r* / *t*), *simp*)
apply (*subgoal-tac 0* < *r* / *t*)
apply (*rule hnorm-mult-less*)
apply *assumption*
apply (*simp add*: *InfinitesimalD Reals-divide*)
apply (*simp only*: *divide-pos-pos*)
apply (*erule order-le-less-trans* [*OF hnorm-ge-zero*])
done

lemma *Infinitesimal-scaleR2*:
x ∈ *Infinitesimal* ==> *a* $*_R$ *x* ∈ *Infinitesimal*
apply (*case-tac a* = *0*, *simp*)
apply (*rule InfinitesimalI*)
apply (*drule InfinitesimalD*)
apply (*drule-tac x*=*r* / |*star-of a*| **in** *bspec*)
apply (*simp add*: *Reals-eq-Standard*)
apply (*simp add*: *divide-pos-pos*)
apply (*simp add*: *hnorm-scaleR pos-less-divide-eq mult-commute*)
done

lemma *Compl-HFinite*: − *HFinite* = *HInfinite*
apply (*auto simp add*: *HInfinite-def HFinite-def linorder-not-less*)
apply (*rule-tac y*=*r* + *1* **in** *order-less-le-trans*, *simp*)
apply *simp*
done

lemma *HInfinite-inverse-Infinitesimal*:
fixes *x* :: *'a::real-normed-div-algebra star*
shows *x* ∈ *HInfinite* ==> *inverse x* ∈ *Infinitesimal*
apply (*rule InfinitesimalI*)

apply (*subgoal-tac* $x \neq 0$)
apply (*rule inverse-less-imp-less*)
apply (*simp add*: *nonzero-hnorm-inverse*)
apply (*simp add*: *HInfinite-def Reals-inverse*)
apply *assumption*
apply (*clarify*, *simp add*: *Compl-HFinite* [*symmetric*])
done

lemma *HInfiniteI*: $(\bigwedge r.\ r \in \mathbb{R} \Longrightarrow r < hnorm\ x) \Longrightarrow x \in HInfinite$
by (*simp add*: *HInfinite-def*)

lemma *HInfiniteD*: $[\![x \in HInfinite;\ r \in \mathbb{R}]\!] \Longrightarrow r < hnorm\ x$
by (*simp add*: *HInfinite-def*)

lemma *HInfinite-mult*:
  fixes $x\ y$ :: *'a::real-normed-div-algebra star*
  shows $[|x \in HInfinite;\ y \in HInfinite|] ==> (x*y) \in HInfinite$
apply (*rule HInfiniteI*, *simp only*: *hnorm-mult*)
apply (*subgoal-tac* $r * 1 < hnorm\ x * hnorm\ y$, *simp only*: *mult-1*)
apply (*case-tac* $x = 0$, *simp add*: *HInfinite-def*)
apply (*rule mult-strict-mono*)
apply (*simp-all add*: *HInfiniteD*)
done

lemma *hypreal-add-zero-less-le-mono*: $[|r < x;\ (0::hypreal) \leq y|] ==> r < x+y$
by (*auto dest*: *add-less-le-mono*)

lemma *HInfinite-add-ge-zero*:
  $[|(x::hypreal) \in HInfinite;\ 0 \leq y;\ 0 \leq x|] ==> (x + y)$: *HInfinite*
by (*auto intro*!: *hypreal-add-zero-less-le-mono*
    simp add: *abs-if add-commute add-nonneg-nonneg HInfinite-def*)

lemma *HInfinite-add-ge-zero2*:
  $[|(x::hypreal) \in HInfinite;\ 0 \leq y;\ 0 \leq x|] ==> (y + x)$: *HInfinite*
by (*auto intro*!: *HInfinite-add-ge-zero simp add*: *add-commute*)

lemma *HInfinite-add-gt-zero*:
  $[|(x::hypreal) \in HInfinite;\ 0 < y;\ 0 < x|] ==> (x + y)$: *HInfinite*
by (*blast intro*: *HInfinite-add-ge-zero order-less-imp-le*)

lemma *HInfinite-minus-iff*: $(-x \in HInfinite) = (x \in HInfinite)$
by (*simp add*: *HInfinite-def*)

lemma *HInfinite-add-le-zero*:
  $[|(x::hypreal) \in HInfinite;\ y \leq 0;\ x \leq 0|] ==> (x + y)$: *HInfinite*
apply (*drule HInfinite-minus-iff* [*THEN iffD2*])
apply (*rule HInfinite-minus-iff* [*THEN iffD1*])
apply (*auto intro*: *HInfinite-add-ge-zero*)
done

lemma *HInfinite-add-lt-zero*:
  $[|(x::hypreal) \in HInfinite;\ y < 0;\ x < 0|] ==> (x + y): HInfinite$
by (*blast intro*: *HInfinite-add-le-zero order-less-imp-le*)

lemma *HFinite-sum-squares*:
 fixes $a\ b\ c :: 'a::real\text{-}normed\text{-}algebra\ star$
 shows $[|a: HFinite;\ b: HFinite;\ c: HFinite|]$
  $==> a*a + b*b + c*c \in HFinite$
by (*auto intro*: *HFinite-mult HFinite-add*)

lemma *not-Infinitesimal-not-zero*: $x \notin Infinitesimal ==> x \neq 0$
by *auto*

lemma *not-Infinitesimal-not-zero2*: $x \in HFinite - Infinitesimal ==> x \neq 0$
by *auto*

lemma *HFinite-diff-Infinitesimal-hrabs*:
 $(x::hypreal) \in HFinite - Infinitesimal ==> abs\ x \in HFinite - Infinitesimal$
by *blast*

lemma *hnorm-le-Infinitesimal*:
 $[\![e \in Infinitesimal;\ hnorm\ x \leq e]\!] \Longrightarrow x \in Infinitesimal$
by (*auto simp add*: *Infinitesimal-def abs-less-iff*)

lemma *hnorm-less-Infinitesimal*:
 $[\![e \in Infinitesimal;\ hnorm\ x < e]\!] \Longrightarrow x \in Infinitesimal$
by (*erule hnorm-le-Infinitesimal*, *erule order-less-imp-le*)

lemma *hrabs-le-Infinitesimal*:
  $[|\ e \in Infinitesimal;\ abs\ (x::hypreal) \leq e\ |] ==> x \in Infinitesimal$
by (*erule hnorm-le-Infinitesimal*, *simp*)

lemma *hrabs-less-Infinitesimal*:
  $[|\ e \in Infinitesimal;\ abs\ (x::hypreal) < e\ |] ==> x \in Infinitesimal$
by (*erule hnorm-less-Infinitesimal*, *simp*)

lemma *Infinitesimal-interval*:
  $[|\ e \in Infinitesimal;\ e' \in Infinitesimal;\ e' < x\ ;\ x < e\ |]$
  $==> (x::hypreal) \in Infinitesimal$
by (*auto simp add*: *Infinitesimal-def abs-less-iff*)

lemma *Infinitesimal-interval2*:
  $[|\ e \in Infinitesimal;\ e' \in Infinitesimal;$
   $e' \leq x\ ;\ x \leq e\ |] ==> (x::hypreal) \in Infinitesimal$
by (*auto intro*: *Infinitesimal-interval simp add*: *order-le-less*)

lemma *lemma-Infinitesimal-hyperpow*:

[| (*x*::*hypreal*) ∈ *Infinitesimal*; *0* < *N* |] ==> *abs* (*x pow N*) ≤ *abs x*
apply (*unfold Infinitesimal-def*)
apply (*auto intro*!: *hyperpow-Suc-le-self2*
simp add: *hyperpow-hrabs* [*symmetric*] *hypnat-gt-zero-iff2 abs-ge-zero*)
done

lemma *Infinitesimal-hyperpow*:
[| (*x*::*hypreal*) ∈ *Infinitesimal*; *0* < *N* |] ==> *x pow N* ∈ *Infinitesimal*
apply (*rule hrabs-le-Infinitesimal*)
apply (*rule-tac* [*2*] *lemma-Infinitesimal-hyperpow*, *auto*)
done

lemma *hrealpow-hyperpow-Infinitesimal-iff*:
(*x* ˆ *n* ∈ *Infinitesimal*) = (*x pow* (*hypnat-of-nat n*) ∈ *Infinitesimal*)
by (*simp only*: *hyperpow-hypnat-of-nat*)

lemma *Infinitesimal-hrealpow*:
[| (*x*::*hypreal*) ∈ *Infinitesimal*; *0* < *n* |] ==> *x* ˆ *n* ∈ *Infinitesimal*
by (*simp add*: *hrealpow-hyperpow-Infinitesimal-iff Infinitesimal-hyperpow*)

lemma *not-Infinitesimal-mult*:
fixes *x y* :: *'a*::*real-normed-div-algebra star*
shows [| *x* ∉ *Infinitesimal*; *y* ∉ *Infinitesimal*|] ==> (*x*∗*y*) ∉*Infinitesimal*
apply (*unfold Infinitesimal-def*, *clarify*, *rename-tac r s*)
apply (*simp only*: *linorder-not-less hnorm-mult*)
apply (*drule-tac x* = *r* ∗ *s* **in** *bspec*)
apply (*fast intro*: *Reals-mult*)
apply (*drule mp*, *blast intro*: *mult-pos-pos*)
apply (*drule-tac c* = *s* **and** *d* = *hnorm y* **and** *a* = *r* **and** *b* = *hnorm x* **in** *mult-mono*)
apply (*simp-all* (*no-asm-simp*))
done

lemma *Infinitesimal-mult-disj*:
fixes *x y* :: *'a*::*real-normed-div-algebra star*
shows *x*∗*y* ∈ *Infinitesimal* ==> *x* ∈ *Infinitesimal* | *y* ∈ *Infinitesimal*
apply (*rule ccontr*)
apply (*drule de-Morgan-disj* [*THEN iffD1*])
apply (*fast dest*: *not-Infinitesimal-mult*)
done

lemma *HFinite-Infinitesimal-not-zero*: *x* ∈ *HFinite*−*Infinitesimal* ==> *x* ≠ *0*
by *blast*

lemma *HFinite-Infinitesimal-diff-mult*:
fixes *x y* :: *'a*::*real-normed-div-algebra star*
shows [| *x* ∈ *HFinite* − *Infinitesimal*;
y ∈ *HFinite* − *Infinitesimal*
|] ==> (*x*∗*y*) ∈ *HFinite* − *Infinitesimal*

apply *clarify*
apply (*blast dest*: *HFinite-mult not-Infinitesimal-mult*)
done

lemma *Infinitesimal-subset-HFinite*:
Infinitesimal $\subseteq$ *HFinite*
apply (*simp add*: *Infinitesimal-def HFinite-def*, *auto*)
apply (*rule-tac* $x = 1$ **in** *bexI*, *auto*)
done

lemma *Infinitesimal-star-of-mult*:
fixes x :: *'a::real-normed-algebra star*
shows $x \in$ *Infinitesimal* $==>$ $x *$ *star-of* $r \in$ *Infinitesimal*
by (*erule HFinite-star-of* [*THEN* [*2*] *Infinitesimal-HFinite-mult*])

lemma *Infinitesimal-star-of-mult2*:
fixes x :: *'a::real-normed-algebra star*
shows $x \in$ *Infinitesimal* $==>$ *star-of* $r * x \in$ *Infinitesimal*
by (*erule HFinite-star-of* [*THEN* [*2*] *Infinitesimal-HFinite-mult2*])

27.5 The Infinitely Close Relation

lemma *mem-infmal-iff*: ($x \in$ *Infinitesimal*) = (x @= *0*)
by (*simp add*: *Infinitesimal-def approx-def*)

lemma *approx-minus-iff*: (x @= y) = ($x - y$ @= *0*)
by (*simp add*: *approx-def*)

lemma *approx-minus-iff2*: (x @= y) = ($-y + x$ @= *0*)
by (*simp add*: *approx-def diff-minus add-commute*)

lemma *approx-refl* [*iff*]: x @= x
by (*simp add*: *approx-def Infinitesimal-def*)

lemma *hypreal-minus-distrib1*: $-(y + -(x$::*'a::ab-group-add*$)) = x + -y$
by (*simp add*: *add-commute*)

lemma *approx-sym*: x @= y $==>$ y @= x
apply (*simp add*: *approx-def*)
apply (*drule Infinitesimal-minus-iff* [*THEN iffD2*])
apply *simp*
done

lemma *approx-trans*: [| x @= y; y @= z |] $==>$ x @= z
apply (*simp add*: *approx-def*)
apply (*drule* (*1*) *Infinitesimal-add*)
apply (*simp add*: *diff-def*)
done

lemma *approx-trans2*: [| *r* @= *x*; *s* @= *x* |] ==> *r* @= *s*
by (*blast intro*: *approx-sym approx-trans*)

lemma *approx-trans3*: [| *x* @= *r*; *x* @= *s*|] ==> *r* @= *s*
by (*blast intro*: *approx-sym approx-trans*)

lemma *number-of-approx-reorient*: (*number-of w* @= *x*) = (*x* @= *number-of w*)
by (*blast intro*: *approx-sym*)

lemma *zero-approx-reorient*: (*0* @= *x*) = (*x* @= *0*)
by (*blast intro*: *approx-sym*)

lemma *one-approx-reorient*: (*1* @= *x*) = (*x* @= *1*)
by (*blast intro*: *approx-sym*)

ML ⟪
```
local
(*** re−orientation, following HOL/Integ/Bin.ML
    We re−orient x @=y where x is 0, 1 or a numeral, unless y is as well!
 ***)

(*reorientation simprules using ==, for the following simproc*)
val meta-zero-approx-reorient = thm zero-approx-reorient RS eq-reflection;
val meta-one-approx-reorient = thm one-approx-reorient RS eq-reflection;
val meta-number-of-approx-reorient = thm number-of-approx-reorient RS eq-reflection

(*reorientation simplification procedure: reorients (polymorphic)
  0 = x, 1 = x, nnn = x provided x isn't 0, 1 or a numeral.*)
fun reorient-proc sg - (- $ t $ u) =
  case u of
      Const(@{const-name HOL.zero}, -) => NONE
    | Const(@{const-name HOL.one}, -) => NONE
    | Const(@{const-name Numeral.number-of}, -) $ - => NONE
    | - => SOME (case t of
                Const(@{const-name HOL.zero}, -) => meta-zero-approx-reorient
              | Const(@{const-name HOL.one}, -) => meta-one-approx-reorient
              | Const(@{const-name Numeral.number-of}, -) $ - =>
                              meta-number-of-approx-reorient);

in
val approx-reorient-simproc =
  Int-Numeral-Base-Simprocs.prep-simproc
    (reorient-simproc, [0@=x, 1@=x, number-of w @= x], reorient-proc);
end;

Addsimprocs [approx-reorient-simproc];
```
⟫

lemma *Infinitesimal-approx-minus*: (*x*−*y* ∈ *Infinitesimal*) = (*x* @= *y*)
by (*simp add*: *approx-minus-iff* [*symmetric*] *mem-infmal-iff*)

lemma *approx-monad-iff*: (*x* @= *y*) = (*monad*(*x*)=*monad*(*y*))
apply (*simp add*: *monad-def*)
apply (*auto dest*: *approx-sym elim*!: *approx-trans equalityCE*)
done

lemma *Infinitesimal-approx*:
  [| *x* ∈ *Infinitesimal*; *y* ∈ *Infinitesimal* |] ==> *x* @= *y*
apply (*simp add*: *mem-infmal-iff*)
apply (*blast intro*: *approx-trans approx-sym*)
done

lemma *approx-add*: [| *a* @= *b*; *c* @= *d* |] ==> *a*+*c* @= *b*+*d*
proof (*unfold approx-def*)
  assume *inf*: *a* − *b* ∈ *Infinitesimal c* − *d* ∈ *Infinitesimal*
  have *a* + *c* − (*b* + *d*) = (*a* − *b*) + (*c* − *d*) **by** *simp*
  also have ... ∈ *Infinitesimal* **using** *inf* **by** (*rule Infinitesimal-add*)
  finally show *a* + *c* − (*b* + *d*) ∈ *Infinitesimal* **.**
qed

lemma *approx-minus*: *a* @= *b* ==> −*a* @= −*b*
apply (*rule approx-minus-iff* [*THEN iffD2*, *THEN approx-sym*])
apply (*drule approx-minus-iff* [*THEN iffD1*])
apply (*simp add*: *add-commute diff-def*)
done

lemma *approx-minus2*: −*a* @= −*b* ==> *a* @= *b*
by (*auto dest*: *approx-minus*)

lemma *approx-minus-cancel* [*simp*]: (−*a* @= −*b*) = (*a* @= *b*)
by (*blast intro*: *approx-minus approx-minus2*)

lemma *approx-add-minus*: [| *a* @= *b*; *c* @= *d* |] ==> *a* + −*c* @= *b* + −*d*
by (*blast intro*!: *approx-add approx-minus*)

lemma *approx-diff*: [| *a* @= *b*; *c* @= *d* |] ==> *a* − *c* @= *b* − *d*
by (*simp only*: *diff-minus approx-add approx-minus*)

lemma *approx-mult1*:
  fixes *a b c* :: ′*a*::*real-normed-algebra star*
  shows [| *a* @= *b*; *c*: *HFinite*|] ==> *a*∗*c* @= *b*∗*c*
by (*simp add*: *approx-def Infinitesimal-HFinite-mult*
    left-diff-distrib [*symmetric*])

lemma *approx-mult2*:
  fixes *a b c* :: ′*a*::*real-normed-algebra star*
  shows [|*a* @= *b*; *c*: *HFinite*|] ==> *c*∗*a* @= *c*∗*b*

by (*simp add*: *approx-def Infinitesimal-HFinite-mult2*
right-diff-distrib [*symmetric*])

lemma *approx-mult-subst*:
fixes *u v x y* :: *'a::real-normed-algebra star*
shows [|*u* @= *v*∗*x*; *x* @= *y*; *v* ∈ *HFinite*|] ==> *u* @= *v*∗*y*
by (*blast intro*: *approx-mult2 approx-trans*)

lemma *approx-mult-subst2*:
fixes *u v x y* :: *'a::real-normed-algebra star*
shows [| *u* @= *x*∗*v*; *x* @= *y*; *v* ∈ *HFinite* |] ==> *u* @= *y*∗*v*
by (*blast intro*: *approx-mult1 approx-trans*)

lemma *approx-mult-subst-star-of*:
fixes *u x y* :: *'a::real-normed-algebra star*
shows [| *u* @= *x*∗*star-of v*; *x* @= *y* |] ==> *u* @= *y*∗*star-of v*
by (*auto intro*: *approx-mult-subst2*)

lemma *approx-eq-imp*: *a* = *b* ==> *a* @= *b*
by (*simp add*: *approx-def*)

lemma *Infinitesimal-minus-approx*: *x* ∈ *Infinitesimal* ==> −*x* @= *x*
by (*blast intro*: *Infinitesimal-minus-iff* [*THEN iffD2*]
mem-infmal-iff [*THEN iffD1*] *approx-trans2*)

lemma *bex-Infinitesimal-iff*: (∃ *y* ∈ *Infinitesimal*. *x* − *z* = *y*) = (*x* @= *z*)
by (*simp add*: *approx-def*)

lemma *bex-Infinitesimal-iff2*: (∃ *y* ∈ *Infinitesimal*. *x* = *z* + *y*) = (*x* @= *z*)
by (*force simp add*: *bex-Infinitesimal-iff* [*symmetric*])

lemma *Infinitesimal-add-approx*: [| *y* ∈ *Infinitesimal*; *x* + *y* = *z* |] ==> *x* @= *z*
apply (*rule bex-Infinitesimal-iff* [*THEN iffD1*])
apply (*drule Infinitesimal-minus-iff* [*THEN iffD2*])
apply (*auto simp add*: *add-assoc* [*symmetric*])
done

lemma *Infinitesimal-add-approx-self*: *y* ∈ *Infinitesimal* ==> *x* @= *x* + *y*
apply (*rule bex-Infinitesimal-iff* [*THEN iffD1*])
apply (*drule Infinitesimal-minus-iff* [*THEN iffD2*])
apply (*auto simp add*: *add-assoc* [*symmetric*])
done

lemma *Infinitesimal-add-approx-self2*: *y* ∈ *Infinitesimal* ==> *x* @= *y* + *x*
by (*auto dest*: *Infinitesimal-add-approx-self simp add*: *add-commute*)

lemma *Infinitesimal-add-minus-approx-self*: *y* ∈ *Infinitesimal* ==> *x* @= *x* + −*y*
by (*blast intro*!: *Infinitesimal-add-approx-self Infinitesimal-minus-iff* [*THEN iffD2*])

lemma *Infinitesimal-add-cancel*: [| *y* ∈ *Infinitesimal*; *x*+*y* @= *z*|] ==> *x* @= *z*
apply (*drule-tac x* = *x* **in** *Infinitesimal-add-approx-self* [*THEN approx-sym*])
apply (*erule approx-trans3* [*THEN approx-sym*], *assumption*)
done

lemma *Infinitesimal-add-right-cancel*:
[| *y* ∈ *Infinitesimal*; *x* @= *z* + *y*|] ==> *x* @= *z*
apply (*drule-tac x* = *z* **in** *Infinitesimal-add-approx-self2* [*THEN approx-sym*])
apply (*erule approx-trans3* [*THEN approx-sym*])
apply (*simp add*: *add-commute*)
apply (*erule approx-sym*)
done

lemma *approx-add-left-cancel*: *d* + *b* @= *d* + *c* ==> *b* @= *c*
apply (*drule approx-minus-iff* [*THEN iffD1*])
apply (*simp add*: *approx-minus-iff* [*symmetric*] *add-ac*)
done

lemma *approx-add-right-cancel*: *b* + *d* @= *c* + *d* ==> *b* @= *c*
apply (*rule approx-add-left-cancel*)
apply (*simp add*: *add-commute*)
done

lemma *approx-add-mono1*: *b* @= *c* ==> *d* + *b* @= *d* + *c*
apply (*rule approx-minus-iff* [*THEN iffD2*])
apply (*simp add*: *approx-minus-iff* [*symmetric*] *add-ac*)
done

lemma *approx-add-mono2*: *b* @= *c* ==> *b* + *a* @= *c* + *a*
by (*simp add*: *add-commute approx-add-mono1*)

lemma *approx-add-left-iff* [*simp*]: (*a* + *b* @= *a* + *c*) = (*b* @= *c*)
by (*fast elim*: *approx-add-left-cancel approx-add-mono1*)

lemma *approx-add-right-iff* [*simp*]: (*b* + *a* @= *c* + *a*) = (*b* @= *c*)
by (*simp add*: *add-commute*)

lemma *approx-HFinite*: [| *x* ∈ *HFinite*; *x* @= *y* |] ==> *y* ∈ *HFinite*
apply (*drule bex-Infinitesimal-iff2* [*THEN iffD2*], *safe*)
apply (*drule Infinitesimal-subset-HFinite* [*THEN subsetD*, *THEN HFinite-minus-iff* [*THEN iffD2*]])
apply (*drule HFinite-add*)
apply (*auto simp add*: *add-assoc*)
done

lemma *approx-star-of-HFinite*: *x* @= *star-of D* ==> *x* ∈ *HFinite*
by (*rule approx-sym* [*THEN* [*2*] *approx-HFinite*], *auto*)

lemma *approx-mult-HFinite*:

```
  fixes a b c d :: 'a::real-normed-algebra star
  shows [|a @= b; c @= d; b: HFinite; d: HFinite|] ==> a*c @= b*d
apply (rule approx-trans)
apply (rule-tac [2] approx-mult2)
apply (rule approx-mult1)
prefer 2 apply (blast intro: approx-HFinite approx-sym, auto)
done

lemma scaleHR-left-diff-distrib:
  ⋀a b x. scaleHR (a - b) x = scaleHR a x - scaleHR b x
by transfer (rule scaleR-left-diff-distrib)

lemma approx-scaleR1:
  [| a @= star-of b; c: HFinite|] ==> scaleHR a c @= b *R c
apply (unfold approx-def)
apply (drule (1) Infinitesimal-HFinite-scaleHR)
apply (simp only: scaleHR-left-diff-distrib)
apply (simp add: scaleHR-def star-scaleR-def [symmetric])
done

lemma approx-scaleR2:
  a @= b ==> c *R a @= c *R b
by (simp add: approx-def Infinitesimal-scaleR2
            scaleR-right-diff-distrib [symmetric])

lemma approx-scaleR-HFinite:
  [|a @= star-of b; c @= d; d: HFinite|] ==> scaleHR a c @= b *R d
apply (rule approx-trans)
apply (rule-tac [2] approx-scaleR2)
apply (rule approx-scaleR1)
prefer 2 apply (blast intro: approx-HFinite approx-sym, auto)
done

lemma approx-mult-star-of:
  fixes a c :: 'a::real-normed-algebra star
  shows [|a @= star-of b; c @= star-of d |]
     ==> a*c @= star-of b*star-of d
by (blast intro!: approx-mult-HFinite approx-star-of-HFinite HFinite-star-of)

lemma approx-SReal-mult-cancel-zero:
    [| (a::hypreal) ∈ Reals; a ≠ 0; a*x @= 0 |] ==> x @= 0
apply (drule Reals-inverse [THEN SReal-subset-HFinite [THEN subsetD]])
apply (auto dest: approx-mult2 simp add: mult-assoc [symmetric])
done

lemma approx-mult-SReal1: [| (a::hypreal) ∈ Reals; x @= 0 |] ==> x*a @= 0
by (auto dest: SReal-subset-HFinite [THEN subsetD] approx-mult1)

lemma approx-mult-SReal2: [| (a::hypreal) ∈ Reals; x @= 0 |] ==> a*x @= 0
```

by (*auto dest*: *SReal-subset-HFinite* [*THEN subsetD*] *approx-mult2*)

lemma *approx-mult-SReal-zero-cancel-iff* [*simp*]:
[|(*a*::*hypreal*) ∈ *Reals*; *a* ≠ *0* |] ==> (*a***x* @= *0*) = (*x* @= *0*)
by (*blast intro*: *approx-SReal-mult-cancel-zero approx-mult-SReal2*)

lemma *approx-SReal-mult-cancel*:
[| (*a*::*hypreal*) ∈ *Reals*; *a* ≠ *0*; *a** *w* @= *a***z* |] ==> *w* @= *z*
apply (*drule Reals-inverse* [*THEN SReal-subset-HFinite* [*THEN subsetD*]])
apply (*auto dest*: *approx-mult2 simp add*: *mult-assoc* [*symmetric*])
done

lemma *approx-SReal-mult-cancel-iff1* [*simp*]:
[| (*a*::*hypreal*) ∈ *Reals*; *a* ≠ *0*|] ==> (*a** *w* @= *a***z*) = (*w* @= *z*)
by (*auto intro*!: *approx-mult2 SReal-subset-HFinite* [*THEN subsetD*]
intro: *approx-SReal-mult-cancel*)

lemma *approx-le-bound*: [| (*z*::*hypreal*) ≤ *f*; *f* @= *g*; *g* ≤ *z* |] ==> *f* @= *z*
apply (*simp add*: *bex-Infinitesimal-iff2* [*symmetric*], *auto*)
apply (*rule-tac x* = *g*+*y*−*z* **in** *bexI*)
apply (*simp* (*no-asm*))
apply (*rule Infinitesimal-interval2*)
apply (*rule-tac* [*2*] *Infinitesimal-zero*, *auto*)
done

lemma *approx-hnorm*:
fixes *x y* :: *'a*::*real-normed-vector star*
shows $x \approx y \Longrightarrow$ *hnorm* $x \approx$ *hnorm* y
proof (*unfold approx-def*)
assume $x - y \in$ *Infinitesimal*
hence *1*: *hnorm* $(x - y) \in$ *Infinitesimal*
by (*simp only*: *Infinitesimal-hnorm-iff*)
moreover have *2*: (*0*::*real star*) ∈ *Infinitesimal*
by (*rule Infinitesimal-zero*)
moreover have *3*: $0 \le |$*hnorm* $x -$ *hnorm* $y|$
by (*rule abs-ge-zero*)
moreover have *4*: $|$*hnorm* $x -$ *hnorm* $y| \le$ *hnorm* $(x - y)$
by (*rule hnorm-triangle-ineq3*)
ultimately have $|$*hnorm* $x -$ *hnorm* $y| \in$ *Infinitesimal*
by (*rule Infinitesimal-interval2*)
thus *hnorm* $x -$ *hnorm* $y \in$ *Infinitesimal*
by (*simp only*: *Infinitesimal-hrabs-iff*)
qed

27.6 Zero is the Only Infinitesimal that is also a Real

lemma *Infinitesimal-less-SReal*:
[| (*x*::*hypreal*) ∈ *Reals*; *y* ∈ *Infinitesimal*; *0* < *x* |] ==> *y* < *x*
apply (*simp add*: *Infinitesimal-def*)

apply (*rule abs-ge-self* [*THEN order-le-less-trans*], *auto*)
done

lemma *Infinitesimal-less-SReal2*:
  $(y::hypreal) \in Infinitesimal ==> \forall r \in Reals.\ 0 < r --> y < r$
by (*blast intro*: *Infinitesimal-less-SReal*)

lemma *SReal-not-Infinitesimal*:
  $[|\ 0 < y;\ (y::hypreal) \in Reals|] ==> y \notin Infinitesimal$
apply (*simp add*: *Infinitesimal-def*)
apply (*auto simp add*: *abs-if*)
done

lemma *SReal-minus-not-Infinitesimal*:
  $[|\ y < 0;\ (y::hypreal) \in Reals\ |] ==> y \notin Infinitesimal$
apply (*subst Infinitesimal-minus-iff* [*symmetric*])
apply (*rule SReal-not-Infinitesimal*, *auto*)
done

lemma *SReal-Int-Infinitesimal-zero*: $Reals\ Int\ Infinitesimal = \{0::hypreal\}$
apply *auto*
apply (*cut-tac* $x = x$ **and** $y = 0$ **in** *linorder-less-linear*)
apply (*blast dest*: *SReal-not-Infinitesimal SReal-minus-not-Infinitesimal*)
done

lemma *SReal-Infinitesimal-zero*:
  $[|\ (x::hypreal) \in Reals;\ x \in Infinitesimal|] ==> x = 0$
by (*cut-tac SReal-Int-Infinitesimal-zero*, *blast*)

lemma *SReal-HFinite-diff-Infinitesimal*:
  $[|\ (x::hypreal) \in Reals;\ x \neq 0\ |] ==> x \in HFinite - Infinitesimal$
by (*auto dest*: *SReal-Infinitesimal-zero SReal-subset-HFinite* [*THEN subsetD*])

lemma *hypreal-of-real-HFinite-diff-Infinitesimal*:
  $hypreal\text{-}of\text{-}real\ x \neq 0 ==> hypreal\text{-}of\text{-}real\ x \in HFinite - Infinitesimal$
by (*rule SReal-HFinite-diff-Infinitesimal*, *auto*)

lemma *star-of-Infinitesimal-iff-0* [*iff*]:
  $(star\text{-}of\ x \in Infinitesimal) = (x = 0)$
apply (*auto simp add*: *Infinitesimal-def*)
apply (*drule-tac x=hnorm* (*star-of x*) **in** *bspec*)
apply (*simp add*: *SReal-def*)
apply (*rule-tac x=norm x* **in** *exI*, *simp*)
apply *simp*
done

lemma *star-of-HFinite-diff-Infinitesimal*:
  $x \neq 0 ==> star\text{-}of\ x \in HFinite - Infinitesimal$
by *simp*

lemma *number-of-not-Infinitesimal* [*simp*]:
number-of $w \neq$ (*0*::*hypreal*) ==> (*number-of* w :: *hypreal*) $\notin$ *Infinitesimal*
by (*fast dest*: *Reals-number-of* [*THEN SReal-Infinitesimal-zero*])

lemma *one-not-Infinitesimal* [*simp*]:
(*1*::'*a*::{*real-normed-vector*,*zero-neq-one*} *star*) $\notin$ *Infinitesimal*
apply (*simp only*: *star-one-def star-of-Infinitesimal-iff-0*)
apply *simp*
done

lemma *approx-SReal-not-zero*:
[| (*y*::*hypreal*) $\in$ *Reals*; x @= y; $y \neq 0$ |] ==> $x \neq 0$
apply (*cut-tac* $x = 0$ **and** $y = y$ **in** *linorder-less-linear*, *simp*)
apply (*blast dest*: *approx-sym* [*THEN mem-infmal-iff* [*THEN iffD2*]] *SReal-not-Infinitesimal SReal-minus-not-Infinitesimal*)
done

lemma *HFinite-diff-Infinitesimal-approx*:
[| x @= y; $y \in$ *HFinite* $-$ *Infinitesimal* |]
==> $x \in$ *HFinite* $-$ *Infinitesimal*
apply (*auto intro*: *approx-sym* [*THEN* [*2*] *approx-HFinite*]
simp add: *mem-infmal-iff*)
apply (*drule approx-trans3*, *assumption*)
apply (*blast dest*: *approx-sym*)
done

lemma *Infinitesimal-ratio*:
fixes x y :: '*a*::{*real-normed-div-algebra*,*field*} *star*
shows [| $y \neq 0$; $y \in$ *Infinitesimal*; $x/y \in$ *HFinite* |]
==> $x \in$ *Infinitesimal*
apply (*drule Infinitesimal-HFinite-mult2*, *assumption*)
apply (*simp add*: *divide-inverse mult-assoc*)
done

lemma *Infinitesimal-SReal-divide*:
[| (*x*::*hypreal*) $\in$ *Infinitesimal*; $y \in$ *Reals* |] ==> $x/y \in$ *Infinitesimal*
apply (*simp add*: *divide-inverse*)
apply (*auto intro*!: *Infinitesimal-HFinite-mult*
dest!: *Reals-inverse* [*THEN SReal-subset-HFinite* [*THEN subsetD*]])
done

27.7 Uniqueness: Two Infinitely Close Reals are Equal

lemma *star-of-approx-iff* [*simp*]: (*star-of* x @= *star-of* y) = ($x = y$)
apply *safe*
apply (*simp add*: *approx-def*)

apply (*simp only*: *star-of-diff* [*symmetric*])
apply (*simp only*: *star-of-Infinitesimal-iff-0*)
apply *simp*
done

lemma *SReal-approx-iff*:
[|(*x*::*hypreal*) ∈ *Reals*; *y* ∈ *Reals*|] ==> (*x* @= *y*) = (*x* = *y*)
apply *auto*
apply (*simp add*: *approx-def*)
apply (*drule* (*1*) *Reals-diff*)
apply (*drule* (*1*) *SReal-Infinitesimal-zero*)
apply *simp*
done

lemma *number-of-approx-iff* [*simp*]:
(*number-of v* @= (*number-of w* :: ′*a*::{*number*,*real-normed-vector*} *star*)) =
(*number-of v* = (*number-of w* :: ′*a*))
apply (*unfold star-number-def*)
apply (*rule star-of-approx-iff*)
done

lemma [*simp*]:
(*number-of w* @= (*0*::′*a*::{*number*,*real-normed-vector*} *star*)) =
(*number-of w* = (*0*::′*a*))
((*0*::′*a*::{*number*,*real-normed-vector*} *star*) @= *number-of w*) =
(*number-of w* = (*0*::′*a*))
(*number-of w* @= (*1*::′*b*::{*number*,*one*,*real-normed-vector*} *star*)) =
(*number-of w* = (*1*::′*b*))
((*1*::′*b*::{*number*,*one*,*real-normed-vector*} *star*) @= *number-of w*) =
(*number-of w* = (*1*::′*b*))
~ (*0* @= (*1*::′*c*::{*zero-neq-one*,*real-normed-vector*} *star*))
~ (*1* @= (*0*::′*c*::{*zero-neq-one*,*real-normed-vector*} *star*))
apply (*unfold star-number-def star-zero-def star-one-def*)
apply (*unfold star-of-approx-iff*)
by (*auto intro*: *sym*)

lemma *star-of-approx-number-of-iff* [*simp*]:
(*star-of k* @= *number-of w*) = (*k* = *number-of w*)
by (*subst star-of-approx-iff* [*symmetric*], *auto*)

lemma *star-of-approx-zero-iff* [*simp*]: (*star-of k* @= *0*) = (*k* = *0*)
by (*simp-all add*: *star-of-approx-iff* [*symmetric*])

lemma *star-of-approx-one-iff* [*simp*]: (*star-of k* @= *1*) = (*k* = *1*)
by (*simp-all add*: *star-of-approx-iff* [*symmetric*])

lemma *approx-unique-real*:
[| (*r*::*hypreal*) ∈ *Reals*; *s* ∈ *Reals*; *r* @= *x*; *s* @= *x*|] ==> *r* = *s*

by (*blast intro*: *SReal-approx-iff* [*THEN iffD1*] *approx-trans2*)

27.8 Existence of Unique Real Infinitely Close

27.8.1 Lifting of the Ub and Lub Properties

lemma *hypreal-of-real-isUb-iff*:
  (*isUb* (*Reals*) (*hypreal-of-real* ' *Q*) (*hypreal-of-real* *Y*)) =
  (*isUb* (*UNIV* :: *real set*) *Q* *Y*)
by (*simp add*: *isUb-def setle-def*)

lemma *hypreal-of-real-isLub1*:
  isLub Reals (*hypreal-of-real* ' *Q*) (*hypreal-of-real* *Y*)
  ==> *isLub* (*UNIV* :: *real set*) *Q* *Y*
apply (*simp add*: *isLub-def leastP-def*)
apply (*auto intro*: *hypreal-of-real-isUb-iff* [*THEN iffD2*]
    simp add: *hypreal-of-real-isUb-iff setge-def*)
done

lemma *hypreal-of-real-isLub2*:
  isLub (*UNIV* :: *real set*) *Q* *Y*
  ==> *isLub Reals* (*hypreal-of-real* ' *Q*) (*hypreal-of-real* *Y*)
apply (*simp add*: *isLub-def leastP-def*)
apply (*auto simp add*: *hypreal-of-real-isUb-iff setge-def*)
apply (*frule-tac x2* = *x* **in** *isUbD2a* [*THEN SReal-iff* [*THEN iffD1*], *THEN exE*])
prefer *2* **apply** *assumption*
apply (*drule-tac x* = *xa* **in** *spec*)
apply (*auto simp add*: *hypreal-of-real-isUb-iff*)
done

lemma *hypreal-of-real-isLub-iff*:
  (*isLub Reals* (*hypreal-of-real* ' *Q*) (*hypreal-of-real* *Y*)) =
  (*isLub* (*UNIV* :: *real set*) *Q* *Y*)
by (*blast intro*: *hypreal-of-real-isLub1 hypreal-of-real-isLub2*)

lemma *lemma-isUb-hypreal-of-real*:
  isUb Reals P Y ==> ∃ *Yo*. *isUb Reals P* (*hypreal-of-real Yo*)
by (*auto simp add*: *SReal-iff isUb-def*)

lemma *lemma-isLub-hypreal-of-real*:
  isLub Reals P Y ==> ∃ *Yo*. *isLub Reals P* (*hypreal-of-real Yo*)
by (*auto simp add*: *isLub-def leastP-def isUb-def SReal-iff*)

lemma *lemma-isLub-hypreal-of-real2*:
  ∃ *Yo*. *isLub Reals P* (*hypreal-of-real Yo*) ==> ∃ *Y*. *isLub Reals P Y*
by (*auto simp add*: *isLub-def leastP-def isUb-def*)

lemma *SReal-complete*:
  [| *P* ⊆ *Reals*; ∃ *x*. *x* ∈ *P*; ∃ *Y*. *isUb Reals P Y* |]
  ==> ∃ *t*::*hypreal*. *isLub Reals P t*

apply (*frule SReal-hypreal-of-real-image*)
apply (*auto*, *drule lemma-isUb-hypreal-of-real*)
apply (*auto intro*!: *reals-complete lemma-isLub-hypreal-of-real2*
simp add: *hypreal-of-real-isLub-iff hypreal-of-real-isUb-iff*)
done

lemma *hypreal-isLub-unique*:
[| *isLub R S x*; *isLub R S y* |] ==> $x = (y::hypreal)$
apply (*frule isLub-isUb*)
apply (*frule-tac* $x = y$ **in** *isLub-isUb*)
apply (*blast intro*!: *order-antisym dest*!: *isLub-le-isUb*)
done

lemma *lemma-st-part-ub*:
$(x::hypreal) \in HFinite$ ==> $\exists u.$ *isUb Reals* $\{s.\ s \in Reals\ \&\ s < x\}\ u$
apply (*drule HFiniteD*, *safe*)
apply (*rule exI*, *rule isUbI*)
apply (*auto intro*: *setleI isUbI simp add*: *abs-less-iff*)
done

lemma *lemma-st-part-nonempty*:
$(x::hypreal) \in HFinite$ ==> $\exists y.\ y \in \{s.\ s \in Reals\ \&\ s < x\}$
apply (*drule HFiniteD*, *safe*)
apply (*drule Reals-minus*)
apply (*rule-tac* $x = -t$ **in** *exI*)
apply (*auto simp add*: *abs-less-iff*)
done

lemma *lemma-st-part-subset*: $\{s.\ s \in Reals\ \&\ s < x\} \subseteq Reals$
by *auto*

lemma *lemma-st-part-lub*:
$(x::hypreal) \in HFinite$ ==> $\exists t.$ *isLub Reals* $\{s.\ s \in Reals\ \&\ s < x\}\ t$
by (*blast intro*!: *SReal-complete lemma-st-part-ub lemma-st-part-nonempty lemma-st-part-subset*)

lemma *lemma-hypreal-le-left-cancel*: $((t::hypreal) + r \le t) = (r \le 0)$
apply *safe*
apply (*drule-tac* $c = -t$ **in** *add-left-mono*)
apply (*drule-tac* [2] $c = t$ **in** *add-left-mono*)
apply (*auto simp add*: *add-assoc* [*symmetric*])
done

lemma *lemma-st-part-le1*:
[| $(x::hypreal) \in HFinite$; *isLub Reals* $\{s.\ s \in Reals\ \&\ s < x\}\ t$;
$r \in Reals$; $0 < r$ |] ==> $x \le t + r$
apply (*frule isLubD1a*)
apply (*rule ccontr*, *drule linorder-not-le* [*THEN iffD2*])
apply (*drule* (*1*) *Reals-add*)

apply (*drule-tac* $y = r + t$ **in** *isLubD1* [*THEN setleD*], *auto*)
done

lemma *hypreal-setle-less-trans*:
  [| S *<= (x::*hypreal*); $x < y$ |] ==> S *<= y
apply (*simp add*: *setle-def*)
apply (*auto dest*!: *bspec order-le-less-trans intro*: *order-less-imp-le*)
done

lemma *hypreal-gt-isUb*:
  [| *isUb R S* (x::*hypreal*); $x < y$; $y \in R$ |] ==> *isUb R S y*
apply (*simp add*: *isUb-def*)
apply (*blast intro*: *hypreal-setle-less-trans*)
done

lemma *lemma-st-part-gt-ub*:
  [| (x::*hypreal*) $\in$ *HFinite*; $x < y$; $y \in$ *Reals* |]
  ==> *isUb Reals* {s. $s \in$ *Reals* & $s < x$} y
by (*auto dest*: *order-less-trans intro*: *order-less-imp-le intro*!: *isUbI setleI*)

lemma *lemma-minus-le-zero*: $t \leq t + -r$ ==> $r \leq$ (*0*::*hypreal*)
apply (*drule-tac* $c = -t$ **in** *add-left-mono*)
apply (*auto simp add*: *add-assoc* [*symmetric*])
done

lemma *lemma-st-part-le2*:
  [| (x::*hypreal*) $\in$ *HFinite*;
    isLub Reals {s. $s \in$ *Reals* & $s < x$} t;
    $r \in$ *Reals*; $0 < r$ |]
  ==> $t + -r \leq x$
apply (*frule isLubD1a*)
apply (*rule ccontr*, *drule linorder-not-le* [*THEN iffD1*])
apply (*drule Reals-minus*, *drule-tac* $a = t$ **in** *Reals-add*, *assumption*)
apply (*drule lemma-st-part-gt-ub*, *assumption*+)
apply (*drule isLub-le-isUb*, *assumption*)
apply (*drule lemma-minus-le-zero*)
apply (*auto dest*: *order-less-le-trans*)
done

lemma *lemma-st-part1a*:
  [| (x::*hypreal*) $\in$ *HFinite*;
    isLub Reals {s. $s \in$ *Reals* & $s < x$} t;
    $r \in$ *Reals*; $0 < r$ |]
  ==> $x + -t \leq r$
apply (*subgoal-tac* $x \leq t{+}r$)
apply (*auto intro*: *lemma-st-part-le1*)
done

lemma *lemma-st-part2a*:

```
  [| (x::hypreal) ∈ HFinite;
     isLub Reals {s. s ∈ Reals & s < x} t;
     r ∈ Reals;  0 < r |]
   ==> −(x + −t) ≤ r
apply (subgoal-tac (t + −r ≤ x))
apply (auto intro: lemma-st-part-le2)
done

lemma lemma-SReal-ub:
  (x::hypreal) ∈ Reals ==> isUb Reals {s. s ∈ Reals & s < x} x
by (auto intro: isUbI setleI order-less-imp-le)

lemma lemma-SReal-lub:
  (x::hypreal) ∈ Reals ==> isLub Reals {s. s ∈ Reals & s < x} x
apply (auto intro!: isLubI2 lemma-SReal-ub setgeI)
apply (frule isUbD2a)
apply (rule-tac x = x and y = y in linorder-cases)
apply (auto intro!: order-less-imp-le)
apply (drule SReal-dense, assumption, assumption, safe)
apply (drule-tac y = r in isUbD)
apply (auto dest: order-less-le-trans)
done

lemma lemma-st-part-not-eq1:
  [| (x::hypreal) ∈ HFinite;
     isLub Reals {s. s ∈ Reals & s < x} t;
     r ∈ Reals; 0 < r |]
   ==> x + −t ≠ r
apply auto
apply (frule isLubD1a [THEN Reals-minus])
apply (drule Reals-add-cancel, assumption)
apply (drule-tac x = x in lemma-SReal-lub)
apply (drule hypreal-isLub-unique, assumption, auto)
done

lemma lemma-st-part-not-eq2:
  [| (x::hypreal) ∈ HFinite;
     isLub Reals {s. s ∈ Reals & s < x} t;
     r ∈ Reals; 0 < r |]
   ==> −(x + −t) ≠ r
apply (auto)
apply (frule isLubD1a)
apply (drule Reals-add-cancel, assumption)
apply (drule-tac a = −x in Reals-minus, simp)
apply (drule-tac x = x in lemma-SReal-lub)
apply (drule hypreal-isLub-unique, assumption, auto)
done

lemma lemma-st-part-major:
```

[| (*x*::*hypreal*) ∈ *HFinite*;
isLub Reals {*s*. *s* ∈ *Reals* & *s* < *x*} *t*;
r ∈ *Reals*; *0* < *r* |]
==> *abs* (*x* − *t*) < *r*
apply (*frule lemma-st-part1a*)
apply (*frule-tac* [*4*] *lemma-st-part2a*, *auto*)
apply (*drule order-le-imp-less-or-eq*)+
apply (*auto dest*: *lemma-st-part-not-eq1 lemma-st-part-not-eq2 simp add*: *abs-less-iff*)
done

lemma *lemma-st-part-major2*:
[| (*x*::*hypreal*) ∈ *HFinite*; *isLub Reals* {*s*. *s* ∈ *Reals* & *s* < *x*} *t* |]
==> ∀ *r* ∈ *Reals*. *0* < *r* −−> *abs* (*x* − *t*) < *r*
by (*blast dest*!: *lemma-st-part-major*)

Existence of real and Standard Part Theorem

lemma *lemma-st-part-Ex*:
(*x*::*hypreal*) ∈ *HFinite*
==> ∃ *t* ∈ *Reals*. ∀ *r* ∈ *Reals*. *0* < *r* −−> *abs* (*x* − *t*) < *r*
apply (*frule lemma-st-part-lub*, *safe*)
apply (*frule isLubD1a*)
apply (*blast dest*: *lemma-st-part-major2*)
done

lemma *st-part-Ex*:
(*x*::*hypreal*) ∈ *HFinite* ==> ∃ *t* ∈ *Reals*. *x* @= *t*
apply (*simp add*: *approx-def Infinitesimal-def*)
apply (*drule lemma-st-part-Ex*, *auto*)
done

There is a unique real infinitely close

lemma *st-part-Ex1*: *x* ∈ *HFinite* ==> *EX*! *t*::*hypreal*. *t* ∈ *Reals* & *x* @= *t*
apply (*drule st-part-Ex*, *safe*)
apply (*drule-tac* [*2*] *approx-sym*, *drule-tac* [*2*] *approx-sym*, *drule-tac* [*2*] *approx-sym*)
apply (*auto intro*!: *approx-unique-real*)
done

27.9 Finite, Infinite and Infinitesimal

lemma *HFinite-Int-HInfinite-empty* [*simp*]: *HFinite Int HInfinite* = {}
apply (*simp add*: *HFinite-def HInfinite-def*)
apply (*auto dest*: *order-less-trans*)
done

lemma *HFinite-not-HInfinite*:
assumes *x*: *x* ∈ *HFinite* **shows** *x* ∉ *HInfinite*
proof
assume *x'*: *x* ∈ *HInfinite*
with *x* **have** *x* ∈ *HFinite* ∩ *HInfinite* **by** *blast*

```
  thus False by auto
qed

lemma not-HFinite-HInfinite: x∉ HFinite ==> x ∈ HInfinite
apply (simp add: HInfinite-def HFinite-def, auto)
apply (drule-tac x = r + 1 in bspec)
apply (auto)
done

lemma HInfinite-HFinite-disj: x ∈ HInfinite | x ∈ HFinite
by (blast intro: not-HFinite-HInfinite)

lemma HInfinite-HFinite-iff: (x ∈ HInfinite) = (x ∉ HFinite)
by (blast dest: HFinite-not-HInfinite not-HFinite-HInfinite)

lemma HFinite-HInfinite-iff: (x ∈ HFinite) = (x ∉ HInfinite)
by (simp add: HInfinite-HFinite-iff)


lemma HInfinite-diff-HFinite-Infinitesimal-disj:
     x ∉ Infinitesimal ==> x ∈ HInfinite | x ∈ HFinite − Infinitesimal
by (fast intro: not-HFinite-HInfinite)

lemma HFinite-inverse:
  fixes x :: 'a::real-normed-div-algebra star
  shows [| x ∈ HFinite; x ∉ Infinitesimal |] ==> inverse x ∈ HFinite
apply (subgoal-tac x ≠ 0)
apply (cut-tac x = inverse x in HInfinite-HFinite-disj)
apply (auto dest!: HInfinite-inverse-Infinitesimal
            simp add: nonzero-inverse-inverse-eq)
done

lemma HFinite-inverse2:
  fixes x :: 'a::real-normed-div-algebra star
  shows x ∈ HFinite − Infinitesimal ==> inverse x ∈ HFinite
by (blast intro: HFinite-inverse)


lemma Infinitesimal-inverse-HFinite:
  fixes x :: 'a::real-normed-div-algebra star
  shows x ∉ Infinitesimal ==> inverse(x) ∈ HFinite
apply (drule HInfinite-diff-HFinite-Infinitesimal-disj)
apply (blast intro: HFinite-inverse HInfinite-inverse-Infinitesimal Infinitesimal-subset-HFinite
[THEN subsetD])
done

lemma HFinite-not-Infinitesimal-inverse:
  fixes x :: 'a::real-normed-div-algebra star
  shows x ∈ HFinite − Infinitesimal ==> inverse x ∈ HFinite − Infinitesimal
```

```
apply (auto intro: Infinitesimal-inverse-HFinite)
apply (drule Infinitesimal-HFinite-mult2, assumption)
apply (simp add: not-Infinitesimal-not-zero right-inverse)
done

lemma approx-inverse:
  fixes x y :: 'a::real-normed-div-algebra star
  shows
    [| x @= y; y ∈  HFinite − Infinitesimal |]
     ==> inverse x @= inverse y
apply (frule HFinite-diff-Infinitesimal-approx, assumption)
apply (frule not-Infinitesimal-not-zero2)
apply (frule-tac x = x in not-Infinitesimal-not-zero2)
apply (drule HFinite-inverse2)+
apply (drule approx-mult2, assumption, auto)
apply (drule-tac c = inverse x in approx-mult1, assumption)
apply (auto intro: approx-sym simp add: mult-assoc)
done


lemmas star-of-approx-inverse = star-of-HFinite-diff-Infinitesimal [THEN [2] approx-inverse]
lemmas hypreal-of-real-approx-inverse =  hypreal-of-real-HFinite-diff-Infinitesimal
[THEN [2] approx-inverse]

lemma inverse-add-Infinitesimal-approx:
  fixes x h :: 'a::real-normed-div-algebra star
  shows
    [| x ∈ HFinite − Infinitesimal;
       h ∈ Infinitesimal |] ==> inverse(x + h) @= inverse x
apply (auto intro: approx-inverse approx-sym Infinitesimal-add-approx-self)
done

lemma inverse-add-Infinitesimal-approx2:
  fixes x h :: 'a::real-normed-div-algebra star
  shows
    [| x ∈ HFinite − Infinitesimal;
       h ∈ Infinitesimal |] ==> inverse(h + x) @= inverse x
apply (rule add-commute [THEN subst])
apply (blast intro: inverse-add-Infinitesimal-approx)
done

lemma inverse-add-Infinitesimal-approx-Infinitesimal:
  fixes x h :: 'a::real-normed-div-algebra star
  shows
    [| x ∈ HFinite − Infinitesimal;
       h ∈ Infinitesimal |] ==> inverse(x + h) − inverse x @= h
apply (rule approx-trans2)
apply (auto intro: inverse-add-Infinitesimal-approx
          simp add: mem-infmal-iff approx-minus-iff [symmetric])
```

```
done

lemma Infinitesimal-square-iff:
  fixes x :: 'a::real-normed-div-algebra star
  shows (x ∈ Infinitesimal) = (x*x ∈ Infinitesimal)
apply (auto intro: Infinitesimal-mult)
apply (rule ccontr, frule Infinitesimal-inverse-HFinite)
apply (frule not-Infinitesimal-not-zero)
apply (auto dest: Infinitesimal-HFinite-mult simp add: mult-assoc)
done
declare Infinitesimal-square-iff [symmetric, simp]

lemma HFinite-square-iff [simp]:
  fixes x :: 'a::real-normed-div-algebra star
  shows (x*x ∈ HFinite) = (x ∈ HFinite)
apply (auto intro: HFinite-mult)
apply (auto dest: HInfinite-mult simp add: HFinite-HInfinite-iff)
done

lemma HInfinite-square-iff [simp]:
  fixes x :: 'a::real-normed-div-algebra star
  shows (x*x ∈ HInfinite) = (x ∈ HInfinite)
by (auto simp add: HInfinite-HFinite-iff)

lemma approx-HFinite-mult-cancel:
  fixes a w z :: 'a::real-normed-div-algebra star
  shows [| a: HFinite−Infinitesimal; a* w @= a*z |] ==> w @= z
apply safe
apply (frule HFinite-inverse, assumption)
apply (drule not-Infinitesimal-not-zero)
apply (auto dest: approx-mult2 simp add: mult-assoc [symmetric])
done

lemma approx-HFinite-mult-cancel-iff1:
  fixes a w z :: 'a::real-normed-div-algebra star
  shows a: HFinite−Infinitesimal ==> (a * w @= a * z) = (w @= z)
by (auto intro: approx-mult2 approx-HFinite-mult-cancel)

lemma HInfinite-HFinite-add-cancel:
     [| x + y ∈ HInfinite; y ∈ HFinite |] ==> x ∈ HInfinite
apply (rule ccontr)
apply (drule HFinite-HInfinite-iff [THEN iffD2])
apply (auto dest: HFinite-add simp add: HInfinite-HFinite-iff)
done

lemma HInfinite-HFinite-add:
     [| x ∈ HInfinite; y ∈ HFinite |] ==> x + y ∈ HInfinite
apply (rule-tac y = −y in HInfinite-HFinite-add-cancel)
apply (auto simp add: add-assoc HFinite-minus-iff)
```

done

lemma *HInfinite-ge-HInfinite*:
[| (*x*::*hypreal*) ∈ *HInfinite*; *x* ≤ *y*; *0* ≤ *x* |] ==> *y* ∈ *HInfinite*
by (*auto intro*: *HFinite-bounded simp add*: *HInfinite-HFinite-iff*)

lemma *Infinitesimal-inverse-HInfinite*:
fixes *x* :: ′*a*::*real-normed-div-algebra star*
shows [| *x* ∈ *Infinitesimal*; *x* ≠ *0* |] ==> *inverse x* ∈ *HInfinite*
apply (*rule ccontr*, *drule HFinite-HInfinite-iff* [*THEN iffD2*])
apply (*auto dest*: *Infinitesimal-HFinite-mult2*)
done

lemma *HInfinite-HFinite-not-Infinitesimal-mult*:
fixes *x y* :: ′*a*::*real-normed-div-algebra star*
shows [| *x* ∈ *HInfinite*; *y* ∈ *HFinite* − *Infinitesimal* |]
==> *x* ∗ *y* ∈ *HInfinite*
apply (*rule ccontr*, *drule HFinite-HInfinite-iff* [*THEN iffD2*])
apply (*frule HFinite-Infinitesimal-not-zero*)
apply (*drule HFinite-not-Infinitesimal-inverse*)
apply (*safe*, *drule HFinite-mult*)
apply (*auto simp add*: *mult-assoc HFinite-HInfinite-iff*)
done

lemma *HInfinite-HFinite-not-Infinitesimal-mult2*:
fixes *x y* :: ′*a*::*real-normed-div-algebra star*
shows [| *x* ∈ *HInfinite*; *y* ∈ *HFinite* − *Infinitesimal* |]
==> *y* ∗ *x* ∈ *HInfinite*
apply (*rule ccontr*, *drule HFinite-HInfinite-iff* [*THEN iffD2*])
apply (*frule HFinite-Infinitesimal-not-zero*)
apply (*drule HFinite-not-Infinitesimal-inverse*)
apply (*safe*, *drule-tac x*=*inverse y* **in** *HFinite-mult*)
apply *assumption*
apply (*auto simp add*: *mult-assoc* [*symmetric*] *HFinite-HInfinite-iff*)
done

lemma *HInfinite-gt-SReal*:
[| (*x*::*hypreal*) ∈ *HInfinite*; *0* < *x*; *y* ∈ *Reals* |] ==> *y* < *x*
by (*auto dest*!: *bspec simp add*: *HInfinite-def abs-if order-less-imp-le*)

lemma *HInfinite-gt-zero-gt-one*:
[| (*x*::*hypreal*) ∈ *HInfinite*; *0* < *x* |] ==> *1* < *x*
by (*auto intro*: *HInfinite-gt-SReal*)

lemma *not-HInfinite-one* [*simp*]: *1* ∉ *HInfinite*
apply (*simp* (*no-asm*) *add*: *HInfinite-HFinite-iff*)
done

lemma *approx-hrabs-disj*: *abs* (*x*::*hypreal*) @= *x* | *abs x* @= $-x$
by (*cut-tac x* = *x* **in** *hrabs-disj*, *auto*)

27.10 Theorems about Monads

lemma *monad-hrabs-Un-subset*: *monad* (*abs x*) ≤ *monad*(*x*::*hypreal*) *Un monad*($-x$)
by (*rule-tac x1* = *x* **in** *hrabs-disj* [*THEN disjE*], *auto*)

lemma *Infinitesimal-monad-eq*: *e* ∈ *Infinitesimal* ==> *monad* (*x*+*e*) = *monad x*
by (*fast intro*!: *Infinitesimal-add-approx-self* [*THEN approx-sym*] *approx-monad-iff* [*THEN iffD1*])

lemma *mem-monad-iff*: (*u* ∈ *monad x*) = ($-u$ ∈ *monad* ($-x$))
by (*simp add*: *monad-def*)

lemma *Infinitesimal-monad-zero-iff*: (*x* ∈ *Infinitesimal*) = (*x* ∈ *monad 0*)
by (*auto intro*: *approx-sym simp add*: *monad-def mem-infmal-iff*)

lemma *monad-zero-minus-iff*: (*x* ∈ *monad 0*) = ($-x$ ∈ *monad 0*)
apply (*simp* (*no-asm*) *add*: *Infinitesimal-monad-zero-iff* [*symmetric*])
done

lemma *monad-zero-hrabs-iff*: ((*x*::*hypreal*) ∈ *monad 0*) = (*abs x* ∈ *monad 0*)
apply (*rule-tac x1* = *x* **in** *hrabs-disj* [*THEN disjE*])
apply (*auto simp add*: *monad-zero-minus-iff* [*symmetric*])
done

lemma *mem-monad-self* [*simp*]: *x* ∈ *monad x*
by (*simp add*: *monad-def*)

27.11 Proof that $x \approx y$ implies $|x| \approx |y|$

lemma *approx-subset-monad*: *x* @= *y* ==> {*x*,*y*} ≤ *monad x*
apply (*simp* (*no-asm*))
apply (*simp add*: *approx-monad-iff*)
done

lemma *approx-subset-monad2*: *x* @= *y* ==> {*x*,*y*} ≤ *monad y*
apply (*drule approx-sym*)
apply (*fast dest*: *approx-subset-monad*)
done

lemma *mem-monad-approx*: *u* ∈ *monad x* ==> *x* @= *u*
by (*simp add*: *monad-def*)

lemma *approx-mem-monad*: *x* @= *u* ==> *u* ∈ *monad x*
by (*simp add*: *monad-def*)

lemma *approx-mem-monad2*: *x* @= *u* ==> *x* ∈ *monad u*
apply (*simp add*: *monad-def*)

apply (*blast intro*!: *approx-sym*)
done

lemma *approx-mem-monad-zero*: [| x @= y;$x \in$ *monad* 0 |] ==> $y \in$ *monad* 0
apply (*drule mem-monad-approx*)
apply (*fast intro*: *approx-mem-monad approx-trans*)
done

lemma *Infinitesimal-approx-hrabs*:
[| x @= y; (x::*hypreal*) $\in$ *Infinitesimal* |] ==> *abs* x @= *abs* y
apply (*drule Infinitesimal-monad-zero-iff* [*THEN iffD1*])
apply (*blast intro*: *approx-mem-monad-zero monad-zero-hrabs-iff* [*THEN iffD1*] *mem-monad-approx approx-trans3*)
done

lemma *less-Infinitesimal-less*:
[| $0 < x$; (x::*hypreal*) $\notin$*Infinitesimal*; e :*Infinitesimal* |] ==> $e < x$
apply (*rule ccontr*)
apply (*auto intro*: *Infinitesimal-zero* [*THEN* [*2*] *Infinitesimal-interval*]
dest!: *order-le-imp-less-or-eq simp add*: *linorder-not-less*)
done

lemma *Ball-mem-monad-gt-zero*:
[| $0 <$ (x::*hypreal*); $x \notin$ *Infinitesimal*; $u \in$ *monad* x |] ==> $0 < u$
apply (*drule mem-monad-approx* [*THEN approx-sym*])
apply (*erule bex-Infinitesimal-iff2* [*THEN iffD2*, *THEN bexE*])
apply (*drule-tac* $e = -xa$ **in** *less-Infinitesimal-less*, *auto*)
done

lemma *Ball-mem-monad-less-zero*:
[| (x::*hypreal*) < 0; $x \notin$ *Infinitesimal*; $u \in$ *monad* x |] ==> $u < 0$
apply (*drule mem-monad-approx* [*THEN approx-sym*])
apply (*erule bex-Infinitesimal-iff* [*THEN iffD2*, *THEN bexE*])
apply (*cut-tac* $x = -x$ **and** $e = xa$ **in** *less-Infinitesimal-less*, *auto*)
done

lemma *lemma-approx-gt-zero*:
[|$0 <$ (x::*hypreal*); $x \notin$ *Infinitesimal*; x @= y|] ==> $0 < y$
by (*blast dest*: *Ball-mem-monad-gt-zero approx-subset-monad*)

lemma *lemma-approx-less-zero*:
[|(x::*hypreal*) < 0; $x \notin$ *Infinitesimal*; x @= y|] ==> $y < 0$
by (*blast dest*: *Ball-mem-monad-less-zero approx-subset-monad*)

theorem *approx-hrabs*: (x::*hypreal*) @= y ==> *abs* x @= *abs* y
by (*drule approx-hnorm*, *simp*)

lemma *approx-hrabs-zero-cancel*: *abs*(x::*hypreal*) @= 0 ==> x @= 0
apply (*cut-tac* $x = x$ **in** *hrabs-disj*)

apply (*auto dest*: *approx-minus*)
done

lemma *approx-hrabs-add-Infinitesimal*:
$(e::hypreal) \in Infinitesimal ==> abs\ x\ @= abs(x+e)$
by (*fast intro*: *approx-hrabs Infinitesimal-add-approx-self*)

lemma *approx-hrabs-add-minus-Infinitesimal*:
$(e::hypreal) \in Infinitesimal ==> abs\ x\ @= abs(x + -e)$
by (*fast intro*: *approx-hrabs Infinitesimal-add-minus-approx-self*)

lemma *hrabs-add-Infinitesimal-cancel*:
$[| (e::hypreal) \in Infinitesimal;\ e' \in Infinitesimal;$
$abs(x+e) = abs(y+e')|] ==> abs\ x\ @= abs\ y$
apply (*drule-tac* $x = x$ **in** *approx-hrabs-add-Infinitesimal*)
apply (*drule-tac* $x = y$ **in** *approx-hrabs-add-Infinitesimal*)
apply (*auto intro*: *approx-trans2*)
done

lemma *hrabs-add-minus-Infinitesimal-cancel*:
$[| (e::hypreal) \in Infinitesimal;\ e' \in Infinitesimal;$
$abs(x + -e) = abs(y + -e')|] ==> abs\ x\ @= abs\ y$
apply (*drule-tac* $x = x$ **in** *approx-hrabs-add-minus-Infinitesimal*)
apply (*drule-tac* $x = y$ **in** *approx-hrabs-add-minus-Infinitesimal*)
apply (*auto intro*: *approx-trans2*)
done

27.12 More *HFinite* and *Infinitesimal* Theorems

lemma *Infinitesimal-add-hypreal-of-real-less*:
$[| x < y;\ u \in Infinitesimal |]$
$==>$ *hypreal-of-real* $x + u <$ *hypreal-of-real* y
apply (*simp add*: *Infinitesimal-def*)
apply (*drule-tac* $x =$ *hypreal-of-real* $y + -$*hypreal-of-real* x **in** *bspec*, *simp*)
apply (*simp add*: *abs-less-iff*)
done

lemma *Infinitesimal-add-hrabs-hypreal-of-real-less*:
$[| x \in Infinitesimal;$ *abs*(*hypreal-of-real* r) $<$ *hypreal-of-real* y $|]$
$==>$ *abs* (*hypreal-of-real* $r + x$) $<$ *hypreal-of-real* y
apply (*drule-tac* $x =$ *hypreal-of-real* r **in** *approx-hrabs-add-Infinitesimal*)
apply (*drule approx-sym* [*THEN bex-Infinitesimal-iff2* [*THEN iffD2*]])
apply (*auto intro*!: *Infinitesimal-add-hypreal-of-real-less*
simp del: *star-of-abs*
simp add: *star-of-abs* [*symmetric*])
done

lemma *Infinitesimal-add-hrabs-hypreal-of-real-less2*:
$[| x \in Infinitesimal;$ *abs*(*hypreal-of-real* r) $<$ *hypreal-of-real* y $|]$

==> *abs* (*x* + *hypreal-of-real r*) < *hypreal-of-real y*
apply (*rule add-commute* [*THEN subst*])
apply (*erule Infinitesimal-add-hrabs-hypreal-of-real-less*, *assumption*)
done

lemma *hypreal-of-real-le-add-Infininitesimal-cancel*:
[| *u* ∈ *Infinitesimal*; *v* ∈ *Infinitesimal*;
hypreal-of-real x + *u* ≤ *hypreal-of-real y* + *v* |]
==> *hypreal-of-real x* ≤ *hypreal-of-real y*
apply (*simp add*: *linorder-not-less* [*symmetric*], *auto*)
apply (*drule-tac u* = *v*−*u* **in** *Infinitesimal-add-hypreal-of-real-less*)
apply (*auto simp add*: *Infinitesimal-diff*)
done

lemma *hypreal-of-real-le-add-Infininitesimal-cancel2*:
[| *u* ∈ *Infinitesimal*; *v* ∈ *Infinitesimal*;
hypreal-of-real x + *u* ≤ *hypreal-of-real y* + *v* |]
==> *x* ≤ *y*
by (*blast intro*: *star-of-le* [*THEN iffD1*]
intro!: *hypreal-of-real-le-add-Infininitesimal-cancel*)

lemma *hypreal-of-real-less-Infinitesimal-le-zero*:
[| *hypreal-of-real x* < *e*; *e* ∈ *Infinitesimal* |] ==> *hypreal-of-real x* ≤ *0*
apply (*rule linorder-not-less* [*THEN iffD1*], *safe*)
apply (*drule Infinitesimal-interval*)
apply (*drule-tac* [*4*] *SReal-hypreal-of-real* [*THEN SReal-Infinitesimal-zero*], *auto*)
done

lemma *Infinitesimal-add-not-zero*:
[| *h* ∈ *Infinitesimal*; *x* ≠ *0* |] ==> *star-of x* + *h* ≠ *0*
apply *auto*
apply (*subgoal-tac h* = − *star-of x*, *auto intro*: *equals-zero-I* [*symmetric*])
done

lemma *Infinitesimal-square-cancel* [*simp*]:
(*x*::*hypreal*)∗*x* + *y*∗*y* ∈ *Infinitesimal* ==> *x*∗*x* ∈ *Infinitesimal*
apply (*rule Infinitesimal-interval2*)
apply (*rule-tac* [*3*] *zero-le-square*, *assumption*)
apply (*auto*)
done

lemma *HFinite-square-cancel* [*simp*]:
(*x*::*hypreal*)∗*x* + *y*∗*y* ∈ *HFinite* ==> *x*∗*x* ∈ *HFinite*
apply (*rule HFinite-bounded*, *assumption*)
apply (*auto*)
done

lemma *Infinitesimal-square-cancel2* [*simp*]:

(*x*::*hypreal*)∗*x* + *y*∗*y* ∈ *Infinitesimal* ==> *y*∗*y* ∈ *Infinitesimal*
apply (*rule Infinitesimal-square-cancel*)
apply (*rule add-commute* [*THEN subst*])
apply (*simp* (*no-asm*))
done

lemma *HFinite-square-cancel2* [*simp*]:
(*x*::*hypreal*)∗*x* + *y*∗*y* ∈ *HFinite* ==> *y*∗*y* ∈ *HFinite*
apply (*rule HFinite-square-cancel*)
apply (*rule add-commute* [*THEN subst*])
apply (*simp* (*no-asm*))
done

lemma *Infinitesimal-sum-square-cancel* [*simp*]:
(*x*::*hypreal*)∗*x* + *y*∗*y* + *z*∗*z* ∈ *Infinitesimal* ==> *x*∗*x* ∈ *Infinitesimal*
apply (*rule Infinitesimal-interval2*, *assumption*)
apply (*rule-tac* [*2*] *zero-le-square*, *simp*)
apply (*insert zero-le-square* [*of y*])
apply (*insert zero-le-square* [*of z*], *simp del*:*zero-le-square*)
done

lemma *HFinite-sum-square-cancel* [*simp*]:
(*x*::*hypreal*)∗*x* + *y*∗*y* + *z*∗*z* ∈ *HFinite* ==> *x*∗*x* ∈ *HFinite*
apply (*rule HFinite-bounded*, *assumption*)
apply (*rule-tac* [*2*] *zero-le-square*)
apply (*insert zero-le-square* [*of y*])
apply (*insert zero-le-square* [*of z*], *simp del*:*zero-le-square*)
done

lemma *Infinitesimal-sum-square-cancel2* [*simp*]:
(*y*::*hypreal*)∗*y* + *x*∗*x* + *z*∗*z* ∈ *Infinitesimal* ==> *x*∗*x* ∈ *Infinitesimal*
apply (*rule Infinitesimal-sum-square-cancel*)
apply (*simp add*: *add-ac*)
done

lemma *HFinite-sum-square-cancel2* [*simp*]:
(*y*::*hypreal*)∗*y* + *x*∗*x* + *z*∗*z* ∈ *HFinite* ==> *x*∗*x* ∈ *HFinite*
apply (*rule HFinite-sum-square-cancel*)
apply (*simp add*: *add-ac*)
done

lemma *Infinitesimal-sum-square-cancel3* [*simp*]:
(*z*::*hypreal*)∗*z* + *y*∗*y* + *x*∗*x* ∈ *Infinitesimal* ==> *x*∗*x* ∈ *Infinitesimal*
apply (*rule Infinitesimal-sum-square-cancel*)
apply (*simp add*: *add-ac*)
done

lemma *HFinite-sum-square-cancel3* [*simp*]:
(*z*::*hypreal*)∗*z* + *y*∗*y* + *x*∗*x* ∈ *HFinite* ==> *x*∗*x* ∈ *HFinite*

apply (*rule HFinite-sum-square-cancel*)
apply (*simp add*: *add-ac*)
done

lemma *monad-hrabs-less*:
  [| $y \in$ *monad* x; $0 <$ *hypreal-of-real* e |]
  ==> *abs* $(y - x) <$ *hypreal-of-real* e
apply (*drule mem-monad-approx* [*THEN approx-sym*])
apply (*drule bex-Infinitesimal-iff* [*THEN iffD2*])
apply (*auto dest*!: *InfinitesimalD*)
done

lemma *mem-monad-SReal-HFinite*:
  $x \in$ *monad* (*hypreal-of-real* a) ==> $x \in$ *HFinite*
apply (*drule mem-monad-approx* [*THEN approx-sym*])
apply (*drule bex-Infinitesimal-iff2* [*THEN iffD2*])
apply (*safe dest*!: *Infinitesimal-subset-HFinite* [*THEN subsetD*])
apply (*erule SReal-hypreal-of-real* [*THEN SReal-subset-HFinite* [*THEN subsetD*], *THEN HFinite-add*])
done

27.13 Theorems about Standard Part

lemma *st-approx-self*: $x \in$ *HFinite* ==> *st* x @= x
apply (*simp add*: *st-def*)
apply (*frule st-part-Ex*, *safe*)
apply (*rule someI2*)
apply (*auto intro*: *approx-sym*)
done

lemma *st-SReal*: $x \in$ *HFinite* ==> *st* $x \in$ *Reals*
apply (*simp add*: *st-def*)
apply (*frule st-part-Ex*, *safe*)
apply (*rule someI2*)
apply (*auto intro*: *approx-sym*)
done

lemma *st-HFinite*: $x \in$ *HFinite* ==> *st* $x \in$ *HFinite*
by (*erule st-SReal* [*THEN SReal-subset-HFinite* [*THEN subsetD*]])

lemma *st-unique*: $[\![r \in \mathbb{R};\ r \approx x]\!] \Longrightarrow st\ x = r$
apply (*frule SReal-subset-HFinite* [*THEN subsetD*])
apply (*drule* (*1*) *approx-HFinite*)
apply (*unfold st-def*)
apply (*rule some-equality*)
apply (*auto intro*: *approx-unique-real*)
done

lemma *st-SReal-eq*: $x \in$ *Reals* ==> *st* $x = x$

apply (*erule st-unique*)
apply (*rule approx-refl*)
done

lemma *st-hypreal-of-real* [*simp*]: *st* (*hypreal-of-real x*) = *hypreal-of-real x*
by (*rule SReal-hypreal-of-real* [*THEN st-SReal-eq*])

lemma *st-eq-approx*: [| $x \in$ *HFinite*; $y \in$ *HFinite*; *st x* = *st y* |] ==> *x* @= *y*
by (*auto dest*!: *st-approx-self elim*!: *approx-trans3*)

lemma *approx-st-eq*:
  assumes $x \in$ *HFinite* **and** $y \in$ *HFinite* **and** *x* @= *y*
  shows *st x* = *st y*
proof −
  have *st x* @= *x st y* @= *y st x* $\in$ *Reals st y* $\in$ *Reals*
    by (*simp-all add*: *st-approx-self st-SReal prems*)
  with *prems* **show** *?thesis*
    by (*fast elim*: *approx-trans approx-trans2 SReal-approx-iff* [*THEN iffD1*])
qed

lemma *st-eq-approx-iff*:
    [| $x \in$ *HFinite*; $y \in$ *HFinite*|]
        ==> (*x* @= *y*) = (*st x* = *st y*)
by (*blast intro*: *approx-st-eq st-eq-approx*)

lemma *st-Infinitesimal-add-SReal*:
    [| $x \in$ *Reals*; $e \in$ *Infinitesimal* |] ==> *st*(*x* + *e*) = *x*
apply (*erule st-unique*)
apply (*erule Infinitesimal-add-approx-self*)
done

lemma *st-Infinitesimal-add-SReal2*:
    [| $x \in$ *Reals*; $e \in$ *Infinitesimal* |] ==> *st*(*e* + *x*) = *x*
apply (*erule st-unique*)
apply (*erule Infinitesimal-add-approx-self2*)
done

lemma *HFinite-st-Infinitesimal-add*:
    $x \in$ *HFinite* ==> $\exists\, e \in$ *Infinitesimal*. *x* = *st*(*x*) + *e*
by (*blast dest*!: *st-approx-self* [*THEN approx-sym*] *bex-Infinitesimal-iff2* [*THEN iffD2*])

lemma *st-add*: $[\![x \in HFinite;\ y \in HFinite]\!] \Longrightarrow st\ (x + y) = st\ x + st\ y$
by (*simp add*: *st-unique st-SReal st-approx-self approx-add*)

lemma *st-number-of* [*simp*]: *st* (*number-of w*) = *number-of w*
by (*rule Reals-number-of* [*THEN st-SReal-eq*])

lemma [*simp*]: $st\ 0 = 0\ st\ 1 = 1$
by (*simp-all add*: *st-SReal-eq*)

lemma *st-minus*: $x \in HFinite \Longrightarrow st\ (-\ x) = -\ st\ x$
by (*simp add*: *st-unique st-SReal st-approx-self approx-minus*)

lemma *st-diff*: $[\![x \in HFinite;\ y \in HFinite]\!] \Longrightarrow st\ (x - y) = st\ x - st\ y$
by (*simp add*: *st-unique st-SReal st-approx-self approx-diff*)

lemma *st-mult*: $[\![x \in HFinite;\ y \in HFinite]\!] \Longrightarrow st\ (x * y) = st\ x * st\ y$
by (*simp add*: *st-unique st-SReal st-approx-self approx-mult-HFinite*)

lemma *st-Infinitesimal*: $x \in Infinitesimal$ ==> $st\ x = 0$
by (*simp add*: *st-unique mem-infmal-iff*)

lemma *st-not-Infinitesimal*: $st(x) \neq 0$ ==> $x \notin Infinitesimal$
by (*fast intro*: *st-Infinitesimal*)

lemma *st-inverse*:
  [| $x \in HFinite$; $st\ x \neq 0$ |]
  ==> $st(inverse\ x) = inverse\ (st\ x)$
apply (*rule-tac c1 = st x* **in** *hypreal-mult-left-cancel* [*THEN iffD1*])
apply (*auto simp add*: *st-mult* [*symmetric*] *st-not-Infinitesimal HFinite-inverse*)
apply (*subst right-inverse*, *auto*)
done

lemma *st-divide* [*simp*]:
  [| $x \in HFinite$; $y \in HFinite$; $st\ y \neq 0$ |]
  ==> $st(x/y) = (st\ x)\ /\ (st\ y)$
by (*simp add*: *divide-inverse st-mult st-not-Infinitesimal HFinite-inverse st-inverse*)

lemma *st-idempotent* [*simp*]: $x \in HFinite$ ==> $st(st(x)) = st(x)$
by (*blast intro*: *st-HFinite st-approx-self approx-st-eq*)

lemma *Infinitesimal-add-st-less*:
  [| $x \in HFinite$; $y \in HFinite$; $u \in Infinitesimal$; $st\ x < st\ y$ |]
  ==> $st\ x + u < st\ y$
apply (*drule st-SReal*)+
apply (*auto intro*!: *Infinitesimal-add-hypreal-of-real-less simp add*: *SReal-iff*)
done

lemma *Infinitesimal-add-st-le-cancel*:
  [| $x \in HFinite$; $y \in HFinite$;
    $u \in Infinitesimal$; $st\ x \leq st\ y + u$
  |] ==> $st\ x \leq st\ y$
apply (*simp add*: *linorder-not-less* [*symmetric*])
apply (*auto dest*: *Infinitesimal-add-st-less*)
done

lemma *st-le*: [| $x \in HFinite$; $y \in HFinite$; $x \leq y$ |] ==> $st(x) \leq st(y)$
apply (*frule HFinite-st-Infinitesimal-add*)
apply (*rotate-tac 1*)
apply (*frule HFinite-st-Infinitesimal-add*, *safe*)
apply (*rule Infinitesimal-add-st-le-cancel*)
apply (*rule-tac* [*3*] $x = ea$ **and** $y = e$ **in** *Infinitesimal-diff*)
apply (*auto simp add*: *add-assoc* [*symmetric*])
done

lemma *st-zero-le*: [| $0 \leq x$; $x \in HFinite$ |] ==> $0 \leq st\ x$
apply (*subst numeral-0-eq-0* [*symmetric*])
apply (*rule st-number-of* [*THEN subst*])
apply (*rule st-le*, *auto*)
done

lemma *st-zero-ge*: [| $x \leq 0$; $x \in HFinite$ |] ==> $st\ x \leq 0$
apply (*subst numeral-0-eq-0* [*symmetric*])
apply (*rule st-number-of* [*THEN subst*])
apply (*rule st-le*, *auto*)
done

lemma *st-hrabs*: $x \in HFinite$ ==> $abs(st\ x) = st(abs\ x)$
apply (*simp add*: *linorder-not-le st-zero-le abs-if st-minus linorder-not-less*)
apply (*auto dest*!: *st-zero-ge* [*OF order-less-imp-le*])
done

27.14 Alternative Definitions using Free Ultrafilter

27.14.1 *HFinite*

lemma *HFinite-FreeUltrafilterNat*:
star-n X $\in$ *HFinite*
==> $\exists u.$ {*n. norm* (*X n*) < *u*} $\in$ *FreeUltrafilterNat*
apply (*auto simp add*: *HFinite-def SReal-def*)
apply (*rule-tac x=r* **in** *exI*)
apply (*simp add*: *hnorm-def star-of-def starfun-star-n*)
apply (*simp add*: *star-less-def starP2-star-n*)
done

lemma *FreeUltrafilterNat-HFinite*:
$\exists u.$ {*n. norm* (*X n*) < *u*} $\in$ *FreeUltrafilterNat*
==> *star-n X* $\in$ *HFinite*
apply (*auto simp add*: *HFinite-def mem-Rep-star-iff*)
apply (*rule-tac x=star-of u* **in** *bexI*)
apply (*simp add*: *hnorm-def starfun-star-n star-of-def*)
apply (*simp add*: *star-less-def starP2-star-n*)
apply (*simp add*: *SReal-def*)
done

lemma *HFinite-FreeUltrafilterNat-iff*:
  $(star\text{-}n\ X \in HFinite) = (\exists\, u.\ \{n.\ norm\ (X\ n) < u\} \in FreeUltrafilterNat)$
by (*blast intro*!: *HFinite-FreeUltrafilterNat FreeUltrafilterNat-HFinite*)

27.14.2 *HInfinite*

lemma *lemma-Compl-eq*: $-\ \{n.\ u < norm\ (xa\ n)\} = \{n.\ norm\ (xa\ n) \le u\}$
by *auto*

lemma *lemma-Compl-eq2*: $-\ \{n.\ norm\ (xa\ n) < u\} = \{n.\ u \le norm\ (xa\ n)\}$
by *auto*

lemma *lemma-Int-eq1*:
  $\{n.\ norm\ (xa\ n) \le u\}\ Int\ \{n.\ u \le norm\ (xa\ n)\}$
    $= \{n.\ norm(xa\ n) = u\}$
by *auto*

lemma *lemma-FreeUltrafilterNat-one*:
  $\{n.\ norm\ (xa\ n) = u\} \le \{n.\ norm\ (xa\ n) < u + (1::real)\}$
by *auto*

lemma *FreeUltrafilterNat-const-Finite*:
  $\{n.\ norm\ (X\ n) = u\} \in FreeUltrafilterNat$ ==> $star\text{-}n\ X \in HFinite$
apply (*rule FreeUltrafilterNat-HFinite*)
apply (*rule-tac* $x = u + 1$ **in** *exI*)
apply (*erule ultra*, *simp*)
done

lemma *HInfinite-FreeUltrafilterNat*:
  $star\text{-}n\ X \in HInfinite$ ==> $\forall\, u.\ \{n.\ u < norm\ (X\ n)\} \in FreeUltrafilterNat$
apply (*drule HInfinite-HFinite-iff* [*THEN iffD1*])
apply (*simp add*: *HFinite-FreeUltrafilterNat-iff*)
apply (*rule allI*, *drule-tac* $x=u + 1$ **in** *spec*)
apply (*drule FreeUltrafilterNat.not-memD*)
apply (*simp add*: *Collect-neg-eq* [*symmetric*] *linorder-not-less*)
apply (*erule ultra*, *simp*)
done

lemma *lemma-Int-HI*:
  $\{n.\ norm\ (Xa\ n) < u\}\ Int\ \{n.\ X\ n = Xa\ n\} \subseteq \{n.\ norm\ (X\ n) < (u::real)\}$
by *auto*

lemma *lemma-Int-HIa*: $\{n.\ u < norm\ (X\ n)\}\ Int\ \{n.\ norm\ (X\ n) < u\} = \{\}$
by (*auto intro*: *order-less-asym*)

lemma *FreeUltrafilterNat-HInfinite*:
  $\forall\, u.\ \{n.\ u < norm\ (X\ n)\} \in FreeUltrafilterNat$ ==> $star\text{-}n\ X \in HInfinite$
apply (*rule HInfinite-HFinite-iff* [*THEN iffD2*])

apply (*safe*, *drule HFinite-FreeUltrafilterNat*, *safe*)
apply (*drule-tac* $x = u$ **in** *spec*)
apply (*drule* (*1*) *FreeUltrafilterNat.Int*)
apply (*simp add*: *Collect-conj-eq* [*symmetric*])
apply (*subgoal-tac* $\forall n.\ \neg$ (*norm* (X n) $< u \wedge u <$ *norm* (X n)), *auto*)
done

lemma *HInfinite-FreeUltrafilterNat-iff*:
(*star-n* $X \in$ *HInfinite*) = ($\forall u.$ {$n.$ $u <$ *norm* (X n)} $\in$ *FreeUltrafilterNat*)
by (*blast intro*!: *HInfinite-FreeUltrafilterNat FreeUltrafilterNat-HInfinite*)

27.14.3 *Infinitesimal*

lemma *ball-SReal-eq*: ($\forall x$::*hypreal* $\in$ *Reals*. P x) = ($\forall x$::*real*. P (*star-of* x))
by (*unfold SReal-def*, *auto*)

lemma *Infinitesimal-FreeUltrafilterNat*:
star-n $X \in$ *Infinitesimal* ==> $\forall u>0$. {$n.$ *norm* (X n) $< u$} $\in \mathcal{U}$
apply (*simp add*: *Infinitesimal-def ball-SReal-eq*)
apply (*simp add*: *hnorm-def starfun-star-n star-of-def*)
apply (*simp add*: *star-less-def starP2-star-n*)
done

lemma *FreeUltrafilterNat-Infinitesimal*:
$\forall u>0$. {$n.$ *norm* (X n) $< u$} $\in \mathcal{U}$ ==> *star-n* $X \in$ *Infinitesimal*
apply (*simp add*: *Infinitesimal-def ball-SReal-eq*)
apply (*simp add*: *hnorm-def starfun-star-n star-of-def*)
apply (*simp add*: *star-less-def starP2-star-n*)
done

lemma *Infinitesimal-FreeUltrafilterNat-iff*:
(*star-n* $X \in$ *Infinitesimal*) = ($\forall u>0$. {$n.$ *norm* (X n) $< u$} $\in \mathcal{U}$)
by (*blast intro*!: *Infinitesimal-FreeUltrafilterNat FreeUltrafilterNat-Infinitesimal*)

lemma *lemma-Infinitesimal*:
($\forall r.$ $0 < r$ --> $x < r$) = ($\forall n.$ $x <$ *inverse*(*real* (*Suc* n)))
apply (*auto simp add*: *real-of-nat-Suc-gt-zero*)
apply (*blast dest*!: *reals-Archimedean intro*: *order-less-trans*)
done

lemma *lemma-Infinitesimal2*:
($\forall r \in$ *Reals*. $0 < r$ --> $x < r$) =
($\forall n.$ $x <$ *inverse*(*hypreal-of-nat* (*Suc* n)))
apply *safe*
apply (*drule-tac* $x =$ *inverse* (*hypreal-of-real* (*real* (*Suc* n))) **in** *bspec*)
apply (*simp* (*no-asm-use*))
apply (*rule real-of-nat-Suc-gt-zero* [*THEN positive-imp-inverse-positive*, *THEN star-of-less*

[*THEN iffD2*], *THEN* [*2*] *impE*])
prefer *2* **apply** *assumption*
apply (*simp add*: *real-of-nat-def*)
apply (*auto dest*!: *reals-Archimedean simp add*: *SReal-iff*)
apply (*drule star-of-less* [*THEN iffD2*])
apply (*simp add*: *real-of-nat-def*)
apply (*blast intro*: *order-less-trans*)
done

lemma *Infinitesimal-hypreal-of-nat-iff*:
Infinitesimal = $\{x.\ \forall\, n.\ hnorm\ x < inverse\ (hypreal\text{-}of\text{-}nat\ (Suc\ n))\}$
apply (*simp add*: *Infinitesimal-def*)
apply (*auto simp add*: *lemma-Infinitesimal2*)
done

27.15 Proof that ω is an infinite number

It will follow that epsilon is an infinitesimal number.

lemma *Suc-Un-eq*: $\{n.\ n < Suc\ m\} = \{n.\ n < m\}$ *Un* $\{n.\ n = m\}$
by (*auto simp add*: *less-Suc-eq*)

lemma *finite-nat-segment*: *finite* $\{n{::}nat.\ n < m\}$
apply (*induct m*)
apply (*auto simp add*: *Suc-Un-eq*)
done

lemma *finite-real-of-nat-segment*: *finite* $\{n{::}nat.\ real\ n < real\ (m{::}nat)\}$
by (*auto intro*: *finite-nat-segment*)

lemma *finite-real-of-nat-less-real*: *finite* $\{n{::}nat.\ real\ n < u\}$
apply (*cut-tac* $x = u$ **in** *reals-Archimedean2*, *safe*)
apply (*rule finite-real-of-nat-segment* [*THEN* [*2*] *finite-subset*])
apply (*auto dest*: *order-less-trans*)
done

lemma *lemma-real-le-Un-eq*:
$\{n.\ f\ n \le u\} = \{n.\ f\ n < u\}$ *Un* $\{n.\ u = (f\ n :: real)\}$
by (*auto dest*: *order-le-imp-less-or-eq simp add*: *order-less-imp-le*)

lemma *finite-real-of-nat-le-real*: *finite* $\{n{::}nat.\ real\ n \le u\}$
by (*auto simp add*: *lemma-real-le-Un-eq lemma-finite-omega-set finite-real-of-nat-less-real*)

lemma *finite-rabs-real-of-nat-le-real*: *finite* $\{n{::}nat.\ abs(real\ n) \le u\}$
apply (*simp* (*no-asm*) *add*: *real-of-nat-Suc-gt-zero finite-real-of-nat-le-real*)
done

lemma *rabs-real-of-nat-le-real-FreeUltrafilterNat*:

$\{n.\ abs(real\ n) \le u\} \notin FreeUltrafilterNat$
by (*blast intro*!: *FreeUltrafilterNat.finite finite-rabs-real-of-nat-le-real*)

lemma *FreeUltrafilterNat-nat-gt-real*: $\{n.\ u < real\ n\} \in FreeUltrafilterNat$
apply (*rule ccontr*, *drule FreeUltrafilterNat.not-memD*)
apply (*subgoal-tac* $- \{n::nat.\ u < real\ n\} = \{n.\ real\ n \le u\}$)
prefer *2* **apply** *force*
apply (*simp add*: *finite-real-of-nat-le-real* [*THEN FreeUltrafilterNat.finite*])
done

lemma *Compl-real-le-eq*: $- \{n::nat.\ real\ n \le u\} = \{n.\ u < real\ n\}$
by (*auto dest*!: *order-le-less-trans simp add*: *linorder-not-le*)

ω is a member of *HInfinite*

lemma *FreeUltrafilterNat-omega*: $\{n.\ u < real\ n\} \in FreeUltrafilterNat$
apply (*cut-tac* $u = u$ **in** *rabs-real-of-nat-le-real-FreeUltrafilterNat*)
apply (*auto dest*: *FreeUltrafilterNat.not-memD simp add*: *Compl-real-le-eq*)
done

theorem *HInfinite-omega* [*simp*]: $omega \in HInfinite$
apply (*simp add*: *omega-def*)
apply (*rule FreeUltrafilterNat-HInfinite*)
apply (*simp* (*no-asm*) *add*: *real-norm-def real-of-nat-Suc diff-less-eq* [*symmetric*] *FreeUltrafilterNat-omega*)
done

lemma *Infinitesimal-epsilon* [*simp*]: $epsilon \in Infinitesimal$
by (*auto intro*!: *HInfinite-inverse-Infinitesimal HInfinite-omega simp add*: *hypreal-epsilon-inverse-omega*)

lemma *HFinite-epsilon* [*simp*]: $epsilon \in HFinite$
by (*auto intro*: *Infinitesimal-subset-HFinite* [*THEN subsetD*])

lemma *epsilon-approx-zero* [*simp*]: *epsilon* @= *0*
apply (*simp* (*no-asm*) *add*: *mem-infmal-iff* [*symmetric*])
done

lemma *real-of-nat-less-inverse-iff*:
$0 < u \implies (u < inverse\ (real(Suc\ n))) = (real(Suc\ n) < inverse\ u)$
apply (*simp add*: *inverse-eq-divide*)
apply (*subst pos-less-divide-eq*, *assumption*)
apply (*subst pos-less-divide-eq*)
apply (*simp add*: *real-of-nat-Suc-gt-zero*)
apply (*simp add*: *real-mult-commute*)

done

lemma *finite-inverse-real-of-posnat-gt-real*:
  0 < *u* ==> *finite* {*n*. *u* < *inverse*(*real*(*Suc n*))}
apply (*simp* (*no-asm-simp*) *add*: *real-of-nat-less-inverse-iff*)
apply (*simp* (*no-asm-simp*) *add*: *real-of-nat-Suc less-diff-eq* [*symmetric*])
apply (*rule finite-real-of-nat-less-real*)
done

lemma *lemma-real-le-Un-eq2*:
  {*n*. *u* ≤ *inverse*(*real*(*Suc n*))} =
  {*n*. *u* < *inverse*(*real*(*Suc n*))} *Un* {*n*. *u* = *inverse*(*real*(*Suc n*))}
apply (*auto dest*: *order-le-imp-less-or-eq simp add*: *order-less-imp-le*)
done

lemma *real-of-nat-inverse-eq-iff*:
  (*u* = *inverse* (*real*(*Suc n*))) = (*real*(*Suc n*) = *inverse u*)
by (*auto simp add*: *real-of-nat-Suc-gt-zero less-imp-neq* [*THEN not-sym*])

lemma *lemma-finite-omega-set2*: *finite* {*n*::*nat*. *u* = *inverse*(*real*(*Suc n*))}
apply (*simp* (*no-asm-simp*) *add*: *real-of-nat-inverse-eq-iff*)
apply (*cut-tac x* = *inverse u* − *1* **in** *lemma-finite-omega-set*)
apply (*simp add*: *real-of-nat-Suc diff-eq-eq* [*symmetric*] *eq-commute*)
done

lemma *finite-inverse-real-of-posnat-ge-real*:
  0 < *u* ==> *finite* {*n*. *u* ≤ *inverse*(*real*(*Suc n*))}
by (*auto simp add*: *lemma-real-le-Un-eq2 lemma-finite-omega-set2 finite-inverse-real-of-posnat-gt-real*)

lemma *inverse-real-of-posnat-ge-real-FreeUltrafilterNat*:
  0 < *u* ==> {*n*. *u* ≤ *inverse*(*real*(*Suc n*))} ∉ *FreeUltrafilterNat*
by (*blast intro*!: *FreeUltrafilterNat.finite finite-inverse-real-of-posnat-ge-real*)

lemma *Compl-le-inverse-eq*:
  − {*n*. *u* ≤ *inverse*(*real*(*Suc n*))} =
   {*n*. *inverse*(*real*(*Suc n*)) < *u*}
apply (*auto dest*!: *order-le-less-trans simp add*: *linorder-not-le*)
done

lemma *FreeUltrafilterNat-inverse-real-of-posnat*:
  0 < *u* ==>
   {*n*. *inverse*(*real*(*Suc n*)) < *u*} ∈ *FreeUltrafilterNat*
apply (*cut-tac u* = *u* **in** *inverse-real-of-posnat-ge-real-FreeUltrafilterNat*)
apply (*auto dest*: *FreeUltrafilterNat.not-memD simp add*: *Compl-le-inverse-eq*)
done

Example of an hypersequence (i.e. an extended standard sequence) whose term with an hypernatural suffix is an infinitesimal i.e. the whn'nth term of the hypersequence is a member of Infinitesimal

lemma *SEQ-Infinitesimal*:
  (*f* (%n::nat. inverse(real(Suc n)))) whn : Infinitesimal
apply (*simp add*: *hypnat-omega-def starfun-star-n star-n-inverse*)
apply (*simp add*: *Infinitesimal-FreeUltrafilterNat-iff*)
apply (*simp add*: *real-of-nat-Suc-gt-zero FreeUltrafilterNat-inverse-real-of-posnat*)
done

Example where we get a hyperreal from a real sequence for which a particular property holds. The theorem is used in proofs about equivalence of nonstandard and standard neighbourhoods. Also used for equivalence of nonstandard ans standard definitions of pointwise limit.

lemma *real-seq-to-hypreal-Infinitesimal*:
  $\forall$ n. norm(X n − x) < inverse(real(Suc n))
  ==> star-n X − star-of x ∈ Infinitesimal
apply (*auto intro*!: *bexI dest*: *FreeUltrafilterNat-inverse-real-of-posnat FreeUltrafilterNat.Int intro*: *order-less-trans FreeUltrafilterNat.subset simp add*: *star-n-diff star-of-def Infinitesimal-FreeUltrafilterNat-iff star-n-inverse*)
done

lemma *real-seq-to-hypreal-approx*:
  $\forall$ n. norm(X n − x) < inverse(real(Suc n))
  ==> star-n X @= star-of x
apply (*subst approx-minus-iff*)
apply (*rule mem-infmal-iff* [*THEN subst*])
apply (*erule real-seq-to-hypreal-Infinitesimal*)
done

lemma *real-seq-to-hypreal-approx2*:
  $\forall$ n. norm(x − X n) < inverse(real(Suc n))
    ==> star-n X @= star-of x
apply (*rule real-seq-to-hypreal-approx*)
apply (*subst norm-minus-cancel* [*symmetric*])
apply (*simp del*: *norm-minus-cancel*)
done

lemma *real-seq-to-hypreal-Infinitesimal2*:
  $\forall$ n. norm(X n − Y n) < inverse(real(Suc n))
  ==> star-n X − star-n Y ∈ Infinitesimal
by (*auto intro*!: *bexI*
  dest: *FreeUltrafilterNat-inverse-real-of-posnat*
    FreeUltrafilterNat.Int
  intro: *order-less-trans FreeUltrafilterNat.subset*
  simp add: *Infinitesimal-FreeUltrafilterNat-iff star-n-diff*
    star-n-inverse)

end

28 NSComplex: Nonstandard Complex Numbers

theory *NSComplex*
imports *Complex ../Hyperreal/NSA*
begin

types *hcomplex* = *complex star*

abbreviation
hcomplex-of-complex :: *complex* => *complex star* **where**
hcomplex-of-complex == *star-of*

abbreviation
hcmod :: *complex star* => *real star* **where**
hcmod == *hnorm*

definition
hRe :: *hcomplex* => *hypreal* **where**
hRe = **f* Re*

definition
hIm :: *hcomplex* => *hypreal* **where**
hIm = **f* Im*

definition
iii :: *hcomplex* **where**
iii = *star-of ii*

definition
hcnj :: *hcomplex* => *hcomplex* **where**
hcnj = **f* cnj*

definition
hsgn :: *hcomplex* => *hcomplex* **where**
hsgn = **f* sgn*

definition
harg :: *hcomplex* => *hypreal* **where**
harg = **f* arg*

definition

hcis :: *hypreal* => *hcomplex* **where**
hcis = $*f*$ *cis*

abbreviation
hcomplex-of-hypreal :: *hypreal* ⇒ *hcomplex* **where**
hcomplex-of-hypreal ≡ *of-hypreal*

definition

hrcis :: [*hypreal*, *hypreal*] => *hcomplex* **where**
hrcis = $*f2*$ *rcis*

definition
hexpi :: *hcomplex* => *hcomplex* **where**
hexpi = $*f*$ *expi*

definition
HComplex :: [*hypreal*,*hypreal*] => *hcomplex* **where**
HComplex = $*f2*$ *Complex*

lemmas *hcomplex-defs* [*transfer-unfold*] =
hRe-def hIm-def iii-def hcnj-def hsgn-def harg-def hcis-def
hrcis-def hexpi-def HComplex-def

lemma *Standard-hRe* [*simp*]: $x \in Standard \Longrightarrow hRe\ x \in Standard$
by (*simp add*: *hcomplex-defs*)

lemma *Standard-hIm* [*simp*]: $x \in Standard \Longrightarrow hIm\ x \in Standard$
by (*simp add*: *hcomplex-defs*)

lemma *Standard-iii* [*simp*]: $iii \in Standard$
by (*simp add*: *hcomplex-defs*)

lemma *Standard-hcnj* [*simp*]: $x \in Standard \Longrightarrow hcnj\ x \in Standard$
by (*simp add*: *hcomplex-defs*)

lemma *Standard-hsgn* [*simp*]: $x \in Standard \Longrightarrow hsgn\ x \in Standard$
by (*simp add*: *hcomplex-defs*)

lemma *Standard-harg* [*simp*]: $x \in Standard \Longrightarrow harg\ x \in Standard$
by (*simp add*: *hcomplex-defs*)

lemma *Standard-hcis* [*simp*]: $r \in Standard \Longrightarrow hcis\ r \in Standard$

by (*simp add*: *hcomplex-defs*)

lemma *Standard-hexpi* [*simp*]: $x \in Standard \Longrightarrow hexpi\ x \in Standard$
by (*simp add*: *hcomplex-defs*)

lemma *Standard-hrcis* [*simp*]:
$[\![r \in Standard;\ s \in Standard]\!] \Longrightarrow hrcis\ r\ s \in Standard$
by (*simp add*: *hcomplex-defs*)

lemma *Standard-HComplex* [*simp*]:
$[\![r \in Standard;\ s \in Standard]\!] \Longrightarrow HComplex\ r\ s \in Standard$
by (*simp add*: *hcomplex-defs*)

lemma *hcmod-def*: $hcmod = *f*\ cmod$
by (*rule hnorm-def*)

28.1 Properties of Nonstandard Real and Imaginary Parts

lemma *hcomplex-hRe-hIm-cancel-iff*:
$!!w\ z.\ (w{=}z) = (hRe(w) = hRe(z)\ \&\ hIm(w) = hIm(z))$
by *transfer* (*rule complex-Re-Im-cancel-iff*)

lemma *hcomplex-equality* [*intro?*]:
$!!z\ w.\ hRe\ z = hRe\ w ==> hIm\ z = hIm\ w ==> z = w$
by *transfer* (*rule complex-equality*)

lemma *hcomplex-hRe-zero* [*simp*]: $hRe\ 0 = 0$
by *transfer* (*rule complex-Re-zero*)

lemma *hcomplex-hIm-zero* [*simp*]: $hIm\ 0 = 0$
by *transfer* (*rule complex-Im-zero*)

lemma *hcomplex-hRe-one* [*simp*]: $hRe\ 1 = 1$
by *transfer* (*rule complex-Re-one*)

lemma *hcomplex-hIm-one* [*simp*]: $hIm\ 1 = 0$
by *transfer* (*rule complex-Im-one*)

28.2 Addition for Nonstandard Complex Numbers

lemma *hRe-add*: $!!x\ y.\ hRe(x + y) = hRe(x) + hRe(y)$
by *transfer* (*rule complex-Re-add*)

lemma *hIm-add*: $!!x\ y.\ hIm(x + y) = hIm(x) + hIm(y)$
by *transfer* (*rule complex-Im-add*)

28.3 More Minus Laws

lemma *hRe-minus*: $!!z.\ hRe(-z) = -\ hRe(z)$
by *transfer* (*rule complex-Re-minus*)

lemma *hIm-minus*: $!!z.\ hIm(-z) = -\ hIm(z)$
by *transfer* (*rule complex-Im-minus*)

lemma *hcomplex-add-minus-eq-minus*:
$x + y = (0::hcomplex) ==> x = -y$
apply (*drule OrderedGroup.equals-zero-I*)
apply (*simp add*: *minus-equation-iff* [*of x y*])
done

lemma *hcomplex-i-mult-eq* [*simp*]: $iii * iii = -\ 1$
by *transfer* (*rule i-mult-eq2*)

lemma *hcomplex-i-mult-left* [*simp*]: $!!z.\ iii * (iii * z) = -z$
by *transfer* (*rule complex-i-mult-minus*)

lemma *hcomplex-i-not-zero* [*simp*]: $iii \neq 0$
by *transfer* (*rule complex-i-not-zero*)

28.4 More Multiplication Laws

lemma *hcomplex-mult-minus-one*: $-\ 1 * (z::hcomplex) = -z$
by *simp*

lemma *hcomplex-mult-minus-one-right*: $(z::hcomplex) * -\ 1 = -z$
by *simp*

lemma *hcomplex-mult-left-cancel*:
$(c::hcomplex) \neq (0::hcomplex) ==> (c*a=c*b) = (a=b)$
by *simp*

lemma *hcomplex-mult-right-cancel*:
$(c::hcomplex) \neq (0::hcomplex) ==> (a*c=b*c) = (a=b)$
by *simp*

28.5 Subraction and Division

lemma *hcomplex-diff-eq-eq* [*simp*]: $((x::hcomplex) - y = z) = (x = z + y)$

by (*rule OrderedGroup.diff-eq-eq*)

28.6 Embedding Properties for *hcomplex-of-hypreal* Map

lemma *hRe-hcomplex-of-hypreal* [*simp*]: $!!z.\ hRe(hcomplex\text{-}of\text{-}hypreal\ z) = z$
by *transfer* (*rule Re-complex-of-real*)

lemma *hIm-hcomplex-of-hypreal* [*simp*]: $!!z.\ hIm(hcomplex\text{-}of\text{-}hypreal\ z) = 0$
by *transfer* (*rule Im-complex-of-real*)

lemma *hcomplex-of-hypreal-epsilon-not-zero* [*simp*]:

hcomplex-of-hypreal epsilon $\neq$ *0*
by (*simp add*: *hypreal-epsilon-not-zero*)

28.7 HComplex theorems

lemma *hRe-HComplex* [*simp*]: !!*x y*. *hRe* (*HComplex x y*) = *x*
by *transfer* (*rule Re*)

lemma *hIm-HComplex* [*simp*]: !!*x y*. *hIm* (*HComplex x y*) = *y*
by *transfer* (*rule Im*)

lemma *hcomplex-surj* [*simp*]: !!*z*. *HComplex* (*hRe z*) (*hIm z*) = *z*
by *transfer* (*rule complex-surj*)

lemma *hcomplex-induct* [*case-names rect*]:
($\bigwedge$*x y*. *P* (*HComplex x y*)) ==> *P z*
by (*rule hcomplex-surj* [*THEN subst*], *blast*)

28.8 Modulus (Absolute Value) of Nonstandard Complex Number

lemma *hcomplex-of-hypreal-abs*:
hcomplex-of-hypreal (*abs x*) =
hcomplex-of-hypreal(*hcmod*(*hcomplex-of-hypreal x*))
by *simp*

lemma *HComplex-inject* [*simp*]:
!!*x y x′ y′*. *HComplex x y* = *HComplex x′ y′* = (*x*=*x′* & *y*=*y′*)
by *transfer* (*rule complex.inject*)

lemma *HComplex-add* [*simp*]:
!!*x1 y1 x2 y2*. *HComplex x1 y1* + *HComplex x2 y2* = *HComplex* (*x1*+*x2*) (*y1*+*y2*)
by *transfer* (*rule complex-add*)

lemma *HComplex-minus* [*simp*]: !!*x y*. − *HComplex x y* = *HComplex* (−*x*) (−*y*)
by *transfer* (*rule complex-minus*)

lemma *HComplex-diff* [*simp*]:
!!*x1 y1 x2 y2*. *HComplex x1 y1* − *HComplex x2 y2* = *HComplex* (*x1*−*x2*) (*y1*−*y2*)
by *transfer* (*rule complex-diff*)

lemma *HComplex-mult* [*simp*]:
!!*x1 y1 x2 y2*. *HComplex x1 y1* ∗ *HComplex x2 y2* =
HComplex (*x1*∗*x2* − *y1*∗*y2*) (*x1*∗*y2* + *y1*∗*x2*)
by *transfer* (*rule complex-mult*)

lemma *hcomplex-of-hypreal-eq*: !!*r*. *hcomplex-of-hypreal r* = *HComplex r 0*

by *transfer* (*rule complex-of-real-def*)

lemma *HComplex-add-hcomplex-of-hypreal* [*simp*]:
  $!!x\ y\ r.\ HComplex\ x\ y + hcomplex\text{-}of\text{-}hypreal\ r = HComplex\ (x+r)\ y$
by *transfer* (*rule Complex-add-complex-of-real*)

lemma *hcomplex-of-hypreal-add-HComplex* [*simp*]:
  $!!r\ x\ y.\ hcomplex\text{-}of\text{-}hypreal\ r + HComplex\ x\ y = HComplex\ (r+x)\ y$
by *transfer* (*rule complex-of-real-add-Complex*)

lemma *HComplex-mult-hcomplex-of-hypreal*:
  $!!x\ y\ r.\ HComplex\ x\ y * hcomplex\text{-}of\text{-}hypreal\ r = HComplex\ (x*r)\ (y*r)$
by *transfer* (*rule Complex-mult-complex-of-real*)

lemma *hcomplex-of-hypreal-mult-HComplex*:
  $!!r\ x\ y.\ hcomplex\text{-}of\text{-}hypreal\ r * HComplex\ x\ y = HComplex\ (r*x)\ (r*y)$
by *transfer* (*rule complex-of-real-mult-Complex*)

lemma *i-hcomplex-of-hypreal* [*simp*]:
  $!!r.\ iii * hcomplex\text{-}of\text{-}hypreal\ r = HComplex\ 0\ r$
by *transfer* (*rule i-complex-of-real*)

lemma *hcomplex-of-hypreal-i* [*simp*]:
  $!!r.\ hcomplex\text{-}of\text{-}hypreal\ r * iii = HComplex\ 0\ r$
by *transfer* (*rule complex-of-real-i*)

28.9 Conjugation

lemma *hcomplex-hcnj-cancel-iff* [*iff*]: $!!x\ y.\ (hcnj\ x = hcnj\ y) = (x = y)$
by *transfer* (*rule complex-cnj-cancel-iff*)

lemma *hcomplex-hcnj-hcnj* [*simp*]: $!!z.\ hcnj\ (hcnj\ z) = z$
by *transfer* (*rule complex-cnj-cnj*)

lemma *hcomplex-hcnj-hcomplex-of-hypreal* [*simp*]:
  $!!x.\ hcnj\ (hcomplex\text{-}of\text{-}hypreal\ x) = hcomplex\text{-}of\text{-}hypreal\ x$
by *transfer* (*rule complex-cnj-complex-of-real*)

lemma *hcomplex-hmod-hcnj* [*simp*]: $!!z.\ hcmod\ (hcnj\ z) = hcmod\ z$
by *transfer* (*rule complex-mod-cnj*)

lemma *hcomplex-hcnj-minus*: $!!z.\ hcnj\ (-z) = -\ hcnj\ z$
by *transfer* (*rule complex-cnj-minus*)

lemma *hcomplex-hcnj-inverse*: $!!z.\ hcnj(inverse\ z) = inverse(hcnj\ z)$
by *transfer* (*rule complex-cnj-inverse*)

lemma *hcomplex-hcnj-add*: $!!w\ z.\ hcnj(w + z) = hcnj(w) + hcnj(z)$
by *transfer* (*rule complex-cnj-add*)

lemma *hcomplex-hcnj-diff*: $!!w\ z.\ hcnj(w - z) = hcnj(w) - hcnj(z)$
by *transfer* (*rule complex-cnj-diff*)

lemma *hcomplex-hcnj-mult*: $!!w\ z.\ hcnj(w * z) = hcnj(w) * hcnj(z)$
by *transfer* (*rule complex-cnj-mult*)

lemma *hcomplex-hcnj-divide*: $!!w\ z.\ hcnj(w\ /\ z) = (hcnj\ w)/(hcnj\ z)$
by *transfer* (*rule complex-cnj-divide*)

lemma *hcnj-one* [*simp*]: $hcnj\ 1 = 1$
by *transfer* (*rule complex-cnj-one*)

lemma *hcomplex-hcnj-zero* [*simp*]: $hcnj\ 0 = 0$
by *transfer* (*rule complex-cnj-zero*)

lemma *hcomplex-hcnj-zero-iff* [*iff*]: $!!z.\ (hcnj\ z = 0) = (z = 0)$
by *transfer* (*rule complex-cnj-zero-iff*)

lemma *hcomplex-mult-hcnj*:
$!!z.\ z * hcnj\ z = hcomplex\text{-}of\text{-}hypreal\ (hRe(z)\ \hat{}\ 2 + hIm(z)\ \hat{}\ 2)$
by *transfer* (*rule complex-mult-cnj*)

28.10 More Theorems about the Function *hcmod*

lemma *hcmod-hcomplex-of-hypreal-of-nat* [*simp*]:
$hcmod\ (hcomplex\text{-}of\text{-}hypreal(hypreal\text{-}of\text{-}nat\ n)) = hypreal\text{-}of\text{-}nat\ n$
by *simp*

lemma *hcmod-hcomplex-of-hypreal-of-hypnat* [*simp*]:
$hcmod\ (hcomplex\text{-}of\text{-}hypreal(hypreal\text{-}of\text{-}hypnat\ n)) = hypreal\text{-}of\text{-}hypnat\ n$
by *simp*

lemma *hcmod-mult-hcnj*: $!!z.\ hcmod(z * hcnj(z)) = hcmod(z)\ \hat{}\ 2$
by *transfer* (*rule complex-mod-mult-cnj*)

lemma *hcmod-triangle-ineq2* [*simp*]:
$!!a\ b.\ hcmod(b + a) - hcmod\ b \le hcmod\ a$
by *transfer* (*rule complex-mod-triangle-ineq2*)

lemma *hcmod-diff-ineq* [*simp*]: $!!a\ b.\ hcmod(a) - hcmod(b) \le hcmod(a + b)$
by *transfer* (*rule norm-diff-ineq*)

28.11 Exponentiation

lemma *hcomplexpow-0* [*simp*]: $z\ \hat{}\ 0 \quad = (1{::}hcomplex)$
by (*rule power-0*)

lemma *hcomplexpow-Suc* [*simp*]: $z\ \hat{}\ (Suc\ n) = (z{::}hcomplex) * (z\ \hat{}\ n)$
by (*rule power-Suc*)

lemma *hcomplexpow-i-squared* [*simp*]: *iii* ˆ *2* = −*1*
by *transfer* (*rule power2-i*)

lemma *hcomplex-of-hypreal-pow*:
  !!*x*. *hcomplex-of-hypreal* (*x* ˆ *n*) = (*hcomplex-of-hypreal x*) ˆ *n*
by *transfer* (*rule of-real-power*)

lemma *hcomplex-hcnj-pow*: !!*z*. *hcnj*(*z* ˆ *n*) = *hcnj*(*z*) ˆ *n*
by *transfer* (*rule complex-cnj-power*)

lemma *hcmod-hcomplexpow*: !!*x*. *hcmod*(*x* ˆ *n*) = *hcmod*(*x*) ˆ *n*
by *transfer* (*rule norm-power*)

lemma *hcpow-minus*:
  !!*x n*. (−*x*::*hcomplex*) *pow n* =
   (*if* (∗*p*∗ *even*) *n then* (*x pow n*) *else* −(*x pow n*))
by *transfer* (*rule neg-power-if*)

lemma *hcpow-mult*:
 !!*r s n*. ((*r*::*hcomplex*) ∗ *s*) *pow n* = (*r pow n*) ∗ (*s pow n*)
by *transfer* (*rule power-mult-distrib*)

lemma *hcpow-zero2* [*simp*]:
 ⋀*n*. *0 pow* (*hSuc n*) = (*0*::′*a*::{*recpower*,*semiring-0*} *star*)
by *transfer* (*rule power-0-Suc*)

lemma *hcpow-not-zero* [*simp*,*intro*]:
 !!*r n*. *r* ≠ *0* ==> *r pow n* ≠ (*0*::*hcomplex*)
by (*rule hyperpow-not-zero*)

lemma *hcpow-zero-zero*: *r pow n* = (*0*::*hcomplex*) ==> *r* = *0*
by (*blast intro*: *ccontr dest*: *hcpow-not-zero*)

28.12 The Function *hsgn*

lemma *hsgn-zero* [*simp*]: *hsgn 0* = *0*
by *transfer* (*rule sgn-zero*)

lemma *hsgn-one* [*simp*]: *hsgn 1* = *1*
by *transfer* (*rule sgn-one*)

lemma *hsgn-minus*: !!*z*. *hsgn* (−*z*) = − *hsgn*(*z*)
by *transfer* (*rule sgn-minus*)

lemma *hsgn-eq*: !!*z*. *hsgn z* = *z* / *hcomplex-of-hypreal* (*hcmod z*)
by *transfer* (*rule sgn-eq*)

lemma *hcmod-i*: !!*x y*. *hcmod* (*HComplex x y*) = (∗*f*∗ *sqrt*) (*x* ˆ *2* + *y* ˆ *2*)

by *transfer* (*rule complex-norm*)

lemma *hcomplex-eq-cancel-iff1* [*simp*]:
  (*hcomplex-of-hypreal xa* = *HComplex x y*) = (*xa* = *x* & *y* = *0*)
by (*simp add*: *hcomplex-of-hypreal-eq*)

lemma *hcomplex-eq-cancel-iff2* [*simp*]:
  (*HComplex x y* = *hcomplex-of-hypreal xa*) = (*x* = *xa* & *y* = *0*)
by (*simp add*: *hcomplex-of-hypreal-eq*)

lemma *HComplex-eq-0* [*simp*]: !!*x y*. (*HComplex x y* = *0*) = (*x* = *0* & *y* = *0*)
by *transfer* (*rule Complex-eq-0*)

lemma *HComplex-eq-1* [*simp*]: !!*x y*. (*HComplex x y* = *1*) = (*x* = *1* & *y* = *0*)
by *transfer* (*rule Complex-eq-1*)

lemma *i-eq-HComplex-0-1*: *iii* = *HComplex 0 1*
by *transfer* (*rule i-def* [*THEN meta-eq-to-obj-eq*])

lemma *HComplex-eq-i* [*simp*]: !!*x y*. (*HComplex x y* = *iii*) = (*x* = *0* & *y* = *1*)
by *transfer* (*rule Complex-eq-i*)

lemma *hRe-hsgn* [*simp*]: !!*z*. *hRe*(*hsgn z*) = *hRe*(*z*)/*hcmod z*
by *transfer* (*rule Re-sgn*)

lemma *hIm-hsgn* [*simp*]: !!*z*. *hIm*(*hsgn z*) = *hIm*(*z*)/*hcmod z*
by *transfer* (*rule Im-sgn*)

lemma *hcomplex-inverse-complex-split*:
  !!*x y*. *inverse*(*hcomplex-of-hypreal x* + *iii* ∗ *hcomplex-of-hypreal y*) =
   hcomplex-of-hypreal(*x*/(*x* ˆ *2* + *y* ˆ *2*)) −
   iii ∗ *hcomplex-of-hypreal*(*y*/(*x* ˆ *2* + *y* ˆ *2*))
by *transfer* (*rule complex-inverse-complex-split*)

lemma *HComplex-inverse*:
  !!*x y*. *inverse* (*HComplex x y*) =
   HComplex (*x*/(*x* ˆ *2* + *y* ˆ *2*)) (−*y*/(*x* ˆ *2* + *y* ˆ *2*))
by *transfer* (*rule complex-inverse*)

lemma *hRe-mult-i-eq*[*simp*]:
  !!*y*. *hRe* (*iii* ∗ *hcomplex-of-hypreal y*) = *0*
by *transfer simp*

lemma *hIm-mult-i-eq* [*simp*]:
  !!*y*. *hIm* (*iii* ∗ *hcomplex-of-hypreal y*) = *y*
by *transfer simp*

lemma *hcmod-mult-i* [*simp*]: !!*y*. *hcmod* (*iii* ∗ *hcomplex-of-hypreal y*) = *abs y*
by *transfer simp*

lemma *hcmod-mult-i2* [*simp*]: !!*y*. *hcmod* (*hcomplex-of-hypreal y* ∗ *iii*) = *abs y*
by *transfer simp*

lemma *cos-harg-i-mult-zero-pos*:
  !!*y*. *0* < *y* ==> (∗*f*∗ *cos*) (*harg*(*HComplex 0 y*)) = *0*
by *transfer* (*rule cos-arg-i-mult-zero-pos*)

lemma *cos-harg-i-mult-zero-neg*:
  !!*y*. *y* < *0* ==> (∗*f*∗ *cos*) (*harg*(*HComplex 0 y*)) = *0*
by *transfer* (*rule cos-arg-i-mult-zero-neg*)

lemma *cos-harg-i-mult-zero* [*simp*]:
  !!*y*. *y* ≠ *0* ==> (∗*f*∗ *cos*) (*harg*(*HComplex 0 y*)) = *0*
by *transfer* (*rule cos-arg-i-mult-zero*)

lemma *hcomplex-of-hypreal-zero-iff* [*simp*]:
  !!*y*. (*hcomplex-of-hypreal y* = *0*) = (*y* = *0*)
by *transfer* (*rule of-real-eq-0-iff*)

28.13 Polar Form for Nonstandard Complex Numbers

lemma *complex-split-polar2*:
  ∀ *n*. ∃ *r a*. (*z n*) = *complex-of-real r* ∗ (*Complex* (*cos a*) (*sin a*))
by (*blast intro*: *complex-split-polar*)

lemma *hcomplex-split-polar*:
 !!*z*. ∃ *r a*. *z* = *hcomplex-of-hypreal r* ∗ (*HComplex*((∗*f*∗ *cos*) *a*)((∗*f*∗ *sin*) *a*))
by *transfer* (*rule complex-split-polar*)

lemma *hcis-eq*:
 !!*a*. *hcis a* =
  (*hcomplex-of-hypreal*((∗*f*∗ *cos*) *a*) +
  iii ∗ *hcomplex-of-hypreal*((∗*f*∗ *sin*) *a*))
by *transfer* (*simp add*: *cis-def*)

lemma *hrcis-Ex*: !!*z*. ∃ *r a*. *z* = *hrcis r a*
by *transfer* (*rule rcis-Ex*)

lemma *hRe-hcomplex-polar* [*simp*]:
 !!*r a*. *hRe* (*hcomplex-of-hypreal r* ∗ *HComplex* ((∗*f*∗ *cos*) *a*) ((∗*f*∗ *sin*) *a*)) =
   r ∗ (∗*f*∗ *cos*) *a*
by *transfer simp*

lemma *hRe-hrcis* [*simp*]: !!*r a*. *hRe*(*hrcis r a*) = *r* ∗ (∗*f*∗ *cos*) *a*

by *transfer* (*rule Re-rcis*)

lemma *hIm-hcomplex-polar* [*simp*]:
!!*r a*. *hIm* (*hcomplex-of-hypreal r* ∗ *HComplex* ((∗*f*∗ *cos*) *a*) ((∗*f*∗ *sin*) *a*)) =
r ∗ (∗*f*∗ *sin*) *a*
by *transfer simp*

lemma *hIm-hrcis* [*simp*]: !!*r a*. *hIm*(*hrcis r a*) = *r* ∗ (∗*f*∗ *sin*) *a*
by *transfer* (*rule Im-rcis*)

lemma *hcmod-unit-one* [*simp*]:
!!*a*. *hcmod* (*HComplex* ((∗*f*∗ *cos*) *a*) ((∗*f*∗ *sin*) *a*)) = *1*
by *transfer* (*rule cmod-unit-one*)

lemma *hcmod-complex-polar* [*simp*]:
!!*r a*. *hcmod* (*hcomplex-of-hypreal r* ∗ *HComplex* ((∗*f*∗ *cos*) *a*) ((∗*f*∗ *sin*) *a*)) =
abs r
by *transfer* (*rule cmod-complex-polar*)

lemma *hcmod-hrcis* [*simp*]: !!*r a*. *hcmod*(*hrcis r a*) = *abs r*
by *transfer* (*rule complex-mod-rcis*)

lemma *hcis-hrcis-eq*: !!*a*. *hcis a* = *hrcis 1 a*
by *transfer* (*rule cis-rcis-eq*)
declare *hcis-hrcis-eq* [*symmetric*, *simp*]

lemma *hrcis-mult*:
!!*a b r1 r2*. *hrcis r1 a* ∗ *hrcis r2 b* = *hrcis* (*r1*∗*r2*) (*a* + *b*)
by *transfer* (*rule rcis-mult*)

lemma *hcis-mult*: !!*a b*. *hcis a* ∗ *hcis b* = *hcis* (*a* + *b*)
by *transfer* (*rule cis-mult*)

lemma *hcis-zero* [*simp*]: *hcis 0* = *1*
by *transfer* (*rule cis-zero*)

lemma *hrcis-zero-mod* [*simp*]: !!*a*. *hrcis 0 a* = *0*
by *transfer* (*rule rcis-zero-mod*)

lemma *hrcis-zero-arg* [*simp*]: !!*r*. *hrcis r 0* = *hcomplex-of-hypreal r*
by *transfer* (*rule rcis-zero-arg*)

lemma *hcomplex-i-mult-minus* [*simp*]: !!*x*. *iii* ∗ (*iii* ∗ *x*) = − *x*
by *transfer* (*rule complex-i-mult-minus*)

lemma *hcomplex-i-mult-minus2* [*simp*]: $iii * iii * x = - x$
by *simp*

lemma *hcis-hypreal-of-nat-Suc-mult*:
  $!!a.$ *hcis* (*hypreal-of-nat* (*Suc* *n*) $*$ *a*) $=$
    hcis *a* $*$ *hcis* (*hypreal-of-nat* *n* $*$ *a*)
apply *transfer*
apply (*fold real-of-nat-def*)
apply (*rule cis-real-of-nat-Suc-mult*)
done

lemma *NSDeMoivre*: $!!a.$ (*hcis* *a*) ^ *n* $=$ *hcis* (*hypreal-of-nat* *n* $*$ *a*)
apply *transfer*
apply (*fold real-of-nat-def*)
apply (*rule DeMoivre*)
done

lemma *hcis-hypreal-of-hypnat-Suc-mult*:
  $!!$ *a n*. *hcis* (*hypreal-of-hypnat* (*n* + *1*) $*$ *a*) $=$
    hcis *a* $*$ *hcis* (*hypreal-of-hypnat* *n* $*$ *a*)
by *transfer* (*fold real-of-nat-def*, *simp add*: *cis-real-of-nat-Suc-mult*)

lemma *NSDeMoivre-ext*:
  $!!$*a n*. (*hcis* *a*) *pow* *n* $=$ *hcis* (*hypreal-of-hypnat* *n* $*$ *a*)
by *transfer* (*fold real-of-nat-def*, *rule DeMoivre*)

lemma *NSDeMoivre2*:
  $!!$*a r*. (*hrcis* *r* *a*) ^ *n* $=$ *hrcis* (*r* ^ *n*) (*hypreal-of-nat* *n* $*$ *a*)
by *transfer* (*fold real-of-nat-def*, *rule DeMoivre2*)

lemma *DeMoivre2-ext*:
  $!!$ *a r n*. (*hrcis* *r* *a*) *pow* *n* $=$ *hrcis* (*r* *pow* *n*) (*hypreal-of-hypnat* *n* $*$ *a*)
by *transfer* (*fold real-of-nat-def*, *rule DeMoivre2*)

lemma *hcis-inverse* [*simp*]: $!!a.$ *inverse*(*hcis* *a*) $=$ *hcis* ($-a$)
by *transfer* (*rule cis-inverse*)

lemma *hrcis-inverse*: $!!$*a r*. *inverse*(*hrcis* *r* *a*) $=$ *hrcis* (*inverse* *r*) ($-a$)
by *transfer* (*simp add*: *rcis-inverse inverse-eq-divide* [*symmetric*])

lemma *hRe-hcis* [*simp*]: $!!a.$ *hRe*(*hcis* *a*) $=$ ($*f*$ *cos*) *a*
by *transfer* (*rule Re-cis*)

lemma *hIm-hcis* [*simp*]: $!!a.$ *hIm*(*hcis* *a*) $=$ ($*f*$ *sin*) *a*
by *transfer* (*rule Im-cis*)

lemma *cos-n-hRe-hcis-pow-n*: ($*f*$ *cos*) (*hypreal-of-nat* *n* $*$ *a*) $=$ *hRe*(*hcis* *a* ^ *n*)
by (*simp add*: *NSDeMoivre*)

lemma *sin-n-hIm-hcis-pow-n*: (**f* sin*) (*hypreal-of-nat n* * *a*) = *hIm*(*hcis a* ˆ *n*)
by (*simp add*: *NSDeMoivre*)

lemma *cos-n-hRe-hcis-hcpow-n*: (**f* cos*) (*hypreal-of-hypnat n* * *a*) = *hRe*(*hcis a pow n*)
by (*simp add*: *NSDeMoivre-ext*)

lemma *sin-n-hIm-hcis-hcpow-n*: (**f* sin*) (*hypreal-of-hypnat n* * *a*) = *hIm*(*hcis a pow n*)
by (*simp add*: *NSDeMoivre-ext*)

lemma *hexpi-add*: !!*a b*. *hexpi*(*a* + *b*) = *hexpi*(*a*) * *hexpi*(*b*)
by *transfer* (*rule expi-add*)

28.14 *hcomplex-of-complex*: the Injection from type *complex* to to *hcomplex*

lemma *inj-hcomplex-of-complex*: *inj*(*hcomplex-of-complex*)

by (*rule inj-star-of*)

lemma *hcomplex-of-complex-i*: *iii* = *hcomplex-of-complex ii*
by (*rule iii-def*)

lemma *hRe-hcomplex-of-complex*:
hRe (*hcomplex-of-complex z*) = *hypreal-of-real* (*Re z*)
by *transfer* (*rule refl*)

lemma *hIm-hcomplex-of-complex*:
hIm (*hcomplex-of-complex z*) = *hypreal-of-real* (*Im z*)
by *transfer* (*rule refl*)

lemma *hcmod-hcomplex-of-complex*:
hcmod (*hcomplex-of-complex x*) = *hypreal-of-real* (*cmod x*)
by *transfer* (*rule refl*)

28.15 Numerals and Arithmetic

lemma *hcomplex-number-of-def*: (*number-of w* :: *hcomplex*) == *of-int w*
by *transfer* (*rule number-of-eq* [*THEN eq-reflection*])

lemma *hcomplex-of-hypreal-eq-hcomplex-of-complex*:
hcomplex-of-hypreal (*hypreal-of-real x*) =
hcomplex-of-complex (*complex-of-real x*)
by *transfer* (*rule refl*)

lemma *hcomplex-hypreal-number-of*:
hcomplex-of-complex (*number-of w*) = *hcomplex-of-hypreal*(*number-of w*)
by *transfer* (*rule of-real-number-of-eq* [*symmetric*])

lemma *hcomplex-number-of-hcnj* [*simp*]:
  hcnj (*number-of v* :: *hcomplex*) = *number-of v*
by *transfer* (*rule complex-cnj-number-of*)

lemma *hcomplex-number-of-hcmod* [*simp*]:
  hcmod(*number-of v* :: *hcomplex*) = *abs* (*number-of v* :: *hypreal*)
by *transfer* (*rule norm-number-of*)

lemma *hcomplex-number-of-hRe* [*simp*]:
  hRe(*number-of v* :: *hcomplex*) = *number-of v*
by *transfer* (*rule complex-Re-number-of*)

lemma *hcomplex-number-of-hIm* [*simp*]:
  hIm(*number-of v* :: *hcomplex*) = *0*
by *transfer* (*rule complex-Im-number-of*)

end

29 Star: Star-Transforms in Non-Standard Analysis

theory *Star*
imports *NSA*
begin

definition

starset-n :: (*nat* => *'a set*) => *'a star set* (**sn** - [*80*] *80*) **where**
**sn* As* = *Iset* (*star-n As*)

definition
InternalSets :: *'a star set set* **where**
InternalSets = $\{X.\ \exists\, As.\ X = *sn*\ As\}$

definition

is-starext :: [*'a star* => *'a star*, *'a* => *'a*] => *bool* **where**
is-starext F f = ($\forall$ *x y*. $\exists$ *X* $\in$ *Rep-star*(*x*). $\exists$ *Y* $\in$ *Rep-star*(*y*).
  ((*y* = (*F x*)) = ({*n*. *Y n* = *f*(*X n*)} : *FreeUltrafilterNat*)))

definition

starfun-n :: $(nat => ('a => 'b)) => 'a\ star => 'b\ star$ (*$*fn*$* - [*80*] *80*) **where**
$*fn*\ F = Ifun\ (star\text{-}n\ F)$

definition
InternalFuns :: $('a\ star => 'b\ star)\ set$ **where**
$InternalFuns = \{X.\ \exists F.\ X = *fn*\ F\}$

lemma *no-choice*: $\forall x.\ \exists y.\ Q\ x\ y ==> \exists (f :: 'a => nat).\ \forall x.\ Q\ x\ (f\ x)$
apply (*rule-tac* $x = \%x.\ LEAST\ y.\ Q\ x\ y$ **in** *exI*)
apply (*blast intro*: *LeastI*)
done

29.1 Properties of the Star-transform Applied to Sets of Reals

lemma *STAR-star-of-image-subset*: *star-of* ‘ $A <= *s*\ A$
by *auto*

lemma *STAR-hypreal-of-real-Int*: $*s*\ X\ Int\ Reals = hypreal\text{-}of\text{-}real$ ‘ X
by (*auto simp add*: *SReal-def*)

lemma *STAR-star-of-Int*: $*s*\ X\ Int\ Standard = star\text{-}of$ ‘ X
by (*auto simp add*: *Standard-def*)

lemma *lemma-not-hyprealA*: $x \notin hypreal\text{-}of\text{-}real$ ‘ $A ==> \forall y \in A.\ x \neq hypreal\text{-}of\text{-}real\ y$
by *auto*

lemma *lemma-not-starA*: $x \notin star\text{-}of$ ‘ $A ==> \forall y \in A.\ x \neq star\text{-}of\ y$
by *auto*

lemma *lemma-Compl-eq*: $- \{n.\ X\ n = xa\} = \{n.\ X\ n \neq xa\}$
by *auto*

lemma *STAR-real-seq-to-hypreal*:
$\forall n.\ (X\ n) \notin M ==> star\text{-}n\ X \notin *s*\ M$
apply (*unfold starset-def star-of-def*)
apply (*simp add*: *Iset-star-n*)
done

lemma *STAR-singleton*: $*s*\ \{x\} = \{star\text{-}of\ x\}$
by *simp*

lemma *STAR-not-mem*: $x \notin F$ ==> *star-of* $x \notin$ *∗s∗* F
by *transfer*

lemma *STAR-subset-closed*: [| x : *∗s∗* A; A <= B |] ==> x : *∗s∗* B
by (*erule rev-subsetD*, *simp*)

Nonstandard extension of a set (defined using a constant sequence) as a special case of an internal set

lemma *starset-n-starset*: $\forall n.$ (As n = A) ==> *∗sn∗* As = *∗s∗* A
apply (*drule expand-fun-eq* [*THEN iffD2*])
apply (*simp add*: *starset-n-def starset-def star-of-def*)
done

lemma *starfun-n-starfun*: $\forall n.$ (F n = f) ==> *∗fn∗* F = *∗f∗* f
apply (*drule expand-fun-eq* [*THEN iffD2*])
apply (*simp add*: *starfun-n-def starfun-def star-of-def*)
done

lemma *hrabs-is-starext-rabs*: *is-starext abs abs*
apply (*simp add*: *is-starext-def*, *safe*)
apply (*rule-tac x=x* **in** *star-cases*)
apply (*rule-tac x=y* **in** *star-cases*)
apply (*unfold star-n-def*, *auto*)
apply (*rule bexI*, *rule-tac* [*2*] *lemma-starrel-refl*)
apply (*rule bexI*, *rule-tac* [*2*] *lemma-starrel-refl*)
apply (*fold star-n-def*)
apply (*unfold star-abs-def starfun-def star-of-def*)
apply (*simp add*: *Ifun-star-n star-n-eq-iff*)
done

Nonstandard extension of functions

lemma *starfun*:
  (*∗f∗* f) (*star-n* X) = *star-n* (%n. f (X n))
by (*rule starfun-star-n*)

lemma *starfun-if-eq*:
$!!w.\ w \neq star\text{-}of\ x$
$==>\ (\ *f*\ (\lambda z.\ if\ z = x\ then\ a\ else\ g\ z))\ w = (\ *f*\ g)\ w$
by (*transfer*, *simp*)

lemma *starfun-mult*: $!!x.\ (\ *f*\ f)\ x * (\ *f*\ g)\ x = (\ *f*\ (\%x.\ f\ x * g\ x))\ x$
by (*transfer*, *rule refl*)
declare *starfun-mult* [*symmetric*, *simp*]

lemma *starfun-add*: $!!x.\ (\ *f*\ f)\ x + (\ *f*\ g)\ x = (\ *f*\ (\%x.\ f\ x + g\ x))\ x$
by (*transfer*, *rule refl*)
declare *starfun-add* [*symmetric*, *simp*]

lemma *starfun-minus*: $!!x.\ -\ (\ *f*\ f)\ x = (\ *f*\ (\%x.\ -\ f\ x))\ x$
by (*transfer*, *rule refl*)
declare *starfun-minus* [*symmetric*, *simp*]

lemma *starfun-add-minus*: $!!x.\ (\ *f*\ f)\ x + -(\ *f*\ g)\ x = (\ *f*\ (\%x.\ f\ x + -g\ x))\ x$
by (*transfer*, *rule refl*)
declare *starfun-add-minus* [*symmetric*, *simp*]

lemma *starfun-diff*: $!!x.\ (\ *f*\ f)\ x\ -\ (\ *f*\ g)\ x = (\ *f*\ (\%x.\ f\ x - g\ x))\ x$
by (*transfer*, *rule refl*)
declare *starfun-diff* [*symmetric*, *simp*]

lemma *starfun-o2*: $(\%x.\ (\ *f*\ f)\ ((\ *f*\ g)\ x)) = *f*\ (\%x.\ f\ (g\ x))$
by (*transfer*, *rule refl*)

lemma *starfun-o*: $(\ *f*\ f)\ o\ (\ *f*\ g) = (\ *f*\ (f\ o\ g))$
by (*transfer o-def*, *rule refl*)

NS extension of constant function

lemma *starfun-const-fun* [*simp*]: $!!x.\ (\ *f*\ (\%x.\ k))\ x = star\text{-}of\ k$
by (*transfer*, *rule refl*)

the NS extension of the identity function

lemma *starfun-Id* [*simp*]: $!!x.\ (\ *f*\ (\%x.\ x))\ x = x$
by (*transfer*, *rule refl*)

lemma *starfun-Idfun-approx*:
$x\ @=\ star\text{-}of\ a\ ==>\ (\ *f*\ (\%x.\ x))\ x\ @=\ star\text{-}of\ a$

by (*simp only*: *starfun-Id*)

The Star-function is a (nonstandard) extension of the function

lemma *is-starext-starfun*: *is-starext* ($*f*$ f) f
apply (*simp add*: *is-starext-def*, *auto*)
apply (*rule-tac* $x = x$ **in** *star-cases*)
apply (*rule-tac* $x = y$ **in** *star-cases*)
apply (*auto intro*!: *bexI* [*OF* - *Rep-star-star-n*]
simp add: *starfun star-n-eq-iff*)
done

Any nonstandard extension is in fact the Star-function

lemma *is-starfun-starext*: *is-starext* F f ==> $F = *f*\ f$
apply (*simp add*: *is-starext-def*)
apply (*rule ext*)
apply (*rule-tac* $x = x$ **in** *star-cases*)
apply (*drule-tac* $x = x$ **in** *spec*)
apply (*drule-tac* $x = (*f*\ f)\ x$ **in** *spec*)
apply (*auto simp add*: *starfun-star-n*)
apply (*simp add*: *star-n-eq-iff* [*symmetric*])
apply (*simp add*: *starfun-star-n* [*of* f, *symmetric*])
done

lemma *is-starext-starfun-iff*: (*is-starext* F f) = ($F = *f*\ f$)
by (*blast intro*: *is-starfun-starext is-starext-starfun*)

extented function has same solution as its standard version for real arguments. i.e they are the same for all real arguments

lemma *starfun-eq*: ($*f*$ f) (*star-of* a) = *star-of* (f a)
by (*rule starfun-star-of*)

lemma *starfun-approx*: ($*f*$ f) (*star-of* a) @= *star-of* (f a)
by *simp*

lemma *starfun-lambda-cancel*:
!!x'. ($*f*$ (%h. f ($x + h$))) x' = ($*f*$ f) (*star-of* $x + x'$)
by (*transfer*, *rule refl*)

lemma *starfun-lambda-cancel2*:
($*f*$ (%h. f(g($x + h$)))) x' = ($*f*$ (f *o* g)) (*star-of* $x + x'$)
by (*unfold o-def*, *rule starfun-lambda-cancel*)

lemma *starfun-mult-HFinite-approx*:
fixes l m :: $'a$::*real-normed-algebra star*
shows [| ($*f*$ f) x @= l; ($*f*$ g) x @= m;
l: *HFinite*; m: *HFinite*
|] ==> ($*f*$ (%x. f $x * g$ x)) x @= $l * m$
apply (*drule* (*3*) *approx-mult-HFinite*)

apply (*auto intro*: *approx-HFinite* [*OF* - *approx-sym*])
done

lemma *starfun-add-approx*: [| (*f* f) x @= l; (*f* g) x @= m
|] ==> (*f* (%x. f x + g x)) x @= l + m
by (*auto intro*: *approx-add*)

Examples: hrabs is nonstandard extension of rabs inverse is nonstandard extension of inverse

lemma *starfun-rabs-hrabs*: *f* abs = abs
by (*simp only*: *star-abs-def*)

lemma *starfun-inverse-inverse* [*simp*]: (*f* inverse) x = inverse(x)
by (*simp only*: *star-inverse-def*)

lemma *starfun-inverse*: !!x. inverse ((*f* f) x) = (*f* (%x. inverse (f x))) x
by (*transfer*, *rule refl*)
declare *starfun-inverse* [*symmetric*, *simp*]

lemma *starfun-divide*: !!x. (*f* f) x / (*f* g) x = (*f* (%x. f x / g x)) x
by (*transfer*, *rule refl*)
declare *starfun-divide* [*symmetric*, *simp*]

lemma *starfun-inverse2*: !!x. inverse ((*f* f) x) = (*f* (%x. inverse (f x))) x
by (*transfer*, *rule refl*)

General lemma/theorem needed for proofs in elementary topology of the reals

lemma *starfun-mem-starset*:
!!x. (*f* f) x : *s* A ==> x : *s* {x. f x ∈ A}
by (*transfer*, *simp*)

Alternative definition for hrabs with rabs function applied entrywise to equivalence class representative. This is easily proved using starfun and ns extension thm

lemma *hypreal-hrabs*:
abs (star-n X) = star-n (%n. abs (X n))
by (*simp only*: *starfun-rabs-hrabs* [*symmetric*] *starfun*)

nonstandard extension of set through nonstandard extension of rabs function i.e hrabs. A more general result should be where we replace rabs by some arbitrary function f and hrabs by its NS extenson. See second NS set extension below.

lemma *STAR-rabs-add-minus*:
s {x. abs (x + − y) < r} =
{x. abs(x + −star-of y) < star-of r}
by (*transfer*, *rule refl*)

lemma *STAR-starfun-rabs-add-minus*:
$*s* \{x.\ abs\ (f\,x + -\ y) < r\} =$
$\{x.\ abs((\ *f*\ f)\ x + -star\text{-}of\ y) < star\text{-}of\ r\}$
by (*transfer*, *rule refl*)

Another characterization of Infinitesimal and one of @= relation. In this theory since *hypreal-hrabs* proved here. Maybe move both theorems??

lemma *Infinitesimal-FreeUltrafilterNat-iff2*:
(*star-n X* ∈ *Infinitesimal*) =
(∀ *m*. {*n*. *norm*(*X n*) < *inverse*(*real*(*Suc m*))}
∈ *FreeUltrafilterNat*)
by (*simp add*: *Infinitesimal-hypreal-of-nat-iff star-of-def*
hnorm-def star-of-nat-def starfun-star-n
star-n-inverse star-n-less real-of-nat-def)

lemma *HNatInfinite-inverse-Infinitesimal* [*simp*]:
n ∈ *HNatInfinite* ==> *inverse* (*hypreal-of-hypnat n*) ∈ *Infinitesimal*
apply (*cases n*)
apply (*auto simp add*: *of-hypnat-def starfun-star-n real-of-nat-def* [*symmetric*]
star-n-inverse real-norm-def
HNatInfinite-FreeUltrafilterNat-iff
Infinitesimal-FreeUltrafilterNat-iff2)
apply (*drule-tac x=Suc m* **in** *spec*)
apply (*erule ultra*, *simp*)
done

lemma *approx-FreeUltrafilterNat-iff*: *star-n X* @= *star-n Y* =
(∀ *r*>*0*. {*n*. *norm* (*X n* − *Y n*) < *r*} : *FreeUltrafilterNat*)
apply (*subst approx-minus-iff*)
apply (*rule mem-infmal-iff* [*THEN subst*])
apply (*simp add*: *star-n-diff*)
apply (*simp add*: *Infinitesimal-FreeUltrafilterNat-iff*)
done

lemma *approx-FreeUltrafilterNat-iff2*: *star-n X* @= *star-n Y* =
(∀ *m*. {*n*. *norm* (*X n* − *Y n*) <
inverse(*real*(*Suc m*))} : *FreeUltrafilterNat*)
apply (*subst approx-minus-iff*)
apply (*rule mem-infmal-iff* [*THEN subst*])
apply (*simp add*: *star-n-diff*)
apply (*simp add*: *Infinitesimal-FreeUltrafilterNat-iff2*)
done

lemma *inj-starfun*: *inj starfun*
apply (*rule inj-onI*)
apply (*rule ext*, *rule ccontr*)
apply (*drule-tac x* = *star-n* (%*n*. *xa*) **in** *fun-cong*)
apply (*auto simp add*: *starfun star-n-eq-iff*)

done

end

30 NatStar: Star-transforms for the Hypernaturals

theory *NatStar*
imports *Star*
begin

lemma *star-n-eq-starfun-whn*: *star-n X* = (*∗f∗ X*) *whn*
by (*simp add*: *hypnat-omega-def starfun-def star-of-def Ifun-star-n*)

lemma *starset-n-Un*: *∗sn∗* (%*n*. (*A n*) *Un* (*B n*)) = *∗sn∗ A Un ∗sn∗ B*
apply (*simp add*: *starset-n-def star-n-eq-starfun-whn Un-def*)
apply (*rule-tac x*=*whn* **in** *spec*, *transfer*, *simp*)
done

lemma *InternalSets-Un*:
  [| *X* ∈ *InternalSets*; *Y* ∈ *InternalSets* |]
  ==> (*X Un Y*) ∈ *InternalSets*
by (*auto simp add*: *InternalSets-def starset-n-Un* [*symmetric*])

lemma *starset-n-Int*:
  ∗sn∗ (%*n*. (*A n*) *Int* (*B n*)) = *∗sn∗ A Int ∗sn∗ B*
apply (*simp add*: *starset-n-def star-n-eq-starfun-whn Int-def*)
apply (*rule-tac x*=*whn* **in** *spec*, *transfer*, *simp*)
done

lemma *InternalSets-Int*:
  [| *X* ∈ *InternalSets*; *Y* ∈ *InternalSets* |]
  ==> (*X Int Y*) ∈ *InternalSets*
by (*auto simp add*: *InternalSets-def starset-n-Int* [*symmetric*])

lemma *starset-n-Compl*: *∗sn∗* ((%*n*. − *A n*)) = −(*∗sn∗ A*)
apply (*simp add*: *starset-n-def star-n-eq-starfun-whn Compl-def*)
apply (*rule-tac x*=*whn* **in** *spec*, *transfer*, *simp*)
done

lemma *InternalSets-Compl*: *X* ∈ *InternalSets* ==> −*X* ∈ *InternalSets*
by (*auto simp add*: *InternalSets-def starset-n-Compl* [*symmetric*])

lemma *starset-n-diff*: *∗sn∗* (%*n*. (*A n*) − (*B n*)) = *∗sn∗ A* − *∗sn∗ B*
apply (*simp add*: *starset-n-def star-n-eq-starfun-whn set-diff-def*)
apply (*rule-tac x*=*whn* **in** *spec*, *transfer*, *simp*)
done

lemma *InternalSets-diff*:
  $[| X \in InternalSets; Y \in InternalSets |]$
  $==> (X - Y) \in InternalSets$
by (*auto simp add*: *InternalSets-def starset-n-diff* [*symmetric*])

lemma *NatStar-SHNat-subset*: $Nats \leq *s* (UNIV :: nat\ set)$
by *simp*

lemma *NatStar-hypreal-of-real-Int*:
  $*s* X\ Int\ Nats = hypnat\text{-}of\text{-}nat$ ' X
by (*auto simp add*: *SHNat-eq*)

lemma *starset-starset-n-eq*: $*s* X = *sn* (\%n. X)$
by (*simp add*: *starset-n-starset*)

lemma *InternalSets-starset-n* [*simp*]: $(*s* X) \in InternalSets$
by (*auto simp add*: *InternalSets-def starset-starset-n-eq*)

lemma *InternalSets-UNIV-diff*:
  $X \in InternalSets ==> UNIV - X \in InternalSets$
apply (*subgoal-tac* $UNIV - X = - X$)
by (*auto intro*: *InternalSets-Compl*)

30.1 Nonstandard Extensions of Functions

Example of transfer of a property from reals to hyperreals — used for limit comparison of sequences

lemma *starfun-le-mono*:
  $\forall n. N \leq n --> f\ n \leq g\ n$
  $==> \forall n.\ hypnat\text{-}of\text{-}nat\ N \leq n --> (*f* f)\ n \leq (*f* g)\ n$
by *transfer*

lemma *starfun-less-mono*:
  $\forall n. N \leq n --> f\ n < g\ n$
  $==> \forall n.\ hypnat\text{-}of\text{-}nat\ N \leq n --> (*f* f)\ n < (*f* g)\ n$
by *transfer*

Nonstandard extension when we increment the argument by one

lemma *starfun-shift-one*:
  $!!N. (*f* (\%n. f (Suc\ n)))\ N = (*f* f) (N + (1::hypnat))$
by (*transfer*, *simp*)

Nonstandard extension with absolute value

lemma *starfun-abs*: $!!N. (*f* (\%n. abs (f\ n)))\ N = abs((*f* f)\ N)$
by (*transfer*, *rule refl*)

The hyperpow function as a nonstandard extension of realpow

lemma *starfun-pow*: !!N. ($*f*$ (%n. r ^ n)) N = (*hypreal-of-real* r) *pow* N
by (*transfer*, *rule refl*)

lemma *starfun-pow2*:
!!N. ($*f*$ (%n. (X n) ^ m)) N = ($*f*$ X) N *pow hypnat-of-nat* m
by (*transfer*, *rule refl*)

lemma *starfun-pow3*: !!R. ($*f*$ (%r. r ^ n)) R = (R) *pow hypnat-of-nat* n
by (*transfer*, *rule refl*)

The *hypreal-of-hypnat* function as a nonstandard extension of *real-of-nat*

lemma *starfunNat-real-of-nat*: ($*f*$ *real*) = *hypreal-of-hypnat*
by *transfer* (*simp add*: *expand-fun-eq real-of-nat-def*)

lemma *starfun-inverse-real-of-nat-eq*:
N ∈ *HNatInfinite*
==> ($*f*$ (%x::*nat*. *inverse*(*real* x))) N = *inverse*(*hypreal-of-hypnat* N)
apply (*rule-tac f1* = *inverse* **in** *starfun-o2* [*THEN subst*])
apply (*subgoal-tac hypreal-of-hypnat* N ~= *0*)
apply (*simp-all add*: *zero-less-HNatInfinite starfunNat-real-of-nat starfun-inverse-inverse*)
done

Internal functions - some redundancy with *f* now

lemma *starfun-n*: ($*fn*$ f) (*star-n* X) = *star-n* (%n. f n (X n))
by (*simp add*: *starfun-n-def Ifun-star-n*)

Multiplication: ($*fn$) x ($*gn$) = $*$(fn x gn)

lemma *starfun-n-mult*:
($*fn*$ f) z $*$ ($*fn*$ g) z = ($*fn*$ (% i x. f i x $*$ g i x)) z
apply (*cases* z)
apply (*simp add*: *starfun-n star-n-mult*)
done

Addition: ($*fn$) + ($*gn$) = $*$(fn + gn)

lemma *starfun-n-add*:
($*fn*$ f) z + ($*fn*$ g) z = ($*fn*$ (%i x. f i x + g i x)) z
apply (*cases* z)
apply (*simp add*: *starfun-n star-n-add*)
done

Subtraction: ($*fn$) − ($*gn$) = $*$(fn + − gn)

lemma *starfun-n-add-minus*:
($*fn*$ f) z + −($*fn*$ g) z = ($*fn*$ (%i x. f i x + −g i x)) z
apply (*cases* z)
apply (*simp add*: *starfun-n star-n-minus star-n-add*)
done

Composition: ($*fn$) o ($*gn$) = $*$(fn o gn)

lemma *starfun-n-const-fun* [*simp*]:
(*fn* (%i x. k)) z = star-of k
apply (*cases z*)
apply (*simp add*: *starfun-n star-of-def*)
done

lemma *starfun-n-minus*: − (*fn* f) x = (*fn* (%i x. − (f i) x)) x
apply (*cases x*)
apply (*simp add*: *starfun-n star-n-minus*)
done

lemma *starfun-n-eq* [*simp*]:
(*fn* f) (star-of n) = star-n (%i. f i n)
by (*simp add*: *starfun-n star-of-def*)

lemma *starfun-eq-iff*: ((*f* f) = (*f* g)) = (f = g)
by (*transfer*, *rule refl*)

lemma *starfunNat-inverse-real-of-nat-Infinitesimal* [*simp*]:
N ∈ HNatInfinite ==> (*f* (%x. inverse (real x))) N ∈ Infinitesimal
apply (*rule-tac f1 = inverse* **in** *starfun-o2* [*THEN subst*])
apply (*subgoal-tac hypreal-of-hypnat N ~= 0*)
apply (*simp-all add*: *zero-less-HNatInfinite starfunNat-real-of-nat*)
done

30.2 Nonstandard Characterization of Induction

lemma *hypnat-induct-obj*:
!!n. ((*p* P) (0::hypnat) &
(∀ n. (*p* P)(n) −−> (*p* P)(n + 1)))
−−> (*p* P)(n)
by (*transfer*, *induct-tac n*, *auto*)

lemma *hypnat-induct*:
!!n. [| (*p* P) (0::hypnat);
!!n. (*p* P)(n) ==> (*p* P)(n + 1)|]
==> (*p* P)(n)
by (*transfer*, *induct-tac n*, *auto*)

lemma *starP2-eq-iff*: (*p2* (op =)) = (op =)
by *transfer* (*rule refl*)

lemma *starP2-eq-iff2*: (*p2* (%x y. x = y)) X Y = (X = Y)
by (*simp add*: *starP2-eq-iff*)

lemma *nonempty-nat-set-Least-mem*:
c ∈ (S :: nat set) ==> (LEAST n. n ∈ S) ∈ S
by (*erule LeastI*)

lemma *nonempty-set-star-has-least*:
  !!*S*::*nat set star*. *Iset S* $\neq$ {} ==> $\exists$ *n* $\in$ *Iset S*. $\forall$ *m* $\in$ *Iset S*. *n* $\leq$ *m*
apply (*transfer empty-def*)
apply (*rule-tac x*=*LEAST n*. *n* $\in$ *S* **in** *bexI*)
apply (*simp add*: *Least-le*)
apply (*rule LeastI-ex*, *auto*)
done

lemma *nonempty-InternalNatSet-has-least*:
  [| (*S*::*hypnat set*) $\in$ *InternalSets*; *S* $\neq$ {} |] ==> $\exists$ *n* $\in$ *S*. $\forall$ *m* $\in$ *S*. *n* $\leq$ *m*
apply (*clarsimp simp add*: *InternalSets-def starset-n-def*)
apply (*erule nonempty-set-star-has-least*)
done

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lemma *internal-induct-lemma*:
  !!*X*::*nat set star*. [| (*0*::*hypnat*) $\in$ *Iset X*; $\forall$ *n*. *n* $\in$ *Iset X* −−> *n* + *1* $\in$ *Iset X* |]
   ==> *Iset X* = (*UNIV*:: *hypnat set*)
apply (*transfer UNIV-def*)
apply (*rule equalityI* [*OF subset-UNIV subsetI*])
apply (*induct-tac x*, *auto*)
done

lemma *internal-induct*:
  [| *X* $\in$ *InternalSets*; (*0*::*hypnat*) $\in$ *X*; $\forall$ *n*. *n* $\in$ *X* −−> *n* + *1* $\in$ *X* |]
   ==> *X* = (*UNIV*:: *hypnat set*)
apply (*clarsimp simp add*: *InternalSets-def starset-n-def*)
apply (*erule* (*1*) *internal-induct-lemma*)
done

end

31 HSEQ: Sequences and Convergence (Nonstandard)

theory *HSEQ*
imports *SEQ NatStar*
begin

definition
 NSLIMSEQ :: [*nat* => '*a*::*real-normed-vector*, '*a*] => *bool*
  (((-)/ −−−−*NS*> (-)) [*60*, *60*] *60*) **where**
  — Nonstandard definition of convergence of sequence
 X −−−−*NS*> *L* = ($\forall$ *N* $\in$ *HNatInfinite*. (**f** *X*) *N* $\approx$ *star-of L*)

definition
nslim :: (*nat* => $'a$::*real-normed-vector*) => $'a$ **where**
— Nonstandard definition of limit using choice operator
nslim X = (*THE* L. X ----*NS*> L)

definition
NSconvergent :: (*nat* => $'a$::*real-normed-vector*) => *bool* **where**
— Nonstandard definition of convergence
NSconvergent X = ($\exists$ L. X ----*NS*> L)

definition
NSBseq :: (*nat* => $'a$::*real-normed-vector*) => *bool* **where**
— Nonstandard definition for bounded sequence
NSBseq X = ($\forall$ N $\in$ *HNatInfinite*. (*f* X) N : *HFinite*)

definition
NSCauchy :: (*nat* => $'a$::*real-normed-vector*) => *bool* **where**
— Nonstandard definition
NSCauchy X = ($\forall$ M $\in$ *HNatInfinite*. $\forall$ N $\in$ *HNatInfinite*. (*f* X) M $\approx$ (*f* X) N)

31.1 Limits of Sequences

lemma *NSLIMSEQ-iff*:
(X ----*NS*> L) = ($\forall$ N $\in$ *HNatInfinite*. (*f* X) N $\approx$ *star-of* L)
by (*simp add*: *NSLIMSEQ-def*)

lemma *NSLIMSEQ-I*:
($\bigwedge N$. N $\in$ *HNatInfinite* $\Longrightarrow$ *starfun* X N $\approx$ *star-of* L) $\Longrightarrow$ X ----*NS*> L
by (*simp add*: *NSLIMSEQ-def*)

lemma *NSLIMSEQ-D*:
$[\![X$ ----*NS*> L; N $\in$ *HNatInfinite*$]\!]$ $\Longrightarrow$ *starfun* X N $\approx$ *star-of* L
by (*simp add*: *NSLIMSEQ-def*)

lemma *NSLIMSEQ-const*: (%n. k) ----*NS*> k
by (*simp add*: *NSLIMSEQ-def*)

lemma *NSLIMSEQ-add*:
[| X ----*NS*> a; Y ----*NS*> b |] ==> (%n. X n + Y n) ----*NS*> a + b
by (*auto intro*: *approx-add simp add*: *NSLIMSEQ-def starfun-add* [*symmetric*])

lemma *NSLIMSEQ-add-const*: f ----*NS*> a ==> (%n.(f n + b)) ----*NS*> a + b
by (*simp only*: *NSLIMSEQ-add NSLIMSEQ-const*)

lemma *NSLIMSEQ-mult*:
fixes a b :: $'a$::*real-normed-algebra*

shows [| X ———$NS>$ a; Y ———$NS>$ b |] ==> (%n. X n ∗ Y n) ———$NS>$ a ∗ b
by (*auto intro*!: *approx-mult-HFinite simp add*: *NSLIMSEQ-def*)

lemma *NSLIMSEQ-minus*: X ———$NS>$ a ==> (%n. −(X n)) ———$NS>$ −a
by (*auto simp add*: *NSLIMSEQ-def*)

lemma *NSLIMSEQ-minus-cancel*: (%n. −(X n)) ———$NS>$ −a ==> X ———$NS>$ a
by (*drule NSLIMSEQ-minus*, *simp*)

lemma *NSLIMSEQ-add-minus*:
[| X ———$NS>$ a; Y ———$NS>$ b |] ==> (%n. X n + −Y n) ———$NS>$ a + −b
by (*simp add*: *NSLIMSEQ-add NSLIMSEQ-minus*)

lemma *NSLIMSEQ-diff*:
[| X ———$NS>$ a; Y ———$NS>$ b |] ==> (%n. X n − Y n) ———$NS>$ a − b
by (*simp add*: *diff-minus NSLIMSEQ-add NSLIMSEQ-minus*)

lemma *NSLIMSEQ-diff-const*: f ———$NS>$ a ==> (%n.(f n − b)) ———$NS>$ a − b
by (*simp add*: *NSLIMSEQ-diff NSLIMSEQ-const*)

lemma *NSLIMSEQ-inverse*:
fixes a :: $'a$::*real-normed-div-algebra*
shows [| X ———$NS>$ a; a ~= *0* |] ==> (%n. *inverse*(X n)) ———$NS>$ *inverse*(a)
by (*simp add*: *NSLIMSEQ-def star-of-approx-inverse*)

lemma *NSLIMSEQ-mult-inverse*:
fixes a b :: $'a$::*real-normed-field*
shows
[| X ———$NS>$ a; Y ———$NS>$ b; b ~= *0* |] ==> (%n. X n / Y n) ———$NS>$ a/b
by (*simp add*: *NSLIMSEQ-mult NSLIMSEQ-inverse divide-inverse*)

lemma *starfun-hnorm*: $\bigwedge x$. *hnorm* ((∗f∗ f) x) = (∗f∗ (λx. *norm* (f x))) x
by *transfer simp*

lemma *NSLIMSEQ-norm*: X ———$NS>$ a $\Longrightarrow$ (λn. *norm* (X n)) ———$NS>$ *norm* a
by (*simp add*: *NSLIMSEQ-def starfun-hnorm* [*symmetric*] *approx-hnorm*)

Uniqueness of limit

lemma *NSLIMSEQ-unique*: [| X ———$NS>$ a; X ———$NS>$ b |] ==> a = b
apply (*simp add*: *NSLIMSEQ-def*)

apply (*drule HNatInfinite-whn* [*THEN* [*2*] *bspec*])+
apply (*auto dest*: *approx-trans3*)
done

lemma *NSLIMSEQ-pow* [*rule-format*]:
fixes *a* :: *'a*::{*real-normed-algebra*,*recpower*}
shows (*X* ----*NS*> *a*) --> ((%*n*. (*X n*) ^ *m*) ----*NS*> *a* ^ *m*)
apply (*induct m*)
apply (*auto simp add*: *power-Suc intro*: *NSLIMSEQ-mult NSLIMSEQ-const*)
done

We can now try and derive a few properties of sequences, starting with the limit comparison property for sequences.

lemma *NSLIMSEQ-le*:
[| *f* ----*NS*> *l*; *g* ----*NS*> *m*;
$\exists N. \forall n \geq N. f(n) \leq g(n)$
|] ==> $l \leq$ (*m*::*real*)
apply (*simp add*: *NSLIMSEQ-def*, *safe*)
apply (*drule starfun-le-mono*)
apply (*drule HNatInfinite-whn* [*THEN* [*2*] *bspec*])+
apply (*drule-tac x* = *whn* **in** *spec*)
apply (*drule bex-Infinitesimal-iff2* [*THEN iffD2*])+
apply *clarify*
apply (*auto intro*: *hypreal-of-real-le-add-Infininitesimal-cancel2*)
done

lemma *NSLIMSEQ-le-const*: [| *X* ----*NS*> (*r*::*real*); $\forall n.$ $a \leq X\ n$ |] ==> *a* $\leq$ *r*
by (*erule NSLIMSEQ-le* [*OF NSLIMSEQ-const*], *auto*)

lemma *NSLIMSEQ-le-const2*: [| *X* ----*NS*> (*r*::*real*); $\forall n.$ $X\ n \leq a$ |] ==> *r* $\leq$ *a*
by (*erule NSLIMSEQ-le* [*OF - NSLIMSEQ-const*], *auto*)

Shift a convergent series by 1: By the equivalence between Cauchiness and convergence and because the successor of an infinite hypernatural is also infinite.

lemma *NSLIMSEQ-Suc*: *f* ----*NS*> *l* ==> (%*n*. *f*(*Suc n*)) ----*NS*> *l*
apply (*unfold NSLIMSEQ-def*, *safe*)
apply (*drule-tac x*=*N* + *1* **in** *bspec*)
apply (*erule HNatInfinite-add*)
apply (*simp add*: *starfun-shift-one*)
done

lemma *NSLIMSEQ-imp-Suc*: (%*n*. *f*(*Suc n*)) ----*NS*> *l* ==> *f* ----*NS*> *l*
apply (*unfold NSLIMSEQ-def*, *safe*)
apply (*drule-tac x*=*N* − *1* **in** *bspec*)
apply (*erule Nats-1* [*THEN* [*2*] *HNatInfinite-diff*])

apply (*simp add*: *starfun-shift-one one-le-HNatInfinite*)
done

lemma *NSLIMSEQ-Suc-iff*: ((%n. f (Suc n)) ----NS> l) = (f ----NS> l)
by (*blast intro*: *NSLIMSEQ-imp-Suc NSLIMSEQ-Suc*)

31.1.1 Equivalence of *LIMSEQ* and *NSLIMSEQ*

lemma *LIMSEQ-NSLIMSEQ*:
  assumes *X*: *X* ----> *L* **shows** *X* ----NS> *L*
proof (*rule NSLIMSEQ-I*)
  fix *N* **assume** *N*: *N* ∈ *HNatInfinite*
  have *starfun X N* − *star-of L* ∈ *Infinitesimal*
  proof (*rule InfinitesimalI2*)
    fix *r*::*real* **assume** *r*: *0* < *r*
    from *LIMSEQ-D* [*OF X r*]
    obtain *no* **where** ∀ *n*≥*no*. *norm* (*X n* − *L*) < *r* **..**
    hence ∀ *n*≥*star-of no*. *hnorm* (*starfun X n* − *star-of L*) < *star-of r*
      by *transfer*
    thus *hnorm* (*starfun X N* − *star-of L*) < *star-of r*
      using *N* **by** (*simp add*: *star-of-le-HNatInfinite*)
  qed
  thus *starfun X N* ≈ *star-of L*
    by (*unfold approx-def*)
qed

lemma *NSLIMSEQ-LIMSEQ*:
  assumes *X*: *X* ----NS> *L* **shows** *X* ----> *L*
proof (*rule LIMSEQ-I*)
  fix *r*::*real* **assume** *r*: *0* < *r*
  have ∃ *no*. ∀ *n*≥*no*. *hnorm* (*starfun X n* − *star-of L*) < *star-of r*
  proof (*intro exI allI impI*)
    fix *n* **assume** *whn* ≤ *n*
    with *HNatInfinite-whn* **have** *n* ∈ *HNatInfinite*
      by (*rule HNatInfinite-upward-closed*)
    with *X* **have** *starfun X n* ≈ *star-of L*
      by (*rule NSLIMSEQ-D*)
    hence *starfun X n* − *star-of L* ∈ *Infinitesimal*
      by (*unfold approx-def*)
    thus *hnorm* (*starfun X n* − *star-of L*) < *star-of r*
      using *r* **by** (*rule InfinitesimalD2*)
  qed
  thus ∃ *no*. ∀ *n*≥*no*. *norm* (*X n* − *L*) < *r*
    by *transfer*
qed

theorem *LIMSEQ-NSLIMSEQ-iff*: (*f* ----> *L*) = (*f* ----NS> *L*)
by (*blast intro*: *LIMSEQ-NSLIMSEQ NSLIMSEQ-LIMSEQ*)

lemma *NSLIMSEQ-finite-set*:
!!(*f*::*nat*=>*nat*). ∀ *n*. *n* ≤ *f n* ==> *finite* {*n*. *f n* ≤ *u*}
by (*rule-tac B*={..*u*} **in** *finite-subset*, *auto intro*: *order-trans*)

31.1.2 Derived theorems about *NSLIMSEQ*

We prove the NS version from the standard one, since the NS proof seems more complicated than the standard one above!

lemma *NSLIMSEQ-norm-zero*: ((λ*n*. *norm* (*X n*)) −−−−*NS*> *0*) = (*X* −−−−*NS*> *0*)
by (*simp add*: *LIMSEQ-NSLIMSEQ-iff* [*symmetric*] *LIMSEQ-norm-zero*)

lemma *NSLIMSEQ-rabs-zero*: ((%*n*. |*f n*|) −−−−*NS*> *0*) = (*f* −−−−*NS*> (*0*::*real*))
by (*simp add*: *LIMSEQ-NSLIMSEQ-iff* [*symmetric*] *LIMSEQ-rabs-zero*)

Generalization to other limits

lemma *NSLIMSEQ-imp-rabs*: *f* −−−−*NS*> (*l*::*real*) ==> (%*n*. |*f n*|) −−−−*NS*> |*l*|
apply (*simp add*: *NSLIMSEQ-def*)
apply (*auto intro*: *approx-hrabs*
simp add: *starfun-abs*)
done

lemma *NSLIMSEQ-inverse-zero*:
∀ *y*::*real*. ∃ *N*. ∀ *n* ≥ *N*. *y* < *f*(*n*)
==> (%*n*. *inverse*(*f n*)) −−−−*NS*> *0*
by (*simp add*: *LIMSEQ-NSLIMSEQ-iff* [*symmetric*] *LIMSEQ-inverse-zero*)

lemma *NSLIMSEQ-inverse-real-of-nat*: (%*n*. *inverse*(*real*(*Suc n*))) −−−−*NS*> *0*
by (*simp add*: *LIMSEQ-NSLIMSEQ-iff* [*symmetric*] *LIMSEQ-inverse-real-of-nat*)

lemma *NSLIMSEQ-inverse-real-of-nat-add*:
(%*n*. *r* + *inverse*(*real*(*Suc n*))) −−−−*NS*> *r*
by (*simp add*: *LIMSEQ-NSLIMSEQ-iff* [*symmetric*] *LIMSEQ-inverse-real-of-nat-add*)

lemma *NSLIMSEQ-inverse-real-of-nat-add-minus*:
(%*n*. *r* + −*inverse*(*real*(*Suc n*))) −−−−*NS*> *r*
by (*simp add*: *LIMSEQ-NSLIMSEQ-iff* [*symmetric*] *LIMSEQ-inverse-real-of-nat-add-minus*)

lemma *NSLIMSEQ-inverse-real-of-nat-add-minus-mult*:
(%*n*. *r*∗(*1* + −*inverse*(*real*(*Suc n*)))) −−−−*NS*> *r*
by (*simp add*: *LIMSEQ-NSLIMSEQ-iff* [*symmetric*] *LIMSEQ-inverse-real-of-nat-add-minus-mult*)

31.2 Convergence

lemma *nslimI*: *X* −−−−*NS*> *L* ==> *nslim X* = *L*
apply (*simp add*: *nslim-def*)
apply (*blast intro*: *NSLIMSEQ-unique*)

done

lemma *lim-nslim-iff*: *lim X = nslim X*
by (*simp add*: *lim-def nslim-def LIMSEQ-NSLIMSEQ-iff*)

lemma *NSconvergentD*: *NSconvergent X ==>* $\exists$ *L*. (*X ----NS> L*)
by (*simp add*: *NSconvergent-def*)

lemma *NSconvergentI*: (*X ----NS> L*) *==> NSconvergent X*
by (*auto simp add*: *NSconvergent-def*)

lemma *convergent-NSconvergent-iff*: *convergent X = NSconvergent X*
by (*simp add*: *convergent-def NSconvergent-def LIMSEQ-NSLIMSEQ-iff*)

lemma *NSconvergent-NSLIMSEQ-iff*: *NSconvergent X =* (*X ----NS> nslim X*)
by (*auto intro*: *theI NSLIMSEQ-unique simp add*: *NSconvergent-def nslim-def*)

31.3 Bounded Monotonic Sequences

lemma *NSBseqD*: [| *NSBseq X*; *N*: *HNatInfinite* |] *==>* (**f* X*) *N* : *HFinite*
by (*simp add*: *NSBseq-def*)

lemma *Standard-subset-HFinite*: *Standard* $\subseteq$ *HFinite*
unfolding *Standard-def* **by** *auto*

lemma *NSBseqD2*: *NSBseq X* $\Longrightarrow$ (**f* X*) *N* $\in$ *HFinite*
apply (*cases N* $\in$ *HNatInfinite*)
apply (*erule* (*1*) *NSBseqD*)
apply (*rule subsetD* [*OF Standard-subset-HFinite*])
apply (*simp add*: *HNatInfinite-def Nats-eq-Standard*)
done

lemma *NSBseqI*: $\forall$ *N* $\in$ *HNatInfinite*. (**f* X*) *N* : *HFinite ==> NSBseq X*
by (*simp add*: *NSBseq-def*)

The standard definition implies the nonstandard definition

lemma *Bseq-NSBseq*: *Bseq X ==> NSBseq X*
proof (*unfold NSBseq-def*, *safe*)
  assume *X*: *Bseq X*
  fix *N* **assume** *N*: *N* $\in$ *HNatInfinite*
  from *BseqD* [*OF X*] **obtain** *K* **where** $\forall$ *n*. *norm* (*X n*) $\le$ *K* **by** *fast*
  hence $\forall$ *N*. *hnorm* (*starfun X N*) $\le$ *star-of K* **by** *transfer*
  hence *hnorm* (*starfun X N*) $\le$ *star-of K* **by** *simp*
  also have *star-of K < star-of* (*K + 1*) **by** *simp*
  finally have $\exists$ *x*$\in$*Reals*. *hnorm* (*starfun X N*) *< x* **by** (*rule bexI*, *simp*)
  thus *starfun X N* $\in$ *HFinite* **by** (*simp add*: *HFinite-def*)
qed

The nonstandard definition implies the standard definition

lemma *SReal-less-omega*: $r \in \mathbb{R} \Longrightarrow r < \omega$
apply (*insert HInfinite-omega*)
apply (*simp add*: *HInfinite-def*)
apply (*simp add*: *order-less-imp-le*)
done

lemma *NSBseq-Bseq*: *NSBseq X* $\Longrightarrow$ *Bseq X*
proof (*rule ccontr*)
  let *?n* = λK. *LEAST n*. $K <$ *norm* (*X n*)
  assume *NSBseq X*
  hence *finite*: (**f* X*) ((**f* ?n*) ω) $\in$ *HFinite*
    by (*rule NSBseqD2*)
  assume $\neg$ *Bseq X*
  hence $\forall K>0.\ \exists n.\ K <$ *norm* (*X n*)
    by (*simp add*: *Bseq-def linorder-not-le*)
  hence $\forall K>0.\ K <$ *norm* (*X* (*?n K*))
    by (*auto intro*: *LeastI-ex*)
  hence $\forall K>0.\ K <$ *hnorm* ((**f* X*) ((**f* ?n*) *K*))
    by *transfer*
  hence $\omega <$ *hnorm* ((**f* X*) ((**f* ?n*) ω))
    by *simp*
  hence $\forall r\in\mathbb{R}.\ r <$ *hnorm* ((**f* X*) ((**f* ?n*) ω))
    by (*simp add*: *order-less-trans* [*OF SReal-less-omega*])
  hence (**f* X*) ((**f* ?n*) ω) $\in$ *HInfinite*
    by (*simp add*: *HInfinite-def*)
  with *finite* **show** *False*
    by (*simp add*: *HFinite-HInfinite-iff*)
qed

Equivalence of nonstandard and standard definitions for a bounded sequence

lemma *Bseq-NSBseq-iff*: (*Bseq X*) = (*NSBseq X*)
by (*blast intro*!: *NSBseq-Bseq Bseq-NSBseq*)

A convergent sequence is bounded: Boundedness as a necessary condition for convergence. The nonstandard version has no existential, as usual

lemma *NSconvergent-NSBseq*: *NSconvergent X* ==> *NSBseq X*
apply (*simp add*: *NSconvergent-def NSBseq-def NSLIMSEQ-def*)
apply (*blast intro*: *HFinite-star-of approx-sym approx-HFinite*)
done

Standard Version: easily now proved using equivalence of NS and standard definitions

lemma *convergent-Bseq*: *convergent X* ==> *Bseq X*
by (*simp add*: *NSconvergent-NSBseq convergent-NSconvergent-iff Bseq-NSBseq-iff*)

31.3.1 Upper Bounds and Lubs of Bounded Sequences

lemma *NSBseq-isUb*: *NSBseq X* ==> $\exists$ *U*::*real*. *isUb UNIV* $\{x.\ \exists n.\ X\ n = x\}$ *U*

by (*simp add*: *Bseq-NSBseq-iff* [*symmetric*] *Bseq-isUb*)

lemma *NSBseq-isLub*: *NSBseq X* ==> $\exists$ *U*::*real*. *isLub UNIV* {*x*. $\exists$ *n*. *X n* = *x*} *U*
by (*simp add*: *Bseq-NSBseq-iff* [*symmetric*] *Bseq-isLub*)

31.3.2 A Bounded and Monotonic Sequence Converges

The best of both worlds: Easier to prove this result as a standard theorem and then use equivalence to "transfer" it into the equivalent nonstandard form if needed!

lemma *Bmonoseq-NSLIMSEQ*: $\forall$ *n* $\geq$ *m*. *X n* = *X m* ==> $\exists$ *L*. (*X* ----*NS*> *L*)
by (*auto dest*!: *Bmonoseq-LIMSEQ simp add*: *LIMSEQ-NSLIMSEQ-iff*)

lemma *NSBseq-mono-NSconvergent*:
  [| *NSBseq X*; $\forall$ *m*. $\forall$ *n* $\geq$ *m*. *X m* $\leq$ *X n* |] ==> *NSconvergent* (*X*::*nat*=>*real*)
by (*auto intro*: *Bseq-mono-convergent*
    simp add: *convergent-NSconvergent-iff* [*symmetric*]
      Bseq-NSBseq-iff [*symmetric*])

31.4 Cauchy Sequences

lemma *NSCauchyI*:
  ($\bigwedge$*M N*. $[\![$*M* $\in$ *HNatInfinite*; *N* $\in$ *HNatInfinite*$]\!]$ $\Longrightarrow$ *starfun X M* $\approx$ *starfun X N*)
  $\Longrightarrow$ *NSCauchy X*
by (*simp add*: *NSCauchy-def*)

lemma *NSCauchyD*:
  $[\![$*NSCauchy X*; *M* $\in$ *HNatInfinite*; *N* $\in$ *HNatInfinite*$]\!]$
  $\Longrightarrow$ *starfun X M* $\approx$ *starfun X N*
by (*simp add*: *NSCauchy-def*)

31.4.1 Equivalence Between NS and Standard

lemma *Cauchy-NSCauchy*:
 assumes *X*: *Cauchy X* **shows** *NSCauchy X*
proof (*rule NSCauchyI*)
 fix *M* **assume** *M*: *M* $\in$ *HNatInfinite*
 fix *N* **assume** *N*: *N* $\in$ *HNatInfinite*
 have *starfun X M* $-$ *starfun X N* $\in$ *Infinitesimal*
 proof (*rule InfinitesimalI2*)
  fix *r* :: *real* **assume** *r*: *0* < *r*
  from *CauchyD* [*OF X r*]
  obtain *k* **where** $\forall$ *m*$\geq$*k*. $\forall$ *n*$\geq$*k*. *norm* (*X m* $-$ *X n*) < *r* **..**
  hence $\forall$ *m*$\geq$*star-of k*. $\forall$ *n*$\geq$*star-of k*.
      hnorm (*starfun X m* $-$ *starfun X n*) < *star-of r*
   by *transfer*

```
    thus hnorm (starfun X M − starfun X N) < star-of r
      using M N by (simp add: star-of-le-HNatInfinite)
  qed
  thus starfun X M ≈ starfun X N
    by (unfold approx-def)
qed

lemma NSCauchy-Cauchy:
  assumes X: NSCauchy X shows Cauchy X
proof (rule CauchyI)
  fix r::real assume r: 0 < r
  have ∃ k. ∀ m≥k. ∀ n≥k. hnorm (starfun X m − starfun X n) < star-of r
  proof (intro exI allI impI)
    fix M assume whn ≤ M
    with HNatInfinite-whn have M: M ∈ HNatInfinite
      by (rule HNatInfinite-upward-closed)
    fix N assume whn ≤ N
    with HNatInfinite-whn have N: N ∈ HNatInfinite
      by (rule HNatInfinite-upward-closed)
    from X M N have starfun X M ≈ starfun X N
      by (rule NSCauchyD)
    hence starfun X M − starfun X N ∈ Infinitesimal
      by (unfold approx-def)
    thus hnorm (starfun X M − starfun X N) < star-of r
      using r by (rule InfinitesimalD2)
  qed
  thus ∃ k. ∀ m≥k. ∀ n≥k. norm (X m − X n) < r
    by transfer
qed

theorem NSCauchy-Cauchy-iff: NSCauchy X = Cauchy X
by (blast intro!: NSCauchy-Cauchy Cauchy-NSCauchy)
```

31.4.2 Cauchy Sequences are Bounded

A Cauchy sequence is bounded – nonstandard version

```
lemma NSCauchy-NSBseq: NSCauchy X ==> NSBseq X
by (simp add: Cauchy-Bseq Bseq-NSBseq-iff [symmetric] NSCauchy-Cauchy-iff)
```

31.4.3 Cauchy Sequences are Convergent

Equivalence of Cauchy criterion and convergence: We will prove this using our NS formulation which provides a much easier proof than using the standard definition. We do not need to use properties of subsequences such as boundedness, monotonicity etc... Compare with Harrison's corresponding proof in HOL which is much longer and more complicated. Of course, we do not have problems which he encountered with guessing the right instantiations for his 'espsilon-delta' proof(s) in this case since the NS formulations do not involve existential quantifiers.

```
lemma NSconvergent-NSCauchy: NSconvergent X ⟹ NSCauchy X
apply (simp add: NSconvergent-def NSLIMSEQ-def NSCauchy-def, safe)
apply (auto intro: approx-trans2)
done

lemma real-NSCauchy-NSconvergent:
  fixes X :: nat ⇒ real
  shows NSCauchy X ⟹ NSconvergent X
apply (simp add: NSconvergent-def NSLIMSEQ-def)
apply (frule NSCauchy-NSBseq)
apply (simp add: NSBseq-def NSCauchy-def)
apply (drule HNatInfinite-whn [THEN [2] bspec])
apply (drule HNatInfinite-whn [THEN [2] bspec])
apply (auto dest!: st-part-Ex simp add: SReal-iff)
apply (blast intro: approx-trans3)
done

lemma NSCauchy-NSconvergent:
  fixes X :: nat ⇒ 'a::banach
  shows NSCauchy X ⟹ NSconvergent X
apply (drule NSCauchy-Cauchy [THEN Cauchy-convergent])
apply (erule convergent-NSconvergent-iff [THEN iffD1])
done

lemma NSCauchy-NSconvergent-iff:
  fixes X :: nat ⇒ 'a::banach
  shows NSCauchy X = NSconvergent X
by (fast intro: NSCauchy-NSconvergent NSconvergent-NSCauchy)
```

31.5 Power Sequences

The sequence $x \mathbin{\hat{}} n$ tends to 0 if $(0::'a) \le x$ and $x < (1::'a)$. Proof will use (NS) Cauchy equivalence for convergence and also fact that bounded and monotonic sequence converges.

We now use NS criterion to bring proof of theorem through

```
lemma NSLIMSEQ-realpow-zero:
  [| 0 ≤ (x::real); x < 1 |] ==> (%n. x ^ n) ----NS> 0
apply (simp add: NSLIMSEQ-def)
apply (auto dest!: convergent-realpow simp add: convergent-NSconvergent-iff)
apply (frule NSconvergentD)
apply (auto simp add: NSLIMSEQ-def NSCauchy-NSconvergent-iff [symmetric]
NSCauchy-def starfun-pow)
apply (frule HNatInfinite-add-one)
apply (drule bspec, assumption)
apply (drule bspec, assumption)
apply (drule-tac x = N + (1::hypnat) in bspec, assumption)
```

apply (*simp add*: *hyperpow-add*)
apply (*drule approx-mult-subst-star-of*, *assumption*)
apply (*drule approx-trans3*, *assumption*)
apply (*auto simp del*: *star-of-mult simp add*: *star-of-mult* [*symmetric*])
done

lemma *NSLIMSEQ-rabs-realpow-zero*: $|c| < (1::real) ==> (\%n. |c| \hat{}\ n) ----NS> 0$
by (*simp add*: *LIMSEQ-rabs-realpow-zero LIMSEQ-NSLIMSEQ-iff* [*symmetric*])

lemma *NSLIMSEQ-rabs-realpow-zero2*: $|c| < (1::real) ==> (\%n.\ c \hat{}\ n) ----NS> 0$
by (*simp add*: *LIMSEQ-rabs-realpow-zero2 LIMSEQ-NSLIMSEQ-iff* [*symmetric*])

end

32 HSeries: Finite Summation and Infinite Series for Hyperreals

theory *HSeries*
imports *Series HSEQ*
begin

definition
  sumhr :: (*hypnat* ∗ *hypnat* ∗ (*nat=>real*)) => *hypreal* **where**
  sumhr =
     (%(*M*,*N*,*f*). *starfun2* (%*m n*. *setsum f* {*m*..<*n*}) *M N*)

definition
  NSsums :: [*nat=>real*,*real*] => *bool*   (**infixr** *NSsums 80*) **where**
  f NSsums s = (%*n*. *setsum f* {*0*..<*n*}) −−−−*NS*> *s*

definition
  NSsummable :: (*nat=>real*) => *bool* **where**
  NSsummable f = (∃ *s*. *f NSsums s*)

definition
  NSsuminf :: (*nat=>real*) => *real* **where**
  NSsuminf f = (*THE s*. *f NSsums s*)

lemma *sumhr-app*: *sumhr*(*M*,*N*,*f*) = (∗*f2*∗ (λ*m n*. *setsum f* {*m*..<*n*})) *M N*
by (*simp add*: *sumhr-def*)

Base case in definition of *sumr*

lemma *sumhr-zero* [*simp*]: !!*m*. *sumhr* (*m*,*0*,*f*) = *0*

unfolding *sumhr-app* **by** *transfer simp*

Recursive case in definition of *sumr*

lemma *sumhr-if*:
  $!!m\ n.\ sumhr(m,n+1,f) =$
  $(if\ n + 1 \leq m\ then\ 0\ else\ sumhr(m,n,f) + (\ {*}f{*}\ f)\ n)$
unfolding *sumhr-app* **by** *transfer simp*

lemma *sumhr-Suc-zero* [*simp*]: $!!n.\ sumhr\ (n + 1,\ n,\ f) = 0$
unfolding *sumhr-app* **by** *transfer simp*

lemma *sumhr-eq-bounds* [*simp*]: $!!n.\ sumhr\ (n,n,f) = 0$
unfolding *sumhr-app* **by** *transfer simp*

lemma *sumhr-Suc* [*simp*]: $!!m.\ sumhr\ (m,m + 1,f) = (\ {*}f{*}\ f)\ m$
unfolding *sumhr-app* **by** *transfer simp*

lemma *sumhr-add-lbound-zero* [*simp*]: $!!k\ m.\ sumhr(m+k,k,f) = 0$
unfolding *sumhr-app* **by** *transfer simp*

lemma *sumhr-add*:
  $!!m\ n.\ sumhr\ (m,n,f) + sumhr(m,n,g) = sumhr(m,n,\%i.\ f\ i + g\ i)$
unfolding *sumhr-app* **by** *transfer* (*rule setsum-addf* [*symmetric*])

lemma *sumhr-mult*:
  $!!m\ n.\ hypreal\text{-}of\text{-}real\ r * sumhr(m,n,f) = sumhr(m,n,\%n.\ r * f\ n)$
unfolding *sumhr-app* **by** *transfer* (*rule setsum-right-distrib*)

lemma *sumhr-split-add*:
  $!!n\ p.\ n < p ==> sumhr(0,n,f) + sumhr(n,p,f) = sumhr(0,p,f)$
unfolding *sumhr-app* **by** *transfer* (*simp add*: *setsum-add-nat-ivl*)

lemma *sumhr-split-diff*: $n<p ==> sumhr(0,p,f) - sumhr(0,n,f) = sumhr(n,p,f)$
by (*drule-tac* $f = f$ **in** *sumhr-split-add* [*symmetric*], *simp*)

lemma *sumhr-hrabs*: $!!m\ n.\ abs(sumhr(m,n,f)) \leq sumhr(m,n,\%i.\ abs(f\ i))$
unfolding *sumhr-app* **by** *transfer* (*rule setsum-abs*)

other general version also needed

lemma *sumhr-fun-hypnat-eq*:
  $(\forall r.\ m \leq r\ \&\ r < n\ {-}{-}{>}\ f\ r = g\ r)\ {-}{-}{>}$
  $sumhr(hypnat\text{-}of\text{-}nat\ m,\ hypnat\text{-}of\text{-}nat\ n,\ f) =$
  $sumhr(hypnat\text{-}of\text{-}nat\ m,\ hypnat\text{-}of\text{-}nat\ n,\ g)$
unfolding *sumhr-app* **by** *transfer simp*

lemma *sumhr-const*:
  $!!n.\ sumhr(0,\ n,\ \%i.\ r) = hypreal\text{-}of\text{-}hypnat\ n * hypreal\text{-}of\text{-}real\ r$
unfolding *sumhr-app* **by** *transfer* (*simp add*: *real-of-nat-def*)

lemma *sumhr-less-bounds-zero* [*simp*]: !!*m n*. *n* < *m* ==> *sumhr*(*m*,*n*,*f*) = *0*
unfolding *sumhr-app* **by** *transfer simp*

lemma *sumhr-minus*: !!*m n*. *sumhr*(*m*, *n*, %*i*. − *f i*) = − *sumhr*(*m*, *n*, *f*)
unfolding *sumhr-app* **by** *transfer* (*rule setsum-negf*)

lemma *sumhr-shift-bounds*:
  !!*m n*. *sumhr*(*m*+*hypnat-of-nat k*,*n*+*hypnat-of-nat k*,*f*) =
        sumhr(*m*,*n*,%*i*. *f*(*i* + *k*))
unfolding *sumhr-app* **by** *transfer* (*rule setsum-shift-bounds-nat-ivl*)

32.1 Nonstandard Sums

Infinite sums are obtained by summing to some infinite hypernatural (such as *whn*)

lemma *sumhr-hypreal-of-hypnat-omega*:
    sumhr(*0*,*whn*,%*i*. *1*) = *hypreal-of-hypnat whn*
by (*simp add*: *sumhr-const*)

lemma *sumhr-hypreal-omega-minus-one*: *sumhr*(*0*, *whn*, %*i*. *1*) = *omega* − *1*
apply (*simp add*: *sumhr-const*)

apply (*unfold star-class-defs omega-def hypnat-omega-def*
        of-hypnat-def star-of-def)
apply (*simp add*: *starfun-star-n starfun2-star-n real-of-nat-def*)
done

lemma *sumhr-minus-one-realpow-zero* [*simp*]:
    !!*N*. *sumhr*(*0*, *N* + *N*, %*i*. (−*1*) ^ (*i*+*1*)) = *0*
unfolding *sumhr-app*
by *transfer* (*simp del*: *realpow-Suc add*: *nat-mult-2* [*symmetric*])

lemma *sumhr-interval-const*:
    (∀ *n*. *m* ≤ *Suc n* −−> *f n* = *r*) & *m* ≤ *na*
    ==> *sumhr*(*hypnat-of-nat m*,*hypnat-of-nat na*,*f*) =
        (*hypreal-of-nat* (*na* − *m*) ∗ *hypreal-of-real r*)
unfolding *sumhr-app* **by** *transfer simp*

lemma *starfunNat-sumr*: !!*N*. (∗*f*∗ (%*n*. *setsum f* {*0*..<*n*})) *N* = *sumhr*(*0*,*N*,*f*)
unfolding *sumhr-app* **by** *transfer* (*rule refl*)

lemma *sumhr-hrabs-approx* [*simp*]: *sumhr*(*0*, *M*, *f*) @= *sumhr*(*0*, *N*, *f*)
    ==> *abs* (*sumhr*(*M*, *N*, *f*)) @= *0*
apply (*cut-tac x* = *M* **and** *y* = *N* **in** *linorder-less-linear*)
apply (*auto simp add*: *approx-refl*)
apply (*drule approx-sym* [*THEN approx-minus-iff* [*THEN iffD1*]])
apply (*auto dest*: *approx-hrabs*
        simp add: *sumhr-split-diff diff-minus* [*symmetric*])

done

lemma *sums-NSsums-iff*: $(f\ sums\ l) = (f\ NSsums\ l)$
by (*simp add*: *sums-def NSsums-def LIMSEQ-NSLIMSEQ-iff*)

lemma *summable-NSsummable-iff*: $(summable\ f) = (NSsummable\ f)$
by (*simp add*: *summable-def NSsummable-def sums-NSsums-iff*)

lemma *suminf-NSsuminf-iff*: $(suminf\ f) = (NSsuminf\ f)$
by (*simp add*: *suminf-def NSsuminf-def sums-NSsums-iff*)

lemma *NSsums-NSsummable*: $f\ NSsums\ l ==> NSsummable\ f$
by (*simp add*: *NSsums-def NSsummable-def*, *blast*)

lemma *NSsummable-NSsums*: $NSsummable\ f ==> f\ NSsums\ (NSsuminf\ f)$
apply (*simp add*: *NSsummable-def NSsuminf-def NSsums-def*)
apply (*blast intro*: *theI NSLIMSEQ-unique*)
done

lemma *NSsums-unique*: $f\ NSsums\ s ==> (s = NSsuminf\ f)$
by (*simp add*: *suminf-NSsuminf-iff* [*symmetric*] *sums-NSsums-iff sums-unique*)

lemma *NSseries-zero*:
$\forall m.\ n \le Suc\ m --> f(m) = 0 ==> f\ NSsums\ (setsum\ f\ \{0..<n\})$
by (*simp add*: *sums-NSsums-iff* [*symmetric*] *series-zero*)

lemma *NSsummable-NSCauchy*:
$NSsummable\ f =$
$(\forall M \in HNatInfinite.\ \forall N \in HNatInfinite.\ abs\ (sumhr(M,N,f))\ @{=}\ 0)$
apply (*auto simp add*: *summable-NSsummable-iff* [*symmetric*]
summable-convergent-sumr-iff convergent-NSconvergent-iff
NSCauchy-NSconvergent-iff [*symmetric*] *NSCauchy-def starfunNat-sumr*)
apply (*cut-tac* $x = M$ **and** $y = N$ **in** *linorder-less-linear*)
apply (*auto simp add*: *approx-refl*)
apply (*rule approx-minus-iff* [*THEN iffD2*, *THEN approx-sym*])
apply (*rule-tac* [*2*] *approx-minus-iff* [*THEN iffD2*])
apply (*auto dest*: *approx-hrabs-zero-cancel*
simp add: *sumhr-split-diff diff-minus* [*symmetric*])
done

Terms of a convergent series tend to zero

lemma *NSsummable-NSLIMSEQ-zero*: $NSsummable\ f ==> f\ ----NS>\ 0$
apply (*auto simp add*: *NSLIMSEQ-def NSsummable-NSCauchy*)
apply (*drule bspec*, *auto*)
apply (*drule-tac* $x = N + 1$ **in** *bspec*)
apply (*auto intro*: *HNatInfinite-add-one approx-hrabs-zero-cancel*)
done

Nonstandard comparison test

lemma *NSsummable-comparison-test*:
  $[| \exists N. \forall n. N \le n \dashrightarrow abs(f\, n) \le g\, n;$ *NSsummable g* $|] ==>$ *NSsummable f*
apply (*fold summable-NSsummable-iff*)
apply (*rule summable-comparison-test*, *simp*, *assumption*)
done

lemma *NSsummable-rabs-comparison-test*:
  $[| \exists N. \forall n. N \le n \dashrightarrow abs(f\, n) \le g\, n;$ *NSsummable g* $|]$
  $==>$ *NSsummable* (%k. *abs* (*f k*))
apply (*rule NSsummable-comparison-test*)
apply (*auto*)
done

end

33 HLim: Limits and Continuity (Nonstandard)

theory *HLim*
imports *Star Lim*
begin

Nonstandard Definitions

definition
  NSLIM :: ['*a*::*real-normed-vector* => '*b*::*real-normed-vector*, '*a*, '*b*] => *bool*
    (((-)/ −− (-)/ −−*NS*> (-)) [*60*, *0*, *60*] *60*) **where**
  f −− *a* −−*NS*> *L* =
    ($\forall x.$ ($x \neq$ *star-of a* & *x* @= *star-of a* −−> (**f* f*) *x* @= *star-of L*))

definition
  isNSCont :: ['*a*::*real-normed-vector* => '*b*::*real-normed-vector*, '*a*] => *bool* **where**
    — NS definition dispenses with limit notions
  isNSCont f a = ($\forall y.$ *y* @= *star-of a* −−>
      (**f* f*) *y* @= *star-of* (*f a*))

definition
  isNSUCont :: ['*a*::*real-normed-vector* => '*b*::*real-normed-vector*] => *bool* **where**
  isNSUCont f = ($\forall x\, y.$ *x* @= *y* −−> (**f* f*) *x* @= (**f* f*) *y*)

33.1 Limits of Functions

lemma *NSLIM-I*:
  $(\bigwedge x. [\![x \neq$ *star-of a*; $x \approx$ *star-of a*$]\!] \Longrightarrow$ *starfun f x* $\approx$ *star-of L*)
  $\Longrightarrow$ *f* −− *a* −−*NS*> *L*
by (*simp add*: *NSLIM-def*)

lemma *NSLIM-D*:
  $[\![f$ −− *a* −−*NS*> *L*; $x \neq$ *star-of a*; $x \approx$ *star-of a*$]\!]$

$\Longrightarrow$ *starfun f x* $\approx$ *star-of L*
by (*simp add*: *NSLIM-def*)

Proving properties of limits using nonstandard definition. The properties hold for standard limits as well!

lemma *NSLIM-mult*:
  fixes *l m* :: *'a::real-normed-algebra*
  shows [| *f* −− *x* −−*NS*> *l*; *g* −− *x* −−*NS*> *m* |]
    ==> (%*x*. *f*(*x*) ∗ *g*(*x*)) −− *x* −−*NS*> (*l* ∗ *m*)
by (*auto simp add*: *NSLIM-def intro*!: *approx-mult-HFinite*)

lemma *starfun-scaleR* [*simp*]:
  starfun (λx. *f x* $*_R$ *g x*) = (λx. *scaleHR* (*starfun f x*) (*starfun g x*))
by *transfer* (*rule refl*)

lemma *NSLIM-scaleR*:
  [| *f* −− *x* −−*NS*> *l*; *g* −− *x* −−*NS*> *m* |]
    ==> (%*x*. *f*(*x*) $*_R$ *g*(*x*)) −− *x* −−*NS*> (*l* $*_R$ *m*)
by (*auto simp add*: *NSLIM-def intro*!: *approx-scaleR-HFinite*)

lemma *NSLIM-add*:
    [| *f* −− *x* −−*NS*> *l*; *g* −− *x* −−*NS*> *m* |]
    ==> (%*x*. *f*(*x*) + *g*(*x*)) −− *x* −−*NS*> (*l* + *m*)
by (*auto simp add*: *NSLIM-def intro*!: *approx-add*)

lemma *NSLIM-const* [*simp*]: (%*x*. *k*) −− *x* −−*NS*> *k*
by (*simp add*: *NSLIM-def*)

lemma *NSLIM-minus*: *f* −− *a* −−*NS*> *L* ==> (%*x*. −*f*(*x*)) −− *a* −−*NS*> −*L*
by (*simp add*: *NSLIM-def*)

lemma *NSLIM-diff*:
  ⟦*f* −− *x* −−*NS*> *l*; *g* −− *x* −−*NS*> *m*⟧ $\Longrightarrow$ (λx. *f x* − *g x*) −− *x* −−*NS*> (*l* − *m*)
by (*simp only*: *diff-def NSLIM-add NSLIM-minus*)

lemma *NSLIM-add-minus*: [| *f* −− *x* −−*NS*> *l*; *g* −− *x* −−*NS*> *m* |] ==> (%*x*. *f*(*x*) + −*g*(*x*)) −− *x* −−*NS*> (*l* + −*m*)
by (*simp only*: *NSLIM-add NSLIM-minus*)

lemma *NSLIM-inverse*:
  fixes *L* :: *'a::real-normed-div-algebra*
  shows [| *f* −− *a* −−*NS*> *L*; *L* $\neq$ *0* |]
    ==> (%*x*. *inverse*(*f*(*x*))) −− *a* −−*NS*> (*inverse L*)
apply (*simp add*: *NSLIM-def*, *clarify*)
apply (*drule spec*)
apply (*auto simp add*: *star-of-approx-inverse*)
done

lemma *NSLIM-zero*:
assumes *f*: *f* −− *a* −−*NS*> *l* **shows** (%*x*. *f*(*x*) − *l*) −− *a* −−*NS*> *0*
proof −
have (λ*x*. *f x* − *l*) −− *a* −−*NS*> *l* − *l*
by (*rule NSLIM-diff* [*OF f NSLIM-const*])
thus *?thesis* **by** *simp*
qed

lemma *NSLIM-zero-cancel*: (%*x*. *f*(*x*) − *l*) −− *x* −−*NS*> *0* ==> *f* −− *x* −−*NS*> *l*
apply (*drule-tac g* = %*x*. *l* **and** *m* = *l* **in** *NSLIM-add*)
apply (*auto simp add*: *diff-minus add-assoc*)
done

lemma *NSLIM-const-not-eq*:
fixes *a* :: ′*a*::*real-normed-algebra-1*
shows *k* ≠ *L* ⟹ ¬ (λ*x*. *k*) −− *a* −−*NS*> *L*
apply (*simp add*: *NSLIM-def*)
apply (*rule-tac x*=*star-of a* + *of-hypreal epsilon* **in** *exI*)
apply (*simp add*: *hypreal-epsilon-not-zero approx-def*)
done

lemma *NSLIM-not-zero*:
fixes *a* :: ′*a*::*real-normed-algebra-1*
shows *k* ≠ *0* ⟹ ¬ (λ*x*. *k*) −− *a* −−*NS*> *0*
by (*rule NSLIM-const-not-eq*)

lemma *NSLIM-const-eq*:
fixes *a* :: ′*a*::*real-normed-algebra-1*
shows (λ*x*. *k*) −− *a* −−*NS*> *L* ⟹ *k* = *L*
apply (*rule ccontr*)
apply (*blast dest*: *NSLIM-const-not-eq*)
done

lemma *NSLIM-unique*:
fixes *a* :: ′*a*::*real-normed-algebra-1*
shows ⟦*f* −− *a* −−*NS*> *L*; *f* −− *a* −−*NS*> *M*⟧ ⟹ *L* = *M*
apply (*drule* (*1*) *NSLIM-diff*)
apply (*auto dest*!: *NSLIM-const-eq*)
done

lemma *NSLIM-mult-zero*:
fixes *f g* :: ′*a*::*real-normed-vector* ⇒ ′*b*::*real-normed-algebra*
shows [| *f* −− *x* −−*NS*> *0*; *g* −− *x* −−*NS*> *0* |] ==> (%*x*. *f*(*x*)∗*g*(*x*)) −− *x* −−*NS*> *0*
by (*drule NSLIM-mult*, *auto*)

lemma *NSLIM-self*: (%*x*. *x*) −− *a* −−*NS*> *a*
by (*simp add*: *NSLIM-def*)

33.1.1 Equivalence of *LIM* and *NSLIM*

lemma *LIM-NSLIM*:
  assumes *f*: *f* −− *a* −−> *L* **shows** *f* −− *a* −−*NS*> *L*
proof (*rule NSLIM-I*)
  fix *x*
  assume *neq*: $x \neq$ *star-of a*
  assume *approx*: $x \approx$ *star-of a*
  have *starfun f x* − *star-of L* ∈ *Infinitesimal*
  proof (*rule InfinitesimalI2*)
    fix *r*::*real* **assume** *r*: *0* < *r*
    from *LIM-D* [*OF f r*]
    obtain *s* **where** *s*: *0* < *s* **and**
      less-r: $\bigwedge x.$ ⟦$x \neq a$; *norm* $(x - a) < s$⟧ ⟹ *norm* $(f\,x - L) < r$
      by *fast*
    from *less-r* **have** *less-r′*:
      $\bigwedge x.$ ⟦$x \neq$ *star-of a*; *hnorm* (x − *star-of a*) < *star-of s*⟧
       ⟹ *hnorm* (*starfun f x* − *star-of L*) < *star-of r*
      by *transfer*
    from *approx* **have** x − *star-of a* ∈ *Infinitesimal*
      by (*unfold approx-def*)
    hence *hnorm* (x − *star-of a*) < *star-of s*
      using *s* **by** (*rule InfinitesimalD2*)
    with *neq* **show** *hnorm* (*starfun f x* − *star-of L*) < *star-of r*
      by (*rule less-r′*)
  qed
  thus *starfun f x* ≈ *star-of L*
    by (*unfold approx-def*)
qed

lemma *NSLIM-LIM*:
  assumes *f*: *f* −− *a* −−*NS*> *L* **shows** *f* −− *a* −−> *L*
proof (*rule LIM-I*)
  fix *r*::*real* **assume** *r*: *0* < *r*
  have ∃ *s*>*0*. ∀ *x*. $x \neq$ *star-of a* ∧ *hnorm* (x − *star-of a*) < *s*
      ⟶ *hnorm* (*starfun f x* − *star-of L*) < *star-of r*
  proof (*rule exI*, *safe*)
    show *0* < *epsilon* **by** (*rule hypreal-epsilon-gt-zero*)
  next
    fix *x* **assume** *neq*: $x \neq$ *star-of a*
    assume *hnorm* (x − *star-of a*) < *epsilon*
    with *Infinitesimal-epsilon*
    have x − *star-of a* ∈ *Infinitesimal*
      by (*rule hnorm-less-Infinitesimal*)
    hence $x \approx$ *star-of a*
      by (*unfold approx-def*)
    with *f neq* **have** *starfun f x* ≈ *star-of L*
      by (*rule NSLIM-D*)
    hence *starfun f x* − *star-of L* ∈ *Infinitesimal*
      by (*unfold approx-def*)

thus *hnorm* (*starfun f x* − *star-of L*) < *star-of r*
using *r* **by** (*rule InfinitesimalD2*)
qed
thus $\exists\, s{>}0.\ \forall x.\ x \neq a \wedge norm\ (x - a) < s \longrightarrow norm\ (f\,x - L) < r$
by *transfer*
qed

theorem *LIM-NSLIM-iff*: (*f* −− *x* −−> *L*) = (*f* −− *x* −−*NS*> *L*)
by (*blast intro*: *LIM-NSLIM NSLIM-LIM*)

33.2 Continuity

lemma *isNSContD*:
⟦*isNSCont f a*; *y* ≈ *star-of a*⟧ ⟹ (*∗f∗ f*) *y* ≈ *star-of* (*f a*)
by (*simp add*: *isNSCont-def*)

lemma *isNSCont-NSLIM*: *isNSCont f a* ==> *f* −− *a* −−*NS*> (*f a*)
by (*simp add*: *isNSCont-def NSLIM-def*)

lemma *NSLIM-isNSCont*: *f* −− *a* −−*NS*> (*f a*) ==> *isNSCont f a*
apply (*simp add*: *isNSCont-def NSLIM-def*, *auto*)
apply (*case-tac y* = *star-of a*, *auto*)
done

NS continuity can be defined using NS Limit in similar fashion to standard def of continuity

lemma *isNSCont-NSLIM-iff*: (*isNSCont f a*) = (*f* −− *a* −−*NS*> (*f a*))
by (*blast intro*: *isNSCont-NSLIM NSLIM-isNSCont*)

Hence, NS continuity can be given in terms of standard limit

lemma *isNSCont-LIM-iff*: (*isNSCont f a*) = (*f* −− *a* −−> (*f a*))
by (*simp add*: *LIM-NSLIM-iff isNSCont-NSLIM-iff*)

Moreover, it's trivial now that NS continuity is equivalent to standard continuity

lemma *isNSCont-isCont-iff*: (*isNSCont f a*) = (*isCont f a*)
apply (*simp add*: *isCont-def*)
apply (*rule isNSCont-LIM-iff*)
done

Standard continuity ==¿ NS continuity

lemma *isCont-isNSCont*: *isCont f a* ==> *isNSCont f a*
by (*erule isNSCont-isCont-iff* [*THEN iffD2*])

NS continuity ==¿ Standard continuity

lemma *isNSCont-isCont*: *isNSCont f a* ==> *isCont f a*
by (*erule isNSCont-isCont-iff* [*THEN iffD1*])

Alternative definition of continuity

lemma *NSLIM-h-iff*: $(f\ --\ a\ --NS{>}\ L) = ((\%h.\ f(a + h))\ --\ 0\ --NS{>}\ L)$
apply (*simp add*: *NSLIM-def*, *auto*)
apply (*drule-tac* x = *star-of* $a + x$ **in** *spec*)
apply (*drule-tac* [*2*] $x = -$ *star-of* $a + x$ **in** *spec*, *safe*, *simp*)
apply (*erule mem-infmal-iff* [*THEN iffD2*, *THEN Infinitesimal-add-approx-self* [*THEN approx-sym*]])
apply (*erule-tac* [*3*] *approx-minus-iff2* [*THEN iffD1*])
prefer *2* **apply** (*simp add*: *add-commute diff-def* [*symmetric*])
apply (*rule-tac* $x = x$ **in** *star-cases*)
apply (*rule-tac* [*2*] $x = x$ **in** *star-cases*)
apply (*auto simp add*: *starfun star-of-def star-n-minus star-n-add add-assoc approx-refl star-n-zero-num*)
done

lemma *NSLIM-isCont-iff*: $(f\ --\ a\ --NS{>}\ f\ a) = ((\%h.\ f(a + h))\ --\ 0\ --NS{>}\ f\ a)$
by (*rule NSLIM-h-iff*)

lemma *isNSCont-minus*: *isNSCont f a* $==>$ *isNSCont* $(\%x.\ - f\ x)\ a$
by (*simp add*: *isNSCont-def*)

lemma *isNSCont-inverse*:
  fixes f :: $'a$::*real-normed-vector* $\Rightarrow$ $'b$::*real-normed-div-algebra*
  shows [| *isNSCont f x*; $f\ x \neq 0$ |] $==>$ *isNSCont* $(\%x.\ inverse\ (f\ x))\ x$
by (*auto intro*: *isCont-inverse simp add*: *isNSCont-isCont-iff*)

lemma *isNSCont-const* [*simp*]: *isNSCont* $(\%x.\ k)\ a$
by (*simp add*: *isNSCont-def*)

lemma *isNSCont-abs* [*simp*]: *isNSCont abs* (a::*real*)
apply (*simp add*: *isNSCont-def*)
apply (*auto intro*: *approx-hrabs simp add*: *starfun-rabs-hrabs*)
done

33.3 Uniform Continuity

lemma *isNSUContD*: [| *isNSUCont f*; $x \approx y$|] $==>$ $(\ {*}f{*}\ f)\ x \approx (\ {*}f{*}\ f)\ y$
by (*simp add*: *isNSUCont-def*)

lemma *isUCont-isNSUCont*:
  fixes f :: $'a$::*real-normed-vector* $\Rightarrow$ $'b$::*real-normed-vector*
  assumes f: *isUCont f* **shows** *isNSUCont f*
proof (*unfold isNSUCont-def*, *safe*)
  fix $x\ y$ :: $'a$ *star*
  assume *approx*: $x \approx y$
  have *starfun f x* $-$ *starfun f y* $\in$ *Infinitesimal*
  proof (*rule InfinitesimalI2*)
    fix r::*real* **assume** r: $0 < r$

```
    with f obtain s where s: 0 < s and
      less-r: ⋀x y. norm (x − y) < s ⟹ norm (f x − f y) < r
      by (auto simp add: isUCont-def)
    from less-r have less-r':
       ⋀x y. hnorm (x − y) < star-of s
        ⟹ hnorm (starfun f x − starfun f y) < star-of r
      by transfer
    from approx have x − y ∈ Infinitesimal
      by (unfold approx-def)
    hence hnorm (x − y) < star-of s
      using s by (rule InfinitesimalD2)
    thus hnorm (starfun f x − starfun f y) < star-of r
      by (rule less-r')
  qed
  thus starfun f x ≈ starfun f y
    by (unfold approx-def)
qed

lemma isNSUCont-isUCont:
  fixes f :: 'a::real-normed-vector ⇒ 'b::real-normed-vector
  assumes f: isNSUCont f shows isUCont f
proof (unfold isUCont-def, safe)
  fix r::real assume r: 0 < r
  have ∃s>0. ∀x y. hnorm (x − y) < s
        ⟶ hnorm (starfun f x − starfun f y) < star-of r
  proof (rule exI, safe)
    show 0 < epsilon by (rule hypreal-epsilon-gt-zero)
  next
    fix x y :: 'a star
    assume hnorm (x − y) < epsilon
    with Infinitesimal-epsilon
    have x − y ∈ Infinitesimal
      by (rule hnorm-less-Infinitesimal)
    hence x ≈ y
      by (unfold approx-def)
    with f have starfun f x ≈ starfun f y
      by (simp add: isNSUCont-def)
    hence starfun f x − starfun f y ∈ Infinitesimal
      by (unfold approx-def)
    thus hnorm (starfun f x − starfun f y) < star-of r
      using r by (rule InfinitesimalD2)
  qed
  thus ∃s>0. ∀x y. norm (x − y) < s ⟶ norm (f x − f y) < r
    by transfer
qed

end
```

34 HDeriv: Differentiation (Nonstandard)

theory *HDeriv*
imports *Deriv HLim*
begin

Nonstandard Definitions

definition
  nsderiv :: [′a::real-normed-field ⇒ ′a, ′a, ′a] ⇒ bool
    ((NSDERIV (-)/ (-)/ :> (-)) [1000, 1000, 60] 60) **where**
  NSDERIV f x :> D = (∀ h ∈ Infinitesimal − {0}.
    ((*f* f)(star-of x + h)
     − star-of (f x))/h @= star-of D)

definition
  NSdifferentiable :: [′a::real-normed-field ⇒ ′a, ′a] ⇒ bool
   (**infixl** NSdifferentiable 60) **where**
  f NSdifferentiable x = (∃ D. NSDERIV f x :> D)

definition
  increment :: [real=>real,real,hypreal] => hypreal **where**
  increment f x h = (@inc. f NSdifferentiable x &
      inc = (*f* f)(hypreal-of-real x + h) − hypreal-of-real (f x))

34.1 Derivatives

lemma *DERIV-NS-iff*:
  (DERIV f x :> D) = ((%h. (f(x + h) − f(x))/h) −− 0 −−NS> D)
by (simp add: deriv-def LIM-NSLIM-iff)

lemma *NS-DERIV-D*: DERIV f x :> D ==> (%h. (f(x + h) − f(x))/h) −− 0 −−NS> D
by (simp add: deriv-def LIM-NSLIM-iff)

lemma *hnorm-of-hypreal*:
  ⋀r. hnorm ((*f* of-real) r::′a::real-normed-div-algebra star) = |r|
by transfer (rule norm-of-real)

lemma *Infinitesimal-of-hypreal*:
  x ∈ Infinitesimal ⟹
   ((*f* of-real) x::′a::real-normed-div-algebra star) ∈ Infinitesimal
apply (rule InfinitesimalI2)
apply (drule (1) InfinitesimalD2)
apply (simp add: hnorm-of-hypreal)
done

lemma *of-hypreal-eq-0-iff*:
  ⋀x. ((*f* of-real) x = (0::′a::real-algebra-1 star)) = (x = 0)
by transfer (rule of-real-eq-0-iff)

```
lemma NSDeriv-unique:
    [| NSDERIV f x :> D; NSDERIV f x :> E |] ==> D = E
apply (subgoal-tac ( *f* of-real) epsilon ∈ Infinitesimal − {0::'a star})
apply (simp only: nsderiv-def)
apply (drule (1) bspec)+
apply (drule (1) approx-trans3)
apply simp
apply (simp add: Infinitesimal-of-hypreal Infinitesimal-epsilon)
apply (simp add: of-hypreal-eq-0-iff hypreal-epsilon-not-zero)
done
```

First NSDERIV in terms of NSLIM

first equivalence

```
lemma NSDERIV-NSLIM-iff:
     (NSDERIV f x :> D) = ((%h. (f(x + h) − f(x))/h) −− 0 −−NS> D)
apply (simp add: nsderiv-def NSLIM-def, auto)
apply (drule-tac x = xa in bspec)
apply (rule-tac [3] ccontr)
apply (drule-tac [3] x = h in spec)
apply (auto simp add: mem-infmal-iff starfun-lambda-cancel)
done
```

second equivalence

```
lemma NSDERIV-NSLIM-iff2:
    (NSDERIV f x :> D) = ((%z. (f(z) − f(x)) / (z−x)) −− x −−NS> D)
by (simp add: NSDERIV-NSLIM-iff DERIV-LIM-iff diff-minus [symmetric]
          LIM-NSLIM-iff [symmetric])
```

```
lemma NSDERIV-iff2:
    (NSDERIV f x :> D) =
     (∀ w.
       w ≠ star-of x & w ≈ star-of x −−>
       ( *f* (%z. (f z − f x) / (z−x))) w ≈ star-of D)
by (simp add: NSDERIV-NSLIM-iff2 NSLIM-def)
```

```
lemma hypreal-not-eq-minus-iff:
  (x ≠ a) = (x − a ≠ (0::'a::ab-group-add))
by auto
```

```
lemma NSDERIVD5:
  (NSDERIV f x :> D) ==>
   (∀ u. u ≈ hypreal-of-real x −−>
     ( *f* (%z. f z − f x)) u ≈ hypreal-of-real D * (u − hypreal-of-real x))
apply (auto simp add: NSDERIV-iff2)
apply (case-tac u = hypreal-of-real x, auto)
```

```
apply (drule-tac x = u in spec, auto)
apply (drule-tac c = u − hypreal-of-real x and b = hypreal-of-real D in approx-mult1)
apply (drule-tac [!] hypreal-not-eq-minus-iff [THEN iffD1])
apply (subgoal-tac [2] ( *f* (%z. z−x)) u ≠ (0::hypreal) )
apply (auto simp add:
        approx-minus-iff [THEN iffD1, THEN mem-infmal-iff [THEN iffD2]]
        Infinitesimal-subset-HFinite [THEN subsetD])
done

lemma NSDERIVD4:
    (NSDERIV f x :> D) ==>
     (∀ h ∈ Infinitesimal.
            (( *f* f)(hypreal-of-real x + h) −
              hypreal-of-real (f x))≈ (hypreal-of-real D) ∗ h)
apply (auto simp add: nsderiv-def)
apply (case-tac h = (0::hypreal) )
apply (auto simp add: diff-minus)
apply (drule-tac x = h in bspec)
apply (drule-tac [2] c = h in approx-mult1)
apply (auto intro: Infinitesimal-subset-HFinite [THEN subsetD]
           simp add: diff-minus)
done

lemma NSDERIVD3:
    (NSDERIV f x :> D) ==>
     (∀ h ∈ Infinitesimal − {0}.
            (( *f* f)(hypreal-of-real x + h) −
              hypreal-of-real (f x))≈ (hypreal-of-real D) ∗ h)
apply (auto simp add: nsderiv-def)
apply (rule ccontr, drule-tac x = h in bspec)
apply (drule-tac [2] c = h in approx-mult1)
apply (auto intro: Infinitesimal-subset-HFinite [THEN subsetD]
           simp add: mult-assoc diff-minus)
done
```

Differentiability implies continuity nice and simple "algebraic" proof

```
lemma NSDERIV-isNSCont: NSDERIV f x :> D ==> isNSCont f x
apply (auto simp add: nsderiv-def isNSCont-NSLIM-iff NSLIM-def)
apply (drule approx-minus-iff [THEN iffD1])
apply (drule hypreal-not-eq-minus-iff [THEN iffD1])
apply (drule-tac x = xa − star-of x in bspec)
 prefer 2 apply (simp add: add-assoc [symmetric])
apply (auto simp add: mem-infmal-iff [symmetric] add-commute)
apply (drule-tac c = xa − star-of x in approx-mult1)
apply (auto intro: Infinitesimal-subset-HFinite [THEN subsetD]
           simp add: mult-assoc nonzero-mult-divide-cancel-right)
apply (drule-tac x3=D in
          HFinite-star-of [THEN [2] Infinitesimal-HFinite-mult,
            THEN mem-infmal-iff [THEN iffD1]])
```

apply (*auto simp add*: *mult-commute*
 intro: *approx-trans approx-minus-iff* [*THEN iffD2*])
done

Differentiation rules for combinations of functions follow from clear, straightforard, algebraic manipulations

Constant function

lemma *NSDERIV-const* [*simp*]: (*NSDERIV* (%*x*. *k*) *x* :> *0*)
by (*simp add*: *NSDERIV-NSLIM-iff*)

Sum of functions- proved easily

lemma *NSDERIV-add*: [| *NSDERIV f x* :> *Da*; *NSDERIV g x* :> *Db* |]
 ==> *NSDERIV* (%*x*. *f x* + *g x*) *x* :> *Da* + *Db*
apply (*auto simp add*: *NSDERIV-NSLIM-iff NSLIM-def*)
apply (*auto simp add*: *add-divide-distrib diff-divide-distrib dest*!: *spec*)
apply (*drule-tac b* = *star-of Da* **and** *d* = *star-of Db* **in** *approx-add*)
apply (*auto simp add*: *diff-def add-ac*)
done

Product of functions - Proof is trivial but tedious and long due to rearrangement of terms

lemma *lemma-nsderiv1*:
 fixes *a b c d* :: *'a*::*comm-ring star*
 shows (*a***b*) − (*c***d*) = (*b**(*a* − *c*)) + (*c**(*b* − *d*))
by (*simp add*: *right-diff-distrib mult-ac*)

lemma *lemma-nsderiv2*:
 fixes *x y z* :: *'a*::*real-normed-field star*
 shows [| (*x* − *y*) / *z* = *star-of D* + *yb*; *z* ≠ *0*;
 z ∈ *Infinitesimal*; *yb* ∈ *Infinitesimal* |]
 ==> *x* − *y* ≈ *0*
apply (*simp add*: *nonzero-divide-eq-eq*)
apply (*auto intro*!: *Infinitesimal-HFinite-mult2 HFinite-add*
 simp add: *mult-assoc mem-infmal-iff* [*symmetric*])
apply (*erule Infinitesimal-subset-HFinite* [*THEN subsetD*])
done

lemma *NSDERIV-mult*: [| *NSDERIV f x* :> *Da*; *NSDERIV g x* :> *Db* |]
 ==> *NSDERIV* (%*x*. *f x* * *g x*) *x* :> (*Da* * *g*(*x*)) + (*Db* * *f*(*x*))
apply (*auto simp add*: *NSDERIV-NSLIM-iff NSLIM-def*)
apply (*auto dest*!: *spec*
 simp add: *starfun-lambda-cancel lemma-nsderiv1*)
apply (*simp* (*no-asm*) *add*: *add-divide-distrib diff-divide-distrib*)
apply (*drule bex-Infinitesimal-iff2* [*THEN iffD2*])+
apply (*auto simp add*: *times-divide-eq-right* [*symmetric*]
 simp del: *times-divide-eq*)
apply (*drule-tac D* = *Db* **in** *lemma-nsderiv2*, *assumption*+)
apply (*drule-tac*

approx-minus-iff [*THEN iffD2*, *THEN bex-Infinitesimal-iff2* [*THEN iffD2*]])
apply (*auto intro*!: *approx-add-mono1*
simp add: *left-distrib right-distrib mult-commute add-assoc*)
apply (*rule-tac b1* = *star-of Db* ∗ *star-of* (*f x*)
in *add-commute* [*THEN subst*])
apply (*auto intro*!: *Infinitesimal-add-approx-self2* [*THEN approx-sym*]
Infinitesimal-add Infinitesimal-mult
Infinitesimal-star-of-mult
Infinitesimal-star-of-mult2
simp add: *add-assoc* [*symmetric*])
done

Multiplying by a constant

lemma *NSDERIV-cmult*: *NSDERIV f x* :> *D*
==> *NSDERIV* (%*x*. *c* ∗ *f x*) *x* :> *c*∗*D*
apply (*simp only*: *times-divide-eq-right* [*symmetric*] *NSDERIV-NSLIM-iff*
minus-mult-right right-diff-distrib [*symmetric*])
apply (*erule NSLIM-const* [*THEN NSLIM-mult*])
done

Negation of function

lemma *NSDERIV-minus*: *NSDERIV f x* :> *D* ==> *NSDERIV* (%*x*. −(*f x*)) *x* :> −*D*
proof (*simp add*: *NSDERIV-NSLIM-iff*)
assume (λ*h*. (*f* (*x* + *h*) − *f x*) / *h*) −− *0* −−*NS*> *D*
hence *deriv*: (λ*h*. − ((*f*(*x*+*h*) − *f x*) / *h*)) −− *0* −−*NS*> − *D*
by (*rule NSLIM-minus*)
have ∀ *h*. − ((*f* (*x* + *h*) − *f x*) / *h*) = (− *f* (*x* + *h*) + *f x*) / *h*
by (*simp add*: *minus-divide-left diff-def*)
with *deriv*
show (λ*h*. (− *f* (*x* + *h*) + *f x*) / *h*) −− *0* −−*NS*> − *D* **by** *simp*
qed

Subtraction

lemma *NSDERIV-add-minus*: [| *NSDERIV f x* :> *Da*; *NSDERIV g x* :> *Db* |]
==> *NSDERIV* (%*x*. *f x* + −*g x*) *x* :> *Da* + −*Db*
by (*blast dest*: *NSDERIV-add NSDERIV-minus*)

lemma *NSDERIV-diff*:
[| *NSDERIV f x* :> *Da*; *NSDERIV g x* :> *Db* |]
==> *NSDERIV* (%*x*. *f x* − *g x*) *x* :> *Da*−*Db*
apply (*simp add*: *diff-minus*)
apply (*blast intro*: *NSDERIV-add-minus*)
done

Similarly to the above, the chain rule admits an entirely straightforward derivation. Compare this with Harrison's HOL proof of the chain rule, which proved to be trickier and required an alternative characterisation of differentiability- the so-called Carathedory derivative. Our main problem is manipulation of terms.

lemma *NSDERIV-zero*:
  [| *NSDERIV g x* :> *D*;
    (**f** *g*) (*star-of x* + *xa*) = *star-of* (*g x*);
    xa ∈ *Infinitesimal*;
    xa ≠ *0*
  |] ==> *D* = *0*
apply (*simp add*: *nsderiv-def*)
apply (*drule bspec*, *auto*)
done

lemma *NSDERIV-approx*:
  [| *NSDERIV f x* :> *D*; *h* ∈ *Infinitesimal*; *h* ≠ *0* |]
  ==> (**f** *f*) (*star-of x* + *h*) − *star-of* (*f x*) ≈ *0*
apply (*simp add*: *nsderiv-def*)
apply (*simp add*: *mem-infmal-iff* [*symmetric*])
apply (*rule Infinitesimal-ratio*)
apply (*rule-tac* [*3*] *approx-star-of-HFinite*, *auto*)
done

lemma *NSDERIVD1*: [| *NSDERIV f* (*g x*) :> *Da*;
  (**f** *g*) (*star-of*(*x*) + *xa*) ≠ *star-of* (*g x*);
  (**f** *g*) (*star-of*(*x*) + *xa*) ≈ *star-of* (*g x*)
  |] ==> ((**f** *f*) ((**f** *g*) (*star-of*(*x*) + *xa*))
    − *star-of* (*f* (*g x*)))
    / ((**f** *g*) (*star-of*(*x*) + *xa*) − *star-of* (*g x*))
    ≈ *star-of*(*Da*)
by (*auto simp add*: *NSDERIV-NSLIM-iff2 NSLIM-def diff-minus* [*symmetric*])

lemma *NSDERIVD2*: [| *NSDERIV g x* :> *Db*; *xa* ∈ *Infinitesimal*; *xa* ≠ *0* |]
  ==> ((**f** *g*) (*star-of*(*x*) + *xa*) − *star-of*(*g x*)) / *xa*
    ≈ *star-of*(*Db*)
by (*auto simp add*: *NSDERIV-NSLIM-iff NSLIM-def mem-infmal-iff starfun-lambda-cancel*)

lemma *lemma-chain*: (*z*::′*a*::*real-normed-field star*) ≠ *0* ==> *x***y* = (*x***inverse*(*z*))*(*z***y*)
proof −
  assume *z*: *z* ≠ *0*
  have *x* * *y* = *x* * (*inverse z* * *z*) * *y* **by** (*simp add*: *z*)
  thus *?thesis* **by** (*simp add*: *mult-assoc*)
qed

This proof uses both definitions of differentiability.

lemma *NSDERIV-chain*: [| *NSDERIV f* (*g x*) :> *Da*; *NSDERIV g x* :> *Db* |]

==> NSDERIV (f o g) x :> Da * Db
apply (simp (no-asm-simp) add: NSDERIV-NSLIM-iff NSLIM-def
mem-infmal-iff [symmetric])
apply clarify
apply (frule-tac f = g **in** NSDERIV-approx)
apply (auto simp add: starfun-lambda-cancel2 starfun-o [symmetric])
apply (case-tac (*f* g) (star-of (x) + xa) = star-of (g x))
apply (drule-tac g = g **in** NSDERIV-zero)
apply (auto simp add: divide-inverse)
apply (rule-tac z1 = (*f* g) (star-of (x) + xa) − star-of (g x) **and** y1 = inverse xa **in** lemma-chain [THEN ssubst])
apply (erule hypreal-not-eq-minus-iff [THEN iffD1])
apply (rule approx-mult-star-of)
apply (simp-all add: divide-inverse [symmetric])
apply (blast intro: NSDERIVD1 approx-minus-iff [THEN iffD2])
apply (blast intro: NSDERIVD2)
done

Differentiation of natural number powers

lemma NSDERIV-Id [simp]: NSDERIV (%x. x) x :> 1
by (simp add: NSDERIV-NSLIM-iff NSLIM-def del: divide-self-if)

lemma NSDERIV-cmult-Id [simp]: NSDERIV (op * c) x :> c
by (cut-tac c = c **and** x = x **in** NSDERIV-Id [THEN NSDERIV-cmult], simp)

lemma NSDERIV-inverse:
fixes x :: 'a::{real-normed-field,recpower}
shows $x \neq 0$ ==> NSDERIV (%x. inverse(x)) x :> (− (inverse x ^ Suc (Suc 0)))
apply (simp add: nsderiv-def)
apply (rule ballI, simp, clarify)
apply (frule (1) Infinitesimal-add-not-zero)
apply (simp add: add-commute)

apply (simp add: inverse-add nonzero-inverse-mult-distrib [symmetric] power-Suc
nonzero-inverse-minus-eq [symmetric] add-ac mult-ac diff-def
del: inverse-mult-distrib inverse-minus-eq
minus-mult-left [symmetric] minus-mult-right [symmetric])
apply (subst mult-commute, simp add: nonzero-mult-divide-cancel-right)
apply (simp (no-asm-simp) add: mult-assoc [symmetric] left-distrib
del: minus-mult-left [symmetric] minus-mult-right [symmetric])
apply (rule-tac y = inverse (− (star-of x * star-of x)) **in** approx-trans)
apply (rule inverse-add-Infinitesimal-approx2)
apply (auto dest!: hypreal-of-real-HFinite-diff-Infinitesimal
simp add: inverse-minus-eq [symmetric] HFinite-minus-iff)
apply (rule Infinitesimal-HFinite-mult, auto)
done

34.1.1 Equivalence of NS and Standard definitions

lemma *divideR-eq-divide*: $x\ /_R\ y = x\ /\ y$
by (*simp add*: *real-scaleR-def divide-inverse mult-commute*)

Now equivalence between NSDERIV and DERIV

lemma *NSDERIV-DERIV-iff*: (*NSDERIV f x* :> *D*) = (*DERIV f x* :> *D*)
by (*simp add*: *deriv-def NSDERIV-NSLIM-iff LIM-NSLIM-iff*)

lemma *NSDERIV-pow*: *NSDERIV* (%*x*. *x* ˆ *n*) *x* :> *real n* ∗ (*x* ˆ (*n* − *Suc 0*))
by (*simp add*: *NSDERIV-DERIV-iff DERIV-pow*)

Derivative of inverse

lemma *NSDERIV-inverse-fun*:
  fixes *x* :: *'a*::{*real-normed-field*,*recpower*}
  shows [| *NSDERIV f x* :> *d*; *f*(*x*) ≠ *0* |]
      ==> *NSDERIV* (%*x*. *inverse*(*f x*)) *x* :> (− (*d* ∗ *inverse*(*f*(*x*) ˆ *Suc* (*Suc 0*))))
by (*simp add*: *NSDERIV-DERIV-iff DERIV-inverse-fun del*: *realpow-Suc*)

Derivative of quotient

lemma *NSDERIV-quotient*:
  fixes *x* :: *'a*::{*real-normed-field*,*recpower*}
  shows [| *NSDERIV f x* :> *d*; *NSDERIV g x* :> *e*; *g*(*x*) ≠ *0* |]
      ==> *NSDERIV* (%*y*. *f*(*y*) / (*g y*)) *x* :> (*d*∗*g*(*x*)
                − (*e*∗*f*(*x*))) / (*g*(*x*) ˆ *Suc* (*Suc 0*))
by (*simp add*: *NSDERIV-DERIV-iff DERIV-quotient del*: *realpow-Suc*)

lemma *CARAT-NSDERIV*: *NSDERIV f x* :> *l* ==>
      ∃ *g*. (∀ *z*. *f z* − *f x* = *g z* ∗ (*z*−*x*)) & *isNSCont g x* & *g x* = *l*
by (*auto simp add*: *NSDERIV-DERIV-iff isNSCont-isCont-iff CARAT-DERIV*
      mult-commute)

lemma *hypreal-eq-minus-iff3*: (*x* = *y* + *z*) = (*x* + −*z* = (*y*::*hypreal*))
by *auto*

lemma *CARAT-DERIVD*:
  assumes *all*: ∀ *z*. *f z* − *f x* = *g z* ∗ (*z*−*x*)
    and *nsc*: *isNSCont g x*
  shows *NSDERIV f x* :> *g x*
proof −
  from *nsc*
  have ∀ *w*. *w* ≠ *star-of x* ∧ *w* ≈ *star-of x* ⟶
      (∗*f*∗ *g*) *w* ∗ (*w* − *star-of x*) / (*w* − *star-of x*) ≈
      star-of (*g x*)
    by (*simp add*: *isNSCont-def nonzero-mult-divide-cancel-right*)
  thus *?thesis* **using** *all*
    by (*simp add*: *NSDERIV-iff2 starfun-if-eq cong*: *if-cong*)
qed

34.1.2 Differentiability predicate

lemma *NSdifferentiableD*: *f NSdifferentiable x ==> ∃ D. NSDERIV f x :> D*
by (*simp add*: *NSdifferentiable-def*)

lemma *NSdifferentiableI*: *NSDERIV f x :> D ==> f NSdifferentiable x*
by (*force simp add*: *NSdifferentiable-def*)

34.2 (NS) Increment

lemma *incrementI*:
  f NSdifferentiable x ==>
  *increment f x h = (*f* f) (hypreal-of-real(x) + h) −*
  hypreal-of-real (f x)
by (*simp add*: *increment-def*)

lemma *incrementI2*: *NSDERIV f x :> D ==>*
  *increment f x h = (*f* f) (hypreal-of-real(x) + h) −*
  hypreal-of-real (f x)
apply (*erule NSdifferentiableI* [*THEN incrementI*])
done

lemma *increment-thm*: [| *NSDERIV f x :> D*; *h ∈ Infinitesimal*; *h ≠ 0* |]
  *==> ∃ e ∈ Infinitesimal. increment f x h = hypreal-of-real(D)*h + e*h*
apply (*frule-tac h = h* **in** *incrementI2*, *simp add*: *nsderiv-def*)
apply (*drule bspec*, *auto*)
apply (*drule bex-Infinitesimal-iff2* [*THEN iffD2*], *clarify*)
apply (*frule-tac b1 = hypreal-of-real (D) + y*
  in *hypreal-mult-right-cancel* [*THEN iffD2*])
apply (*erule-tac* [*2*] *V = ((*f* f) (hypreal-of-real (x) + h) − hypreal-of-real (f x)) / h = hypreal-of-real (D) + y* **in** *thin-rl*)
apply *assumption*
apply (*simp add*: *times-divide-eq-right* [*symmetric*])
apply (*auto simp add*: *left-distrib*)
done

lemma *increment-thm2*:
  [| *NSDERIV f x :> D*; *h ≈ 0*; *h ≠ 0* |]
  ==> ∃ e ∈ Infinitesimal. increment f x h =
    *hypreal-of-real(D)*h + e*h*
by (*blast dest!*: *mem-infmal-iff* [*THEN iffD2*] *intro!*: *increment-thm*)

lemma *increment-approx-zero*: [| *NSDERIV f x :> D*; *h ≈ 0*; *h ≠ 0* |]
  ==> increment f x h ≈ 0
apply (*drule increment-thm2*,
  auto intro!: *Infinitesimal-HFinite-mult2 HFinite-add simp add*: *left-distrib* [*symmetric*] *mem-infmal-iff* [*symmetric*])
apply (*erule Infinitesimal-subset-HFinite* [*THEN subsetD*])

done

end

35 HTranscendental: Nonstandard Extensions of Transcendental Functions

theory *HTranscendental*
imports *Transcendental HSeries HDeriv*
begin

definition
  exphr :: *real* => *hypreal* **where**
    — define exponential function using standard part
  exphr x = *st*(*sumhr* (*0*, *whn*, %*n*. *inverse*(*real* (*fact n*)) ∗ (*x* ^ *n*)))

definition
  sinhr :: *real* => *hypreal* **where**
  sinhr x = *st*(*sumhr* (*0*, *whn*, %*n*. (*if even*(*n*) *then 0 else*
        ((−*1*) ^ ((*n* − *1*) *div 2*))/(*real* (*fact n*))) ∗ (*x* ^ *n*)))

definition
  coshr :: *real* => *hypreal* **where**
  coshr x = *st*(*sumhr* (*0*, *whn*, %*n*. (*if even*(*n*) *then*
        ((−*1*) ^ (*n div 2*))/(*real* (*fact n*)) *else 0*) ∗ *x* ^ *n*))

35.1 Nonstandard Extension of Square Root Function

lemma *STAR-sqrt-zero* [*simp*]: (∗*f*∗ *sqrt*) *0* = *0*
by (*simp add*: *starfun star-n-zero-num*)

lemma *STAR-sqrt-one* [*simp*]: (∗*f*∗ *sqrt*) *1* = *1*
by (*simp add*: *starfun star-n-one-num*)

lemma *hypreal-sqrt-pow2-iff*: ((∗*f*∗ *sqrt*)(*x*) ^ *2* = *x*) = (*0* ≤ *x*)
apply (*cases x*)
apply (*auto simp add*: *star-n-le star-n-zero-num starfun hrealpow star-n-eq-iff*
        simp del: *hpowr-Suc realpow-Suc*)
done

lemma *hypreal-sqrt-gt-zero-pow2*: !!*x*. *0* < *x* ==> (∗*f*∗ *sqrt*) (*x*) ^ *2* = *x*
by (*transfer*, *simp*)

lemma *hypreal-sqrt-pow2-gt-zero*: *0* < *x* ==> *0* < (∗*f*∗ *sqrt*) (*x*) ^ *2*
by (*frule hypreal-sqrt-gt-zero-pow2*, *auto*)

lemma *hypreal-sqrt-not-zero*: *0* < *x* ==> (∗*f*∗ *sqrt*) (*x*) ≠ *0*

```
apply (frule hypreal-sqrt-pow2-gt-zero)
apply (auto simp add: numeral-2-eq-2)
done

lemma hypreal-inverse-sqrt-pow2:
    0 < x ==> inverse (( *f* sqrt)(x)) ^ 2 = inverse x
apply (cut-tac n = 2 and a = ( *f* sqrt) x in power-inverse [symmetric])
apply (auto dest: hypreal-sqrt-gt-zero-pow2)
done

lemma hypreal-sqrt-mult-distrib:
   !!x y. [|0 < x; 0 <y |] ==>
     ( *f* sqrt)(x*y) = ( *f* sqrt)(x) * ( *f* sqrt)(y)
apply transfer
apply (auto intro: real-sqrt-mult-distrib)
done

lemma hypreal-sqrt-mult-distrib2:
    [|0≤x; 0≤y |] ==>
    ( *f* sqrt)(x*y) = ( *f* sqrt)(x) * ( *f* sqrt)(y)
by (auto intro: hypreal-sqrt-mult-distrib simp add: order-le-less)

lemma hypreal-sqrt-approx-zero [simp]:
    0 < x ==> (( *f* sqrt)(x) @= 0) = (x @= 0)
apply (auto simp add: mem-infmal-iff [symmetric])
apply (rule hypreal-sqrt-gt-zero-pow2 [THEN subst])
apply (auto intro: Infinitesimal-mult
           dest!: hypreal-sqrt-gt-zero-pow2 [THEN ssubst]
           simp add: numeral-2-eq-2)
done

lemma hypreal-sqrt-approx-zero2 [simp]:
    0 ≤ x ==> (( *f* sqrt)(x) @= 0) = (x @= 0)
by (auto simp add: order-le-less)

lemma hypreal-sqrt-sum-squares [simp]:
    (( *f* sqrt)(x*x + y*y + z*z) @= 0) = (x*x + y*y + z*z @= 0)
apply (rule hypreal-sqrt-approx-zero2)
apply (rule add-nonneg-nonneg)+
apply (auto)
done

lemma hypreal-sqrt-sum-squares2 [simp]:
    (( *f* sqrt)(x*x + y*y) @= 0) = (x*x + y*y @= 0)
apply (rule hypreal-sqrt-approx-zero2)
apply (rule add-nonneg-nonneg)
apply (auto)
done
```

```
lemma hypreal-sqrt-gt-zero: !!x. 0 < x ==> 0 < ( *f* sqrt)(x)
apply transfer
apply (auto intro: real-sqrt-gt-zero)
done

lemma hypreal-sqrt-ge-zero: 0 ≤ x ==> 0 ≤ ( *f* sqrt)(x)
by (auto intro: hypreal-sqrt-gt-zero simp add: order-le-less)

lemma hypreal-sqrt-hrabs [simp]: !!x. ( *f* sqrt)(x ^ 2) = abs(x)
by (transfer, simp)

lemma hypreal-sqrt-hrabs2 [simp]: !!x. ( *f* sqrt)(x*x) = abs(x)
by (transfer, simp)

lemma hypreal-sqrt-hyperpow-hrabs [simp]:
    !!x. ( *f* sqrt)(x pow (hypnat-of-nat 2)) = abs(x)
by (transfer, simp)

lemma star-sqrt-HFinite: ⟦x ∈ HFinite; 0 ≤ x⟧ ⟹ ( *f* sqrt) x ∈ HFinite
apply (rule HFinite-square-iff [THEN iffD1])
apply (simp only: hypreal-sqrt-mult-distrib2 [symmetric], simp)
done

lemma st-hypreal-sqrt:
    [| x ∈ HFinite; 0 ≤ x |] ==> st(( *f* sqrt) x) = ( *f* sqrt)(st x)
apply (rule power-inject-base [where n=1])
apply (auto intro!: st-zero-le hypreal-sqrt-ge-zero)
apply (rule st-mult [THEN subst])
apply (rule-tac [3] hypreal-sqrt-mult-distrib2 [THEN subst])
apply (rule-tac [5] hypreal-sqrt-mult-distrib2 [THEN subst])
apply (auto simp add: st-hrabs st-zero-le star-sqrt-HFinite)
done

lemma hypreal-sqrt-sum-squares-ge1 [simp]: !!x y. x ≤ ( *f* sqrt)(x ^ 2 + y ^ 2)
by transfer (rule real-sqrt-sum-squares-ge1)

lemma HFinite-hypreal-sqrt:
    [| 0 ≤ x; x ∈ HFinite |] ==> ( *f* sqrt) x ∈ HFinite
apply (auto simp add: order-le-less)
apply (rule HFinite-square-iff [THEN iffD1])
apply (drule hypreal-sqrt-gt-zero-pow2)
apply (simp add: numeral-2-eq-2)
done

lemma HFinite-hypreal-sqrt-imp-HFinite:
    [| 0 ≤ x; ( *f* sqrt) x ∈ HFinite |] ==> x ∈ HFinite
apply (auto simp add: order-le-less)
apply (drule HFinite-square-iff [THEN iffD2])
apply (drule hypreal-sqrt-gt-zero-pow2)
```

apply (*simp add*: *numeral-2-eq-2 del*: *HFinite-square-iff*)
done

lemma *HFinite-hypreal-sqrt-iff* [*simp*]:
$0 \leq x$ ==> ((*f* *sqrt*) $x \in$ *HFinite*) = ($x \in$ *HFinite*)
by (*blast intro*: *HFinite-hypreal-sqrt HFinite-hypreal-sqrt-imp-HFinite*)

lemma *HFinite-sqrt-sum-squares* [*simp*]:
((*f* *sqrt*)($x*x$ + $y*y$) $\in$ *HFinite*) = ($x*x$ + $y*y$ $\in$ *HFinite*)
apply (*rule HFinite-hypreal-sqrt-iff*)
apply (*rule add-nonneg-nonneg*)
apply (*auto*)
done

lemma *Infinitesimal-hypreal-sqrt*:
[| $0 \leq x$; $x \in$ *Infinitesimal* |] ==> (*f* *sqrt*) $x \in$ *Infinitesimal*
apply (*auto simp add*: *order-le-less*)
apply (*rule Infinitesimal-square-iff* [*THEN iffD2*])
apply (*drule hypreal-sqrt-gt-zero-pow2*)
apply (*simp add*: *numeral-2-eq-2*)
done

lemma *Infinitesimal-hypreal-sqrt-imp-Infinitesimal*:
[| $0 \leq x$; (*f* *sqrt*) $x \in$ *Infinitesimal* |] ==> $x \in$ *Infinitesimal*
apply (*auto simp add*: *order-le-less*)
apply (*drule Infinitesimal-square-iff* [*THEN iffD1*])
apply (*drule hypreal-sqrt-gt-zero-pow2*)
apply (*simp add*: *numeral-2-eq-2 del*: *Infinitesimal-square-iff* [*symmetric*])
done

lemma *Infinitesimal-hypreal-sqrt-iff* [*simp*]:
$0 \leq x$ ==> ((*f* *sqrt*) $x \in$ *Infinitesimal*) = ($x \in$ *Infinitesimal*)
by (*blast intro*: *Infinitesimal-hypreal-sqrt-imp-Infinitesimal Infinitesimal-hypreal-sqrt*)

lemma *Infinitesimal-sqrt-sum-squares* [*simp*]:
((*f* *sqrt*)($x*x$ + $y*y$) $\in$ *Infinitesimal*) = ($x*x$ + $y*y$ $\in$ *Infinitesimal*)
apply (*rule Infinitesimal-hypreal-sqrt-iff*)
apply (*rule add-nonneg-nonneg*)
apply (*auto*)
done

lemma *HInfinite-hypreal-sqrt*:
[| $0 \leq x$; $x \in$ *HInfinite* |] ==> (*f* *sqrt*) $x \in$ *HInfinite*
apply (*auto simp add*: *order-le-less*)
apply (*rule HInfinite-square-iff* [*THEN iffD1*])
apply (*drule hypreal-sqrt-gt-zero-pow2*)
apply (*simp add*: *numeral-2-eq-2*)
done

lemma *HInfinite-hypreal-sqrt-imp-HInfinite*:
  [| $0 \leq x$; (*f* *sqrt*) $x \in$ *HInfinite* |] ==> $x \in$ *HInfinite*
apply (*auto simp add*: *order-le-less*)
apply (*drule HInfinite-square-iff* [*THEN iffD2*])
apply (*drule hypreal-sqrt-gt-zero-pow2*)
apply (*simp add*: *numeral-2-eq-2 del*: *HInfinite-square-iff*)
done

lemma *HInfinite-hypreal-sqrt-iff* [*simp*]:
  $0 \leq x$ ==> ((*f* *sqrt*) $x \in$ *HInfinite*) = ($x \in$ *HInfinite*)
by (*blast intro*: *HInfinite-hypreal-sqrt HInfinite-hypreal-sqrt-imp-HInfinite*)

lemma *HInfinite-sqrt-sum-squares* [*simp*]:
  ((*f* *sqrt*)($x*x + y*y$) $\in$ *HInfinite*) = ($x*x + y*y \in$ *HInfinite*)
apply (*rule HInfinite-hypreal-sqrt-iff*)
apply (*rule add-nonneg-nonneg*)
apply (*auto*)
done

lemma *HFinite-exp* [*simp*]:
  sumhr (*0*, *whn*, %*n*. *inverse* (*real* (*fact n*)) * x ^ n) $\in$ *HFinite*
unfolding *sumhr-app*
apply (*simp only*: *star-zero-def starfun2-star-of*)
apply (*rule NSBseqD2*)
apply (*rule NSconvergent-NSBseq*)
apply (*rule convergent-NSconvergent-iff* [*THEN iffD1*])
apply (*rule summable-convergent-sumr-iff* [*THEN iffD1*])
apply (*rule summable-exp*)
done

lemma *exphr-zero* [*simp*]: *exphr 0* = *1*
apply (*simp add*: *exphr-def sumhr-split-add*
    [*OF hypnat-one-less-hypnat-omega*, *symmetric*])
apply (*rule st-unique*, *simp*)
apply (*rule subst* [**where** $P=\lambda x.\ 1 \approx x$, *OF* - *approx-refl*])
apply (*rule rev-mp* [*OF hypnat-one-less-hypnat-omega*])
apply (*rule-tac x*=*whn* **in** *spec*)
apply (*unfold sumhr-app*, *transfer*, *simp*)
done

lemma *coshr-zero* [*simp*]: *coshr 0* = *1*
apply (*simp add*: *coshr-def sumhr-split-add*
    [*OF hypnat-one-less-hypnat-omega*, *symmetric*])
apply (*rule st-unique*, *simp*)
apply (*rule subst* [**where** $P=\lambda x.\ 1 \approx x$, *OF* - *approx-refl*])
apply (*rule rev-mp* [*OF hypnat-one-less-hypnat-omega*])
apply (*rule-tac x*=*whn* **in** *spec*)
apply (*unfold sumhr-app*, *transfer*, *simp*)
done

lemma *STAR-exp-zero-approx-one* [*simp*]: (*∗f∗ exp*) (*0*::*hypreal*) @= *1*
apply (*subgoal-tac* (*∗f∗ exp*) (*0*::*hypreal*) = *1*, *simp*)
apply (*transfer*, *simp*)
done

lemma *STAR-exp-Infinitesimal*: *x* ∈ *Infinitesimal* ==> (*∗f∗ exp*) (*x*::*hypreal*) @= *1*
apply (*case-tac x* = *0*)
apply (*cut-tac* [*2*] *x* = *0* **in** *DERIV-exp*)
apply (*auto simp add*: *NSDERIV-DERIV-iff* [*symmetric*] *nsderiv-def*)
apply (*drule-tac x* = *x* **in** *bspec*, *auto*)
apply (*drule-tac c* = *x* **in** *approx-mult1*)
apply (*auto intro*: *Infinitesimal-subset-HFinite* [*THEN subsetD*]
simp add: *mult-assoc*)
apply (*rule approx-add-right-cancel* [**where** *d*=−*1*])
apply (*rule approx-sym* [*THEN* [*2*] *approx-trans2*])
apply (*auto simp add*: *diff-def mem-infmal-iff*)
done

lemma *STAR-exp-epsilon* [*simp*]: (*∗f∗ exp*) *epsilon* @= *1*
by (*auto intro*: *STAR-exp-Infinitesimal*)

lemma *STAR-exp-add*: !!*x y*. (*∗f∗ exp*)(*x* + *y*) = (*∗f∗ exp*) *x* ∗ (*∗f∗ exp*) *y*
by *transfer* (*rule exp-add*)

lemma *exphr-hypreal-of-real-exp-eq*: *exphr x* = *hypreal-of-real* (*exp x*)
apply (*simp add*: *exphr-def*)
apply (*rule st-unique*, *simp*)
apply (*subst starfunNat-sumr* [*symmetric*])
apply (*rule NSLIMSEQ-D* [*THEN approx-sym*])
apply (*rule LIMSEQ-NSLIMSEQ*)
apply (*subst sums-def* [*symmetric*])
apply (*cut-tac exp-converges* [**where** *x*=*x*], *simp*)
apply (*rule HNatInfinite-whn*)
done

lemma *starfun-exp-ge-add-one-self* [*simp*]: !!*x*::*hypreal*. *0* ≤ *x* ==> (*1* + *x*) ≤ (*∗f∗ exp*) *x*
by *transfer* (*rule exp-ge-add-one-self-aux*)

lemma *starfun-exp-HInfinite*:
[| *x* ∈ *HInfinite*; *0* ≤ *x* |] ==> (*∗f∗ exp*) (*x*::*hypreal*) ∈ *HInfinite*
apply (*frule starfun-exp-ge-add-one-self*)
apply (*rule HInfinite-ge-HInfinite*, *assumption*)
apply (*rule order-trans* [*of - 1*+*x*], *auto*)
done

lemma *starfun-exp-minus*: !!x. ($*f*$ *exp*) $(-x)$ = *inverse*(($*f*$ *exp*) x)
by *transfer* (*rule exp-minus*)

lemma *starfun-exp-Infinitesimal*:
[| $x \in$ *HInfinite*; $x \leq 0$ |] ==> ($*f*$ *exp*) (x::*hypreal*) $\in$ *Infinitesimal*
apply (*subgoal-tac* $\exists\, y.\ x = -\ y$)
apply (*rule-tac* [2] $x = -\ x$ **in** *exI*)
apply (*auto intro*!: *HInfinite-inverse-Infinitesimal starfun-exp-HInfinite*
simp add: *starfun-exp-minus HInfinite-minus-iff*)
done

lemma *starfun-exp-gt-one* [*simp*]: !!x::*hypreal*. $0 < x$ ==> $1 <$ ($*f*$ *exp*) x
by *transfer* (*rule exp-gt-one*)

lemma *starfun-ln-exp* [*simp*]: !!x. ($*f*$ *ln*) (($*f*$ *exp*) x) = x
by *transfer* (*rule ln-exp*)

lemma *starfun-exp-ln-iff* [*simp*]: !!x. (($*f*$ *exp*)(($*f*$ *ln*) x) = x) = ($0 < x$)
by *transfer* (*rule exp-ln-iff*)

lemma *starfun-exp-ln-eq*: !!u x. ($*f*$ *exp*) $u = x$ ==> ($*f*$ *ln*) $x = u$
by *transfer* (*rule exp-ln-eq*)

lemma *starfun-ln-less-self* [*simp*]: !!x. $0 < x$ ==> ($*f*$ *ln*) $x < x$
by *transfer* (*rule ln-less-self*)

lemma *starfun-ln-ge-zero* [*simp*]: !!x. $1 \leq x$ ==> $0 \leq$ ($*f*$ *ln*) x
by *transfer* (*rule ln-ge-zero*)

lemma *starfun-ln-gt-zero* [*simp*]: !!x .$1 < x$ ==> $0 <$ ($*f*$ *ln*) x
by *transfer* (*rule ln-gt-zero*)

lemma *starfun-ln-not-eq-zero* [*simp*]: !!x. [| $0 < x$; $x \neq 1$ |] ==> ($*f*$ *ln*) $x \neq 0$
by *transfer simp*

lemma *starfun-ln-HFinite*: [| $x \in$ *HFinite*; $1 \leq x$ |] ==> ($*f*$ *ln*) $x \in$ *HFinite*
apply (*rule HFinite-bounded*)
apply *assumption*
apply (*simp-all add*: *starfun-ln-less-self order-less-imp-le*)
done

lemma *starfun-ln-inverse*: !!x. $0 < x$ ==> ($*f*$ *ln*) (*inverse* x) = $-$($*f*$ *ln*) x
by *transfer* (*rule ln-inverse*)

lemma *starfun-abs-exp-cancel*: $\bigwedge x$. |($*f*$ *exp*) (x::*hypreal*)| = ($*f*$ *exp*) x

by *transfer* (*rule abs-exp-cancel*)

lemma *starfun-exp-less-mono*: $\bigwedge x\ y$::*hypreal*. $x < y \Longrightarrow$ (*∗f∗ exp*) $x <$ (*∗f∗ exp*) y
by *transfer* (*rule exp-less-mono*)

lemma *starfun-exp-HFinite*: $x \in$ *HFinite* ==> (*∗f∗ exp*) (x::*hypreal*) $\in$ *HFinite*
apply (*auto simp add*: *HFinite-def*, *rename-tac u*)
apply (*rule-tac x*=(*∗f∗ exp*) *u* **in** *rev-bexI*)
apply (*simp add*: *Reals-eq-Standard*)
apply (*simp add*: *starfun-abs-exp-cancel*)
apply (*simp add*: *starfun-exp-less-mono*)
done

lemma *starfun-exp-add-HFinite-Infinitesimal-approx*:
[|$x \in$ *Infinitesimal*; $z \in$ *HFinite* |] ==> (*∗f∗ exp*) ($z + x$::*hypreal*) @= (*∗f∗ exp*) z
apply (*simp add*: *STAR-exp-add*)
apply (*frule STAR-exp-Infinitesimal*)
apply (*drule approx-mult2*)
apply (*auto intro*: *starfun-exp-HFinite*)
done

lemma *starfun-ln-HInfinite*:
[| $x \in$ *HInfinite*; $0 < x$ |] ==> (*∗f∗ ln*) $x \in$ *HInfinite*
apply (*rule ccontr*, *drule HFinite-HInfinite-iff* [*THEN iffD2*])
apply (*drule starfun-exp-HFinite*)
apply (*simp add*: *starfun-exp-ln-iff* [*THEN iffD2*] *HFinite-HInfinite-iff*)
done

lemma *starfun-exp-HInfinite-Infinitesimal-disj*:
$x \in$ *HInfinite* ==> (*∗f∗ exp*) $x \in$ *HInfinite* | (*∗f∗ exp*) (x::*hypreal*) $\in$ *Infinitesimal*
apply (*insert linorder-linear* [*of x 0*])
apply (*auto intro*: *starfun-exp-HInfinite starfun-exp-Infinitesimal*)
done

lemma *starfun-ln-HFinite-not-Infinitesimal*:
[| $x \in$ *HFinite* $-$ *Infinitesimal*; $0 < x$ |] ==> (*∗f∗ ln*) $x \in$ *HFinite*
apply (*rule ccontr*, *drule HInfinite-HFinite-iff* [*THEN iffD2*])
apply (*drule starfun-exp-HInfinite-Infinitesimal-disj*)
apply (*simp add*: *starfun-exp-ln-iff* [*symmetric*] *HInfinite-HFinite-iff*
del: *starfun-exp-ln-iff*)
done

lemma *starfun-ln-Infinitesimal-HInfinite*:
[| $x \in$ *Infinitesimal*; $0 < x$ |] ==> (*∗f∗ ln*) $x \in$ *HInfinite*

apply (*drule Infinitesimal-inverse-HInfinite*)
apply (*frule positive-imp-inverse-positive*)
apply (*drule-tac* [*2*] *starfun-ln-HInfinite*)
apply (*auto simp add*: *starfun-ln-inverse HInfinite-minus-iff*)
done

lemma *starfun-ln-less-zero*: !!*x*. [| *0* < *x*; *x* < *1* |] ==> (**f* ln*) *x* < *0*
by *transfer* (*rule ln-less-zero*)

lemma *starfun-ln-Infinitesimal-less-zero*:
  [| *x* ∈ *Infinitesimal*; *0* < *x* |] ==> (**f* ln*) *x* < *0*
by (*auto intro*!: *starfun-ln-less-zero simp add*: *Infinitesimal-def*)

lemma *starfun-ln-HInfinite-gt-zero*:
  [| *x* ∈ *HInfinite*; *0* < *x* |] ==> *0* < (**f* ln*) *x*
by (*auto intro*!: *starfun-ln-gt-zero simp add*: *HInfinite-def*)

lemma *HFinite-sin* [*simp*]:
  sumhr (*0*, *whn*, %*n*. (*if even*(*n*) *then 0 else*
      (−*1* ˆ ((*n* − *1*) *div 2*))/(*real* (*fact n*))) ∗ *x* ˆ *n*)
      ∈ *HFinite*
unfolding *sumhr-app*
apply (*simp only*: *star-zero-def starfun2-star-of*)
apply (*rule NSBseqD2*)
apply (*rule NSconvergent-NSBseq*)
apply (*rule convergent-NSconvergent-iff* [*THEN iffD1*])
apply (*rule summable-convergent-sumr-iff* [*THEN iffD1*])
apply (*simp only*: *One-nat-def summable-sin*)
done

lemma *STAR-sin-zero* [*simp*]: (**f* sin*) *0* = *0*
by *transfer* (*rule sin-zero*)

lemma *STAR-sin-Infinitesimal* [*simp*]: *x* ∈ *Infinitesimal* ==> (**f* sin*) *x* @= *x*
apply (*case-tac x* = *0*)
apply (*cut-tac* [*2*] *x* = *0* **in** *DERIV-sin*)
apply (*auto simp add*: *NSDERIV-DERIV-iff* [*symmetric*] *nsderiv-def*)
apply (*drule bspec* [**where** *x* = *x*], *auto*)
apply (*drule approx-mult1* [**where** *c* = *x*])
apply (*auto intro*: *Infinitesimal-subset-HFinite* [*THEN subsetD*]
    simp add: *mult-assoc*)
done

lemma *HFinite-cos* [*simp*]:
  sumhr (*0*, *whn*, %*n*. (*if even*(*n*) *then*
    (−*1* ˆ (*n div 2*))/(*real* (*fact n*)) *else*

0) ∗ *x* ˆ *n*) ∈ *HFinite*
unfolding *sumhr-app*
apply (*simp only*: *star-zero-def starfun2-star-of*)
apply (*rule NSBseqD2*)
apply (*rule NSconvergent-NSBseq*)
apply (*rule convergent-NSconvergent-iff* [*THEN iffD1*])
apply (*rule summable-convergent-sumr-iff* [*THEN iffD1*])
apply (*rule summable-cos*)
done

lemma *STAR-cos-zero* [*simp*]: (∗*f*∗ *cos*) *0* = *1*
by *transfer* (*rule cos-zero*)

lemma *STAR-cos-Infinitesimal* [*simp*]: *x* ∈ *Infinitesimal* ==> (∗*f*∗ *cos*) *x* @= *1*
apply (*case-tac x* = *0*)
apply (*cut-tac* [*2*] *x* = *0* **in** *DERIV-cos*)
apply (*auto simp add*: *NSDERIV-DERIV-iff* [*symmetric*] *nsderiv-def*)
apply (*drule bspec* [**where** *x* = *x*])
apply *auto*
apply (*drule approx-mult1* [**where** *c* = *x*])
apply (*auto intro*: *Infinitesimal-subset-HFinite* [*THEN subsetD*]
simp add: *mult-assoc*)
apply (*rule approx-add-right-cancel* [**where** *d* = −*1*])
apply (*simp add*: *diff-def*)
done

lemma *STAR-tan-zero* [*simp*]: (∗*f*∗ *tan*) *0* = *0*
by *transfer* (*rule tan-zero*)

lemma *STAR-tan-Infinitesimal*: *x* ∈ *Infinitesimal* ==> (∗*f*∗ *tan*) *x* @= *x*
apply (*case-tac x* = *0*)
apply (*cut-tac* [*2*] *x* = *0* **in** *DERIV-tan*)
apply (*auto simp add*: *NSDERIV-DERIV-iff* [*symmetric*] *nsderiv-def*)
apply (*drule bspec* [**where** *x* = *x*], *auto*)
apply (*drule approx-mult1* [**where** *c* = *x*])
apply (*auto intro*: *Infinitesimal-subset-HFinite* [*THEN subsetD*]
simp add: *mult-assoc*)
done

lemma *STAR-sin-cos-Infinitesimal-mult*:
x ∈ *Infinitesimal* ==> (∗*f*∗ *sin*) *x* ∗ (∗*f*∗ *cos*) *x* @= *x*
apply (*insert approx-mult-HFinite* [*of* (∗*f*∗ *sin*) *x* - (∗*f*∗ *cos*) *x 1*])
apply (*simp add*: *Infinitesimal-subset-HFinite* [*THEN subsetD*])
done

lemma *HFinite-pi*: *hypreal-of-real pi* ∈ *HFinite*
by *simp*

lemma *lemma-split-hypreal-of-real*:
N ∈ *HNatInfinite*
==> *hypreal-of-real a* =
hypreal-of-hypnat N ∗ (*inverse*(*hypreal-of-hypnat N*) ∗ *hypreal-of-real a*)
by (*simp add*: *mult-assoc* [*symmetric*] *zero-less-HNatInfinite*)

lemma *STAR-sin-Infinitesimal-divide*:
[|*x* ∈ *Infinitesimal*; *x* ≠ *0* |] ==> (∗*f*∗ *sin*) *x*/*x* @= *1*
apply (*cut-tac x* = *0* **in** *DERIV-sin*)
apply (*simp add*: *NSDERIV-DERIV-iff* [*symmetric*] *nsderiv-def*)
done

lemma *lemma-sin-pi*:
n ∈ *HNatInfinite*
==> (∗*f*∗ *sin*) (*inverse* (*hypreal-of-hypnat n*))/(*inverse* (*hypreal-of-hypnat n*)) @= *1*
apply (*rule STAR-sin-Infinitesimal-divide*)
apply (*auto simp add*: *zero-less-HNatInfinite*)
done

lemma *STAR-sin-inverse-HNatInfinite*:
n ∈ *HNatInfinite*
==> (∗*f*∗ *sin*) (*inverse* (*hypreal-of-hypnat n*)) ∗ *hypreal-of-hypnat n* @= *1*
apply (*frule lemma-sin-pi*)
apply (*simp add*: *divide-inverse*)
done

lemma *Infinitesimal-pi-divide-HNatInfinite*:
N ∈ *HNatInfinite*
==> *hypreal-of-real pi*/(*hypreal-of-hypnat N*) ∈ *Infinitesimal*
apply (*simp add*: *divide-inverse*)
apply (*auto intro*: *Infinitesimal-HFinite-mult2*)
done

lemma *pi-divide-HNatInfinite-not-zero* [*simp*]:
N ∈ *HNatInfinite* ==> *hypreal-of-real pi*/(*hypreal-of-hypnat N*) ≠ *0*
by (*simp add*: *zero-less-HNatInfinite*)

lemma *STAR-sin-pi-divide-HNatInfinite-approx-pi*:
n ∈ *HNatInfinite*
==> (∗*f*∗ *sin*) (*hypreal-of-real pi*/(*hypreal-of-hypnat n*)) ∗ *hypreal-of-hypnat n*
@= *hypreal-of-real pi*
apply (*frule STAR-sin-Infinitesimal-divide*

```
                [OF Infinitesimal-pi-divide-HNatInfinite
                    pi-divide-HNatInfinite-not-zero])
apply (auto)
apply (rule approx-SReal-mult-cancel [of inverse (hypreal-of-real pi)])
apply (auto intro: Reals-inverse simp add: divide-inverse mult-ac)
done

lemma STAR-sin-pi-divide-HNatInfinite-approx-pi2:
     n ∈ HNatInfinite
      ==> hypreal-of-hypnat n *
          ( *f* sin) (hypreal-of-real pi/(hypreal-of-hypnat n))
          @= hypreal-of-real pi
apply (rule mult-commute [THEN subst])
apply (erule STAR-sin-pi-divide-HNatInfinite-approx-pi)
done

lemma starfunNat-pi-divide-n-Infinitesimal:
     N ∈ HNatInfinite ==> ( *f* (%x. pi / real x)) N ∈ Infinitesimal
by (auto intro!: Infinitesimal-HFinite-mult2
         simp add: starfun-mult [symmetric] divide-inverse
                   starfun-inverse [symmetric] starfunNat-real-of-nat)

lemma STAR-sin-pi-divide-n-approx:
     N ∈ HNatInfinite ==>
     ( *f* sin) (( *f* (%x. pi / real x)) N) @=
      hypreal-of-real pi/(hypreal-of-hypnat N)
apply (simp add: starfunNat-real-of-nat [symmetric])
apply (rule STAR-sin-Infinitesimal)
apply (simp add: divide-inverse)
apply (rule Infinitesimal-HFinite-mult2)
apply (subst starfun-inverse)
apply (erule starfunNat-inverse-real-of-nat-Infinitesimal)
apply simp
done

lemma NSLIMSEQ-sin-pi: (%n. real n * sin (pi / real n)) ----NS> pi
apply (auto simp add: NSLIMSEQ-def starfun-mult [symmetric] starfunNat-real-of-nat)
apply (rule-tac f1 = sin in starfun-o2 [THEN subst])
apply (auto simp add: starfun-mult [symmetric] starfunNat-real-of-nat divide-inverse)
apply (rule-tac f1 = inverse in starfun-o2 [THEN subst])
apply (auto dest: STAR-sin-pi-divide-HNatInfinite-approx-pi
            simp add: starfunNat-real-of-nat mult-commute divide-inverse)
done

lemma NSLIMSEQ-cos-one: (%n. cos (pi / real n))----NS> 1
apply (simp add: NSLIMSEQ-def, auto)
apply (rule-tac f1 = cos in starfun-o2 [THEN subst])
apply (rule STAR-cos-Infinitesimal)
apply (auto intro!: Infinitesimal-HFinite-mult2
```

simp add: *starfun-mult* [*symmetric*] *divide-inverse*
starfun-inverse [*symmetric*] *starfunNat-real-of-nat*)
done

lemma *NSLIMSEQ-sin-cos-pi*:
(%*n*. *real n* ∗ *sin* (*pi* / *real n*) ∗ *cos* (*pi* / *real n*)) −−−−*NS*> *pi*
by (*insert NSLIMSEQ-mult* [*OF NSLIMSEQ-sin-pi NSLIMSEQ-cos-one*], *simp*)

A familiar approximation to *cos x* when *x* is small

lemma *STAR-cos-Infinitesimal-approx*:
x ∈ *Infinitesimal* ==> (∗*f*∗ *cos*) *x* @= *1* − *x* ˆ *2*
apply (*rule STAR-cos-Infinitesimal* [*THEN approx-trans*])
apply (*auto simp add*: *Infinitesimal-approx-minus* [*symmetric*]
diff-minus add-assoc [*symmetric*] *numeral-2-eq-2*)
done

lemma *STAR-cos-Infinitesimal-approx2*:
x ∈ *Infinitesimal* ==> (∗*f*∗ *cos*) *x* @= *1* − (*x* ˆ *2*)/*2*
apply (*rule STAR-cos-Infinitesimal* [*THEN approx-trans*])
apply (*auto intro*: *Infinitesimal-SReal-divide*
simp add: *Infinitesimal-approx-minus* [*symmetric*] *numeral-2-eq-2*)
done

end

36 NSCA: Non-Standard Complex Analysis

theory *NSCA*
imports *NSComplex* ../*Hyperreal*/*HTranscendental*
begin

abbreviation

SComplex :: *hcomplex set* **where**
SComplex ≡ *Standard*

definition
stc :: *hcomplex* => *hcomplex* **where**
— standard part map
stc x = (*SOME r*. *x* ∈ *HFinite* & *r*:*SComplex* & *r* @= *x*)

36.1 Closure Laws for SComplex, the Standard Complex Numbers

lemma *SComplex-minus-iff* [*simp*]: (−*x* ∈ *SComplex*) = (*x* ∈ *SComplex*)
by (*auto*, *drule Standard-minus*, *auto*)

lemma *SComplex-add-cancel*:

$[| \ x + y \in SComplex; \ y \in SComplex \ |] ==> x \in SComplex$
by (*drule* (*1*) *Standard-diff*, *simp*)

lemma *SReal-hcmod-hcomplex-of-complex* [*simp*]:
hcmod (*hcomplex-of-complex r*) $\in$ *Reals*
by (*simp add*: *Reals-eq-Standard*)

lemma *SReal-hcmod-number-of* [*simp*]: *hcmod* (*number-of w* ::*hcomplex*) $\in$ *Reals*
by (*simp add*: *Reals-eq-Standard*)

lemma *SReal-hcmod-SComplex*: $x \in SComplex ==> hcmod\ x \in Reals$
by (*simp add*: *Reals-eq-Standard*)

lemma *SComplex-divide-number-of*:
$r \in SComplex ==> r/(number\text{-}of\ w::hcomplex) \in SComplex$
by *simp*

lemma *SComplex-UNIV-complex*:
$\{x.\ hcomplex\text{-}of\text{-}complex\ x \in SComplex\} = (UNIV::complex\ set)$
by *simp*

lemma *SComplex-iff*: $(x \in SComplex) = (\exists\, y.\ x = hcomplex\text{-}of\text{-}complex\ y)$
by (*simp add*: *Standard-def image-def*)

lemma *hcomplex-of-complex-image*:
hcomplex-of-complex ‘(*UNIV*::*complex set*) = *SComplex*
by (*simp add*: *Standard-def*)

lemma *inv-hcomplex-of-complex-image*: *inv hcomplex-of-complex* ‘*SComplex* = *UNIV*
apply (*auto simp add*: *Standard-def image-def*)
apply (*rule inj-hcomplex-of-complex* [*THEN inv-f-f*, *THEN subst*], *blast*)
done

lemma *SComplex-hcomplex-of-complex-image*:
$[| \ \exists\, x.\ x: P;\ P \le SComplex \ |] ==> \exists\, Q.\ P = hcomplex\text{-}of\text{-}complex$ ‘ Q
apply (*simp add*: *Standard-def*, *blast*)
done

lemma *SComplex-SReal-dense*:
$[| \ x \in SComplex;\ y \in SComplex;\ hcmod\ x < hcmod\ y$
$|] ==> \exists\, r \in Reals.\ hcmod\ x < r\ \&\ r < hcmod\ y$
apply (*auto intro*: *SReal-dense simp add*: *SReal-hcmod-SComplex*)
done

lemma *SComplex-hcmod-SReal*:
$z \in SComplex ==> hcmod\ z \in Reals$
by (*simp add*: *Reals-eq-Standard*)

36.2 The Finite Elements form a Subring

lemma *HFinite-hcmod-hcomplex-of-complex* [*simp*]:
hcmod (*hcomplex-of-complex r*) ∈ *HFinite*
by (*auto intro*!: *SReal-subset-HFinite* [*THEN subsetD*])

lemma *HFinite-hcmod-iff*: (*x* ∈ *HFinite*) = (*hcmod x* ∈ *HFinite*)
by (*simp add*: *HFinite-def*)

lemma *HFinite-bounded-hcmod*:
[|*x* ∈ *HFinite*; *y* ≤ *hcmod x*; *0* ≤ *y* |] ==> *y*: *HFinite*
by (*auto intro*: *HFinite-bounded simp add*: *HFinite-hcmod-iff*)

36.3 The Complex Infinitesimals form a Subring

lemma *hcomplex-sum-of-halves*: *x*/(*2*::*hcomplex*) + *x*/(*2*::*hcomplex*) = *x*
by *auto*

lemma *Infinitesimal-hcmod-iff*:
(*z* ∈ *Infinitesimal*) = (*hcmod z* ∈ *Infinitesimal*)
by (*simp add*: *Infinitesimal-def*)

lemma *HInfinite-hcmod-iff*: (*z* ∈ *HInfinite*) = (*hcmod z* ∈ *HInfinite*)
by (*simp add*: *HInfinite-def*)

lemma *HFinite-diff-Infinitesimal-hcmod*:
x ∈ *HFinite* − *Infinitesimal* ==> *hcmod x* ∈ *HFinite* − *Infinitesimal*
by (*simp add*: *HFinite-hcmod-iff Infinitesimal-hcmod-iff*)

lemma *hcmod-less-Infinitesimal*:
[| *e* ∈ *Infinitesimal*; *hcmod x* < *hcmod e* |] ==> *x* ∈ *Infinitesimal*
by (*auto elim*: *hrabs-less-Infinitesimal simp add*: *Infinitesimal-hcmod-iff*)

lemma *hcmod-le-Infinitesimal*:
[| *e* ∈ *Infinitesimal*; *hcmod x* ≤ *hcmod e* |] ==> *x* ∈ *Infinitesimal*
by (*auto elim*: *hrabs-le-Infinitesimal simp add*: *Infinitesimal-hcmod-iff*)

lemma *Infinitesimal-interval-hcmod*:
[| *e* ∈ *Infinitesimal*;
e' ∈ *Infinitesimal*;
hcmod e' < *hcmod x* ; *hcmod x* < *hcmod e*
|] ==> *x* ∈ *Infinitesimal*
by (*auto intro*: *Infinitesimal-interval simp add*: *Infinitesimal-hcmod-iff*)

lemma *Infinitesimal-interval2-hcmod*:
[| *e* ∈ *Infinitesimal*;
e' ∈ *Infinitesimal*;
hcmod e' ≤ *hcmod x* ; *hcmod x* ≤ *hcmod e*
|] ==> *x* ∈ *Infinitesimal*
by (*auto intro*: *Infinitesimal-interval2 simp add*: *Infinitesimal-hcmod-iff*)

36.4 The "Infinitely Close" Relation

lemma *approx-SComplex-mult-cancel-zero*:
[| $a \in$ *SComplex*; $a \neq 0$; $a*x$ @= *0* |] ==> x @= *0*
apply (*drule Standard-inverse* [*THEN Standard-subset-HFinite* [*THEN subsetD*]])
apply (*auto dest*: *approx-mult2 simp add*: *mult-assoc* [*symmetric*])
done

lemma *approx-mult-SComplex1*: [| $a \in$ *SComplex*; x @= *0* |] ==> $x*a$ @= *0*
by (*auto dest*: *Standard-subset-HFinite* [*THEN subsetD*] *approx-mult1*)

lemma *approx-mult-SComplex2*: [| $a \in$ *SComplex*; x @= *0* |] ==> $a*x$ @= *0*
by (*auto dest*: *Standard-subset-HFinite* [*THEN subsetD*] *approx-mult2*)

lemma *approx-mult-SComplex-zero-cancel-iff* [*simp*]:
[|$a \in$ *SComplex*; $a \neq 0$ |] ==> ($a*x$ @= *0*) = (x @= *0*)
by (*blast intro*: *approx-SComplex-mult-cancel-zero approx-mult-SComplex2*)

lemma *approx-SComplex-mult-cancel*:
[| $a \in$ *SComplex*; $a \neq 0$; $a*$ w @= $a*z$ |] ==> w @= z
apply (*drule Standard-inverse* [*THEN Standard-subset-HFinite* [*THEN subsetD*]])
apply (*auto dest*: *approx-mult2 simp add*: *mult-assoc* [*symmetric*])
done

lemma *approx-SComplex-mult-cancel-iff1* [*simp*]:
[| $a \in$ *SComplex*; $a \neq 0$|] ==> ($a*$ w @= $a*z$) = (w @= z)
by (*auto intro*!: *approx-mult2 Standard-subset-HFinite* [*THEN subsetD*]
intro: *approx-SComplex-mult-cancel*)

lemma *approx-hcmod-approx-zero*: (x @= y) = (*hcmod* ($y - x$) @= *0*)
apply (*subst hnorm-minus-commute*)
apply (*simp add*: *approx-def Infinitesimal-hcmod-iff diff-minus*)
done

lemma *approx-approx-zero-iff*: (x @= *0*) = (*hcmod* x @= *0*)
by (*simp add*: *approx-hcmod-approx-zero*)

lemma *approx-minus-zero-cancel-iff* [*simp*]: ($-x$ @= *0*) = (x @= *0*)
by (*simp add*: *approx-def*)

lemma *Infinitesimal-hcmod-add-diff*:
u @= *0* ==> *hcmod*($x + u$) $-$ *hcmod* $x \in$ *Infinitesimal*
apply (*drule approx-approx-zero-iff* [*THEN iffD1*])
apply (*rule-tac* e = *hcmod* u **and** e' = $-$ *hcmod* u **in** *Infinitesimal-interval2*)
apply (*auto simp add*: *mem-infmal-iff* [*symmetric*] *diff-def*)
apply (*rule-tac c1* = *hcmod* x **in** *add-le-cancel-left* [*THEN iffD1*])
apply (*auto simp add*: *diff-minus* [*symmetric*])
done

lemma *approx-hcmod-add-hcmod*: u @= 0 ==> $hcmod(x + u)$ @= $hcmod\ x$
apply (*rule approx-minus-iff* [*THEN iffD2*])
apply (*auto intro*: *Infinitesimal-hcmod-add-diff simp add*: *mem-infmal-iff* [*symmetric*] *diff-minus* [*symmetric*])
done

36.5 Zero is the Only Infinitesimal Complex Number

lemma *Infinitesimal-less-SComplex*:
[| $x \in SComplex$; $y \in Infinitesimal$; $0 < hcmod\ x$ |] ==> $hcmod\ y < hcmod\ x$
by (*auto intro*: *Infinitesimal-less-SReal SComplex-hcmod-SReal simp add*: *Infinitesimal-hcmod-iff*)

lemma *SComplex-Int-Infinitesimal-zero*: $SComplex\ Int\ Infinitesimal = \{0\}$
by (*auto simp add*: *Standard-def Infinitesimal-hcmod-iff*)

lemma *SComplex-Infinitesimal-zero*:
[| $x \in SComplex$; $x \in Infinitesimal$|] ==> $x = 0$
by (*cut-tac SComplex-Int-Infinitesimal-zero*, *blast*)

lemma *SComplex-HFinite-diff-Infinitesimal*:
[| $x \in SComplex$; $x \neq 0$ |] ==> $x \in HFinite - Infinitesimal$
by (*auto dest*: *SComplex-Infinitesimal-zero Standard-subset-HFinite* [*THEN subsetD*])

lemma *hcomplex-of-complex-HFinite-diff-Infinitesimal*:
hcomplex-of-complex $x \neq 0$
==> *hcomplex-of-complex* $x \in HFinite - Infinitesimal$
by (*rule SComplex-HFinite-diff-Infinitesimal*, *auto*)

lemma *number-of-not-Infinitesimal* [*simp*]:
number-of $w \neq (0$::*hcomplex*) ==> (*number-of* w::*hcomplex*) $\notin$ *Infinitesimal*
by (*fast dest*: *Standard-number-of* [*THEN SComplex-Infinitesimal-zero*])

lemma *approx-SComplex-not-zero*:
[| $y \in SComplex$; x @= y; $y \neq 0$ |] ==> $x \neq 0$
by (*auto dest*: *SComplex-Infinitesimal-zero approx-sym* [*THEN mem-infmal-iff* [*THEN iffD2*]])

lemma *SComplex-approx-iff*:
[|$x \in SComplex$; $y \in SComplex$|] ==> (x @= y) = ($x = y$)
by (*auto simp add*: *Standard-def*)

lemma *number-of-Infinitesimal-iff* [*simp*]:
((*number-of* w :: *hcomplex*) $\in$ *Infinitesimal*) =
(*number-of* w = (0::*hcomplex*))
apply (*rule iffI*)
apply (*fast dest*: *Standard-number-of* [*THEN SComplex-Infinitesimal-zero*])
apply (*simp* (*no-asm-simp*))

done

lemma *approx-unique-complex*:
  [| $r \in$ *SComplex*; $s \in$ *SComplex*; r @= x; s @= x|] ==> $r = s$
by (*blast intro*: *SComplex-approx-iff* [*THEN iffD1*] *approx-trans2*)

36.6 Properties of *hRe*, *hIm* and *HComplex*

lemma *abs-Re-le-cmod*: $|Re\ x| \leq cmod\ x$
by (*induct x*) *simp*

lemma *abs-Im-le-cmod*: $|Im\ x| \leq cmod\ x$
by (*induct x*) *simp*

lemma *abs-hRe-le-hcmod*: $\bigwedge x.\ |hRe\ x| \leq hcmod\ x$
by *transfer* (*rule abs-Re-le-cmod*)

lemma *abs-hIm-le-hcmod*: $\bigwedge x.\ |hIm\ x| \leq hcmod\ x$
by *transfer* (*rule abs-Im-le-cmod*)

lemma *Infinitesimal-hRe*: $x \in$ *Infinitesimal* $\Longrightarrow$ *hRe* $x \in$ *Infinitesimal*
apply (*rule InfinitesimalI2*, *simp*)
apply (*rule order-le-less-trans* [*OF abs-hRe-le-hcmod*])
apply (*erule* (*1*) *InfinitesimalD2*)
done

lemma *Infinitesimal-hIm*: $x \in$ *Infinitesimal* $\Longrightarrow$ *hIm* $x \in$ *Infinitesimal*
apply (*rule InfinitesimalI2*, *simp*)
apply (*rule order-le-less-trans* [*OF abs-hIm-le-hcmod*])
apply (*erule* (*1*) *InfinitesimalD2*)
done

lemma *real-sqrt-lessI*: $[\![0 < u;\ x < u^2]\!] \Longrightarrow sqrt\ x < u$

by (*frule real-sqrt-less-mono*) *simp*

lemma *hypreal-sqrt-lessI*:
  $\bigwedge x\ u.\ [\![0 < u;\ x < u^2]\!] \Longrightarrow (\ *f*\ sqrt)\ x < u$
by *transfer* (*rule real-sqrt-lessI*)

lemma *hypreal-sqrt-ge-zero*: $\bigwedge x.\ 0 \leq x \Longrightarrow 0 \leq (\ *f*\ sqrt)\ x$
by *transfer* (*rule real-sqrt-ge-zero*)

lemma *Infinitesimal-sqrt*:
  $[\![x \in$ *Infinitesimal*; $0 \leq x]\!] \Longrightarrow (\ *f*\ sqrt)\ x \in$ *Infinitesimal*
apply (*rule InfinitesimalI2*)
apply (*drule-tac* $r{=}r^2$ **in** *InfinitesimalD2*, *simp*)
apply (*simp add*: *hypreal-sqrt-ge-zero*)
apply (*rule hypreal-sqrt-lessI*, *simp-all*)

done

lemma *Infinitesimal-HComplex*:
$[\![x \in \mathit{Infinitesimal};\ y \in \mathit{Infinitesimal}]\!] \Longrightarrow \mathit{HComplex}\ x\ y \in \mathit{Infinitesimal}$
apply (*rule Infinitesimal-hcmod-iff* [*THEN iffD2*])
apply (*simp add*: *hcmod-i*)
apply (*rule Infinitesimal-sqrt*)
apply (*rule Infinitesimal-add*)
apply (*erule Infinitesimal-hrealpow*, *simp*)
apply (*erule Infinitesimal-hrealpow*, *simp*)
apply (*rule add-nonneg-nonneg*)
apply (*rule zero-le-power2*)
apply (*rule zero-le-power2*)
done

lemma *hcomplex-Infinitesimal-iff*:
$(x \in \mathit{Infinitesimal}) = (\mathit{hRe}\ x \in \mathit{Infinitesimal} \wedge \mathit{hIm}\ x \in \mathit{Infinitesimal})$
apply (*safe intro*!: *Infinitesimal-hRe Infinitesimal-hIm*)
apply (*drule* (*1*) *Infinitesimal-HComplex*, *simp*)
done

lemma *hRe-diff* [*simp*]: $\bigwedge x\ y.\ \mathit{hRe}\ (x - y) = \mathit{hRe}\ x - \mathit{hRe}\ y$
by *transfer* (*rule complex-Re-diff*)

lemma *hIm-diff* [*simp*]: $\bigwedge x\ y.\ \mathit{hIm}\ (x - y) = \mathit{hIm}\ x - \mathit{hIm}\ y$
by *transfer* (*rule complex-Im-diff*)

lemma *approx-hRe*: $x \approx y \Longrightarrow \mathit{hRe}\ x \approx \mathit{hRe}\ y$
unfolding *approx-def* **by** (*drule Infinitesimal-hRe*) *simp*

lemma *approx-hIm*: $x \approx y \Longrightarrow \mathit{hIm}\ x \approx \mathit{hIm}\ y$
unfolding *approx-def* **by** (*drule Infinitesimal-hIm*) *simp*

lemma *approx-HComplex*:
$[\![a \approx b;\ c \approx d]\!] \Longrightarrow \mathit{HComplex}\ a\ c \approx \mathit{HComplex}\ b\ d$
unfolding *approx-def* **by** (*simp add*: *Infinitesimal-HComplex*)

lemma *hcomplex-approx-iff*:
$(x \approx y) = (\mathit{hRe}\ x \approx \mathit{hRe}\ y \wedge \mathit{hIm}\ x \approx \mathit{hIm}\ y)$
unfolding *approx-def* **by** (*simp add*: *hcomplex-Infinitesimal-iff*)

lemma *HFinite-hRe*: $x \in \mathit{HFinite} \Longrightarrow \mathit{hRe}\ x \in \mathit{HFinite}$
apply (*auto simp add*: *HFinite-def SReal-def*)
apply (*rule-tac x=star-of r* **in** *exI*, *simp*)
apply (*erule order-le-less-trans* [*OF abs-hRe-le-hcmod*])
done

lemma *HFinite-hIm*: $x \in \mathit{HFinite} \Longrightarrow \mathit{hIm}\ x \in \mathit{HFinite}$
apply (*auto simp add*: *HFinite-def SReal-def*)

apply (*rule-tac x=star-of r* **in** *exI*, *simp*)
apply (*erule order-le-less-trans* [*OF abs-hIm-le-hcmod*])
done

lemma *HFinite-HComplex*:
$[\![x \in HFinite;\ y \in HFinite]\!] \Longrightarrow HComplex\ x\ y \in HFinite$
apply (*subgoal-tac HComplex x 0 + HComplex 0 y* ∈ *HFinite*, *simp*)
apply (*rule HFinite-add*)
apply (*simp add*: *HFinite-hcmod-iff hcmod-i*)
apply (*simp add*: *HFinite-hcmod-iff hcmod-i*)
done

lemma *hcomplex-HFinite-iff*:
($x \in HFinite$) = ($hRe\ x \in HFinite \wedge hIm\ x \in HFinite$)
apply (*safe intro*!: *HFinite-hRe HFinite-hIm*)
apply (*drule* (*1*) *HFinite-HComplex*, *simp*)
done

lemma *hcomplex-HInfinite-iff*:
($x \in HInfinite$) = ($hRe\ x \in HInfinite \vee hIm\ x \in HInfinite$)
by (*simp add*: *HInfinite-HFinite-iff hcomplex-HFinite-iff*)

lemma *hcomplex-of-hypreal-approx-iff* [*simp*]:
(*hcomplex-of-hypreal x* @= *hcomplex-of-hypreal z*) = (*x* @= *z*)
by (*simp add*: *hcomplex-approx-iff*)

lemma *Standard-HComplex*:
$[\![x \in Standard;\ y \in Standard]\!] \Longrightarrow HComplex\ x\ y \in Standard$
by (*simp add*: *HComplex-def*)

lemma *stc-part-Ex*: *x*:*HFinite* ==> ∃ *t* ∈ *SComplex*. *x* @= *t*
apply (*simp add*: *hcomplex-HFinite-iff hcomplex-approx-iff*)
apply (*rule-tac x=HComplex* (*st* (*hRe x*)) (*st* (*hIm x*)) **in** *bexI*)
apply (*simp add*: *st-approx-self* [*THEN approx-sym*])
apply (*simp add*: *Standard-HComplex st-SReal* [*unfolded Reals-eq-Standard*])
done

lemma *stc-part-Ex1*: *x*:*HFinite* ==> *EX*! *t*. *t* ∈ *SComplex* & *x* @= *t*
apply (*drule stc-part-Ex*, *safe*)
apply (*drule-tac* [*2*] *approx-sym*, *drule-tac* [*2*] *approx-sym*, *drule-tac* [*2*] *approx-sym*)
apply (*auto intro*!: *approx-unique-complex*)
done

lemmas *hcomplex-of-complex-approx-inverse* =
hcomplex-of-complex-HFinite-diff-Infinitesimal [*THEN* [*2*] *approx-inverse*]

36.7 Theorems About Monads

lemma *monad-zero-hcmod-iff*: $(x \in monad\ 0) = (hcmod\ x{:}monad\ 0)$
by (*simp add*: *Infinitesimal-monad-zero-iff* [*symmetric*] *Infinitesimal-hcmod-iff*)

36.8 Theorems About Standard Part

lemma *stc-approx-self*: $x \in HFinite ==> stc\ x\ @= x$
apply (*simp add*: *stc-def*)
apply (*frule stc-part-Ex*, *safe*)
apply (*rule someI2*)
apply (*auto intro*: *approx-sym*)
done

lemma *stc-SComplex*: $x \in HFinite ==> stc\ x \in SComplex$
apply (*simp add*: *stc-def*)
apply (*frule stc-part-Ex*, *safe*)
apply (*rule someI2*)
apply (*auto intro*: *approx-sym*)
done

lemma *stc-HFinite*: $x \in HFinite ==> stc\ x \in HFinite$
by (*erule stc-SComplex* [*THEN Standard-subset-HFinite* [*THEN subsetD*]])

lemma *stc-unique*: $[\![y \in SComplex;\ y \approx x]\!] \Longrightarrow stc\ x = y$
apply (*frule Standard-subset-HFinite* [*THEN subsetD*])
apply (*drule* (*1*) *approx-HFinite*)
apply (*unfold stc-def*)
apply (*rule some-equality*)
apply (*auto intro*: *approx-unique-complex*)
done

lemma *stc-SComplex-eq* [*simp*]: $x \in SComplex ==> stc\ x = x$
apply (*erule stc-unique*)
apply (*rule approx-refl*)
done

lemma *stc-hcomplex-of-complex*:
  $stc\ (hcomplex\text{-}of\text{-}complex\ x) = hcomplex\text{-}of\text{-}complex\ x$
by *auto*

lemma *stc-eq-approx*:
  $[|\ x \in HFinite;\ y \in HFinite;\ stc\ x = stc\ y\ |] ==> x\ @= y$
by (*auto dest*!: *stc-approx-self elim*!: *approx-trans3*)

lemma *approx-stc-eq*:
  $[|\ x \in HFinite;\ y \in HFinite;\ x\ @= y\ |] ==> stc\ x = stc\ y$
by (*blast intro*: *approx-trans approx-trans2 SComplex-approx-iff* [*THEN iffD1*]
    dest: *stc-approx-self stc-SComplex*)

lemma *stc-eq-approx-iff*:
$[| \ x \in HFinite; \ y \in HFinite|] ==> (x \ @= \ y) = (stc \ x = stc \ y)$
by (*blast intro*: *approx-stc-eq stc-eq-approx*)

lemma *stc-Infinitesimal-add-SComplex*:
$[| \ x \in SComplex; \ e \in Infinitesimal \ |] ==> stc(x + e) = x$
apply (*erule stc-unique*)
apply (*erule Infinitesimal-add-approx-self*)
done

lemma *stc-Infinitesimal-add-SComplex2*:
$[| \ x \in SComplex; \ e \in Infinitesimal \ |] ==> stc(e + x) = x$
apply (*erule stc-unique*)
apply (*erule Infinitesimal-add-approx-self2*)
done

lemma *HFinite-stc-Infinitesimal-add*:
$x \in HFinite ==> \exists \, e \in Infinitesimal. \ x = stc(x) + e$
by (*blast dest*!: *stc-approx-self* [*THEN approx-sym*] *bex-Infinitesimal-iff2* [*THEN iffD2*])

lemma *stc-add*:
$[| \ x \in HFinite; \ y \in HFinite \ |] ==> stc \ (x + y) = stc(x) + stc(y)$
by (*simp add*: *stc-unique stc-SComplex stc-approx-self approx-add*)

lemma *stc-number-of* [*simp*]: *stc* (*number-of w*) = *number-of w*
by (*rule Standard-number-of* [*THEN stc-SComplex-eq*])

lemma *stc-zero* [*simp*]: *stc 0* = *0*
by *simp*

lemma *stc-one* [*simp*]: *stc 1* = *1*
by *simp*

lemma *stc-minus*: $y \in HFinite ==> stc(-y) = -stc(y)$
by (*simp add*: *stc-unique stc-SComplex stc-approx-self approx-minus*)

lemma *stc-diff*:
$[| \ x \in HFinite; \ y \in HFinite \ |] ==> stc \ (x-y) = stc(x) - stc(y)$
by (*simp add*: *stc-unique stc-SComplex stc-approx-self approx-diff*)

lemma *stc-mult*:
$[| \ x \in HFinite; \ y \in HFinite \ |]$
$==> stc \ (x * y) = stc(x) * stc(y)$
by (*simp add*: *stc-unique stc-SComplex stc-approx-self approx-mult-HFinite*)

lemma *stc-Infinitesimal*: $x \in Infinitesimal ==> stc \ x = 0$
by (*simp add*: *stc-unique mem-infmal-iff*)

lemma *stc-not-Infinitesimal*: *stc*(*x*) ≠ *0* ==> *x* ∉ *Infinitesimal*
by (*fast intro*: *stc-Infinitesimal*)

lemma *stc-inverse*:
  [| *x* ∈ *HFinite*; *stc x* ≠ *0* |]
  ==> *stc*(*inverse x*) = *inverse* (*stc x*)
apply (*drule stc-not-Infinitesimal*)
apply (*simp add*: *stc-unique stc-SComplex stc-approx-self approx-inverse*)
done

lemma *stc-divide* [*simp*]:
  [| *x* ∈ *HFinite*; *y* ∈ *HFinite*; *stc y* ≠ *0* |]
  ==> *stc*(*x*/*y*) = (*stc x*) / (*stc y*)
by (*simp add*: *divide-inverse stc-mult stc-not-Infinitesimal HFinite-inverse stc-inverse*)

lemma *stc-idempotent* [*simp*]: *x* ∈ *HFinite* ==> *stc*(*stc*(*x*)) = *stc*(*x*)
by (*blast intro*: *stc-HFinite stc-approx-self approx-stc-eq*)

lemma *HFinite-HFinite-hcomplex-of-hypreal*:
  z ∈ *HFinite* ==> *hcomplex-of-hypreal z* ∈ *HFinite*
by (*simp add*: *hcomplex-HFinite-iff*)

lemma *SComplex-SReal-hcomplex-of-hypreal*:
  x ∈ *Reals* ==> *hcomplex-of-hypreal x* ∈ *SComplex*
apply (*rule Standard-of-hypreal*)
apply (*simp add*: *Reals-eq-Standard*)
done

lemma *stc-hcomplex-of-hypreal*:
 z ∈ *HFinite* ==> *stc*(*hcomplex-of-hypreal z*) = *hcomplex-of-hypreal* (*st z*)
apply (*rule stc-unique*)
apply (*rule SComplex-SReal-hcomplex-of-hypreal*)
apply (*erule st-SReal*)
apply (*simp add*: *hcomplex-of-hypreal-approx-iff st-approx-self*)
done

lemma *Infinitesimal-hcnj-iff* [*simp*]:
  (*hcnj z* ∈ *Infinitesimal*) = (*z* ∈ *Infinitesimal*)
by (*simp add*: *Infinitesimal-hcmod-iff*)

lemma *Infinitesimal-hcomplex-of-hypreal-epsilon* [*simp*]:
  hcomplex-of-hypreal epsilon ∈ *Infinitesimal*
by (*simp add*: *Infinitesimal-hcmod-iff*)

end

37 CStar: Star-transforms in NSA, Extending Sets of Complex Numbers and Complex Functions

theory *CStar*
imports *NSCA*
begin

37.1 Properties of the *-Transform Applied to Sets of Reals

lemma *STARC-hcomplex-of-complex-Int*:
  $*s*$ *X Int SComplex* = *hcomplex-of-complex* ' *X*
by (*auto simp add*: *Standard-def*)

lemma *lemma-not-hcomplexA*:
  $x \notin$ *hcomplex-of-complex* ' A ==> $\forall\, y \in A.\ x \neq$ *hcomplex-of-complex* y
by *auto*

37.2 Theorems about Nonstandard Extensions of Functions

lemma *starfunC-hcpow*: !!Z. ($*f*$ (%z. z ^ n)) Z = Z *pow hypnat-of-nat* n
by *transfer* (*rule refl*)

lemma *starfunCR-cmod*: $*f*$ *cmod* = *hcmod*
by *transfer* (*rule refl*)

37.3 Internal Functions - Some Redundancy With *f* Now

lemma *starfun-Re*: ($*f*$ (λx. *Re* ($f\,x$))) = (λx. *hRe* (($*f*$ f) x))
by *transfer* (*rule refl*)

lemma *starfun-Im*: ($*f*$ (λx. *Im* ($f\,x$))) = (λx. *hIm* (($*f*$ f) x))
by *transfer* (*rule refl*)

lemma *starfunC-eq-Re-Im-iff*:
  (($*f*$ f) x = z) = ((($*f*$ (%x. *Re*($f\,x$))) x = *hRe* (z)) &
    (($*f*$ (%x. *Im*($f\,x$))) x = *hIm* (z)))
by (*simp add*: *hcomplex-hRe-hIm-cancel-iff starfun-Re starfun-Im*)

lemma *starfunC-approx-Re-Im-iff*:
  (($*f*$ f) x @= z) = ((($*f*$ (%x. *Re*($f\,x$))) x @= *hRe* (z)) &
    (($*f*$ (%x. *Im*($f\,x$))) x @= *hIm* (z)))
by (*simp add*: *hcomplex-approx-iff starfun-Re starfun-Im*)

end

38 CLim: Limits, Continuity and Differentiation for Complex Functions

theory *CLim*
imports *CStar*
begin

declare *hypreal-epsilon-not-zero* [*simp*]

lemma *lemma-complex-mult-inverse-squared* [*simp*]:
$x \neq (0::complex) \Longrightarrow (x * inverse(x)\ \hat{}\ 2) = inverse\ x$
by (*simp add*: *numeral-2-eq-2*)

Changing the quantified variable. Install earlier?

lemma *all-shift*: $(\forall\, x::'a::comm\text{-}ring\text{-}1.\ P\ x) = (\forall\, x.\ P\ (x-a))$
apply *auto*
apply (*drule-tac* $x=x+a$ **in** *spec*)
apply (*simp add*: *diff-minus add-assoc*)
done

lemma *complex-add-minus-iff* [*simp*]: $(x + - a = (0::complex)) = (x=a)$
by (*simp add*: *diff-eq-eq diff-minus* [*symmetric*])

lemma *complex-add-eq-0-iff* [*iff*]: $(x+y = (0::complex)) = (y = -x)$
apply *auto*
apply (*drule sym* [*THEN diff-eq-eq* [*THEN iffD2*]], *auto*)
done

38.1 Limit of Complex to Complex Function

lemma *NSLIM-Re*: f −− a −−NS> L ==> (%x. $Re(f\,x)$) −− a −−NS> $Re(L)$
by (*simp add*: *NSLIM-def starfunC-approx-Re-Im-iff*
hRe-hcomplex-of-complex)

lemma *NSLIM-Im*: f −− a −−NS> L ==> (%x. $Im(f\,x)$) −− a −−NS> $Im(L)$
by (*simp add*: *NSLIM-def starfunC-approx-Re-Im-iff*
hIm-hcomplex-of-complex)

lemma *LIM-Re*: f −− a −−> L ==> (%x. $Re(f\,x)$) −− a −−> $Re(L)$
by (*simp add*: *LIM-NSLIM-iff NSLIM-Re*)

lemma *LIM-Im*: f −− a −−> L ==> (%x. $Im(f\,x)$) −− a −−> $Im(L)$
by (*simp add*: *LIM-NSLIM-iff NSLIM-Im*)

lemma *LIM-cnj*: f −− a −−> L ==> (%x. *cnj* $(f\,x)$) −− a −−> *cnj* L
by (*simp add*: *LIM-def complex-cnj-diff* [*symmetric*])

lemma *LIM-cnj-iff*: ((%x. cnj (f x)) -- a --> cnj L) = (f -- a --> L)
by (*simp add*: *LIM-def complex-cnj-diff* [*symmetric*])

lemma *starfun-norm*: (*f* (λx. norm (f x))) = (λx. hnorm ((*f* f) x))
by *transfer* (*rule refl*)

lemma *star-of-Re* [*simp*]: *star-of* (Re x) = hRe (star-of x)
by *transfer* (*rule refl*)

lemma *star-of-Im* [*simp*]: *star-of* (Im x) = hIm (star-of x)
by *transfer* (*rule refl*)

lemma *NSCLIM-NSCRLIM-iff*:
(f -- x --NS> L) = ((%y. cmod(f y - L)) -- x --NS> 0)
by (*simp add*: *NSLIM-def starfun-norm*
approx-approx-zero-iff [*symmetric*] *approx-minus-iff* [*symmetric*])

lemma *CLIM-CRLIM-iff*: (f -- x --> L) = ((%y. cmod(f y - L)) -- x -->
0)
by (*simp add*: *LIM-def*)

lemma *NSCLIM-NSCRLIM-iff2*:
(f -- x --NS> L) = ((%y. cmod(f y - L)) -- x --NS> 0)
by (*simp add*: *LIM-NSLIM-iff* [*symmetric*] *CLIM-CRLIM-iff*)

lemma *NSLIM-NSCRLIM-Re-Im-iff*:
(f -- a --NS> L) = ((%x. Re(f x)) -- a --NS> Re(L) &
(%x. Im(f x)) -- a --NS> Im(L))
apply (*auto intro*: *NSLIM-Re NSLIM-Im*)
apply (*auto simp add*: *NSLIM-def starfun-Re starfun-Im*)
apply (*auto dest*!: *spec*)
apply (*simp add*: *hcomplex-approx-iff*)
done

lemma *LIM-CRLIM-Re-Im-iff*:
(f -- a --> L) = ((%x. Re(f x)) -- a --> Re(L) &
(%x. Im(f x)) -- a --> Im(L))
by (*simp add*: *LIM-NSLIM-iff NSLIM-NSCRLIM-Re-Im-iff*)

38.2 Continuity

lemma *NSLIM-isContc-iff*:
(f -- a --NS> f a) = ((%h. f(a + h)) -- 0 --NS> f a)
by (*rule NSLIM-h-iff*)

38.3 Functions from Complex to Reals

lemma *isNSContCR-cmod* [*simp*]: *isNSCont cmod* (*a*)
by (*auto intro*: *approx-hnorm*
simp add: *starfunCR-cmod hcmod-hcomplex-of-complex* [*symmetric*]
isNSCont-def)

lemma *isContCR-cmod* [*simp*]: *isCont cmod* (*a*)
by (*simp add*: *isNSCont-isCont-iff* [*symmetric*])

lemma *isCont-Re*: *isCont f a ==> isCont* (%*x*. *Re* (*f x*)) *a*
by (*simp add*: *isCont-def LIM-Re*)

lemma *isCont-Im*: *isCont f a ==> isCont* (%*x*. *Im* (*f x*)) *a*
by (*simp add*: *isCont-def LIM-Im*)

38.4 Differentiation of Natural Number Powers

lemma *CDERIV-pow* [*simp*]:
DERIV (%*x*. *x* ^ *n*) *x* :> (*complex-of-real* (*real n*)) ∗ (*x* ^ (*n* − *Suc 0*))
apply (*induct-tac n*)
apply (*drule-tac* [*2*] *DERIV-ident* [*THEN DERIV-mult*])
apply (*auto simp add*: *left-distrib real-of-nat-Suc*)
apply (*case-tac n*)
apply (*auto simp add*: *mult-ac add-commute*)
done

Nonstandard version

lemma *NSCDERIV-pow*:
NSDERIV (%*x*. *x* ^ *n*) *x* :> *complex-of-real* (*real n*) ∗ (*x* ^ (*n* − *1*))
by (*simp add*: *NSDERIV-DERIV-iff*)

Can't relax the premise $x \neq (0::'a)$: it isn't continuous at zero

lemma *NSCDERIV-inverse*:
(*x*::*complex*) ≠ *0* ==> *NSDERIV* (%*x*. *inverse*(*x*)) *x* :> (− (*inverse x* ^ *2*))
unfolding *numeral-2-eq-2*
by (*rule NSDERIV-inverse*)

lemma *CDERIV-inverse*:
(*x*::*complex*) ≠ *0* ==> *DERIV* (%*x*. *inverse*(*x*)) *x* :> (−(*inverse x* ^ *2*))
unfolding *numeral-2-eq-2*
by (*rule DERIV-inverse*)

38.5 Derivative of Reciprocals (Function *inverse*)

lemma *CDERIV-inverse-fun*:
[| *DERIV f x* :> *d*; *f*(*x*) ≠ (*0*::*complex*) |]
==> *DERIV* (%*x*. *inverse*(*f x*)) *x* :> (− (*d* ∗ *inverse*(*f*(*x*) ^ *2*)))
unfolding *numeral-2-eq-2*
by (*rule DERIV-inverse-fun*)

lemma *NSCDERIV-inverse-fun*:
  [| *NSDERIV f x* :> *d*; $f(x) \neq (0{::}complex)$ |]
   ==> *NSDERIV* (%*x*. *inverse*(*f x*)) *x* :> (− (*d* ∗ *inverse*(*f*(*x*) ^ *2*)))
unfolding *numeral-2-eq-2*
by (*rule NSDERIV-inverse-fun*)

38.6 Derivative of Quotient

lemma *CDERIV-quotient*:
  [| *DERIV f x* :> *d*; *DERIV g x* :> *e*; $g(x) \neq (0{::}complex)$ |]
   ==> *DERIV* (%*y*. *f*(*y*) / (*g y*)) *x* :> (*d*∗*g*(*x*) − (*e*∗*f*(*x*))) / (*g*(*x*) ^ *2*)
unfolding *numeral-2-eq-2*
by (*rule DERIV-quotient*)

lemma *NSCDERIV-quotient*:
  [| *NSDERIV f x* :> *d*; *NSDERIV g x* :> *e*; $g(x) \neq (0{::}complex)$ |]
   ==> *NSDERIV* (%*y*. *f*(*y*) / (*g y*)) *x* :> (*d*∗*g*(*x*) − (*e*∗*f*(*x*))) / (*g*(*x*) ^ *2*)
unfolding *numeral-2-eq-2*
by (*rule NSDERIV-quotient*)

38.7 Caratheodory Formulation of Derivative at a Point: Standard Proof

lemma *CARAT-CDERIVD*:
  (∀ *z*. *f z* − *f x* = *g z* ∗ (*z* − *x*)) & *isNSCont g x* & *g x* = *l*
   ==> *NSDERIV f x* :> *l*
by *clarify* (*rule CARAT-DERIVD*)

end

39 Ln: Properties of ln

theory *Ln*
imports *Transcendental*
begin

lemma *exp-first-two-terms*: *exp x* = *1* + *x* + *suminf* (%*n*.
 inverse(*real* (*fact* (*n*+*2*))) ∗ (*x* ^ (*n*+*2*)))
proof −
 have *exp x* = *suminf* (%*n*. *inverse*(*real* (*fact n*)) ∗ (*x* ^ *n*))
  by (*simp add*: *exp-def*)
 also from *summable-exp* **have** ... = (*SUM n* : {*0*..<*2*}.
   inverse(*real* (*fact n*)) ∗ (*x* ^ *n*)) + *suminf* (%*n*.
   inverse(*real* (*fact* (*n*+*2*))) ∗ (*x* ^ (*n*+*2*))) (**is** - = *?a* + -)
  by (*rule suminf-split-initial-segment*)
 also have *?a* = *1* + *x*
  by (*simp add*: *numerals*)

```
  finally show ?thesis .
qed

lemma exp-tail-after-first-two-terms-summable:
  summable (%n. inverse(real (fact (n+2))) * (x ^ (n+2)))
proof -
  note summable-exp
  thus ?thesis
    by (frule summable-ignore-initial-segment)
qed

lemma aux1: assumes a: 0 <= x and b: x <= 1
    shows inverse (real (fact (n + 2))) * x ^ (n + 2) <= (x^2/2) * ((1/2)^n)
proof (induct n)
  show inverse (real (fact (0 + 2))) * x ^ (0 + 2) <=
      x ^ 2 / 2 * (1 / 2) ^ 0
    by (simp add: real-of-nat-Suc power2-eq-square)
next
  fix n
  assume c: inverse (real (fact (n + 2))) * x ^ (n + 2)
      <= x ^ 2 / 2 * (1 / 2) ^ n
  show inverse (real (fact (Suc n + 2))) * x ^ (Suc n + 2)
          <= x ^ 2 / 2 * (1 / 2) ^ Suc n
  proof -
    have inverse(real (fact (Suc n + 2))) <=
        (1 / 2) *inverse (real (fact (n+2)))
    proof -
      have Suc n + 2 = Suc (n + 2) by simp
      then have fact (Suc n + 2) = Suc (n + 2) * fact (n + 2)
        by simp
      then have real(fact (Suc n + 2)) = real(Suc (n + 2) * fact (n + 2))
        apply (rule subst)
        apply (rule refl)
        done
      also have ... = real(Suc (n + 2)) * real(fact (n + 2))
        by (rule real-of-nat-mult)
      finally have real (fact (Suc n + 2)) =
         real (Suc (n + 2)) * real (fact (n + 2)) .
      then have inverse(real (fact (Suc n + 2))) =
         inverse(real (Suc (n + 2))) * inverse(real (fact (n + 2)))
        apply (rule ssubst)
        apply (rule inverse-mult-distrib)
        done
      also have ... <= (1/2) * inverse(real (fact (n + 2)))
        apply (rule mult-right-mono)
        apply (subst inverse-eq-divide)
        apply simp
        apply (rule inv-real-of-nat-fact-ge-zero)
        done
```

```
      finally show ?thesis .
    qed
    moreover have x ^ (Suc n + 2) <= x ^ (n + 2)
      apply (simp add: mult-compare-simps)
      apply (simp add: prems)
      apply (subgoal-tac 0 <= x * (x * x^n))
      apply force
      apply (rule mult-nonneg-nonneg, rule a)+
      apply (rule zero-le-power, rule a)
      done
    ultimately have inverse (real (fact (Suc n + 2))) * x ^ (Suc n + 2) <=
        (1 / 2 * inverse (real (fact (n + 2)))) * x ^ (n + 2)
      apply (rule mult-mono)
      apply (rule mult-nonneg-nonneg)
      apply simp
      apply (subst inverse-nonnegative-iff-nonnegative)
      apply (rule real-of-nat-fact-ge-zero)
      apply (rule zero-le-power)
      apply (rule a)
      done
    also have ... = 1 / 2 * (inverse (real (fact (n + 2))) * x ^ (n + 2))
      by simp
    also have ... <= 1 / 2 * (x ^ 2 / 2 * (1 / 2) ^ n)
      apply (rule mult-left-mono)
      apply (rule prems)
      apply simp
      done
    also have ... = x ^ 2 / 2 * (1 / 2 * (1 / 2) ^ n)
      by auto
    also have (1::real) / 2 * (1 / 2) ^ n = (1 / 2) ^ (Suc n)
      by (rule realpow-Suc [THEN sym])
    finally show ?thesis .
  qed
qed

lemma aux2: (%n. (x::real) ^ 2 / 2 * (1 / 2) ^ n) sums x^2
proof -
  have (%n. (1 / 2::real)^n) sums (1 / (1 - (1/2)))
    apply (rule geometric-sums)
    by (simp add: abs-less-iff)
  also have (1::real) / (1 - 1/2) = 2
    by simp
  finally have (%n. (1 / 2::real)^n) sums 2 .
  then have (%n. x ^ 2 / 2 * (1 / 2) ^ n) sums (x^2 / 2 * 2)
    by (rule sums-mult)
  also have x^2 / 2 * 2 = x^2
    by simp
  finally show ?thesis .
qed
```

```
lemma exp-bound: 0 <= (x::real) ==> x <= 1 ==> exp x <= 1 + x + x^2
proof –
  assume a: 0 <= x
  assume b: x <= 1
  have c: exp x = 1 + x + suminf (%n. inverse(real (fact (n+2))) *
      (x ^ (n+2)))
    by (rule exp-first-two-terms)
  moreover have suminf (%n. inverse(real (fact (n+2))) * (x ^ (n+2))) <= x^2
  proof –
    have suminf (%n. inverse(real (fact (n+2))) * (x ^ (n+2))) <=
        suminf (%n. (x^2/2) * ((1/2)^n))
      apply (rule summable-le)
      apply (auto simp only: aux1 prems)
      apply (rule exp-tail-after-first-two-terms-summable)
      by (rule sums-summable, rule aux2)
    also have ... = x^2
      by (rule sums-unique [THEN sym], rule aux2)
    finally show ?thesis .
  qed
  ultimately show ?thesis
    by auto
qed

lemma aux4: 0 <= (x::real) ==> x <= 1 ==> exp (x – x^2) <= 1 + x
proof –
  assume a: 0 <= x and b: x <= 1
  have exp (x – x^2) = exp x / exp (x^2)
    by (rule exp-diff)
  also have ... <= (1 + x + x^2) / exp (x ^2)
    apply (rule divide-right-mono)
    apply (rule exp-bound)
    apply (rule a, rule b)
    apply simp
    done
  also have ... <= (1 + x + x^2) / (1 + x^2)
    apply (rule divide-left-mono)
    apply (auto simp add: exp-ge-add-one-self-aux)
    apply (rule add-nonneg-nonneg)
    apply (insert prems, auto)
    apply (rule mult-pos-pos)
    apply auto
    apply (rule add-pos-nonneg)
    apply auto
    done
  also from a have ... <= 1 + x
    by(simp add:field-simps zero-compare-simps)
  finally show ?thesis .
qed
```

```
lemma ln-one-plus-pos-lower-bound: 0 <= x ==> x <= 1 ==>
    x - x^2 <= ln (1 + x)
proof -
  assume a: 0 <= x and b: x <= 1
  then have exp (x - x^2) <= 1 + x
    by (rule aux4)
  also have ... = exp (ln (1 + x))
  proof -
    from a have 0 < 1 + x by auto
    thus ?thesis
      by (auto simp only: exp-ln-iff [THEN sym])
  qed
  finally have exp (x - x ^ 2) <= exp (ln (1 + x)) .
  thus ?thesis by (auto simp only: exp-le-cancel-iff)
qed

lemma ln-one-minus-pos-upper-bound: 0 <= x ==> x < 1 ==> ln (1 - x) <=
- x
proof -
  assume a: 0 <= (x::real) and b: x < 1
  have (1 - x) * (1 + x + x^2) = (1 - x^3)
    by (simp add: ring-simps power2-eq-square power3-eq-cube)
  also have ... <= 1
    by (auto intro: zero-le-power simp add: a)
  finally have (1 - x) * (1 + x + x ^ 2) <= 1 .
  moreover have 0 < 1 + x + x^2
    apply (rule add-pos-nonneg)
    apply (insert a, auto)
    done
  ultimately have 1 - x <= 1 / (1 + x + x^2)
    by (elim mult-imp-le-div-pos)
  also have ... <= 1 / exp x
    apply (rule divide-left-mono)
    apply (rule exp-bound, rule a)
    apply (insert prems, auto)
    apply (rule mult-pos-pos)
    apply (rule add-pos-nonneg)
    apply auto
    done
  also have ... = exp (-x)
    by (auto simp add: exp-minus real-divide-def)
  finally have 1 - x <= exp (- x) .
  also have 1 - x = exp (ln (1 - x))
  proof -
    have 0 < 1 - x
      by (insert b, auto)
    thus ?thesis
      by (auto simp only: exp-ln-iff [THEN sym])
```

```
  qed
  finally have exp (ln (1 - x)) <= exp (- x) .
  thus ?thesis by (auto simp only: exp-le-cancel-iff)
qed

lemma aux5: x < 1 ==> ln(1 - x) = - ln(1 + x / (1 - x))
proof -
  assume a: x < 1
  have ln(1 - x) = - ln(1 / (1 - x))
  proof -
    have ln(1 - x) = - (- ln (1 - x))
      by auto
    also have - ln(1 - x) = ln 1 - ln(1 - x)
      by simp
    also have ... = ln(1 / (1 - x))
      apply (rule ln-div [THEN sym])
      by (insert a, auto)
    finally show ?thesis .
  qed
  also have  1 / (1 - x) = 1 + x / (1 - x) using a by(simp add:field-simps)
  finally show ?thesis .
qed

lemma ln-one-minus-pos-lower-bound: 0 <= x ==> x <= (1 / 2) ==>
    - x - 2 * x^2 <= ln (1 - x)
proof -
  assume a: 0 <= x and b: x <= (1 / 2)
  from b have c: x < 1
    by auto
  then have ln (1 - x) = - ln (1 + x / (1 - x))
    by (rule aux5)
  also have - (x / (1 - x)) <= ...
  proof -
    have ln (1 + x / (1 - x)) <= x / (1 - x)
      apply (rule ln-add-one-self-le-self)
      apply (rule divide-nonneg-pos)
      by (insert a c, auto)
    thus ?thesis
      by auto
  qed
  also have - (x / (1 - x)) = -x / (1 - x)
    by auto
  finally have d: - x / (1 - x) <= ln (1 - x) .
  have 0 < 1 - x using prems by simp
  hence e: -x - 2 * x^2 <= - x / (1 - x)
    using mult-right-le-one-le[of x*x 2*x] prems
    by(simp add:field-simps power2-eq-square)
  from e d show - x - 2 * x^2 <= ln (1 - x)
    by (rule order-trans)
```

```
qed

lemma exp-ge-add-one-self [simp]: 1 + (x::real) <= exp x
  apply (case-tac 0 <= x)
  apply (erule exp-ge-add-one-self-aux)
  apply (case-tac x <= -1)
  apply (subgoal-tac 1 + x <= 0)
  apply (erule order-trans)
  apply simp
  apply simp
  apply (subgoal-tac 1 + x = exp(ln (1 + x)))
  apply (erule ssubst)
  apply (subst exp-le-cancel-iff)
  apply (subgoal-tac ln (1 - (- x)) <= - (- x))
  apply simp
  apply (rule ln-one-minus-pos-upper-bound)
  apply auto
done

lemma ln-add-one-self-le-self2: -1 < x ==> ln(1 + x) <= x
  apply (subgoal-tac x = ln (exp x))
  apply (erule ssubst)back
  apply (subst ln-le-cancel-iff)
  apply auto
done

lemma abs-ln-one-plus-x-minus-x-bound-nonneg:
    0 <= x ==> x <= 1 ==> abs(ln (1 + x) - x) <= x^2
proof -
  assume x: 0 <= x
  assume x <= 1
  from x have ln (1 + x) <= x
    by (rule ln-add-one-self-le-self)
  then have ln (1 + x) - x <= 0
    by simp
  then have abs(ln(1 + x) - x) = - (ln(1 + x) - x)
    by (rule abs-of-nonpos)
  also have ... = x - ln (1 + x)
    by simp
  also have ... <= x^2
  proof -
    from prems have x - x^2 <= ln (1 + x)
      by (intro ln-one-plus-pos-lower-bound)
    thus ?thesis
      by simp
  qed
  finally show ?thesis .
qed
```

```
lemma abs-ln-one-plus-x-minus-x-bound-nonpos:
    -(1 / 2) <= x ==> x <= 0 ==> abs(ln (1 + x) - x) <= 2 * x^2
proof -
  assume -(1 / 2) <= x
  assume x <= 0
  have abs(ln (1 + x) - x) = x - ln(1 - (-x))
    apply (subst abs-of-nonpos)
    apply simp
    apply (rule ln-add-one-self-le-self2)
    apply (insert prems, auto)
    done
  also have ... <= 2 * x^2
    apply (subgoal-tac - (-x) - 2 * (-x)^2 <= ln (1 - (-x)))
    apply (simp add: compare-rls)
    apply (rule ln-one-minus-pos-lower-bound)
    apply (insert prems, auto)
    done
  finally show ?thesis .
qed

lemma abs-ln-one-plus-x-minus-x-bound:
    abs x <= 1 / 2 ==> abs(ln (1 + x) - x) <= 2 * x^2
  apply (case-tac 0 <= x)
  apply (rule order-trans)
  apply (rule abs-ln-one-plus-x-minus-x-bound-nonneg)
  apply auto
  apply (rule abs-ln-one-plus-x-minus-x-bound-nonpos)
  apply auto
done

lemma DERIV-ln: 0 < x ==> DERIV ln x :> 1 / x
  apply (unfold deriv-def, unfold LIM-def, clarsimp)
  apply (rule exI)
  apply (rule conjI)
  prefer 2
  apply clarsimp
  apply (subgoal-tac (ln (x + xa) - ln x) / xa - (1 / x) =
      (ln (1 + xa / x) - xa / x) / xa)
  apply (erule ssubst)
  apply (subst abs-divide)
  apply (rule mult-imp-div-pos-less)
  apply force
  apply (rule order-le-less-trans)
  apply (rule abs-ln-one-plus-x-minus-x-bound)
  apply (subst abs-divide)
  apply (subst abs-of-pos, assumption)
  apply (erule mult-imp-div-pos-le)
  apply (subgoal-tac abs xa < min (x / 2) (r * x^2 / 2))
  apply force
```

```
  apply assumption
  apply (simp add: power2-eq-square mult-compare-simps)
  apply (rule mult-imp-div-pos-less)
  apply (rule mult-pos-pos, assumption, assumption)
  apply (subgoal-tac xa * xa = abs xa * abs xa)
  apply (erule ssubst)
  apply (subgoal-tac abs xa * (abs xa * 2) < abs xa * (r * (x * x)))
  apply (simp only: mult-ac)
  apply (rule mult-strict-left-mono)
  apply (erule conjE, assumption)
  apply force
  apply simp
  apply (subst ln-div [THEN sym])
  apply arith
  apply (auto simp add: ring-simps add-frac-eq frac-eq-eq
    add-divide-distrib power2-eq-square)
  apply (rule mult-pos-pos, assumption)+
  apply assumption
done

lemma ln-x-over-x-mono: exp 1 <= x ==> x <= y ==> (ln y / y) <= (ln x /
x)
proof -
  assume exp 1 <= x and x <= y
  have a: 0 < x and b: 0 < y
    apply (insert prems)
    apply (subgoal-tac 0 < exp (1::real))
    apply arith
    apply auto
    apply (subgoal-tac 0 < exp (1::real))
    apply arith
    apply auto
    done
  have x * ln y - x * ln x = x * (ln y - ln x)
    by (simp add: ring-simps)
  also have ... = x * ln(y / x)
    apply (subst ln-div)
    apply (rule b, rule a, rule refl)
    done
  also have y / x = (x + (y - x)) / x
    by simp
  also have ... = 1 + (y - x) / x using a prems by(simp add:field-simps)
  also have x * ln(1 + (y - x) / x) <= x * ((y - x) / x)
    apply (rule mult-left-mono)
    apply (rule ln-add-one-self-le-self)
    apply (rule divide-nonneg-pos)
    apply (insert prems a, simp-all)
    done
  also have ... = y - x using a by simp
```

```
  also have ... = (y − x) ∗ ln (exp 1) by simp
  also have ... <= (y − x) ∗ ln x
    apply (rule mult-left-mono)
    apply (subst ln-le-cancel-iff)
    apply force
    apply (rule a)
    apply (rule prems)
    apply (insert prems, simp)
    done
  also have ... = y ∗ ln x − x ∗ ln x
    by (rule left-diff-distrib)
  finally have x ∗ ln y <= y ∗ ln x
    by arith
  then have ln y <= (y ∗ ln x) / x using a by(simp add:field-simps)
  also have ... = y ∗ (ln x / x)  by simp
  finally show ?thesis using b by(simp add:field-simps)
qed

end
```

40 Poly: Univariate Real Polynomials

```
theory Poly
imports Deriv
begin
```

Application of polynomial as a real function.

```
consts poly :: real list => real => real
primrec
  poly-Nil:  poly [] x = 0
  poly-Cons: poly (h#t) x = h + x ∗ poly t x
```

40.1 Arithmetic Operations on Polynomials

addition

```
consts padd :: [real list, real list] => real list  (infixl +++ 65)
primrec
  padd-Nil:  [] +++ l2 = l2
  padd-Cons: (h#t) +++ l2 = (if l2 = [] then h#t
                             else (h + hd l2)#(t +++ tl l2))
```

Multiplication by a constant

```
consts cmult :: [real, real list] => real list  (infixl %∗ 70)
primrec
   cmult-Nil:  c %∗ [] = []
   cmult-Cons: c %∗ (h#t) = (c ∗ h)#(c %∗ t)
```

Multiplication by a polynomial

consts *pmult* :: [*real list*, *real list*] => *real list* (**infixl** *** *70*)
primrec
  pmult-Nil: [] *** *l2* = []
  pmult-Cons: (*h*#*t*) *** *l2* = (*if t* = [] *then h* %* *l2*
    else (*h* %* *l2*) +++ ((*0*) # (*t* *** *l2*)))

Repeated multiplication by a polynomial

consts *mulexp* :: [*nat*, *real list*, *real list*] => *real list*
primrec
  mulexp-zero: *mulexp 0 p q* = *q*
  mulexp-Suc: *mulexp* (*Suc n*) *p q* = *p* *** *mulexp n p q*

Exponential

consts *pexp* :: [*real list*, *nat*] => *real list* (**infixl** %ˆ *80*)
primrec
  pexp-0: *p* %ˆ *0* = [*1*]
  pexp-Suc: *p* %ˆ (*Suc n*) = *p* *** (*p* %ˆ *n*)

Quotient related value of dividing a polynomial by x + a

consts *pquot* :: [*real list*, *real*] => *real list*
primrec
  pquot-Nil: *pquot* [] *a*= []
  pquot-Cons: *pquot* (*h*#*t*) *a* = (*if t* = [] *then* [*h*]
    else (*inverse*(*a*) * (*h* − *hd*(*pquot t a*)))#(*pquot t a*))

Differentiation of polynomials (needs an auxiliary function).

consts *pderiv-aux* :: *nat* => *real list* => *real list*
primrec
  pderiv-aux-Nil: *pderiv-aux n* [] = []
  pderiv-aux-Cons: *pderiv-aux n* (*h*#*t*) =
    (*real n* * *h*)#(*pderiv-aux* (*Suc n*) *t*)

normalization of polynomials (remove extra 0 coeff)

consts *pnormalize* :: *real list* => *real list*
primrec
  pnormalize-Nil: *pnormalize* [] = []
  pnormalize-Cons: *pnormalize* (*h*#*p*) = (*if* ((*pnormalize p*) = [])
    then (*if* (*h* = *0*) *then* [] *else* [*h*])
    else (*h*#(*pnormalize p*)))

definition *pnormal p* = ((*pnormalize p* = *p*) ∧ *p* ≠ [])
definition *nonconstant p* = (*pnormal p* ∧ (∀ *x*. *p* ≠ [*x*]))

Other definitions

definition
  poly-minus :: *real list* => *real list* (−− - [*80*] *80*) **where**

$-- \ p = (- \ 1) \ \%* \ p$

definition
pderiv :: *real list* => *real list* **where**
pderiv p = (*if p* = [] *then* [] *else pderiv-aux 1* (*tl p*))

definition
divides :: [*real list*,*real list*] => *bool* (**infixl** *divides 70*) **where**
p1 divides p2 = ($\exists$ *q*. *poly p2* = *poly*(*p1* *** *q*))

definition
order :: *real* => *real list* => *nat* **where**
— order of a polynomial
order a p = (*SOME n*. ([$-a$, *1*] %^ *n*) *divides p* &
~ (([$-a$, *1*] %^ (*Suc n*)) *divides p*))

definition
degree :: *real list* => *nat* **where**
— degree of a polynomial
degree p = *length* (*pnormalize p*) − *1*

definition
rsquarefree :: *real list* => *bool* **where**
— squarefree polynomials — NB with respect to real roots only.
rsquarefree p = (*poly p* $\neq$ *poly* [] &
($\forall$ *a*. (*order a p* = *0*) | (*order a p* = *1*)))

lemma *padd-Nil2*: *p* +++ [] = *p*
by (*induct p*) *auto*
declare *padd-Nil2* [*simp*]

lemma *padd-Cons-Cons*: (*h1* # *p1*) +++ (*h2* # *p2*) = (*h1* + *h2*) # (*p1* +++ *p2*)
by *auto*

lemma *pminus-Nil*: −− [] = []
by (*simp add*: *poly-minus-def*)
declare *pminus-Nil* [*simp*]

lemma *pmult-singleton*: [*h1*] *** *p1* = *h1* %* *p1*
by *simp*

lemma *poly-ident-mult*: *1* %* *t* = *t*
by (*induct t*, *auto*)
declare *poly-ident-mult* [*simp*]

lemma *poly-simple-add-Cons*: [*a*] +++ ((*0*)#*t*) = (*a*#*t*)

by *simp*
declare *poly-simple-add-Cons* [*simp*]

Handy general properties

lemma *padd-commut*: b +++ $a = a$ +++ b
apply (*subgoal-tac* $\forall a$. b +++ $a = a$ +++ b)
apply (*induct-tac* [*2*] *b*, *auto*)
apply (*rule padd-Cons* [*THEN ssubst*])
apply (*case-tac aa*, *auto*)
done

lemma *padd-assoc* [*rule-format*]: $\forall b\ c$. (a +++ b) +++ $c = a$ +++ (b +++ c)
apply (*induct a*, *simp*, *clarify*)
apply (*case-tac b*, *simp-all*)
done

lemma *poly-cmult-distr* [*rule-format*]:
  $\forall q$. a %* (p +++ q) = (a %* p +++ a %* q)
apply (*induct p*, *simp*, *clarify*)
apply (*case-tac q*)
apply (*simp-all add*: *right-distrib*)
done

lemma *pmult-by-x*: [*0*, *1*] *** t = ((*0*)#t)
apply (*induct t*, *simp*)
apply (*auto simp add*: *poly-ident-mult padd-commut*)
done
declare *pmult-by-x* [*simp*]

properties of evaluation of polynomials.

lemma *poly-add*: *poly* (*p1* +++ *p2*) x = *poly p1 x* + *poly p2 x*
apply (*subgoal-tac* $\forall$*p2*. *poly* (*p1* +++ *p2*) x = *poly* (*p1*) x + *poly* (*p2*) x)
apply (*induct-tac* [*2*] *p1*, *auto*)
apply (*case-tac p2*)
apply (*auto simp add*: *right-distrib*)
done

lemma *poly-cmult*: *poly* (c %* p) $x = c$ * *poly p x*
apply (*induct p*)
apply (*case-tac* [*2*] *x=0*)
apply (*auto simp add*: *right-distrib mult-ac*)
done

lemma *poly-minus*: *poly* (−− p) x = − (*poly p x*)
apply (*simp add*: *poly-minus-def*)
apply (*auto simp add*: *poly-cmult*)
done

lemma *poly-mult*: *poly* (*p1* *** *p2*) x = *poly p1 x* * *poly p2 x*

apply (*subgoal-tac* $\forall$ *p2. poly* (*p1* $***$ *p2*) *x* = *poly p1 x* $*$ *poly p2 x*)
apply (*simp* (*no-asm-simp*))
apply (*induct p1*)
apply (*auto simp add*: *poly-cmult*)
apply (*case-tac p1*)
apply (*auto simp add*: *poly-cmult poly-add left-distrib right-distrib mult-ac*)
done

lemma *poly-exp*: *poly* (*p* %^ *n*) *x* = (*poly p x*) ^ *n*
apply (*induct n*)
apply (*auto simp add*: *poly-cmult poly-mult*)
done

More Polynomial Evaluation Lemmas

lemma *poly-add-rzero*: *poly* (*a* +++ []) *x* = *poly a x*
by *simp*
declare *poly-add-rzero* [*simp*]

lemma *poly-mult-assoc*: *poly* ((*a* $***$ *b*) $***$ *c*) *x* = *poly* (*a* $***$ (*b* $***$ *c*)) *x*
by (*simp add*: *poly-mult real-mult-assoc*)

lemma *poly-mult-Nil2*: *poly* (*p* $***$ []) *x* = *0*
by (*induct p*, *auto*)
declare *poly-mult-Nil2* [*simp*]

lemma *poly-exp-add*: *poly* (*p* %^ (*n* + *d*)) *x* = *poly*(*p* %^ *n* $***$ *p* %^ *d*) *x*
apply (*induct n*)
apply (*auto simp add*: *poly-mult real-mult-assoc*)
done

The derivative

lemma *pderiv-Nil*: *pderiv* [] = []

apply (*simp add*: *pderiv-def*)
done
declare *pderiv-Nil* [*simp*]

lemma *pderiv-singleton*: *pderiv* [*c*] = []
by (*simp add*: *pderiv-def*)
declare *pderiv-singleton* [*simp*]

lemma *pderiv-Cons*: *pderiv* (*h*#*t*) = *pderiv-aux 1 t*
by (*simp add*: *pderiv-def*)

lemma *DERIV-cmult2*: *DERIV f x* :> *D* ==> *DERIV* (%*x*. (*f x*) $*$ *c* :: *real*) *x* :> *D* $*$ *c*
by (*simp add*: *DERIV-cmult mult-commute* [*of - c*])

lemma *DERIV-pow2*: *DERIV* (%*x*. *x* ^ *Suc n*) *x* :> *real* (*Suc n*) $*$ (*x* ^ *n*)

by (*rule lemma-DERIV-subst*, *rule DERIV-pow*, *simp*)
declare *DERIV-pow2* [*simp*] *DERIV-pow* [*simp*]

lemma *lemma-DERIV-poly1*: ∀ *n*. *DERIV* (%*x*. (*x* ^ (*Suc n*) ∗ *poly p x*)) *x* :>
x ^ *n* ∗ *poly* (*pderiv-aux* (*Suc n*) *p*) *x*
apply (*induct p*)
apply (*auto intro*!: *DERIV-add DERIV-cmult2*
simp add: *pderiv-def right-distrib real-mult-assoc* [*symmetric*]
simp del: *realpow-Suc*)
apply (*subst mult-commute*)
apply (*simp del*: *realpow-Suc*)
apply (*simp add*: *mult-commute realpow-Suc* [*symmetric*] *del*: *realpow-Suc*)
done

lemma *lemma-DERIV-poly*: *DERIV* (%*x*. (*x* ^ (*Suc n*) ∗ *poly p x*)) *x* :>
x ^ *n* ∗ *poly* (*pderiv-aux* (*Suc n*) *p*) *x*
by (*simp add*: *lemma-DERIV-poly1 del*: *realpow-Suc*)

lemma *DERIV-add-const*: *DERIV f x* :> *D* ==> *DERIV* (%*x*. *a* + *f x* :: *real*) *x* :> *D*
by (*rule lemma-DERIV-subst*, *rule DERIV-add*, *auto*)

lemma *poly-DERIV*: *DERIV* (%*x*. *poly p x*) *x* :> *poly* (*pderiv p*) *x*
apply (*induct p*)
apply (*auto simp add*: *pderiv-Cons*)
apply (*rule DERIV-add-const*)
apply (*rule lemma-DERIV-subst*)
apply (*rule lemma-DERIV-poly* [**where** *n=0*, *simplified*], *simp*)
done
declare *poly-DERIV* [*simp*]

Consequences of the derivative theorem above

lemma *poly-differentiable*: (%*x*. *poly p x*) *differentiable x*

apply (*simp add*: *differentiable-def*)
apply (*blast intro*: *poly-DERIV*)
done
declare *poly-differentiable* [*simp*]

lemma *poly-isCont*: *isCont* (%*x*. *poly p x*) *x*
by (*rule poly-DERIV* [*THEN DERIV-isCont*])
declare *poly-isCont* [*simp*]

lemma *poly-IVT-pos*: [| *a* < *b*; *poly p a* < *0*; *0* < *poly p b* |]
==> ∃ *x*. *a* < *x* & *x* < *b* & (*poly p x* = *0*)
apply (*cut-tac f* = %*x*. *poly p x* **and** *a* = *a* **and** *b* = *b* **and** *y* = *0* **in** *IVT-objl*)
apply (*auto simp add*: *order-le-less*)
done

lemma *poly-IVT-neg*: [| $a < b$; $0 < poly\ p\ a$; $poly\ p\ b < 0$ |]
==> $\exists x.\ a < x\ \&\ x < b\ \&\ (poly\ p\ x = 0)$
apply (*insert poly-IVT-pos* [**where** $p = --\ p$])
apply (*simp add*: *poly-minus neg-less-0-iff-less*)
done

lemma *poly-MVT*: $a < b$ ==>
$\exists x.\ a < x\ \&\ x < b\ \&\ (poly\ p\ b - poly\ p\ a = (b - a) * poly\ (pderiv\ p)\ x)$
apply (*drule-tac* $f = poly\ p$ **in** *MVT*, *auto*)
apply (*rule-tac* $x = z$ **in** *exI*)
apply (*auto simp add*: *real-mult-left-cancel poly-DERIV* [*THEN DERIV-unique*])
done

Lemmas for Derivatives

lemma *lemma-poly-pderiv-aux-add*: $\forall p2\ n.\ poly\ (pderiv\text{-}aux\ n\ (p1\ {+}{+}{+}\ p2))\ x =$
$poly\ (pderiv\text{-}aux\ n\ p1\ {+}{+}{+}\ pderiv\text{-}aux\ n\ p2)\ x$
apply (*induct p1*, *simp*, *clarify*)
apply (*case-tac p2*)
apply (*auto simp add*: *right-distrib*)
done

lemma *poly-pderiv-aux-add*: $poly\ (pderiv\text{-}aux\ n\ (p1\ {+}{+}{+}\ p2))\ x =$
$poly\ (pderiv\text{-}aux\ n\ p1\ {+}{+}{+}\ pderiv\text{-}aux\ n\ p2)\ x$
apply (*simp add*: *lemma-poly-pderiv-aux-add*)
done

lemma *lemma-poly-pderiv-aux-cmult*: $\forall n.\ poly\ (pderiv\text{-}aux\ n\ (c\ \%{*}\ p))\ x = poly\ (c\ \%{*}\ pderiv\text{-}aux\ n\ p)\ x$
apply (*induct p*)
apply (*auto simp add*: *poly-cmult mult-ac*)
done

lemma *poly-pderiv-aux-cmult*: $poly\ (pderiv\text{-}aux\ n\ (c\ \%{*}\ p))\ x = poly\ (c\ \%{*}\ pderiv\text{-}aux\ n\ p)\ x$
by (*simp add*: *lemma-poly-pderiv-aux-cmult*)

lemma *poly-pderiv-aux-minus*:
$poly\ (pderiv\text{-}aux\ n\ (--\ p))\ x = poly\ (--\ pderiv\text{-}aux\ n\ p)\ x$
apply (*simp add*: *poly-minus-def poly-pderiv-aux-cmult*)
done

lemma *lemma-poly-pderiv-aux-mult1*: $\forall n.\ poly\ (pderiv\text{-}aux\ (Suc\ n)\ p)\ x = poly\ ((pderiv\text{-}aux\ n\ p)\ {+}{+}{+}\ p)\ x$
apply (*induct p*)
apply (*auto simp add*: *real-of-nat-Suc left-distrib*)
done

lemma *lemma-poly-pderiv-aux-mult*: $poly\ (pderiv\text{-}aux\ (Suc\ n)\ p)\ x = poly\ ((pderiv\text{-}aux\ n\ p)\ {+}{+}{+}\ p)\ x$

by (*simp add*: *lemma-poly-pderiv-aux-mult1*)

lemma *lemma-poly-pderiv-add*: $\forall q$. *poly* (*pderiv* (*p* +++ *q*)) *x* = *poly* (*pderiv p* +++ *pderiv q*) *x*
apply (*induct p*, *simp*, *clarify*)
apply (*case-tac q*)
apply (*auto simp add*: *poly-pderiv-aux-add poly-add pderiv-def*)
done

lemma *poly-pderiv-add*: *poly* (*pderiv* (*p* +++ *q*)) *x* = *poly* (*pderiv p* +++ *pderiv q*) *x*
by (*simp add*: *lemma-poly-pderiv-add*)

lemma *poly-pderiv-cmult*: *poly* (*pderiv* (*c* %* *p*)) *x* = *poly* (*c* %* (*pderiv p*)) *x*
apply (*induct p*)
apply (*auto simp add*: *poly-pderiv-aux-cmult poly-cmult pderiv-def*)
done

lemma *poly-pderiv-minus*: *poly* (*pderiv* (−−*p*)) *x* = *poly* (−−(*pderiv p*)) *x*
by (*simp add*: *poly-minus-def poly-pderiv-cmult*)

lemma *lemma-poly-mult-pderiv*:
poly (*pderiv* (*h*#*t*)) *x* = *poly* ((*0* # (*pderiv t*)) +++ *t*) *x*
apply (*simp add*: *pderiv-def*)
apply (*induct t*)
apply (*auto simp add*: *poly-add lemma-poly-pderiv-aux-mult*)
done

lemma *poly-pderiv-mult*: $\forall q$. *poly* (*pderiv* (*p* *** *q*)) *x* =
poly (*p* *** (*pderiv q*) +++ *q* *** (*pderiv p*)) *x*
apply (*induct p*)
apply (*auto simp add*: *poly-add poly-cmult poly-pderiv-cmult poly-pderiv-add poly-mult*)
apply (*rule lemma-poly-mult-pderiv* [*THEN ssubst*])
apply (*rule lemma-poly-mult-pderiv* [*THEN ssubst*])
apply (*rule poly-add* [*THEN ssubst*])
apply (*rule poly-add* [*THEN ssubst*])
apply (*simp* (*no-asm-simp*) *add*: *poly-mult right-distrib add-ac mult-ac*)
done

lemma *poly-pderiv-exp*: *poly* (*pderiv* (*p* %^ (*Suc n*))) *x* =
poly ((*real* (*Suc n*)) %* (*p* %^ *n*) *** *pderiv p*) *x*
apply (*induct n*)
apply (*auto simp add*: *poly-add poly-pderiv-cmult poly-cmult poly-pderiv-mult*
real-of-nat-zero poly-mult real-of-nat-Suc
right-distrib left-distrib mult-ac)
done

lemma *poly-pderiv-exp-prime*: *poly* (*pderiv* ([−*a*, *1*] %^ (*Suc n*))) *x* =
poly (*real* (*Suc n*) %* ([−*a*, *1*] %^ *n*)) *x*

apply (*simp add*: *poly-pderiv-exp poly-mult del*: *pexp-Suc*)
apply (*simp add*: *poly-cmult pderiv-def*)
done

40.2 Key Property: if $f\ a = (0::'a)$ then $x - a$ divides $p\ x$

lemma *lemma-poly-linear-rem*: $\forall h.\ \exists q\ r.\ h\#t = [r] +\!\!+\!\!+ [-a,\ 1] *** q$
apply (*induct t*, *safe*)
apply (*rule-tac* $x = []$ **in** *exI*)
apply (*rule-tac* $x = h$ **in** *exI*, *simp*)
apply (*drule-tac* $x = aa$ **in** *spec*, *safe*)
apply (*rule-tac* $x = r\#q$ **in** *exI*)
apply (*rule-tac* $x = a*r + h$ **in** *exI*)
apply (*case-tac q*, *auto*)
done

lemma *poly-linear-rem*: $\exists q\ r.\ h\#t = [r] +\!\!+\!\!+ [-a,\ 1] *** q$
by (*cut-tac* $t = t$ **and** $a = a$ **in** *lemma-poly-linear-rem*, *auto*)

lemma *poly-linear-divides*: $(poly\ p\ a = 0) = ((p = []) \mid (\exists q.\ p = [-a,\ 1] *** q))$
apply (*auto simp add*: *poly-add poly-cmult right-distrib*)
apply (*case-tac p*, *simp*)
apply (*cut-tac* $h = aa$ **and** $t = list$ **and** $a = a$ **in** *poly-linear-rem*, *safe*)
apply (*case-tac q*, *auto*)
apply (*drule-tac* $x = []$ **in** *spec*, *simp*)
apply (*auto simp add*: *poly-add poly-cmult add-assoc*)
apply (*drule-tac* $x = aa\#lista$ **in** *spec*, *auto*)
done

lemma *lemma-poly-length-mult*: $\forall h\ k\ a.\ length\ (k\ \%*\ p +\!\!+\!\!+\ (h\ \#\ (a\ \%*\ p))) = Suc\ (length\ p)$
by (*induct p*, *auto*)
declare *lemma-poly-length-mult* [*simp*]

lemma *lemma-poly-length-mult2*: $\forall h\ k.\ length\ (k\ \%*\ p +\!\!+\!\!+\ (h\ \#\ p)) = Suc\ (length\ p)$
by (*induct p*, *auto*)
declare *lemma-poly-length-mult2* [*simp*]

lemma *poly-length-mult*: $length([-a,1] *** q) = Suc\ (length\ q)$
by *auto*
declare *poly-length-mult* [*simp*]

40.3 Polynomial length

lemma *poly-cmult-length*: $length\ (a\ \%*\ p) = length\ p$
by (*induct p*, *auto*)
declare *poly-cmult-length* [*simp*]

lemma *poly-add-length* [*rule-format*]:
$\forall$ *p2*. *length* (*p1* +++ *p2*) =
(*if* (*length p1* < *length p2*) *then length p2 else length p1*)
apply (*induct p1*, *simp-all*)
apply *arith*
done

lemma *poly-root-mult-length*: *length*([*a*,*b*] $***$ *p*) = *Suc* (*length p*)
by (*simp add*: *poly-cmult-length poly-add-length*)
declare *poly-root-mult-length* [*simp*]

lemma *poly-mult-not-eq-poly-Nil*: (*poly* (*p* $***$ *q*) *x* $\neq$ *poly* [] *x*) =
(*poly p x* $\neq$ *poly* [] *x* & *poly q x* $\neq$ *poly* [] *x*)
apply (*auto simp add*: *poly-mult*)
done
declare *poly-mult-not-eq-poly-Nil* [*simp*]

lemma *poly-mult-eq-zero-disj*: (*poly* (*p* $***$ *q*) *x* = *0*) = (*poly p x* = *0* | *poly q x* = *0*)
by (*auto simp add*: *poly-mult*)

Normalisation Properties

lemma *poly-normalized-nil*: (*pnormalize p* = []) −−> (*poly p x* = *0*)
by (*induct p*, *auto*)

A nontrivial polynomial of degree n has no more than n roots

lemma *poly-roots-index-lemma* [*rule-format*]:
$\forall$ *p x*. *poly p x* $\neq$ *poly* [] *x* & *length p* = *n*
−−> ($\exists$ *i*. $\forall$ *x*. (*poly p x* = (*0*::*real*)) −−> ($\exists$ *m*. (*m* $\leq$ *n* & *x* = *i m*)))
apply (*induct n*, *safe*)
apply (*rule ccontr*)
apply (*subgoal-tac* $\exists$ *a*. *poly p a* = *0*, *safe*)
apply (*drule poly-linear-divides* [*THEN iffD1*], *safe*)
apply (*drule-tac x* = *q* **in** *spec*)
apply (*drule-tac x* = *x* **in** *spec*)
apply (*simp del*: *poly-Nil pmult-Cons*)
apply (*erule exE*)
apply (*drule-tac x* = %*m*. *if m* = *Suc n then a else i m* **in** *spec*, *safe*)
apply (*drule poly-mult-eq-zero-disj* [*THEN iffD1*], *safe*)
apply (*drule-tac x* = *Suc* (*length q*) **in** *spec*)
apply *simp*
apply (*drule-tac x* = *xa* **in** *spec*, *safe*)
apply (*drule-tac x* = *m* **in** *spec*, *simp*, *blast*)
done
lemmas *poly-roots-index-lemma2* = *conjI* [*THEN poly-roots-index-lemma*, *standard*]

lemma *poly-roots-index-length*: *poly p x* $\neq$ *poly* [] *x* ==>
$\exists$ *i*. $\forall$ *x*. (*poly p x* = *0*) −−> ($\exists$ *n*. *n* $\leq$ *length p* & *x* = *i n*)

by (*blast intro*: *poly-roots-index-lemma2*)

lemma *poly-roots-finite-lemma*: *poly p x* $\neq$ *poly* [] *x* ==>
  $\exists$ *N i*. $\forall$ *x*. (*poly p x* = *0*) --> ($\exists$ *n*. (*n*::*nat*) < *N* & *x* = *i n*)
apply (*drule poly-roots-index-length*, *safe*)
apply (*rule-tac x* = *Suc* (*length p*) **in** *exI*)
apply (*rule-tac x* = *i* **in** *exI*)
apply (*simp add*: *less-Suc-eq-le*)
done

lemma *real-finite-lemma* [*rule-format* (*no-asm*)]:
  $\forall$ *P*. ($\forall$ *x*. *P x* --> ($\exists$ *n*. *n* < *N* & *x* = (*j*::*nat*=>*real*) *n*))
  --> ($\exists$ *a*. $\forall$ *x*. *P x* --> *x* < *a*)
apply (*induct N*, *simp*, *safe*)
apply (*drule-tac x* = %*z*. *P z* & (*z* $\neq$ *j N*) **in** *spec*)
apply (*auto simp add*: *less-Suc-eq*)
apply (*rename-tac N P a*)
apply (*rule-tac x* = *abs a* + *abs* (*j N*) + *1* **in** *exI*)
apply *safe*
apply (*drule-tac x* = *x* **in** *spec*, *safe*)
apply (*drule-tac x* = *j n* **in** *spec*)
apply *arith*
apply *arith*
done

lemma *poly-roots-finite*: (*poly p* $\neq$ *poly* []) =
  ($\exists$ *N j*. $\forall$ *x*. *poly p x* = *0* --> ($\exists$ *n*. (*n*::*nat*) < *N* & *x* = *j n*))
apply *safe*
apply (*erule contrapos-np*, *rule ext*)
apply (*rule ccontr*)
apply (*clarify dest*!: *poly-roots-finite-lemma*)
apply (*clarify dest*!: *real-finite-lemma*)
apply (*drule-tac x* = *a* **in** *fun-cong*, *auto*)
done

Entirety and Cancellation for polynomials

lemma *poly-entire-lemma*: [| *poly p* $\neq$ *poly* [] ; *poly q* $\neq$ *poly* [] |]
  ==> *poly* (*p* *** *q*) $\neq$ *poly* []
apply (*auto simp add*: *poly-roots-finite*)
apply (*rule-tac x* = *N* + *Na* **in** *exI*)
apply (*rule-tac x* = %*n*. *if n* < *N then j n else ja* (*n* − *N*) **in** *exI*)
apply (*auto simp add*: *poly-mult-eq-zero-disj*, *force*)
done

lemma *poly-entire*: (*poly* (*p* *** *q*) = *poly* []) = ((*poly p* = *poly* []) | (*poly q* = *poly* []))
apply (*auto intro*: *ext dest*: *fun-cong simp add*: *poly-entire-lemma poly-mult*)
apply (*blast intro*: *ccontr dest*: *poly-entire-lemma poly-mult* [*THEN subst*])

done

lemma *poly-entire-neg*: $(poly\ (p\ {*}{*}{*}\ q) \neq poly\ []) = ((poly\ p \neq poly\ []) \ \&\ (poly\ q \neq poly\ []))$
by (*simp add*: *poly-entire*)

lemma *fun-eq*: $(f = g) = (\forall x.\ f\ x = g\ x)$
by (*auto intro*!: *ext*)

lemma *poly-add-minus-zero-iff*: $(poly\ (p\ {+}{+}{+}\ {-}{-}\ q) = poly\ []) = (poly\ p = poly\ q)$
by (*auto simp add*: *poly-add poly-minus-def fun-eq poly-cmult*)

lemma *poly-add-minus-mult-eq*: $poly\ (p\ {*}{*}{*}\ q\ {+}{+}{+}\ {-}{-}(p\ {*}{*}{*}\ r)) = poly\ (p\ {*}{*}{*}\ (q\ {+}{+}{+}\ {-}{-}\ r))$
by (*auto simp add*: *poly-add poly-minus-def fun-eq poly-mult poly-cmult right-distrib*)

lemma *poly-mult-left-cancel*: $(poly\ (p\ {*}{*}{*}\ q) = poly\ (p\ {*}{*}{*}\ r)) = (poly\ p = poly\ []\ |\ poly\ q = poly\ r)$
apply (*rule-tac p1* = $p\ {*}{*}{*}\ q$ **in** *poly-add-minus-zero-iff* [*THEN subst*])
apply (*auto intro*: *ext simp add*: *poly-add-minus-mult-eq poly-entire poly-add-minus-zero-iff*)
done

lemma *real-mult-zero-disj-iff*: $(x * y = 0) = (x = (0::real)\ |\ y = 0)$
by *simp*

lemma *poly-exp-eq-zero*:
$(poly\ (p\ \%\hat{}\ n) = poly\ []) = (poly\ p = poly\ []\ \&\ n \neq 0)$
apply (*simp only*: *fun-eq add*: *all-simps* [*symmetric*])
apply (*rule arg-cong* [**where** $f = All$])
apply (*rule ext*)
apply (*induct-tac n*)
apply (*auto simp add*: *poly-mult real-mult-zero-disj-iff*)
done
declare *poly-exp-eq-zero* [*simp*]

lemma *poly-prime-eq-zero*: $poly\ [a,1] \neq poly\ []$
apply (*simp add*: *fun-eq*)
apply (*rule-tac* $x = 1 - a$ **in** *exI*, *simp*)
done
declare *poly-prime-eq-zero* [*simp*]

lemma *poly-exp-prime-eq-zero*: $(poly\ ([a,\ 1]\ \%\hat{}\ n) \neq poly\ [])$
by *auto*
declare *poly-exp-prime-eq-zero* [*simp*]

A more constructive notion of polynomials being trivial

lemma *poly-zero-lemma*: $poly\ (h\ \#\ t) = poly\ [] \Longrightarrow h = 0\ \&\ poly\ t = poly\ []$
apply (*simp add*: *fun-eq*)

apply (*case-tac h = 0*)
apply (*drule-tac* [*2*] *x = 0* **in** *spec*, *auto*)
apply (*case-tac poly t = poly* [], *simp*)
apply (*auto simp add*: *poly-roots-finite real-mult-zero-disj-iff*)
apply (*drule real-finite-lemma*, *safe*)
apply (*drule-tac x = abs a + 1* **in** *spec*)+
apply *arith*
done

lemma *poly-zero*: (*poly p = poly* []) = *list-all* (%*c*. *c = 0*) *p*
apply (*induct p*, *simp*)
apply (*rule iffI*)
apply (*drule poly-zero-lemma*, *auto*)
done

declare *real-mult-zero-disj-iff* [*simp*]

lemma *pderiv-aux-iszero* [*rule-format*, *simp*]:
∀ *n*. *list-all* (%*c*. *c = 0*) (*pderiv-aux* (*Suc n*) *p*) = *list-all* (%*c*. *c = 0*) *p*
by (*induct p*, *auto*)

lemma *pderiv-aux-iszero-num*: (*number-of n* :: *nat*) ≠ *0*
==> (*list-all* (%*c*. *c = 0*) (*pderiv-aux* (*number-of n*) *p*) =
list-all (%*c*. *c = 0*) *p*)
apply(*drule not0-implies-Suc*, *clarify*)
apply (*rule-tac n1 = m* **in** *pderiv-aux-iszero* [*THEN subst*])
apply (*simp* (*no-asm-simp*) *del*: *pderiv-aux-iszero*)
done

lemma *pderiv-iszero* [*rule-format*]:
poly (*pderiv p*) = *poly* [] --> (∃ *h*. *poly p = poly* [*h*])
apply (*simp add*: *poly-zero*)
apply (*induct p*, *force*)
apply (*simp add*: *pderiv-Cons pderiv-aux-iszero-num del*: *poly-Cons*)
apply (*auto simp add*: *poly-zero* [*symmetric*])
done

lemma *pderiv-zero-obj*: *poly p = poly* [] --> (*poly* (*pderiv p*) = *poly* [])
apply (*simp add*: *poly-zero*)
apply (*induct p*, *force*)
apply (*simp add*: *pderiv-Cons pderiv-aux-iszero-num del*: *poly-Cons*)
done

lemma *pderiv-zero*: *poly p = poly* [] ==> (*poly* (*pderiv p*) = *poly* [])
by (*blast elim*: *pderiv-zero-obj* [*THEN impE*])
declare *pderiv-zero* [*simp*]

lemma *poly-pderiv-welldef*: *poly p = poly q* ==> (*poly* (*pderiv p*) = *poly* (*pderiv*

q))
apply (*cut-tac p* = *p* +++ −−*q* **in** *pderiv-zero-obj*)
apply (*simp add*: *fun-eq poly-add poly-minus poly-pderiv-add poly-pderiv-minus del*: *pderiv-zero*)
done

Basics of divisibility.

lemma *poly-primes*: ([*a*, *1*] *divides* (*p* ∗∗∗ *q*)) = ([*a*, *1*] *divides p* | [*a*, *1*] *divides q*)
apply (*auto simp add*: *divides-def fun-eq poly-mult poly-add poly-cmult left-distrib* [*symmetric*])
apply (*drule-tac x* = −*a* **in** *spec*)
apply (*auto simp add*: *poly-linear-divides poly-add poly-cmult left-distrib* [*symmetric*])
apply (*rule-tac x* = *qa* ∗∗∗ *q* **in** *exI*)
apply (*rule-tac* [*2*] *x* = *p* ∗∗∗ *qa* **in** *exI*)
apply (*auto simp add*: *poly-add poly-mult poly-cmult mult-ac*)
done

lemma *poly-divides-refl*: *p divides p*
apply (*simp add*: *divides-def*)
apply (*rule-tac x* = [*1*] **in** *exI*)
apply (*auto simp add*: *poly-mult fun-eq*)
done
declare *poly-divides-refl* [*simp*]

lemma *poly-divides-trans*: [| *p divides q*; *q divides r* |] ==> *p divides r*
apply (*simp add*: *divides-def*, *safe*)
apply (*rule-tac x* = *qa* ∗∗∗ *qaa* **in** *exI*)
apply (*auto simp add*: *poly-mult fun-eq real-mult-assoc*)
done

lemma *poly-divides-exp*: $m \le n$ ==> (*p* %ˆ *m*) *divides* (*p* %ˆ *n*)
apply (*auto simp add*: *le-iff-add*)
apply (*induct-tac k*)
apply (*rule-tac* [*2*] *poly-divides-trans*)
apply (*auto simp add*: *divides-def*)
apply (*rule-tac x* = *p* **in** *exI*)
apply (*auto simp add*: *poly-mult fun-eq mult-ac*)
done

lemma *poly-exp-divides*: [| (*p* %ˆ *n*) *divides q*; $m \le n$ |] ==> (*p* %ˆ *m*) *divides q*
by (*blast intro*: *poly-divides-exp poly-divides-trans*)

lemma *poly-divides-add*:
[| *p divides q*; *p divides r* |] ==> *p divides* (*q* +++ *r*)
apply (*simp add*: *divides-def*, *auto*)
apply (*rule-tac x* = *qa* +++ *qaa* **in** *exI*)
apply (*auto simp add*: *poly-add fun-eq poly-mult right-distrib*)
done

lemma *poly-divides-diff*:
[| *p divides q*; *p divides* (*q* +++ *r*) |] ==> *p divides r*
apply (*simp add*: *divides-def*, *auto*)
apply (*rule-tac x* = *qaa* +++ −− *qa* **in** *exI*)
apply (*auto simp add*: *poly-add fun-eq poly-mult poly-minus right-diff-distrib compare-rls add-ac*)
done

lemma *poly-divides-diff2*: [| *p divides r*; *p divides* (*q* +++ *r*) |] ==> *p divides q*
apply (*erule poly-divides-diff*)
apply (*auto simp add*: *poly-add fun-eq poly-mult divides-def add-ac*)
done

lemma *poly-divides-zero*: *poly p* = *poly* [] ==> *q divides p*
apply (*simp add*: *divides-def*)
apply (*auto simp add*: *fun-eq poly-mult*)
done

lemma *poly-divides-zero2*: *q divides* []
apply (*simp add*: *divides-def*)
apply (*rule-tac x* = [] **in** *exI*)
apply (*auto simp add*: *fun-eq*)
done
declare *poly-divides-zero2* [*simp*]

At last, we can consider the order of a root.

lemma *poly-order-exists-lemma* [*rule-format*]:
$\forall$ *p*. *length p* = *d* −−> *poly p* $\neq$ *poly* []
−−> ($\exists$ *n q*. *p* = *mulexp n* [−*a*, *1*] *q* & *poly q a* $\neq$ *0*)
apply (*induct d*)
apply (*simp add*: *fun-eq*, *safe*)
apply (*case-tac poly p a* = *0*)
apply (*drule-tac poly-linear-divides* [*THEN iffD1*], *safe*)
apply (*drule-tac x* = *q* **in** *spec*)
apply (*drule-tac poly-entire-neg* [*THEN iffD1*], *safe*, *force*, *blast*)
apply (*rule-tac x* = *Suc n* **in** *exI*)
apply (*rule-tac x* = *qa* **in** *exI*)
apply (*simp del*: *pmult-Cons*)
apply (*rule-tac x* = *0* **in** *exI*, *force*)
done

lemma *poly-order-exists*:
[| *length p* = *d*; *poly p* $\neq$ *poly* [] |]
==> $\exists$ *n*. ([−*a*, *1*] %^ *n*) *divides p* &
~(([−*a*, *1*] %^ (*Suc n*)) *divides p*)
apply (*drule poly-order-exists-lemma* [**where** *a*=*a*], *assumption*, *clarify*)
apply (*rule-tac x* = *n* **in** *exI*, *safe*)

apply (*unfold divides-def*)
apply (*rule-tac x = q* **in** *exI*)
apply (*induct-tac n*, *simp*)
apply (*simp* (*no-asm-simp*) *add*: *poly-add poly-cmult poly-mult right-distrib mult-ac*)
apply *safe*
apply (*subgoal-tac poly* (*mulexp n* [− *a*, *1*] *q*) ≠ *poly* ([− *a*, *1*] %^ *Suc n* *** *qa*))

apply *simp*
apply (*induct-tac n*)
apply (*simp del*: *pmult-Cons pexp-Suc*)
apply (*erule-tac Q* = *poly q a* = *0* **in** *contrapos-np*)
apply (*simp add*: *poly-add poly-cmult*)
apply (*rule pexp-Suc* [*THEN ssubst*])
apply (*rule ccontr*)
apply (*simp add*: *poly-mult-left-cancel poly-mult-assoc del*: *pmult-Cons pexp-Suc*)
done

lemma *poly-one-divides*: [*1*] *divides p*
by (*simp add*: *divides-def*, *auto*)
declare *poly-one-divides* [*simp*]

lemma *poly-order*: *poly p* ≠ *poly* []
    ==> *EX*! *n*. ([−*a*, *1*] %^ *n*) *divides p* &
        ~(([−*a*, *1*] %^ (*Suc n*)) *divides p*)
apply (*auto intro*: *poly-order-exists simp add*: *less-linear simp del*: *pmult-Cons pexp-Suc*)
apply (*metis Suc-leI Nat.less-linear poly-exp-divides*)
done

Order

lemma *some1-equalityD*: [| *n* = (@*n*. *P n*); *EX*! *n*. *P n* |] ==> *P n*
by (*blast intro*: *someI2*)

lemma *order*:
    (([−*a*, *1*] %^ *n*) *divides p* &
     ~(([−*a*, *1*] %^ (*Suc n*)) *divides p*)) =
     ((*n* = *order a p*) & ~(*poly p* = *poly* []))
apply (*unfold order-def*)
apply (*rule iffI*)
apply (*blast dest*: *poly-divides-zero intro*!: *some1-equality* [*symmetric*] *poly-order*)
apply (*blast intro*!: *poly-order* [*THEN* [*2*] *some1-equalityD*])
done

lemma *order2*: [| *poly p* ≠ *poly* [] |]
    ==> ([−*a*, *1*] %^ (*order a p*)) *divides p* &
        ~(([−*a*, *1*] %^ (*Suc*(*order a p*))) *divides p*)
by (*simp add*: *order del*: *pexp-Suc*)

lemma *order-unique*: [| *poly p* ≠ *poly* []; ([−*a*, *1*] %^ *n*) *divides p*;

```
    ~(([−a, 1] %^ (Suc n)) divides p)
  |] ==> (n = order a p)
by (insert order [of a n p], auto)

lemma order-unique-lemma: (poly p ≠ poly [] & ([−a, 1] %^ n) divides p &
    ~(([−a, 1] %^ (Suc n)) divides p))
  ==> (n = order a p)
by (blast intro: order-unique)

lemma order-poly: poly p = poly q ==> order a p = order a q
by (auto simp add: fun-eq divides-def poly-mult order-def)

lemma pexp-one: p %^ (Suc 0) = p
apply (induct p)
apply (auto simp add: numeral-1-eq-1)
done
declare pexp-one [simp]

lemma lemma-order-root [rule-format]:
  ∀ p a. n > 0 & [− a, 1] %^ n divides p & ~ [− a, 1] %^ (Suc n) divides p
      −−> poly p a = 0
apply (induct n, blast)
apply (auto simp add: divides-def poly-mult simp del: pmult-Cons)
done

lemma order-root: (poly p a = 0) = ((poly p = poly []) | order a p ≠ 0)
apply (case-tac poly p = poly [], auto)
apply (simp add: poly-linear-divides del: pmult-Cons, safe)
apply (drule-tac [!] a = a in order2)
apply (rule ccontr)
apply (simp add: divides-def poly-mult fun-eq del: pmult-Cons, blast)
apply (blast intro: lemma-order-root)
done

lemma order-divides: (([−a, 1] %^ n) divides p) = ((poly p = poly []) | n ≤ order
a p)
apply (case-tac poly p = poly [], auto)
apply (simp add: divides-def fun-eq poly-mult)
apply (rule-tac x = [] in exI)
apply (auto dest!: order2 [where a=a]
        intro: poly-exp-divides simp del: pexp-Suc)
done

lemma order-decomp:
  poly p ≠ poly []
  ==> ∃ q. (poly p = poly (([−a, 1] %^ (order a p)) *** q)) &
        ~([−a, 1] divides q)
apply (unfold divides-def)
apply (drule order2 [where a = a])
```

apply (*simp add*: *divides-def del*: *pexp-Suc pmult-Cons*, *safe*)
apply (*rule-tac x* = *q* **in** *exI*, *safe*)
apply (*drule-tac x* = *qa* **in** *spec*)
apply (*auto simp add*: *poly-mult fun-eq poly-exp mult-ac simp del*: *pmult-Cons*)
done

Important composition properties of orders.

lemma *order-mult*: *poly* (*p* *** *q*) ≠ *poly* []
  ==> *order a* (*p* *** *q*) = *order a p* + *order a q*
apply (*cut-tac a* = *a* **and** *p* = *p***q* **and** *n* = *order a p* + *order a q* **in** *order*)
apply (*auto simp add*: *poly-entire simp del*: *pmult-Cons*)
apply (*drule-tac a* = *a* **in** *order2*)+
apply *safe*
apply (*simp add*: *divides-def fun-eq poly-exp-add poly-mult del*: *pmult-Cons*, *safe*)
apply (*rule-tac x* = *qa* *** *qaa* **in** *exI*)
apply (*simp add*: *poly-mult mult-ac del*: *pmult-Cons*)
apply (*drule-tac a* = *a* **in** *order-decomp*)+
apply *safe*
apply (*subgoal-tac* [−*a*,*1*] *divides* (*qa* *** *qaa*))
apply (*simp add*: *poly-primes del*: *pmult-Cons*)
apply (*auto simp add*: *divides-def simp del*: *pmult-Cons*)
apply (*rule-tac x* = *qb* **in** *exI*)
apply (*subgoal-tac poly* ([−*a*, *1*] %^ (*order a p*) *** (*qa* *** *qaa*)) = *poly* ([−*a*, *1*] %^ (*order a p*) *** ([−*a*, *1*] *** *qb*)))
apply (*drule poly-mult-left-cancel* [*THEN iffD1*], *force*)
apply (*subgoal-tac poly* ([−*a*, *1*] %^ (*order a q*) *** ([−*a*, *1*] %^ (*order a p*) *** (*qa* *** *qaa*))) = *poly* ([−*a*, *1*] %^ (*order a q*) *** ([−*a*, *1*] %^ (*order a p*) *** ([−*a*, *1*] *** *qb*))))
apply (*drule poly-mult-left-cancel* [*THEN iffD1*], *force*)
apply (*simp add*: *fun-eq poly-exp-add poly-mult mult-ac del*: *pmult-Cons*)
done

lemma *lemma-order-pderiv* [*rule-format*]:
  ∀ *p q a*. *n* > *0* &
   poly (*pderiv p*) ≠ *poly* [] &
   poly p = *poly* ([− *a*, *1*] %^ *n* *** *q*) & ~ [− *a*, *1*] *divides q*
   −−> *n* = *Suc* (*order a* (*pderiv p*))
apply (*induct n*, *safe*)
apply (*rule order-unique-lemma*, *rule conjI*, *assumption*)
apply (*subgoal-tac* ∀ *r*. *r divides* (*pderiv p*) = *r divides* (*pderiv* ([−*a*, *1*] %^ *Suc n* *** *q*)))
apply (*drule-tac* [*2*] *poly-pderiv-welldef*)
prefer *2* **apply** (*simp add*: *divides-def del*: *pmult-Cons pexp-Suc*)
apply (*simp del*: *pmult-Cons pexp-Suc*)
apply (*rule conjI*)
apply (*simp add*: *divides-def fun-eq del*: *pmult-Cons pexp-Suc*)
apply (*rule-tac x* = [−*a*, *1*] *** (*pderiv q*) +++ *real* (*Suc n*) %* *q* **in** *exI*)
apply (*simp add*: *poly-pderiv-mult poly-pderiv-exp-prime poly-add poly-mult poly-cmult*

right-distrib mult-ac del: *pmult-Cons pexp-Suc*)
apply (*simp add*: *poly-mult right-distrib left-distrib mult-ac del*: *pmult-Cons*)
apply (*erule-tac V* = ∀ *r*. *r divides pderiv p* = *r divides pderiv* ([− *a*, *1*] %^ *Suc n* ∗∗∗ *q*) **in** *thin-rl*)
apply (*unfold divides-def*)
apply (*simp* (*no-asm*) *add*: *poly-pderiv-mult poly-pderiv-exp-prime fun-eq poly-add poly-mult del*: *pmult-Cons pexp-Suc*)
apply (*rule contrapos-np*, *assumption*)
apply (*rotate-tac 3*, *erule contrapos-np*)
apply (*simp del*: *pmult-Cons pexp-Suc*, *safe*)
apply (*rule-tac x* = *inverse* (*real* (*Suc n*)) %∗ (*qa* +++ −− (*pderiv q*)) **in** *exI*)
apply (*subgoal-tac poly* ([−*a*, *1*] %^ *n* ∗∗∗ *q*) = *poly* ([−*a*, *1*] %^ *n* ∗∗∗ ([−*a*, *1*] ∗∗∗ (*inverse* (*real* (*Suc n*)) %∗ (*qa* +++ −− (*pderiv q*))))))
apply (*drule poly-mult-left-cancel* [*THEN iffD1*], *simp*)
apply (*simp add*: *fun-eq poly-mult poly-add poly-cmult poly-minus del*: *pmult-Cons mult-cancel-left*, *safe*)
apply (*rule-tac c1* = *real* (*Suc n*) **in** *real-mult-left-cancel* [*THEN iffD1*])
apply (*simp* (*no-asm*))
apply (*subgoal-tac real* (*Suc n*) ∗ (*poly* ([− *a*, *1*] %^ *n*) *xa* ∗ *poly q xa*) =
(*poly qa xa* + − *poly* (*pderiv q*) *xa*) ∗
(*poly* ([− *a*, *1*] %^ *n*) *xa* ∗
((− *a* + *xa*) ∗ (*inverse* (*real* (*Suc n*)) ∗ *real* (*Suc n*)))))
apply (*simp only*: *mult-ac*)
apply (*rotate-tac 2*)
apply (*drule-tac x* = *xa* **in** *spec*)
apply (*simp add*: *left-distrib mult-ac del*: *pmult-Cons*)
done

lemma *order-pderiv*: [| *poly* (*pderiv p*) ≠ *poly* []; *order a p* ≠ *0* |]
==> (*order a p* = *Suc* (*order a* (*pderiv p*)))
apply (*case-tac poly p* = *poly* [])
apply (*auto dest*: *pderiv-zero*)
apply (*drule-tac a* = *a* **and** *p* = *p* **in** *order-decomp*)
apply (*metis lemma-order-pderiv length-0-conv length-greater-0-conv*)
done

Now justify the standard squarefree decomposition, i.e. f / gcd(f,f’). *) (* ‘a la Harrison

lemma *poly-squarefree-decomp-order*: [| *poly* (*pderiv p*) ≠ *poly* [];
poly p = *poly* (*q* ∗∗∗ *d*);
poly (*pderiv p*) = *poly* (*e* ∗∗∗ *d*);
poly d = *poly* (*r* ∗∗∗ *p* +++ *s* ∗∗∗ *pderiv p*)
|] ==> *order a q* = (*if order a p* = *0 then 0 else 1*)
apply (*subgoal-tac order a p* = *order a q* + *order a d*)
apply (*rule-tac* [*2*] *s* = *order a* (*q* ∗∗∗ *d*) **in** *trans*)
prefer *2* **apply** (*blast intro*: *order-poly*)
apply (*rule-tac* [*2*] *order-mult*)
prefer *2* **apply** *force*
apply (*case-tac order a p* = *0*, *simp*)

apply (*subgoal-tac order a* (*pderiv p*) = *order a e* + *order a d*)
apply (*rule-tac* [*2*] *s* = *order a* (*e* *** *d*) **in** *trans*)
prefer *2* **apply** (*blast intro*: *order-poly*)
apply (*rule-tac* [*2*] *order-mult*)
prefer *2* **apply** *force*
apply (*case-tac poly p* = *poly* [])
apply (*drule-tac p* = *p* **in** *pderiv-zero*, *simp*)
apply (*drule order-pderiv*, *assumption*)
apply (*subgoal-tac order a* (*pderiv p*) ≤ *order a d*)
apply (*subgoal-tac* [*2*] ([−*a*, *1*] %^ (*order a* (*pderiv p*))) *divides d*)
prefer *2* **apply** (*simp add*: *poly-entire order-divides*)
apply (*subgoal-tac* [*2*] ([−*a*, *1*] %^ (*order a* (*pderiv p*))) *divides p* & ([−*a*, *1*] %^ (*order a* (*pderiv p*))) *divides* (*pderiv p*))
prefer *3* **apply** (*simp add*: *order-divides*)
prefer *2* **apply** (*simp add*: *divides-def del*: *pexp-Suc pmult-Cons*, *safe*)
apply (*rule-tac x* = *r* *** *qa* +++ *s* *** *qaa* **in** *exI*)
apply (*simp add*: *fun-eq poly-add poly-mult left-distrib right-distrib mult-ac del*: *pexp-Suc pmult-Cons*, *auto*)
done

lemma *poly-squarefree-decomp-order2*: [| *poly* (*pderiv p*) ≠ *poly* [];
poly p = *poly* (*q* *** *d*);
poly (*pderiv p*) = *poly* (*e* *** *d*);
poly d = *poly* (*r* *** *p* +++ *s* *** *pderiv p*)
|] ==> ∀ *a*. *order a q* = (*if order a p* = *0 then 0 else 1*)
apply (*blast intro*: *poly-squarefree-decomp-order*)
done

lemma *order-root2*: *poly p* ≠ *poly* [] ==> (*poly p a* = *0*) = (*order a p* ≠ *0*)
by (*rule order-root* [*THEN ssubst*], *auto*)

lemma *order-pderiv2*: [| *poly* (*pderiv p*) ≠ *poly* []; *order a p* ≠ *0* |]
==> (*order a* (*pderiv p*) = *n*) = (*order a p* = *Suc n*)
by (*metis Suc-Suc-eq order-pderiv*)

lemma *rsquarefree-roots*:
rsquarefree p = (∀ *a*. ~(*poly p a* = *0* & *poly* (*pderiv p*) *a* = *0*))
apply (*simp add*: *rsquarefree-def*)
apply (*case-tac poly p* = *poly* [], *simp*, *simp*)
apply (*case-tac poly* (*pderiv p*) = *poly* [])
apply *simp*
apply (*drule pderiv-iszero*, *clarify*)
apply (*subgoal-tac* ∀ *a*. *order a p* = *order a* [*h*])
apply (*simp add*: *fun-eq*)
apply (*rule allI*)
apply (*cut-tac p* = [*h*] **and** *a* = *a* **in** *order-root*)
apply (*simp add*: *fun-eq*)
apply (*blast intro*: *order-poly*)

apply (*metis One-nat-def order-pderiv2 order-root rsquarefree-def*)
done

lemma *pmult-one*: $[1]$ $***$ $p = p$
by *auto*
declare *pmult-one* [*simp*]

lemma *poly-Nil-zero*: *poly* [] = *poly* [*0*]
by (*simp add*: *fun-eq*)

lemma *rsquarefree-decomp*:
  [| *rsquarefree p*; *poly p a* = *0* |]
  ==> $\exists$ *q*. (*poly p* = *poly* ([$-a$, *1*] $***$ *q*)) & *poly q a* $\neq$ *0*
apply (*simp add*: *rsquarefree-def*, *safe*)
apply (*frule-tac a* = *a* **in** *order-decomp*)
apply (*drule-tac x* = *a* **in** *spec*)
apply (*drule-tac a* = *a* **in** *order-root2* [*symmetric*])
apply (*auto simp del*: *pmult-Cons*)
apply (*rule-tac x* = *q* **in** *exI*, *safe*)
apply (*simp add*: *poly-mult fun-eq*)
apply (*drule-tac p1* = *q* **in** *poly-linear-divides* [*THEN iffD1*])
apply (*simp add*: *divides-def del*: *pmult-Cons*, *safe*)
apply (*drule-tac x* = [] **in** *spec*)
apply (*auto simp add*: *fun-eq*)
done

lemma *poly-squarefree-decomp*: [| *poly* (*pderiv p*) $\neq$ *poly* [];
    poly p = *poly* (*q* $***$ *d*);
    poly (*pderiv p*) = *poly* (*e* $***$ *d*);
    poly d = *poly* (*r* $***$ *p* +++ *s* $***$ *pderiv p*)
  |] ==> *rsquarefree q* & ($\forall$ *a*. (*poly q a* = *0*) = (*poly p a* = *0*))
apply (*frule poly-squarefree-decomp-order2*, *assumption+*)
apply (*case-tac poly p* = *poly* [])
apply (*blast dest*: *pderiv-zero*)
apply (*simp* (*no-asm*) *add*: *rsquarefree-def order-root del*: *pmult-Cons*)
apply (*simp add*: *poly-entire del*: *pmult-Cons*)
done

Normalization of a polynomial.

lemma *poly-normalize*: *poly* (*pnormalize p*) = *poly p*
apply (*induct p*)
apply (*auto simp add*: *fun-eq*)
done
declare *poly-normalize* [*simp*]

The degree of a polynomial.

lemma *lemma-degree-zero*:
  list-all (%*c*. *c* = *0*) *p* $\longleftrightarrow$ *pnormalize p* = []
by (*induct p*, *auto*)

lemma *degree-zero*: $(poly\ p = poly\ []) \Longrightarrow (degree\ p = 0)$
apply (*simp add*: *degree-def*)
apply (*case-tac pnormalize p* = [])
apply (*auto simp add*: *poly-zero lemma-degree-zero*)
done

lemma *pnormalize-sing*: $(pnormalize\ [x] = [x]) \longleftrightarrow x \neq 0$ **by** *simp*
lemma *pnormalize-pair*: $y \neq 0 \longleftrightarrow (pnormalize\ [x, y] = [x, y])$ **by** *simp*
lemma *pnormal-cons*: $pnormal\ p \Longrightarrow pnormal\ (c\#p)$
  unfolding *pnormal-def* **by** *simp*
lemma *pnormal-tail*: $p \neq [] \Longrightarrow pnormal\ (c\#p) \Longrightarrow pnormal\ p$
  unfolding *pnormal-def*
  apply (*cases pnormalize p* = [], *auto*)
  by (*cases c* = *0*, *auto*)
lemma *pnormal-last-nonzero*: *pnormal p* ==> *last p* $\neq$ *0*
  apply (*induct p*, *auto simp add*: *pnormal-def*)
  apply (*case-tac pnormalize p* = [], *auto*)
  by (*case-tac a*=*0*, *auto*)
lemma *pnormal-length*: $pnormal\ p \Longrightarrow 0 < length\ p$
  unfolding *pnormal-def length-greater-0-conv* **by** *blast*
lemma *pnormal-last-length*: $[\![0 < length\ p\ ;\ last\ p \neq 0]\!] \Longrightarrow pnormal\ p$
  apply (*induct p*, *auto*)
  apply (*case-tac p* = [], *auto*)
  apply (*simp add*: *pnormal-def*)
  by (*rule pnormal-cons*, *auto*)
lemma *pnormal-id*: $pnormal\ p \longleftrightarrow (0 < length\ p \wedge last\ p \neq 0)$
  using *pnormal-last-length pnormal-length pnormal-last-nonzero* **by** *blast*

Tidier versions of finiteness of roots.

lemma *poly-roots-finite-set*: *poly p* $\neq$ *poly* [] ==> *finite* {*x*. *poly p x* = *0*}
apply (*auto simp add*: *poly-roots-finite*)
apply (*rule-tac B* = {*x*::*real*. $\exists$ *n*. (*n*::*nat*) < *N* & (*x* = *j n*) } **in** *finite-subset*)
apply (*induct-tac* [*2*] *N*, *auto*)
apply (*subgoal-tac* {*x*::*real*. $\exists$ *na*. *na* < *Suc n* & (*x* = *j na*) } = { (*j n*) } *Un* {*x*. $\exists$ *na*. *na* < *n* & (*x* = *j na*) })
apply (*auto simp add*: *less-Suc-eq*)
done

bound for polynomial.

lemma *poly-mono*: *abs*(*x*) $\leq$ *k* ==> *abs*(*poly p x*) $\leq$ *poly* (*map abs p*) *k*
apply (*induct p*, *auto*)
apply (*rule-tac j* = *abs a* + *abs* (*x* $*$ *poly p x*) **in** *real-le-trans*)
apply (*rule abs-triangle-ineq*)
apply (*auto intro*!: *mult-mono simp add*: *abs-mult*)
done

lemma *poly-Sing*: *poly* [*c*] *x* = *c* **by** *simp*
end

41 MacLaurin: MacLaurin Series

theory *MacLaurin*
imports *Transcendental*
begin

41.1 Maclaurin's Theorem with Lagrange Form of Remainder

This is a very long, messy proof even now that it's been broken down into lemmas.

lemma *Maclaurin-lemma*:
  $0 < h \Longrightarrow$
   $\exists B.\ f\, h = (\sum m{=}0..{<}n.\ (j\ m\ /\ real\ (fact\ m)) * (h\hat{}m)) +$
       $(B * ((h\hat{}n)\ /\ real(fact\ n)))$
apply $(rule\text{-}tac\ x = (f\, h - (\sum m{=}0..{<}n.\ (j\ m\ /\ real\ (fact\ m)) * h\hat{}m)) *$
       $real(fact\ n)\ /\ (h\hat{}n)$
   in *exI*)
apply (*simp*)
done

lemma *eq-diff-eq'*: $(x = y - z) = (y = x + (z{::}real))$
by *arith*

A crude tactic to differentiate by proof.

lemmas *deriv-rulesI* =
 DERIV-ident DERIV-const DERIV-cos DERIV-cmult
 DERIV-sin DERIV-exp DERIV-inverse DERIV-pow
 DERIV-add DERIV-diff DERIV-mult DERIV-minus
 DERIV-inverse-fun DERIV-quotient DERIV-fun-pow
 DERIV-fun-exp DERIV-fun-sin DERIV-fun-cos
 DERIV-ident DERIV-const DERIV-cos

ML
```
⟪
local
exception DERIV-name;
fun get-fun-name (- $ (Const (Lim.deriv,-) $ Abs(-,-, Const (f,-) $ -) $ - $ -)) = f
|  get-fun-name (- $ (- $ (Const (Lim.deriv,-) $ Abs(-,-, Const (f,-) $ -) $ - $ -)))
= f
|  get-fun-name - = raise DERIV-name;

in

val deriv-tac =
  SUBGOAL (fn (prem,i) =>
```

```
  (resolve-tac @{thms deriv-rulesI} i) ORELSE
   ((rtac (read-instantiate [(f,get-fun-name prem)]
                 @{thm DERIV-chain2}) i) handle DERIV-name => no-tac));;

val DERIV-tac = ALLGOALS(fn i => REPEAT(deriv-tac i));

end
⟫
```

lemma *Maclaurin-lemma2*:

```
     [| ∀m t. m < n ∧ 0≤t ∧ t≤h ⟶ DERIV (diff m) t :> diff (Suc m) t;
        n = Suc k;
       difg =
       (λm t. diff m t −
             ((∑p = 0..<n − m. diff (m + p) 0 / real (fact p) ∗ t ˆ p) +
              B ∗ (t ˆ (n − m) / real (fact (n − m)))))|] ==>
       ∀m t. m < n & 0 ≤ t & t ≤ h −−>
                 DERIV (difg m) t :> difg (Suc m) t
apply clarify
apply (rule DERIV-diff)
apply (simp (no-asm-simp))
apply (tactic DERIV-tac)
apply (tactic DERIV-tac)
apply (rule-tac [2] lemma-DERIV-subst)
apply (rule-tac [2] DERIV-quotient)
apply (rule-tac [3] DERIV-const)
apply (rule-tac [2] DERIV-pow)
  prefer 3 apply (simp add: fact-diff-Suc)
 prefer 2 apply simp
apply (frule-tac m = m in less-add-one, clarify)
apply (simp del: setsum-op-ivl-Suc)
apply (insert sumr-offset4 [of 1])
apply (simp del: setsum-op-ivl-Suc fact-Suc realpow-Suc)
apply (rule lemma-DERIV-subst)
apply (rule DERIV-add)
apply (rule-tac [2] DERIV-const)
apply (rule DERIV-sumr, clarify)
 prefer 2 apply simp
apply (simp (no-asm) add: divide-inverse mult-assoc del: fact-Suc realpow-Suc)
apply (rule DERIV-cmult)
apply (rule lemma-DERIV-subst)
apply (best intro: DERIV-chain2 intro!: DERIV-intros)
apply (subst fact-Suc)
apply (subst real-of-nat-mult)
apply (simp add: mult-ac)
done
```

lemma *Maclaurin-lemma3*:

```
  fixes difg :: nat => real => real shows
    [|∀k t. k < Suc m ∧ 0≤t & t≤h ⟶ DERIV (difg k) t :> difg (Suc k) t;
      ∀k<Suc m. difg k 0 = 0; DERIV (difg n) t :> 0;  n < m; 0 < t;
      t < h|]
    ==> ∃ta. 0 < ta & ta < t & DERIV (difg (Suc n)) ta :> 0
apply (rule Rolle, assumption, simp)
apply (drule-tac x = n and P=%k. k<Suc m --> difg k 0 = 0 in spec)
apply (rule DERIV-unique)
prefer 2 apply assumption
apply force
apply (metis DERIV-isCont dlo-simps(4) dlo-simps(9) less-trans-Suc nat-less-le
not-less-eq real-le-trans)
apply (metis Suc-less-eq differentiableI dlo-simps(7) dlo-simps(8) dlo-simps(9)
real-le-trans xt1(8))
done

lemma Maclaurin:
  [| 0 < h; n > 0; diff 0 = f;
    ∀m t. m < n & 0 ≤ t & t ≤ h --> DERIV (diff m) t :> diff (Suc m) t |]
   ==> ∃t. 0 < t &
           t < h &
           f h =
           setsum (%m. (diff m 0 / real (fact m)) * h ^ m) {0..<n} +
           (diff n t / real (fact n)) * h ^ n
apply (case-tac n = 0, force)
apply (drule not0-implies-Suc)
apply (erule exE)
apply (frule-tac f=f and n=n and j=%m. diff m 0 in Maclaurin-lemma)
apply (erule exE)
apply (subgoal-tac ∃g.
    g = (%t. f t - (setsum (%m. (diff m 0 / real(fact m)) * t^m) {0..<n} + (B
* (t^n / real(fact n))))))
 prefer 2 apply blast
apply (erule exE)
apply (subgoal-tac g 0 = 0 & g h =0)
 prefer 2
 apply (simp del: setsum-op-ivl-Suc)
 apply (cut-tac n = m and k = 1 in sumr-offset2)
 apply (simp add: eq-diff-eq' del: setsum-op-ivl-Suc)
apply (subgoal-tac ∃difg. difg = (%m t. diff m t - (setsum (%p. (diff (m + p)
0 / real (fact p)) * (t ^ p)) {0..<n-m} + (B * ((t ^ (n - m)) / real (fact (n -
m)))))))
 prefer 2 apply blast
apply (erule exE)
apply (subgoal-tac difg 0 = g)
 prefer 2 apply simp
apply (frule Maclaurin-lemma2, assumption+)
apply (subgoal-tac ∀ma. ma < n --> (∃t. 0 < t & t < h & difg (Suc ma) t =
0) )
```

apply (drule-tac x = m **and** P=%m. m<n −−> (∃ t. ?QQ m t) **in** spec)
apply (erule impE)
apply (simp (no-asm-simp))
apply (erule exE)
apply (rule-tac x = t **in** exI)
apply (simp del: realpow-Suc fact-Suc)
apply (subgoal-tac ∀ m. m < n −−> difg m 0 = 0)
prefer 2
apply clarify
apply simp
apply (frule-tac m = ma **in** less-add-one, clarify)
apply (simp del: setsum-op-ivl-Suc)
apply (insert sumr-offset4 [of 1])
apply (simp del: setsum-op-ivl-Suc fact-Suc realpow-Suc)
apply (subgoal-tac ∀ m. m < n −−> (∃ t. 0 < t & t < h & DERIV (difg m) t :> 0))
apply (rule allI, rule impI)
apply (drule-tac x = ma **and** P=%m. m<n −−> (∃ t. ?QQ m t) **in** spec)
apply (erule impE, assumption)
apply (erule exE)
apply (rule-tac x = t **in** exI)

apply (erule-tac [!] V= difg = (%m t. diff m t − (setsum (%p. diff (m + p) 0 / real (fact p) ∗ t ˆ p) {0..<n−m} + B ∗ (t ˆ (n − m) / real (fact (n − m)))))
in thin-rl)
apply (erule-tac [!] V=g = (%t. f t − (setsum (%m. diff m 0 / real (fact m) ∗ t ˆ m) {0..<n} + B ∗ (t ˆ n / real (fact n))))
in thin-rl)
apply (erule-tac [!] V=f h = setsum (%m. diff m 0 / real (fact m) ∗ h ˆ m) {0..<n} + B ∗ (h ˆ n / real (fact n))
in thin-rl)

apply (simp (no-asm-simp))
apply (rule DERIV-unique)
prefer 2 **apply** blast
apply force
apply (rule allI, induct-tac ma)
apply (rule impI, rule Rolle, assumption, simp, simp)
apply (metis DERIV-isCont zero-less-Suc)
apply (metis One-nat-def differentiableI dlo-simps(7))
apply safe
apply force
apply (frule Maclaurin-lemma3, assumption+, safe)
apply (rule-tac x = ta **in** exI, force)
done

lemma Maclaurin-objl:
0 < h & n>0 & diff 0 = f &
(∀ m t. m < n & 0 ≤ t & t ≤ h −−> DERIV (diff m) t :> diff (Suc m) t)

```
  --> (∃ t. 0 < t & t < h &
          f h = (∑ m=0..<n. diff m 0 / real (fact m) * h ^ m) +
                diff n t / real (fact n) * h ^ n)
by (blast intro: Maclaurin)


lemma Maclaurin2:
  [| 0 < h; diff 0 = f;
     ∀ m t.
        m < n & 0 ≤ t & t ≤ h --> DERIV (diff m) t :> diff (Suc m) t |]
   ==> ∃ t. 0 < t &
            t ≤ h &
            f h =
            (∑ m=0..<n. diff m 0 / real (fact m) * h ^ m) +
            diff n t / real (fact n) * h ^ n
apply (case-tac n, auto)
apply (drule Maclaurin, auto)
done

lemma Maclaurin2-objl:
     0 < h & diff 0 = f &
       (∀ m t.
          m < n & 0 ≤ t & t ≤ h --> DERIV (diff m) t :> diff (Suc m) t)
   --> (∃ t. 0 < t &
            t ≤ h &
            f h =
            (∑ m=0..<n. diff m 0 / real (fact m) * h ^ m) +
            diff n t / real (fact n) * h ^ n)
by (blast intro: Maclaurin2)

lemma Maclaurin-minus:
  [| h < 0; n > 0; diff 0 = f;
     ∀ m t. m < n & h ≤ t & t ≤ 0 --> DERIV (diff m) t :> diff (Suc m) t |]
   ==> ∃ t. h < t &
            t < 0 &
            f h =
            (∑ m=0..<n. diff m 0 / real (fact m) * h ^ m) +
            diff n t / real (fact n) * h ^ n
apply (cut-tac f = %x. f (-x)
       and diff = %n x. (-1 ^ n) * diff n (-x)
       and h = -h and n = n in Maclaurin-objl)
apply (simp)
apply safe
apply (subst minus-mult-right)
apply (rule DERIV-cmult)
apply (rule lemma-DERIV-subst)
apply (rule DERIV-chain2 [where g=uminus])
apply (rule-tac [2] DERIV-minus, rule-tac [2] DERIV-ident)
prefer 2 apply force
```

apply *force*
apply (*rule-tac x* = −*t* **in** *exI*, *auto*)
apply (*subgoal-tac* (∑ *m* = *0*..<*n*. −*1* ˆ *m* ∗ *diff m 0* ∗ (−*h*)ˆ*m* / *real*(*fact m*))
=
(∑ *m* = *0*..<*n*. *diff m 0* ∗ *h* ˆ *m* / *real*(*fact m*)))
apply (*rule-tac* [*2*] *setsum-cong*[*OF refl*])
apply (*auto simp add*: *divide-inverse power-mult-distrib* [*symmetric*])
done

lemma *Maclaurin-minus-objl*:
(*h* < *0* & *n* > *0* & *diff 0* = *f* &
(∀ *m t*.
m < *n* & *h* ≤ *t* & *t* ≤ *0* −−> *DERIV* (*diff m*) *t* :> *diff* (*Suc m*) *t*))
−−> (∃ *t*. *h* < *t* &
t < *0* &
f h =
(∑ *m*=*0*..<*n*. *diff m 0* / *real* (*fact m*) ∗ *h* ˆ *m*) +
diff n t / *real* (*fact n*) ∗ *h* ˆ *n*)
by (*blast intro*: *Maclaurin-minus*)

41.2 More Convenient "Bidirectional" Version.

lemma *Maclaurin-bi-le-lemma* [*rule-format*]:
n>*0* ⟶
diff 0 0 =
(∑ *m* = *0*..<*n*. *diff m 0* ∗ *0* ˆ *m* / *real* (*fact m*)) +
diff n 0 ∗ *0* ˆ *n* / *real* (*fact n*)
by (*induct n*, *auto*)

lemma *Maclaurin-bi-le*:
[| *diff 0* = *f*;
∀ *m t*. *m* < *n* & *abs t* ≤ *abs x* −−> *DERIV* (*diff m*) *t* :> *diff* (*Suc m*) *t* |]
==> ∃ *t*. *abs t* ≤ *abs x* &
f x =
(∑ *m*=*0*..<*n*. *diff m 0* / *real* (*fact m*) ∗ *x* ˆ *m*) +
diff n t / *real* (*fact n*) ∗ *x* ˆ *n*
apply (*case-tac n* = *0*, *force*)
apply (*case-tac x* = *0*)
apply (*rule-tac x* = *0* **in** *exI*)
apply (*force simp add*: *Maclaurin-bi-le-lemma*)
apply (*cut-tac x* = *x* **and** *y* = *0* **in** *linorder-less-linear*, *auto*)

Case 1, where *x* < *0*

apply (*cut-tac f* = *diff 0* **and** *diff* = *diff* **and** *h* = *x* **and** *n* = *n* **in** *Maclaurin-minus-objl*, *safe*)
apply (*simp add*: *abs-if*)
apply (*rule-tac x* = *t* **in** *exI*)
apply (*simp add*: *abs-if*)

Case 2, where *0* < *x*

apply (*cut-tac f* = *diff 0* **and** *diff* = *diff* **and** *h* = *x* **and** *n* = *n* **in** *Maclaurin-objl*, *safe*)
apply (*simp add*: *abs-if*)
apply (*rule-tac x* = *t* **in** *exI*)
apply (*simp add*: *abs-if*)
done

lemma *Maclaurin-all-lt*:
[| *diff 0* = *f*;
$\forall$ *m x*. *DERIV* (*diff m*) *x* :> *diff* (*Suc m*) *x*;
x ~= *0*; *n* > *0*
|] ==> $\exists$ *t*. *0* < *abs t* & *abs t* < *abs x* &
f x = ($\sum$ *m*=*0*..<*n*. (*diff m 0* / *real* (*fact m*)) * *x* ^ *m*) +
(*diff n t* / *real* (*fact n*)) * *x* ^ *n*
apply (*rule-tac x* = *x* **and** *y* = *0* **in** *linorder-cases*)
prefer *2* **apply** *blast*
apply (*drule-tac* [*2*] *diff*=*diff* **in** *Maclaurin*)
apply (*drule-tac diff*=*diff* **in** *Maclaurin-minus*, *simp-all*, *safe*)
apply (*rule-tac* [!] *x* = *t* **in** *exI*, *auto*)
done

lemma *Maclaurin-all-lt-objl*:
diff 0 = *f* &
($\forall$ *m x*. *DERIV* (*diff m*) *x* :> *diff* (*Suc m*) *x*) &
x ~= *0* & *n* > *0*
--> ($\exists$ *t*. *0* < *abs t* & *abs t* < *abs x* &
f x = ($\sum$ *m*=*0*..<*n*. (*diff m 0* / *real* (*fact m*)) * *x* ^ *m*) +
(*diff n t* / *real* (*fact n*)) * *x* ^ *n*)
by (*blast intro*: *Maclaurin-all-lt*)

lemma *Maclaurin-zero* [*rule-format*]:
x = (*0*::*real*)
==> *n* $\neq$ *0* -->
($\sum$ *m*=*0*..<*n*. (*diff m* (*0*::*real*) / *real* (*fact m*)) * *x* ^ *m*) =
diff 0 0
by (*induct n*, *auto*)

lemma *Maclaurin-all-le*: [| *diff 0* = *f*;
$\forall$ *m x*. *DERIV* (*diff m*) *x* :> *diff* (*Suc m*) *x*
|] ==> $\exists$ *t*. *abs t* $\leq$ *abs x* &
f x = ($\sum$ *m*=*0*..<*n*. (*diff m 0* / *real* (*fact m*)) * *x* ^ *m*) +
(*diff n t* / *real* (*fact n*)) * *x* ^ *n*
apply(*cases n*=*0*)
apply (*force*)
apply (*case-tac x* = *0*)
apply (*frule-tac diff* = *diff* **and** *n* = *n* **in** *Maclaurin-zero*, *assumption*)
apply (*drule not0-implies-Suc*)
apply (*rule-tac x* = *0* **in** *exI*, *force*)
apply (*frule-tac diff* = *diff* **and** *n* = *n* **in** *Maclaurin-all-lt*, *auto*)

apply (*rule-tac* $x = t$ **in** *exI*, *auto*)
done

lemma *Maclaurin-all-le-objl*: *diff 0* $= f$ &
$(\forall\, m\ x.\ DERIV\ (diff\ m)\ x :> diff\ (Suc\ m)\ x)$
$--> (\exists\, t.\ abs\ t \le abs\ x\ \&$
$f\ x = (\sum m{=}0..{<}n.\ (diff\ m\ 0\ /\ real\ (fact\ m)) * x\ \hat{}\ m) +$
$(diff\ n\ t\ /\ real\ (fact\ n)) * x\ \hat{}\ n)$
by (*blast intro*: *Maclaurin-all-le*)

41.3 Version for Exponential Function

lemma *Maclaurin-exp-lt*: $[|\ x\ {}^{\sim}{=}\ 0;\ n > 0\ |]$
$==> (\exists\, t.\ 0 < abs\ t\ \&$
$abs\ t < abs\ x\ \&$
$exp\ x = (\sum m{=}0..{<}n.\ (x\ \hat{}\ m)\ /\ real\ (fact\ m)) +$
$(exp\ t\ /\ real\ (fact\ n)) * x\ \hat{}\ n)$
by (*cut-tac diff* $= \%n.\ exp$ **and** $f = exp$ **and** $x = x$ **and** $n = n$ **in** *Maclaurin-all-lt-objl*, *auto*)

lemma *Maclaurin-exp-le*:
$\exists\, t.\ abs\ t \le abs\ x\ \&$
$exp\ x = (\sum m{=}0..{<}n.\ (x\ \hat{}\ m)\ /\ real\ (fact\ m)) +$
$(exp\ t\ /\ real\ (fact\ n)) * x\ \hat{}\ n$
by (*cut-tac diff* $= \%n.\ exp$ **and** $f = exp$ **and** $x = x$ **and** $n = n$ **in** *Maclaurin-all-le-objl*, *auto*)

41.4 Version for Sine Function

lemma *MVT2*:
$[|\ a < b;\ \forall x.\ a \le x\ \&\ x \le b\ --> DERIV\ f\ x :> f'(x)\ |]$
$==> \exists\, z{::}real.\ a < z\ \&\ z < b\ \&\ (f\ b - f\ a = (b - a) * f'(z))$
apply (*drule MVT*)
apply (*blast intro*: *DERIV-isCont*)
apply (*force dest*: *order-less-imp-le simp add*: *differentiable-def*)
apply (*blast dest*: *DERIV-unique order-less-imp-le*)
done

lemma *mod-exhaust-less-4*:
$m\ mod\ 4 = 0\ |\ m\ mod\ 4 = 1\ |\ m\ mod\ 4 = 2\ |\ m\ mod\ 4 = (3{::}nat)$
by *auto*

lemma *Suc-Suc-mult-two-diff-two* [*rule-format*, *simp*]:
$n \ne 0\ --> Suc\ (Suc\ (2 * n - 2)) = 2{*}n$
by (*induct n*, *auto*)

lemma *lemma-Suc-Suc-4n-diff-2* [*rule-format*, *simp*]:
$n \ne 0\ --> Suc\ (Suc\ (4{*}n - 2)) = 4{*}n$
by (*induct n*, *auto*)

lemma *Suc-mult-two-diff-one* [*rule-format*, *simp*]:
  n≠*0* −−> *Suc* (*2* ∗ *n* − *1*) = *2*∗*n*
by (*induct n*, *auto*)

It is unclear why so many variant results are needed.

lemma *Maclaurin-sin-expansion2*:
  ∃ *t*. *abs t* ≤ *abs x* &
   sin x =
   (∑ *m*=*0*..<*n*. (*if even m then 0*
          else (−*1* ˆ ((*m* − *Suc 0*) *div 2*)) / *real* (*fact m*)) ∗
          x ˆ *m*)
  + ((*sin*(*t* + *1*/*2* ∗ *real* (*n*) ∗*pi*) / *real* (*fact n*)) ∗ *x* ˆ *n*)
apply (*cut-tac f* = *sin* **and** *n* = *n* **and** *x* = *x*
  and *diff* = %*n x*. *sin* (*x* + *1*/*2*∗*real n* ∗ *pi*) **in** *Maclaurin-all-lt-objl*)
apply *safe*
apply (*simp* (*no-asm*))
apply (*simp* (*no-asm*))
apply (*case-tac n*, *clarify*, *simp*, *simp add*: *lemma-STAR-sin*)
apply (*rule ccontr*, *simp*)
apply (*drule-tac x* = *x* **in** *spec*, *simp*)
apply (*erule ssubst*)
apply (*rule-tac x* = *t* **in** *exI*, *simp*)
apply (*rule setsum-cong*[*OF refl*])
apply (*auto simp add*: *sin-zero-iff odd-Suc-mult-two-ex*)
done

lemma *Maclaurin-sin-expansion*:
  ∃ *t*. *sin x* =
   (∑ *m*=*0*..<*n*. (*if even m then 0*
          else (−*1* ˆ ((*m* − *Suc 0*) *div 2*)) / *real* (*fact m*)) ∗
          x ˆ *m*)
  + ((*sin*(*t* + *1*/*2* ∗ *real* (*n*) ∗*pi*) / *real* (*fact n*)) ∗ *x* ˆ *n*)
apply (*insert Maclaurin-sin-expansion2* [*of x n*])
apply (*blast intro*: *elim*:)
done

lemma *Maclaurin-sin-expansion3*:
  [| *n* > *0*; *0* < *x* |] ==>
   ∃ *t*. *0* < *t* & *t* < *x* &
   sin x =
   (∑ *m*=*0*..<*n*. (*if even m then 0*
          else (−*1* ˆ ((*m* − *Suc 0*) *div 2*)) / *real* (*fact m*)) ∗
          x ˆ *m*)
  + ((*sin*(*t* + *1*/*2* ∗ *real*(*n*) ∗*pi*) / *real* (*fact n*)) ∗ *x* ˆ *n*)
apply (*cut-tac f* = *sin* **and** *n* = *n* **and** *h* = *x* **and** *diff* = %*n x*. *sin* (*x* + *1*/*2*∗*real* (*n*) ∗*pi*) **in** *Maclaurin-objl*)
apply *safe*

apply *simp*
apply (*simp* (*no-asm*))
apply (*erule ssubst*)
apply (*rule-tac x* = *t* **in** *exI*, *simp*)
apply (*rule setsum-cong*[*OF refl*])
apply (*auto simp add*: *sin-zero-iff odd-Suc-mult-two-ex*)
done

lemma *Maclaurin-sin-expansion4*:
  0 < *x* ==>
   ∃ *t*. *0* < *t* & *t* ≤ *x* &
   sin x =
   (∑ *m*=*0*..<*n*. (*if even m then 0*
         else (−*1* ˆ ((*m* − *Suc 0*) *div 2*)) / *real* (*fact m*)) ∗
         x ˆ *m*)
  + ((*sin*(*t* + *1*/*2* ∗ *real* (*n*) ∗*pi*) / *real* (*fact n*)) ∗ *x* ˆ *n*)
apply (*cut-tac f* = *sin* **and** *n* = *n* **and** *h* = *x* **and** *diff* = %*n x*. *sin* (*x* + *1*/*2*∗*real* (*n*) ∗*pi*) **in** *Maclaurin2-objl*)
apply *safe*
apply *simp*
apply (*simp* (*no-asm*))
apply (*erule ssubst*)
apply (*rule-tac x* = *t* **in** *exI*, *simp*)
apply (*rule setsum-cong*[*OF refl*])
apply (*auto simp add*: *sin-zero-iff odd-Suc-mult-two-ex*)
done

41.5 Maclaurin Expansion for Cosine Function

lemma *sumr-cos-zero-one* [*simp*]:
 (∑ *m*=*0*..<(*Suc n*).
  (*if even m then* −*1* ˆ (*m div 2*)/(*real* (*fact m*)) *else 0*) ∗ *0* ˆ *m*) = *1*
by (*induct n*, *auto*)

lemma *Maclaurin-cos-expansion*:
  ∃ *t*. *abs t* ≤ *abs x* &
   cos x =
   (∑ *m*=*0*..<*n*. (*if even m*
         then −*1* ˆ (*m div 2*)/(*real* (*fact m*))
         else 0) ∗
         x ˆ *m*)
  + ((*cos*(*t* + *1*/*2* ∗ *real* (*n*) ∗*pi*) / *real* (*fact n*)) ∗ *x* ˆ *n*)
apply (*cut-tac f* = *cos* **and** *n* = *n* **and** *x* = *x* **and** *diff* = %*n x*. *cos* (*x* + *1*/*2*∗*real* (*n*) ∗*pi*) **in** *Maclaurin-all-lt-objl*)
apply *safe*
apply (*simp* (*no-asm*))
apply (*simp* (*no-asm*))
apply (*case-tac n*, *simp*)
apply (*simp del*: *setsum-op-ivl-Suc*)

```
apply (rule ccontr, simp)
apply (drule-tac x = x in spec, simp)
apply (erule ssubst)
apply (rule-tac x = t in exI, simp)
apply (rule setsum-cong[OF refl])
apply (auto simp add: cos-zero-iff even-mult-two-ex)
done

lemma Maclaurin-cos-expansion2:
   [| 0 < x; n > 0 |] ==>
    ∃t. 0 < t & t < x &
    cos x =
    (∑ m=0..<n. (if even m
                then −1 ^ (m div 2)/(real (fact m))
                else 0) *
                x ^ m)
   + ((cos(t + 1/2 * real (n) *pi) / real (fact n)) * x ^ n)
apply (cut-tac f = cos and n = n and h = x and diff = %n x. cos (x + 1/2*real
(n) *pi) in Maclaurin-objl)
apply safe
apply simp
apply (simp (no-asm))
apply (erule ssubst)
apply (rule-tac x = t in exI, simp)
apply (rule setsum-cong[OF refl])
apply (auto simp add: cos-zero-iff even-mult-two-ex)
done

lemma Maclaurin-minus-cos-expansion:
   [| x < 0; n > 0 |] ==>
    ∃t. x < t & t < 0 &
    cos x =
    (∑ m=0..<n. (if even m
                then −1 ^ (m div 2)/(real (fact m))
                else 0) *
                x ^ m)
   + ((cos(t + 1/2 * real (n) *pi) / real (fact n)) * x ^ n)
apply (cut-tac f = cos and n = n and h = x and diff = %n x. cos (x + 1/2*real
(n) *pi) in Maclaurin-minus-objl)
apply safe
apply simp
apply (simp (no-asm))
apply (erule ssubst)
apply (rule-tac x = t in exI, simp)
apply (rule setsum-cong[OF refl])
apply (auto simp add: cos-zero-iff even-mult-two-ex)
done
```

lemma *sin-bound-lemma*:
  $[|x = y;\ abs\ u \leq (v{::}real)\ |] ==> |(x + u) - y| \leq v$
by *auto*

lemma *Maclaurin-sin-bound*:
  $abs(sin\ x - (\sum m{=}0..{<}n.\ (if\ even\ m\ then\ 0\ else\ (-1\ \hat{}\ ((m - Suc\ 0)\ div\ 2))\ /\ real\ (fact\ m))\ *$
  $x\ \hat{}\ m))\ \leq inverse(real\ (fact\ n))\ *\ |x|\ \hat{}\ n$
proof −
  have !! $x\ (y{::}real).\ x \leq 1 \Longrightarrow 0 \leq y \Longrightarrow x * y \leq 1 * y$
    by (*rule-tac mult-right-mono*,*simp-all*)
  note *est* = *this*[*simplified*]
  let $?diff = \lambda(n{::}nat)\ x.\ if\ n\ mod\ 4 = 0\ then\ sin(x)\ else\ if\ n\ mod\ 4 = 1\ then\ cos(x)\ else\ if\ n\ mod\ 4 = 2\ then\ -sin(x)\ else\ -cos(x)$
  have *diff-0*: *?diff 0 = sin* **by** *simp*
  have *DERIV-diff*: $\forall\ m\ x.\ DERIV\ (?diff\ m)\ x :> ?diff\ (Suc\ m)\ x$
    apply (*clarify*)
    apply (*subst* (*1 2 3*) *mod-Suc-eq-Suc-mod*)
    apply (*cut-tac m=m* **in** *mod-exhaust-less-4*)
    apply (*safe*, *simp-all*)
    apply (*rule DERIV-minus*, *simp*)
    apply (*rule lemma-DERIV-subst*, *rule DERIV-minus*, *rule DERIV-cos*, *simp*)
    done
  from *Maclaurin-all-le* [*OF diff-0 DERIV-diff*]
  obtain *t* **where** *t1*: $|t| \leq |x|$ **and**
    t2: $sin\ x = (\sum m = 0..{<}n.\ ?diff\ m\ 0\ /\ real\ (fact\ m) * x\ \hat{}\ m)\ +$
      $?diff\ n\ t\ /\ real\ (fact\ n) * x\ \hat{}\ n$ **by** *fast*
  have *diff-m-0*:
    $\bigwedge m.\ ?diff\ m\ 0 = (if\ even\ m\ then\ 0$
        $else\ -1\ \hat{}\ ((m - Suc\ 0)\ div\ 2))$
    apply (*subst even-even-mod-4-iff*)
    apply (*cut-tac m=m* **in** *mod-exhaust-less-4*)
    apply (*elim disjE*, *simp-all*)
    apply (*safe dest*!: *mod-eqD*, *simp-all*)
    done
  show *?thesis*
    apply (*subst t2*)
    apply (*rule sin-bound-lemma*)
    apply (*rule setsum-cong*[*OF refl*])
    apply (*subst diff-m-0*, *simp*)
    apply (*auto intro*: *mult-right-mono* [**where** *b=1*, *simplified*] *mult-right-mono*
          simp add: *est mult-nonneg-nonneg mult-ac divide-inverse*
            power-abs [*symmetric*] *abs-mult*)
    done
qed

end

42 Taylor: Taylor series

theory *Taylor*
imports *MacLaurin*
begin

We use MacLaurin and the translation of the expansion point *c* to *0* to prove Taylor's theorem.

lemma *taylor-up*:
  assumes *INIT*: *n>0 diff 0 = f*
  and *DERIV*: (∀ *m t*. *m* < *n* & *a* ≤ *t* & *t* ≤ *b* ⟶ *DERIV* (*diff m*) *t* :> (*diff* (*Suc m*) *t*))
  and *INTERV*: *a* ≤ *c* *c* < *b*
  shows ∃ *t*. *c* < *t* & *t* < *b* &
    f b = *setsum* (%*m*. (*diff m c* / *real* (*fact m*)) ∗ (*b* − *c*)^*m*) {*0..<n*} +
      (*diff n t* / *real* (*fact n*)) ∗ (*b* − *c*)^*n*
proof −
  from *INTERV* **have** *0* < *b*−*c* **by** *arith*
  moreover
  from *INIT* **have** *n>0* ((λ*m x*. *diff m* (*x* + *c*)) *0*) = (λ*x*. *f* (*x* + *c*)) **by** *auto*
  moreover
  have *ALL m t*. *m* < *n* & *0* <= *t* & *t* <= *b* − *c* −−> *DERIV* (%*x*. *diff m* (*x* + *c*)) *t* :> *diff* (*Suc m*) (*t* + *c*)
  proof (*intro strip*)
    fix *m t*
    assume *m* < *n* & *0* <= *t* & *t* <= *b* − *c*
    with *DERIV* **and** *INTERV* **have** *DERIV* (*diff m*) (*t* + *c*) :> *diff* (*Suc m*) (*t* + *c*) **by** *auto*
    moreover
    from *DERIV-ident* **and** *DERIV-const* **have** *DERIV* (%*x*. *x* + *c*) *t* :> *1+0* **by** (*rule DERIV-add*)
    ultimately have *DERIV* (%*x*. *diff m* (*x* + *c*)) *t* :> *diff* (*Suc m*) (*t* + *c*) ∗ (*1+0*)
      by (*rule DERIV-chain2*)
    thus *DERIV* (%*x*. *diff m* (*x* + *c*)) *t* :> *diff* (*Suc m*) (*t* + *c*) **by** *simp*
  qed
  ultimately
  have *EX*:*EX t>0*. *t* < *b* − *c* &
    f (*b* − *c* + *c*) = (*SUM m* = *0..<n*. *diff m* (*0* + *c*) / *real* (*fact m*) ∗ (*b* − *c*) ^ *m*) +
      diff n (*t* + *c*) / *real* (*fact n*) ∗ (*b* − *c*) ^ *n*
    by (*rule Maclaurin*)
  show *?thesis*
  proof −
    from *EX* **obtain** *x* **where**
      X: *0* < *x* & *x* < *b* − *c* &

$f\ (b - c + c) = (\sum m = 0..<n.\ \mathit{diff}\ m\ (0 + c) / \mathit{real}\ (\mathit{fact}\ m) * (b - c)\ \hat{}\ m) +$
$\mathit{diff}\ n\ (x + c) / \mathit{real}\ (\mathit{fact}\ n) * (b - c)\ \hat{}\ n$ **..**
let $?H = x + c$
from X **have** $c<?H\ \&\ ?H<b \wedge f\ b = (\sum m = 0..<n.\ \mathit{diff}\ m\ c / \mathit{real}\ (\mathit{fact}\ m) * (b - c)\ \hat{}\ m) +$
$\mathit{diff}\ n\ ?H / \mathit{real}\ (\mathit{fact}\ n) * (b - c)\ \hat{}\ n$
by *fastsimp*
thus *?thesis* **by** *fastsimp*
qed
qed

lemma *taylor-down*:
assumes *INIT*: $n>0$ $\mathit{diff}\ 0 = f$
and *DERIV*: $(\forall\ m\ t.\ m < n\ \&\ a \le t\ \&\ t \le b \longrightarrow \mathit{DERIV}\ (\mathit{diff}\ m)\ t :> (\mathit{diff}\ (\mathit{Suc}\ m)\ t))$
and *INTERV*: $a < c$ $c \le b$
shows $\exists\ t.\ a < t\ \&\ t < c\ \&$
$f\ a = \mathit{setsum}\ (\%\ m.\ (\mathit{diff}\ m\ c / \mathit{real}\ (\mathit{fact}\ m)) * (a - c)\hat{}m)\ \{0..<n\} +$
$(\mathit{diff}\ n\ t / \mathit{real}\ (\mathit{fact}\ n)) * (a - c)\hat{}n$
proof −
from *INTERV* **have** $a-c < 0$ **by** *arith*
moreover
from *INIT* **have** $n>0$ $((\lambda m\ x.\ \mathit{diff}\ m\ (x + c))\ 0) = (\lambda x.\ f\ (x + c))$ **by** *auto*
moreover
have $ALL\ m\ t.\ m < n\ \&\ a-c <= t\ \&\ t <= 0 \dashrightarrow \mathit{DERIV}\ (\%x.\ \mathit{diff}\ m\ (x + c))\ t :> \mathit{diff}\ (\mathit{Suc}\ m)\ (t + c)$
proof (*rule allI impI*)+
fix *m t*
assume $m < n\ \&\ a-c <= t\ \&\ t <= 0$
with *DERIV* **and** *INTERV* **have** $\mathit{DERIV}\ (\mathit{diff}\ m)\ (t + c) :> \mathit{diff}\ (\mathit{Suc}\ m)\ (t + c)$ **by** *auto*
moreover
from *DERIV-ident* **and** *DERIV-const* **have** $\mathit{DERIV}\ (\%x.\ x + c)\ t :> 1+0$ **by** (*rule DERIV-add*)
ultimately have $\mathit{DERIV}\ (\%x.\ \mathit{diff}\ m\ (x + c))\ t :> \mathit{diff}\ (\mathit{Suc}\ m)\ (t + c) * (1+0)$ **by** (*rule DERIV-chain2*)
thus $\mathit{DERIV}\ (\%x.\ \mathit{diff}\ m\ (x + c))\ t :> \mathit{diff}\ (\mathit{Suc}\ m)\ (t + c)$ **by** *simp*
qed
ultimately
have *EX*: $EX\ t>a - c.\ t < 0\ \&$
$f\ (a - c + c) = (SUM\ m = 0..<n.\ \mathit{diff}\ m\ (0 + c) / \mathit{real}\ (\mathit{fact}\ m) * (a - c)\ \hat{}\ m) +$
$\mathit{diff}\ n\ (t + c) / \mathit{real}\ (\mathit{fact}\ n) * (a - c)\ \hat{}\ n$
by (*rule Maclaurin-minus*)
show *?thesis*
proof −
from *EX* **obtain** *x* **where** *X*: $a - c < x\ \&\ x < 0\ \&$
$f\ (a - c + c) = (SUM\ m = 0..<n.\ \mathit{diff}\ m\ (0 + c) / \mathit{real}\ (\mathit{fact}\ m) * (a - c)$

$\hat{}\ m) +$
$\quad diff\ n\ (x + c)\ /\ real\ (fact\ n) * (a - c)\ \hat{}\ n$ **..**
let $?H = x + c$
from X **have** $a<?H\ \&\ ?H<c \wedge f\ a = (\sum m = 0..<n.\ diff\ m\ c\ /\ real\ (fact\ m) * (a - c)\ \hat{}\ m) +$
$\quad diff\ n\ ?H\ /\ real\ (fact\ n) * (a - c)\ \hat{}\ n$
by *fastsimp*
thus *?thesis* **by** *fastsimp*
qed
qed

lemma *taylor*:
assumes $INIT$: $n>0\ diff\ 0 = f$
and $DERIV$: $(\forall\ m\ t.\ m < n\ \&\ a \leq t\ \&\ t \leq b \longrightarrow DERIV\ (diff\ m)\ t :> (diff\ (Suc\ m)\ t))$
and $INTERV$: $a \leq c\ \ c \leq b\ a \leq x\ x \leq b\ x \neq c$
shows $\exists\ t.\ (if\ x<c\ then\ (x < t\ \&\ t < c)\ else\ (c < t\ \&\ t < x))\ \&$
$f\ x = setsum\ (\%\ m.\ (diff\ m\ c\ /\ real\ (fact\ m)) * (x - c)\hat{}m)\ \{0..<n\} +$
$(diff\ n\ t\ /\ real\ (fact\ n)) * (x - c)\hat{}n$
proof (*cases* $x<c$)
case *True*
note *INIT*
moreover from *DERIV* **and** *INTERV*
have $\forall m\ t.\ m < n \wedge x \leq t \wedge t \leq b \longrightarrow DERIV\ (diff\ m)\ t :> diff\ (Suc\ m)\ t$
by *fastsimp*
moreover note *True*
moreover from *INTERV* **have** $c \leq b$ **by** *simp*
ultimately have EX: $\exists t>x.\ t < c \wedge f\ x =$
$(\sum m = 0..<n.\ diff\ m\ c\ /\ real\ (fact\ m) * (x - c)\ \hat{}\ m) +$
$diff\ n\ t\ /\ real\ (fact\ n) * (x - c)\ \hat{}\ n$
by (*rule taylor-down*)
with *True* **show** *?thesis* **by** *simp*
next
case *False*
note *INIT*
moreover from *DERIV* **and** *INTERV*
have $\forall m\ t.\ m < n \wedge a \leq t \wedge t \leq x \longrightarrow DERIV\ (diff\ m)\ t :> diff\ (Suc\ m)\ t$
by *fastsimp*
moreover from *INTERV* **have** $a \leq c$ **by** *arith*
moreover from *False* **and** *INTERV* **have** $c < x$ **by** *arith*
ultimately have EX: $\exists t>c.\ t < x \wedge f\ x =$
$(\sum m = 0..<n.\ diff\ m\ c\ /\ real\ (fact\ m) * (x - c)\ \hat{}\ m) +$
$diff\ n\ t\ /\ real\ (fact\ n) * (x - c)\ \hat{}\ n$
by (*rule taylor-up*)
with *False* **show** *?thesis* **by** *simp*
qed

end

43 Integration: Theory of Integration

theory *Integration*
imports *MacLaurin*
begin

We follow John Harrison in formalizing the Gauge integral.

definition
— Partitions and tagged partitions etc.

$partition :: [(real*real),nat => real] => bool$ **where**
$partition = (\%(a,b)\ D.\ D\ 0 = a\ \&$
$(\exists N.\ (\forall n < N.\ D(n) < D(Suc\ n))\ \&$
$(\forall n \geq N.\ D(n) = b)))$

definition
$psize :: (nat => real) => nat$ **where**
$psize\ D = (SOME\ N.\ (\forall n < N.\ D(n) < D(Suc\ n))\ \&$
$(\forall n \geq N.\ D(n) = D(N)))$

definition
$tpart :: [(real*real),((nat => real)*(nat =>real))] => bool$ **where**
$tpart = (\%(a,b)\ (D,p).\ partition(a,b)\ D\ \&$
$(\forall n.\ D(n) \leq p(n)\ \&\ p(n) \leq D(Suc\ n)))$

— Gauges and gauge-fine divisions

definition
$gauge :: [real => bool,\ real => real] => bool$ **where**
$gauge\ E\ g = (\forall x.\ E\ x\ {-}{-}{>}\ 0 < g(x))$

definition
$fine :: [real => real,\ ((nat => real)*(nat => real))] => bool$ **where**
$fine = (\%g\ (D,p).\ \forall n.\ n < (psize\ D)\ {-}{-}{>}\ D(Suc\ n) - D(n) < g(p\ n))$

— Riemann sum

definition
$rsum :: [((nat=>real)*(nat=>real)),real=>real] => real$ **where**
$rsum = (\%(D,p)\ f.\ \sum n{=}0..{<}psize(D).\ f(p\ n) * (D(Suc\ n) - D(n)))$

— Gauge integrability (definite)

definition
$Integral :: [(real*real),real=>real,real] => bool$ **where**
$Integral = (\%(a,b)\ f\ k.\ \forall e > 0.$
$(\exists g.\ gauge(\%x.\ a \leq x\ \&\ x \leq b)\ g\ \&$
$(\forall D\ p.\ tpart(a,b)\ (D,p)\ \&\ fine(g)(D,p)\ {-}{-}{>}$
$|rsum(D,p)\ f - k| < e)))$

lemma *partition-zero* [*simp*]: *a* = *b* ==> *psize* (%*n*. *if n* = *0 then a else b*) = *0*
by (*auto simp add*: *psize-def*)

lemma *partition-one* [*simp*]: *a* < *b* ==> *psize* (%*n*. *if n* = *0 then a else b*) = *1*
apply (*simp add*: *psize-def*)
apply (*rule some-equality*, *auto*)
apply (*drule-tac x* = *1* **in** *spec*, *auto*)
done

lemma *partition-single* [*simp*]:
  $a \leq b$ ==> *partition*(*a*,*b*)(%*n*. *if n* = *0 then a else b*)
by (*auto simp add*: *partition-def order-le-less*)

lemma *partition-lhs*: *partition*(*a*,*b*) *D* ==> (*D*(*0*) = *a*)
by (*simp add*: *partition-def*)

lemma *partition*:
  (*partition*(*a*,*b*) *D*) =
  ((*D 0* = *a*) &
  ($\forall$ *n* < *psize D*. *D n* < *D*(*Suc n*)) &
  ($\forall$ *n* $\geq$ *psize D*. *D n* = *b*))
apply (*simp add*: *partition-def*, *auto*)
apply (*subgoal-tac* [!] *psize D* = *N*, *auto*)
apply (*simp-all* (*no-asm*) *add*: *psize-def*)
apply (*rule-tac* [!] *some-equality*, *blast*)
 prefer *2* **apply** *blast*
apply (*rule-tac* [!] *ccontr*)
apply (*simp-all add*: *linorder-neq-iff*, *safe*)
apply (*drule-tac x* = *Na* **in** *spec*)
apply (*rotate-tac 3*)
apply (*drule-tac x* = *Suc Na* **in** *spec*, *simp*)
apply (*rotate-tac 2*)
apply (*drule-tac x* = *N* **in** *spec*, *simp*)
apply (*drule-tac x* = *Na* **in** *spec*)
apply (*drule-tac x* = *Suc Na* **and** *P* = %*n*. *Na*$\leq$*n* $\longrightarrow$ *D n* = *D Na* **in** *spec*, *auto*)
done

lemma *partition-rhs*: *partition*(*a*,*b*) *D* ==> (*D*(*psize D*) = *b*)
by (*simp add*: *partition*)

lemma *partition-rhs2*: [|*partition*(*a*,*b*) *D*; *psize D* $\leq$ *n* |] ==> (*D n* = *b*)
by (*simp add*: *partition*)

lemma *lemma-partition-lt-gen* [*rule-format*]:
 partition(*a*,*b*) *D* & *m* + *Suc d* $\leq$ *n* & *n* $\leq$ (*psize D*) --> *D*(*m*) < *D*(*m* + *Suc d*)

apply (*induct d*, *auto simp add*: *partition*)
apply (*blast dest*: *Suc-le-lessD intro*: *less-le-trans order-less-trans*)
done

lemma *less-eq-add-Suc*: $m < n ==> \exists\, d.\ n = m + Suc\ d$
by (*auto simp add*: *less-iff-Suc-add*)

lemma *partition-lt-gen*:
$[|partition(a,b)\ D;\ m < n;\ n \leq (psize\ D)|] ==> D(m) < D(n)$
by (*auto dest*: *less-eq-add-Suc intro*: *lemma-partition-lt-gen*)

lemma *partition-lt*: $partition(a,b)\ D ==> n < (psize\ D) ==> D(0) < D(Suc\ n)$
apply (*induct n*)
apply (*auto simp add*: *partition*)
done

lemma *partition-le*: $partition(a,b)\ D ==> a \leq b$
apply (*frule partition* [*THEN iffD1*], *safe*)
apply (*drule-tac* $x = psize\ D$ **and** $P{=}\%n.\ psize\ D \leq n\ {-}{-}{>}\ ?P\ n$ **in** *spec*, *safe*)
apply (*case-tac* $psize\ D = 0$)
apply (*drule-tac* [*2*] $n = psize\ D - 1$ **in** *partition-lt*, *auto*)
done

lemma *partition-gt*: $[|partition(a,b)\ D;\ n < (psize\ D)|] ==> D(n) < D(psize\ D)$
by (*auto intro*: *partition-lt-gen*)

lemma *partition-eq*: $partition(a,b)\ D ==> ((a = b) = (psize\ D = 0))$
apply (*frule partition* [*THEN iffD1*], *safe*)
apply (*rotate-tac 2*)
apply (*drule-tac* $x = psize\ D$ **in** *spec*)
apply (*rule ccontr*)
apply (*drule-tac* $n = psize\ D - 1$ **in** *partition-lt*)
apply *auto*
done

lemma *partition-lb*: $partition(a,b)\ D ==> a \leq D(r)$
apply (*frule partition* [*THEN iffD1*], *safe*)
apply (*induct r*)
apply (*cut-tac* [*2*] $y = Suc\ r$ **and** $x = psize\ D$ **in** *linorder-le-less-linear*)
apply (*auto intro*: *partition-le*)
apply (*drule-tac* $x = r$ **in** *spec*)
apply *arith*
done

lemma *partition-lb-lt*: $[|\ partition(a,b)\ D;\ psize\ D\ {\sim}{=}\ 0\ |] ==> a < D(Suc\ n)$
apply (*rule-tac* $t = a$ **in** *partition-lhs* [*THEN subst*], *assumption*)
apply (*cut-tac* $x = Suc\ n$ **and** $y = psize\ D$ **in** *linorder-le-less-linear*)
apply (*frule partition* [*THEN iffD1*], *safe*)
apply (*blast intro*: *partition-lt less-le-trans*)

apply (*rotate-tac 3*)
apply (*drule-tac x = Suc n* **in** *spec*)
apply (*erule impE*)
apply (*erule less-imp-le*)
apply (*frule partition-rhs*)
apply (*drule partition-gt*[*of - - - 0*], *arith*)
apply (*simp* (*no-asm-simp*))
done

lemma *partition-ub*: *partition*(*a*,*b*) *D* ==> *D*(*r*) ≤ *b*
apply (*frule partition* [*THEN iffD1*])
apply (*cut-tac x = psize D* **and** *y = r* **in** *linorder-le-less-linear*, *safe*, *blast*)
apply (*subgoal-tac* ∀ *x*. *D* ((*psize D*) − *x*) ≤ *b*)
apply (*rotate-tac 4*)
apply (*drule-tac x = psize D − r* **in** *spec*)
apply (*subgoal-tac psize D* − (*psize D* − *r*) = *r*)
apply *simp*
apply *arith*
apply *safe*
apply (*induct-tac x*)
apply (*simp* (*no-asm*), *blast*)
apply (*case-tac psize D − Suc n = 0*)
apply (*erule-tac V* = ∀ *n*. *psize D* ≤ *n* −−> *D n* = *b* **in** *thin-rl*)
apply (*simp* (*no-asm-simp*) *add*: *partition-le*)
apply (*rule order-trans*)
prefer *2* **apply** *assumption*
apply (*subgoal-tac psize D − n = Suc* (*psize D − Suc n*))
prefer *2* **apply** *arith*
apply (*drule-tac x = psize D − Suc n* **in** *spec*, *simp*)
done

lemma *partition-ub-lt*: [| *partition*(*a*,*b*) *D*; *n* < *psize D* |] ==> *D*(*n*) < *b*
by (*blast intro*: *partition-rhs* [*THEN subst*] *partition-gt*)

lemma *lemma-partition-append1*:
[| *partition* (*a*, *b*) *D1*; *partition* (*b*, *c*) *D2* |]
==> (∀ *n* < *psize D1* + *psize D2*.
(*if n* < *psize D1 then D1 n else D2* (*n* − *psize D1*))
< (*if Suc n* < *psize D1 then D1* (*Suc n*)
else D2 (*Suc n* − *psize D1*))) &
(∀ *n* ≥ *psize D1* + *psize D2*.
(*if n* < *psize D1 then D1 n else D2* (*n* − *psize D1*)) =
(*if psize D1* + *psize D2* < *psize D1 then D1* (*psize D1* + *psize D2*)
else D2 (*psize D1* + *psize D2* − *psize D1*)))
apply (*auto intro*: *partition-lt-gen*)
apply (*subgoal-tac psize D1 = Suc n*)
apply (*auto intro*!: *partition-lt-gen simp add*: *partition-lhs partition-ub-lt*)
apply (*auto intro*!: *partition-rhs2 simp add*: *partition-rhs*
split: *nat-diff-split*)

done

lemma *lemma-psize1*:
[| *partition* (*a*, *b*) *D1*; *partition* (*b*, *c*) *D2*; *N* < *psize D1* |]
==> *D1*(*N*) < *D2* (*psize D2*)
apply (*rule-tac y* = *D1* (*psize D1*) **in** *order-less-le-trans*)
apply (*erule partition-gt*)
apply (*auto simp add*: *partition-rhs partition-le*)
done

lemma *lemma-partition-append2*:
[| *partition* (*a*, *b*) *D1*; *partition* (*b*, *c*) *D2* |]
==> *psize* (%*n*. *if n* < *psize D1 then D1 n else D2* (*n* − *psize D1*)) =
psize D1 + *psize D2*
apply (*unfold psize-def*
[*of* %*n*. *if n* < *psize D1 then D1 n else D2* (*n* − *psize D1*)])
apply (*rule some1-equality*)
prefer *2* **apply** (*blast intro*!: *lemma-partition-append1*)
apply (*rule ex1I*, *rule lemma-partition-append1*)
apply (*simp-all split*: *split-if-asm*)

The case *N* < *psize D1*

apply (*drule-tac x* = *psize D1* + *psize D2* **and** *P*=%*n*. *?P n* & *?Q n* **in** *spec*)
apply (*force dest*: *lemma-psize1*)
apply (*rule order-antisym*)

The case *psize D1* ≤ *N*: proving *N* ≤ *psize D1* + *psize D2*

apply (*drule-tac x* = *psize D1* + *psize D2* **in** *spec*)
apply (*simp add*: *partition-rhs2*)
apply (*case-tac N* − *psize D1* < *psize D2*)
prefer *2* **apply** *arith*

Proving *psize D1* + *psize D2* ≤ *N*

apply (*drule-tac x* = *psize D1* + *psize D2* **and** *P*=%*n*. *N*≤*n* −−> *?P n* **in** *spec*, *simp*)
apply (*drule-tac a* = *b* **and** *b* = *c* **in** *partition-gt*, *assumption*, *simp*)
done

lemma *tpart-eq-lhs-rhs*: [|*psize D* = *0*; *tpart*(*a*,*b*) (*D*,*p*)|] ==> *a* = *b*
by (*auto simp add*: *tpart-def partition-eq*)

lemma *tpart-partition*: *tpart*(*a*,*b*) (*D*,*p*) ==> *partition*(*a*,*b*) *D*
by (*simp add*: *tpart-def*)

lemma *partition-append*:
[| *tpart*(*a*,*b*) (*D1*,*p1*); *fine*(*g*) (*D1*,*p1*);
tpart(*b*,*c*) (*D2*,*p2*); *fine*(*g*) (*D2*,*p2*) |]
==> ∃ *D p*. *tpart*(*a*,*c*) (*D*,*p*) & *fine*(*g*) (*D*,*p*)
apply (*rule-tac x* = %*n*. *if n* < *psize D1 then D1 n else D2* (*n* − *psize D1*)

in *exI*)
apply (*rule-tac x* = %*n*. *if n* < *psize D1 then p1 n else p2* (*n* − *psize D1*)
in *exI*)
apply (*case-tac psize D1* = *0*)
apply (*auto dest*: *tpart-eq-lhs-rhs*)
prefer *2*
apply (*simp add*: *fine-def*
lemma-partition-append2 [*OF tpart-partition tpart-partition*])
— But must not expand *fine* in other subgoals
apply *auto*
apply (*subgoal-tac psize D1* = *Suc n*)
prefer *2* **apply** *arith*
apply (*drule tpart-partition* [*THEN partition-rhs*])
apply (*drule tpart-partition* [*THEN partition-lhs*])
apply (*auto split*: *nat-diff-split*)
apply (*auto simp add*: *tpart-def*)
defer *1*
apply (*subgoal-tac psize D1* = *Suc n*)
prefer *2* **apply** *arith*
apply (*drule partition-rhs*)
apply (*drule partition-lhs*, *auto*)
apply (*simp split*: *nat-diff-split*)
apply (*subst partition*)
apply (*subst* (*1 2*) *lemma-partition-append2*, *assumption*+)
apply (*rule conjI*)
apply (*simp add*: *partition-lhs*)
apply (*drule lemma-partition-append1*)
apply *assumption*
apply (*simp add*: *partition-rhs*)
done

We can always find a division that is fine wrt any gauge

lemma *partition-exists*:
[| $a \le b$; *gauge*(%*x*. $a \le x$ & $x \le b$) *g* |]
==> ∃ *D p*. *tpart*(*a*,*b*) (*D*,*p*) & *fine g* (*D*,*p*)
apply (*cut-tac P* = %(*u*,*v*). $a \le u$ & $v \le b$ −−>
(∃ *D p*. *tpart* (*u*,*v*) (*D*,*p*) & *fine* (*g*) (*D*,*p*))
in *lemma-BOLZANO2*)
apply *safe*
apply (*blast intro*: *order-trans*)+
apply (*auto intro*: *partition-append*)
apply (*case-tac* $a \le x$ & $x \le b$)
apply (*rule-tac* [*2*] *x* = *1* **in** *exI*, *auto*)
apply (*rule-tac x* = *g x* **in** *exI*)
apply (*auto simp add*: *gauge-def*)
apply (*rule-tac x* = %*n*. *if n* = *0 then aa else ba* **in** *exI*)
apply (*rule-tac x* = %*n*. *if n* = *0 then x else ba* **in** *exI*)
apply (*auto simp add*: *tpart-def fine-def*)
done

Lemmas about combining gauges

lemma *gauge-min*:
$[|$ *gauge*(E) *g1*; *gauge*(E) *g2* $|]$
$==>$ *gauge*(E) (%*x*. *if g1*$(x) < g2(x)$ *then g1*(x) *else g2*(x))
by (*simp add*: *gauge-def*)

lemma *fine-min*:
fine (%*x*. *if g1*$(x) < g2(x)$ *then g1*(x) *else g2*(x)) (D,p)
$==>$ *fine*(*g1*) (D,p) & *fine*(*g2*) (D,p)
by (*auto simp add*: *fine-def split*: *split-if-asm*)

The integral is unique if it exists

lemma *Integral-unique*:
$[|$ $a \leq b$; *Integral*(a,b) *f k1*; *Integral*(a,b) *f k2* $|]$ $==>$ *k1* = *k2*
apply (*simp add*: *Integral-def*)
apply (*drule-tac* x = $|$*k1* $-$ *k2*$|$ /*2* **in** *spec*)+
apply *auto*
apply (*drule gauge-min*, *assumption*)
apply (*drule-tac* g = %*x*. *if g x* $<$ *ga x then g x else ga x*
in *partition-exists*, *assumption*, *auto*)
apply (*drule fine-min*)
apply (*drule spec*)+
apply *auto*
apply (*subgoal-tac* $|$(*rsum* (D,p) *f* $-$ *k2*) $-$ (*rsum* (D,p) *f* $-$ *k1*)$|$ $<$ $|$*k1* $-$ *k2*$|$)
apply *arith*
apply (*drule add-strict-mono*, *assumption*)
apply (*auto simp only*: *left-distrib* [*symmetric*] *mult-2-right* [*symmetric*]
mult-less-cancel-right)
done

lemma *Integral-zero* [*simp*]: *Integral*(a,a) *f 0*
apply (*auto simp add*: *Integral-def*)
apply (*rule-tac* x = %*x*. *1* **in** *exI*)
apply (*auto dest*: *partition-eq simp add*: *gauge-def tpart-def rsum-def*)
done

lemma *sumr-partition-eq-diff-bounds* [*simp*]:
($\sum$ *n*=*0*..$<$*m*. *D* (*Suc n*) $-$ *D n*::*real*) = *D*(m) $-$ *D 0*
by (*induct m*, *auto*)

lemma *Integral-eq-diff-bounds*: $a \leq b$ $==>$ *Integral*(a,b) (%*x*. *1*) $(b - a)$
apply (*auto simp add*: *order-le-less rsum-def Integral-def*)
apply (*rule-tac* x = %*x*. $b - a$ **in** *exI*)
apply (*auto simp add*: *gauge-def abs-less-iff tpart-def partition*)
done

lemma *Integral-mult-const*: $a \leq b$ $==>$ *Integral*(a,b) (%*x*. *c*) (*c*∗$(b - a)$)
apply (*auto simp add*: *order-le-less rsum-def Integral-def*)
apply (*rule-tac* x = %*x*. $b - a$ **in** *exI*)

apply (*auto simp add*: *setsum-right-distrib* [*symmetric*] *gauge-def abs-less-iff*
right-diff-distrib [*symmetric*] *partition tpart-def*)
done

lemma *Integral-mult*:
[| $a \le b$; *Integral*(a,b) f k |] ==> *Integral*(a,b) (%x. $c * f\,x$) $(c * k)$
apply (*auto simp add*: *order-le-less*
dest: *Integral-unique* [*OF order-refl Integral-zero*])
apply (*auto simp add*: *rsum-def Integral-def setsum-right-distrib*[*symmetric*] *mult-assoc*)
apply (*rule-tac a2* = *c* **in** *abs-ge-zero* [*THEN order-le-imp-less-or-eq*, *THEN disjE*])
prefer *2* **apply** *force*
apply (*drule-tac* $x = e/abs\ c$ **in** *spec*, *auto*)
apply (*simp add*: *zero-less-mult-iff divide-inverse*)
apply (*rule exI*, *auto*)
apply (*drule spec*)+
apply *auto*
apply (*rule-tac z1* = *inverse* (*abs c*) **in** *real-mult-less-iff1* [*THEN iffD1*])
apply (*auto simp add*: *abs-mult divide-inverse* [*symmetric*] *right-diff-distrib* [*symmetric*])
done

Fundamental theorem of calculus (Part I)

"Straddle Lemma" : Swartz and Thompson: AMM 95(7) 1988

lemma *choiceP*: $\forall x.\ P(x)$ --> ($\exists y.\ Q\ x\ y$) ==> $\exists f.$ ($\forall x.\ P(x)$ --> $Q\ x\ (f\ x)$)
by (*insert bchoice* [*of Collect P Q*], *simp*)

lemma *strad1*:
$[\![\forall xa{::}real.\ xa \neq x \wedge |xa - x| < s \longrightarrow$
$|(f\,xa - f\,x) / (xa - x) - f'\,x| * 2 < e;$
$0 < e;\ a \le x;\ x \le b;\ 0 < s]\!]$
$\Longrightarrow \forall z.\ |z - x| < s$ -->$|f\,z - f\,x - f'\,x * (z - x)| * 2 \le e * |z - x|$
apply *auto*
apply (*case-tac* $0 < |z - x|$)
prefer *2* **apply** (*simp add*: *zero-less-abs-iff*)
apply (*drule-tac* $x = z$ **in** *spec*)
apply (*rule-tac z1* = |*inverse* $(z - x)$|
in *real-mult-le-cancel-iff2* [*THEN iffD1*])
apply *simp*
apply (*simp del*: *abs-inverse abs-mult add*: *abs-mult* [*symmetric*]
mult-assoc [*symmetric*])
apply (*subgoal-tac inverse* $(z - x) * (f\,z - f\,x - f'\,x * (z - x))$
$= (f\,z - f\,x) / (z - x) - f'\,x$)
apply (*simp add*: *abs-mult* [*symmetric*] *mult-ac diff-minus*)
apply (*subst mult-commute*)
apply (*simp add*: *left-distrib diff-minus*)

apply (*simp add*: *mult-assoc divide-inverse*)
apply (*simp add*: *left-distrib*)
done

lemma *lemma-straddle*:
$[| \forall x.\ a \leq x\ \&\ x \leq b\ {-}{-}{>}\ DERIV\, f\, x :> f'(x);\ 0 < e\ |]$
$==> \exists g.\ gauge(\%x.\ a \leq x\ \&\ x \leq b)\ g\ \&$
$(\forall x\ u\ v.\ a \leq u\ \&\ u \leq x\ \&\ x \leq v\ \&\ v \leq b\ \&\ (v - u) < g(x)$
$-{-}{>}\ |(f(v) - f(u)) - (f'(x) * (v - u))| \leq e * (v - u))$
apply (*simp add*: *gauge-def*)
apply (*subgoal-tac* $\forall x.\ a \leq x\ \&\ x \leq b\ {-}{-}{>}$
$(\exists d > 0.\ \forall u\ v.\ u \leq x\ \&\ x \leq v\ \&\ (v - u) < d\ {-}{-}{>}$
$|(f\ (v) - f\ (u)) - (f'\ (x) * (v - u))| \leq e * (v - u)))$
apply (*drule choiceP*, *auto*)
apply (*drule spec*, *auto*)
apply (*auto simp add*: *DERIV-iff2 LIM-def*)
apply (*drule-tac* $x = e/2$ **in** *spec*, *auto*)
apply (*frule strad1*, *assumption+*)
apply (*rule-tac* $x = s$ **in** *exI*, *auto*)
apply (*rule-tac* $x = u$ **and** $y = v$ **in** *linorder-cases*, *auto*)
apply (*rule-tac* $y = |(f\ (v) - f\ (x)) - (f'\ (x) * (v - x))| +$
$|(f\ (x) - f\ (u)) - (f'\ (x) * (x - u))|$
in *order-trans*)
apply (*rule abs-triangle-ineq* [*THEN* [*2*] *order-trans*])
apply (*simp add*: *right-diff-distrib*)
apply (*rule-tac* $t = e* (v - u)$ **in** *real-sum-of-halves* [*THEN subst*])
apply (*rule add-mono*)
apply (*rule-tac* $y = (e/2) * |v - x|$ **in** *order-trans*)
prefer *2* **apply** *simp*
apply (*erule-tac* [!] $V = \forall x'.\ x' \sim= x\ \&\ |x' - x| < s\ {-}{-}{>}\ ?P\ x'$ **in** *thin-rl*)
apply (*drule-tac* $x = v$ **in** *spec*, *simp add*: *times-divide-eq*)
apply (*drule-tac* $x = u$ **in** *spec*, *auto*)
apply (*subgoal-tac* $|f\, u - f\, x - f'\ x * (u - x)| = |f\, x - f\, u - f'\ x * (x - u)|$)
apply (*rule order-trans*)
apply (*auto simp add*: *abs-le-iff*)
apply (*simp add*: *right-diff-distrib*)
done

lemma *FTC1*: $[|a \leq b; \forall x.\ a \leq x\ \&\ x \leq b\ {-}{-}{>}\ DERIV\, f\, x :> f'(x)\ |]$
$==> Integral(a,b)\ f'\ (f(b) - f(a))$
apply (*drule order-le-imp-less-or-eq*, *auto*)
apply (*auto simp add*: *Integral-def*)
apply (*rule ccontr*)
apply (*subgoal-tac* $\forall e > 0.\ \exists g.\ gauge\ (\%x.\ a \leq x\ \&\ x \leq b)\ g\ \&\ (\forall D\ p.\ tpart\ (a,$
$b)\ (D,\ p)\ \&\ fine\ g\ (D,\ p)\ {-}{-}{>}\ |rsum\ (D,\ p)\ f' - (f\ b - f\ a)| \leq e))$
apply (*rotate-tac 3*)
apply (*drule-tac* $x = e/2$ **in** *spec*, *auto*)
apply (*drule spec*, *auto*)
apply ((*drule spec*)+, *auto*)

apply (drule-tac e = ea/ (b − a) **in** lemma-straddle)
apply (auto simp add: zero-less-divide-iff)
apply (rule exI)
apply (auto simp add: tpart-def rsum-def)
apply (subgoal-tac (∑ n=0..<psize D. f(D(Suc n)) − f(D n)) = f b − f a)
 prefer 2
 apply (cut-tac D = %n. f (D n) **and** m = psize D
    in sumr-partition-eq-diff-bounds)
 apply (simp add: partition-lhs partition-rhs)
apply (drule sym, simp)
apply (simp (no-asm) add: setsum-subtractf[symmetric])
apply (rule setsum-abs [THEN order-trans])
apply (subgoal-tac ea = (∑ n=0..<psize D. (ea / (b − a)) ∗ (D (Suc n) − (D n))))
apply (simp add: abs-minus-commute)
apply (rule-tac t = ea **in** ssubst, assumption)
apply (rule setsum-mono)
apply (rule-tac [2] setsum-right-distrib [THEN subst])
apply (auto simp add: partition-rhs partition-lhs partition-lb partition-ub
    fine-def)
done

lemma Integral-subst: [| Integral(a,b) f k1; k2=k1 |] ==> Integral(a,b) f k2
by simp

lemma Integral-add:
  [| a ≤ b; b ≤ c; Integral(a,b) f′ k1; Integral(b,c) f′ k2;
    ∀x. a ≤ x & x ≤ c −−> DERIV f x :> f′ x |]
  ==> Integral(a,c) f′ (k1 + k2)
apply (rule FTC1 [THEN Integral-subst], auto)
apply (frule FTC1, auto)
apply (frule-tac a = b **in** FTC1, auto)
apply (drule-tac x = x **in** spec, auto)
apply (drule-tac ?k2.0 = f b − f a **in** Integral-unique)
apply (drule-tac [3] ?k2.0 = f c − f b **in** Integral-unique, auto)
done

lemma partition-psize-Least:
  partition(a,b) D ==> psize D = (LEAST n. D(n) = b)
apply (auto intro!: Least-equality [symmetric] partition-rhs)
apply (auto dest: partition-ub-lt simp add: linorder-not-less [symmetric])
done

lemma lemma-partition-bounded: partition (a, c) D ==> ~ (∃n. c < D(n))
apply safe
apply (drule-tac r = n **in** partition-ub, auto)
done

lemma *lemma-partition-eq*:
partition (*a*, *c*) *D* ==> *D* = (%*n*. *if D n* < *c then D n else c*)
apply (*rule ext*, *auto*)
apply (*auto dest*!: *lemma-partition-bounded*)
apply (*drule-tac x* = *n* **in** *spec*, *auto*)
done

lemma *lemma-partition-eq2*:
partition (*a*, *c*) *D* ==> *D* = (%*n*. *if D n* $\leq$ *c then D n else c*)
apply (*rule ext*, *auto*)
apply (*auto dest*!: *lemma-partition-bounded*)
apply (*drule-tac x* = *n* **in** *spec*, *auto*)
done

lemma *partition-lt-Suc*:
[| *partition*(*a*,*b*) *D*; *n* < *psize D* |] ==> *D n* < *D* (*Suc n*)
by (*auto simp add*: *partition*)

lemma *tpart-tag-eq*: *tpart*(*a*,*c*) (*D*,*p*) ==> *p* = (%*n*. *if D n* < *c then p n else c*)
apply (*rule ext*)
apply (*auto simp add*: *tpart-def*)
apply (*drule linorder-not-less* [*THEN iffD1*])
apply (*drule-tac r* = *Suc n* **in** *partition-ub*)
apply (*drule-tac x* = *n* **in** *spec*, *auto*)
done

43.1 Lemmas for Additivity Theorem of Gauge Integral

lemma *lemma-additivity1*:
[| *a* $\leq$ *D n*; *D n* < *b*; *partition*(*a*,*b*) *D* |] ==> *n* < *psize D*
by (*auto simp add*: *partition linorder-not-less* [*symmetric*])

lemma *lemma-additivity2*: [| *a* $\leq$ *D n*; *partition*(*a*,*D n*) *D* |] ==> *psize D* $\leq$ *n*
apply (*rule ccontr*, *drule not-leE*)
apply (*frule partition* [*THEN iffD1*], *safe*)
apply (*frule-tac r* = *Suc n* **in** *partition-ub*)
apply (*auto dest*!: *spec*)
done

lemma *partition-eq-bound*:
[| *partition*(*a*,*b*) *D*; *psize D* < *m* |] ==> *D*(*m*) = *D*(*psize D*)
by (*auto simp add*: *partition*)

lemma *partition-ub2*: [| *partition*(*a*,*b*) *D*; *psize D* < *m* |] ==> *D*(*r*) $\leq$ *D*(*m*)
by (*simp add*: *partition partition-ub*)

lemma *tag-point-eq-partition-point*:
[| *tpart*(*a*,*b*) (*D*,*p*); *psize D* $\leq$ *m* |] ==> *p*(*m*) = *D*(*m*)
apply (*simp add*: *tpart-def*, *auto*)

apply (*drule-tac x = m* **in** *spec*)
apply (*auto simp add*: *partition-rhs2*)
done

lemma *partition-lt-cancel*: [| *partition*(*a*,*b*) *D*; *D m* < *D n* |] ==> *m* < *n*
apply (*cut-tac less-linear* [*of n psize D*], *auto*)
apply (*cut-tac less-linear* [*of m n*])
apply (*cut-tac less-linear* [*of m psize D*])
apply (*auto dest*: *partition-gt*)
apply (*drule-tac n = m* **in** *partition-lt-gen*, *auto*)
apply (*frule partition-eq-bound*)
apply (*drule-tac* [*2*] *partition-gt*, *auto*)
apply (*metis dense-linear-order-class.dlo-simps*(*8*) *le-def partition-rhs partition-rhs2*)
apply (*metis Nat.le-less-trans dense-linear-order-class.dlo-simps*(*8*) *nat-le-linear partition-eq-bound partition-rhs2*)
done

lemma *lemma-additivity4-psize-eq*:
[| *a* $\leq$ *D n*; *D n* < *b*; *partition* (*a*, *b*) *D* |]
==> *psize* (%*x*. *if D x* < *D n then D*(*x*) *else D n*) = *n*
apply (*unfold psize-def*)
apply (*frule lemma-additivity1*)
apply (*assumption*, *assumption*)
apply (*rule some-equality*)
apply (*auto intro*: *partition-lt-Suc*)
apply (*drule-tac n = n* **in** *partition-lt-gen*, *assumption*)
apply (*arith*, *arith*)
apply (*cut-tac m = na* **and** *n = psize D* **in** *Nat.less-linear*)
apply (*auto dest*: *partition-lt-cancel*)
apply (*rule-tac x*=*N* **and** *y*=*n* **in** *linorder-cases*)
apply (*drule-tac x = n* **and** *P*=%*m*. *N* $\leq$ *m* --> *?f m* = *?g m* **in** *spec*, *simp*)
apply (*drule-tac n = n* **in** *partition-lt-gen*, *auto*)
apply (*drule-tac x = n* **in** *spec*)
apply (*simp split*: *split-if-asm*)
done

lemma *lemma-psize-left-less-psize*:
partition (*a*, *b*) *D*
==> *psize* (%*x*. *if D x* < *D n then D*(*x*) *else D n*) $\leq$ *psize D*
apply (*frule-tac r = n* **in** *partition-ub*)
apply (*drule-tac x = D n* **in** *order-le-imp-less-or-eq*)
apply (*auto simp add*: *lemma-partition-eq* [*symmetric*])
apply (*frule-tac r = n* **in** *partition-lb*)
apply (*drule* (*2*) *lemma-additivity4-psize-eq*)
apply (*rule ccontr*, *auto*)
apply (*frule-tac not-leE* [*THEN* [*2*] *partition-eq-bound*])
apply (*auto simp add*: *partition-rhs*)
done

lemma lemma-psize-left-less-psize2:
[| partition(a,b) D; na < psize (%x. if D x < D n then D(x) else D n) |]
==> na < psize D
by (erule lemma-psize-left-less-psize [THEN [2] less-le-trans])

lemma lemma-additivity3:
[| partition(a,b) D; D na < D n; D n < D (Suc na);
n < psize D |]
==> False
by (metis not-less-eq partition-lt-cancel real-of-nat-less-iff)

lemma psize-const [simp]: psize (%x. k) = 0
by (auto simp add: psize-def)

lemma lemma-additivity3a:
[| partition(a,b) D; D na < D n; D n < D (Suc na);
na < psize D |]
==> False
apply (frule-tac m = n **in** partition-lt-cancel)
apply (auto intro: lemma-additivity3)
done

lemma better-lemma-psize-right-eq1:
[| partition(a,b) D; D n < b |] ==> psize (%x. D (x + n)) $\leq$ psize D − n
apply (simp add: psize-def [of (%x. D (x + n))])
apply (rule-tac a = psize D − n **in** someI2, auto)
apply (simp add: partition less-diff-conv)
apply (simp add: le-diff-conv partition-rhs2 split: nat-diff-split)
apply (drule-tac x = psize D − n **in** spec, auto)
apply (frule partition-rhs, safe)
apply (frule partition-lt-cancel, assumption)
apply (drule partition [THEN iffD1], safe)
apply (subgoal-tac ~ D (psize D − n + n) < D (Suc (psize D − n + n)))
apply blast
apply (drule-tac x = Suc (psize D) **and** P=%n. ?P n $\longrightarrow$ D n = D (psize D)
in spec)
apply simp
done

lemma psize-le-n: partition (a, D n) D ==> psize D $\leq$ n
apply (rule ccontr, drule not-leE)
apply (frule partition-lt-Suc, assumption)
apply (frule-tac r = Suc n **in** partition-ub, auto)
done

lemma better-lemma-psize-right-eq1a:
partition(a,D n) D ==> psize (%x. D (x + n)) $\leq$ psize D − n

apply (*simp add*: *psize-def* [*of* (%*x*. *D* (*x* + *n*))])
apply (*rule-tac a* = *psize D* − *n* **in** *someI2*, *auto*)
  apply (*simp add*: *partition less-diff-conv*)
 apply (*simp add*: *le-diff-conv*)
apply (*case-tac psize D* ≤ *n*)
  apply (*force intro*: *partition-rhs2*)
 apply (*simp add*: *partition linorder-not-le*)
apply (*rule ccontr*, *drule not-leE*)
apply (*frule psize-le-n*)
apply (*drule-tac x* = *psize D* − *n* **in** *spec*, *simp*)
apply (*drule partition* [*THEN iffD1*], *safe*)
apply (*drule-tac x* = *Suc n* **and** *P*=%*na*. *?s* ≤ *na* ⟶ *D na* = *D n* **in** *spec*, *auto*)
done

lemma *better-lemma-psize-right-eq*:
    partition(*a*,*b*) *D* ==> *psize* (%*x*. *D* (*x* + *n*)) ≤ *psize D* − *n*
apply (*frule-tac r1* = *n* **in** *partition-ub* [*THEN order-le-imp-less-or-eq*])
apply (*blast intro*: *better-lemma-psize-right-eq1a better-lemma-psize-right-eq1*)
done

lemma *lemma-psize-right-eq1*:
    [| *partition*(*a*,*b*) *D*; *D n* < *b* |] ==> *psize* (%*x*. *D* (*x* + *n*)) ≤ *psize D*
apply (*simp add*: *psize-def* [*of* (%*x*. *D* (*x* + *n*))])
apply (*rule-tac a* = *psize D* − *n* **in** *someI2*, *auto*)
  apply (*simp add*: *partition less-diff-conv*)
 apply (*subgoal-tac n* ≤ *psize D*)
  apply (*simp add*: *partition le-diff-conv*)
 apply (*rule ccontr*, *drule not-leE*)
 apply (*drule-tac less-imp-le* [*THEN* [*2*] *partition-rhs2*], *assumption*, *simp*)
apply (*drule-tac x* = *psize D* **in** *spec*)
apply (*simp add*: *partition*)
done

lemma *lemma-psize-right-eq1a*:
    partition(*a*,*D n*) *D* ==> *psize* (%*x*. *D* (*x* + *n*)) ≤ *psize D*
apply (*simp add*: *psize-def* [*of* (%*x*. *D* (*x* + *n*))])
apply (*rule-tac a* = *psize D* − *n* **in** *someI2*, *auto*)
  apply (*simp add*: *partition less-diff-conv*)
 apply (*case-tac psize D* ≤ *n*)
  apply (*force intro*: *partition-rhs2 simp add*: *le-diff-conv*)
 apply (*simp add*: *partition le-diff-conv*)
apply (*rule ccontr*, *drule not-leE*)
apply (*drule-tac x* = *psize D* **in** *spec*)
apply (*simp add*: *partition*)
done

lemma *lemma-psize-right-eq*:
    [| *partition*(*a*,*b*) *D* |] ==> *psize* (%*x*. *D* (*x* + *n*)) ≤ *psize D*

apply (*frule-tac r1* = *n* **in** *partition-ub* [*THEN order-le-imp-less-or-eq*])
apply (*blast intro*: *lemma-psize-right-eq1a lemma-psize-right-eq1*)
done

lemma *tpart-left1*:
[| $a \le D\ n$; *tpart* $(a,\ b)\ (D,\ p)$ |]
==> *tpart*$(a,\ D\ n)$ (%x. *if* $D\ x < D\ n$ *then* $D(x)$ *else* $D\ n$,
%x. *if* $D\ x < D\ n$ *then* $p(x)$ *else* $D\ n$)
apply (*frule-tac r* = *n* **in** *tpart-partition* [*THEN partition-ub*])
apply (*drule-tac x* = *D n* **in** *order-le-imp-less-or-eq*)
apply (*auto simp add*: *tpart-partition* [*THEN lemma-partition-eq*, *symmetric*] *tpart-tag-eq* [*symmetric*])
apply (*frule-tac tpart-partition* [*THEN* [*3*] *lemma-additivity1*])
apply (*auto simp add*: *tpart-def*)
apply (*drule-tac* [*2*] *linorder-not-less* [*THEN iffD1*, *THEN order-le-imp-less-or-eq*], *auto*)
prefer *3* **apply** (*drule-tac x*=*na* **in** *spec*, *arith*)
prefer *2* **apply** (*blast dest*: *lemma-additivity3*)
apply (*frule* (*2*) *lemma-additivity4-psize-eq*)
apply (*rule partition* [*THEN iffD2*])
apply (*frule partition* [*THEN iffD1*])
apply *safe*
apply (*auto simp add*: *partition-lt-gen*)
apply (*drule* (*1*) *partition-lt-cancel*, *arith*)
done

lemma *fine-left1*:
[| $a \le D\ n$; *tpart* $(a,\ b)\ (D,\ p)$; *gauge* (%x. $a \le x$ & $x \le D\ n$) g;
fine (%x. *if* $x < D\ n$ *then min* $(g\ x)$ $((D\ n - x)/\ 2)$
else if $x = D\ n$ *then min* $(g\ (D\ n))$ $(ga\ (D\ n))$
else min $(ga\ x)$ $((x - D\ n)/\ 2))$ $(D,\ p)$ |]
==> *fine g*
(%x. *if* $D\ x < D\ n$ *then* $D(x)$ *else* $D\ n$,
%x. *if* $D\ x < D\ n$ *then* $p(x)$ *else* $D\ n$)
apply (*auto simp add*: *fine-def tpart-def gauge-def*)
apply (*frule-tac* [!] *na*=*na* **in** *lemma-psize-left-less-psize2*)
apply (*drule-tac* [!] *x* = *na* **in** *spec*, *auto*)
apply (*drule-tac* [!] *x* = *na* **in** *spec*, *auto*)
apply (*auto dest*: *lemma-additivity3a simp add*: *split-if-asm*)
done

lemma *tpart-right1*:
[| $a \le D\ n$; *tpart* $(a,\ b)\ (D,\ p)$ |]
==> *tpart*$(D\ n,\ b)$ (%x. $D(x + n)$,%x. $p(x + n)$)
apply (*simp add*: *tpart-def partition-def*, *safe*)
apply (*rule-tac x* = $N - n$ **in** *exI*, *auto*)
done

lemma *fine-right1*:

[| $a \leq D\ n$; *tpart* $(a,\ b)\ (D,\ p)$; *gauge* (%x. $D\ n \leq x$ & $x \leq b$) *ga*;
fine (%x. *if* $x < D\ n$ *then min* $(g\ x)$ $((D\ n\ -\ x)/\ 2)$
else if $x = D\ n$ *then min* $(g\ (D\ n))$ $(ga\ (D\ n))$
else min $(ga\ x)$ $((x\ -\ D\ n)/\ 2))$ $(D,\ p)$ |]
==> *fine ga* (%x. $D(x + n)$,%x. $p(x + n)$)
apply (*auto simp add*: *fine-def gauge-def*)
apply (*drule-tac* $x = na + n$ **in** *spec*)
apply (*frule-tac* $n = n$ **in** *tpart-partition* [*THEN better-lemma-psize-right-eq*], *auto*)
apply (*simp add*: *tpart-def*, *safe*)
apply (*subgoal-tac* $D\ n \leq p\ (na + n)$)
apply (*drule-tac* $y = p\ (na + n)$ **in** *order-le-imp-less-or-eq*)
apply *safe*
apply (*simp split*: *split-if-asm*, *simp*)
apply (*drule less-le-trans*, *assumption*)
apply (*rotate-tac 5*)
apply (*drule-tac* $x = na + n$ **in** *spec*, *safe*)
apply (*rule-tac* y=$D\ (na + n)$ **in** *order-trans*)
apply (*case-tac* $na = 0$, *auto*)
apply (*erule partition-lt-gen* [*THEN order-less-imp-le*])
apply *arith*
apply *arith*
done

lemma *rsum-add*: *rsum* $(D,\ p)$ (%x. $f\ x + g\ x$) = *rsum* $(D,\ p)\ f$ + *rsum*$(D,\ p)$ g
by (*simp add*: *rsum-def setsum-addf left-distrib*)

Bartle/Sherbert: Theorem 10.1.5 p. 278

lemma *Integral-add-fun*:
[| $a \leq b$; *Integral*(a,b) f *k1*; *Integral*(a,b) g *k2* |]
==> *Integral*(a,b) (%x. $f\ x + g\ x$) $(k1 + k2)$
apply (*simp add*: *Integral-def*, *auto*)
apply ((*drule-tac* $x = e/2$ **in** *spec*)+)
apply *auto*
apply (*drule gauge-min*, *assumption*)
apply (*rule-tac* x = (%x. *if ga* x < *gaa* x *then ga* x *else gaa* x) **in** *exI*)
apply *auto*
apply (*drule fine-min*)
apply ((*drule spec*)+, *auto*)
apply (*drule-tac* $a = |rsum\ (D,\ p)\ f\ -\ k1| * 2$ **and** $c = |rsum\ (D,\ p)\ g\ -\ k2| * 2$ **in** *add-strict-mono*, *assumption*)
apply (*auto simp only*: *rsum-add left-distrib* [*symmetric*]
mult-2-right [*symmetric*] *real-mult-less-iff1*)
done

lemma *partition-lt-gen2*:
[| *partition*(a,b) D; r < *psize* D |] ==> $0 < D\ (Suc\ r) - D\ r$
by (*auto simp add*: *partition*)

lemma *lemma-Integral-le*:
[| ∀ *x*. *a* ≤ *x* & *x* ≤ *b* −−> *f x* ≤ *g x*;
tpart(*a*,*b*) (*D*,*p*)
|] ==> ∀ *n* ≤ *psize D*. *f* (*p n*) ≤ *g* (*p n*)
apply (*simp add*: *tpart-def*)
apply (*auto*, *frule partition* [*THEN iffD1*], *auto*)
apply (*drule-tac x* = *p n* **in** *spec*, *auto*)
apply (*case-tac n* = *0*, *simp*)
apply (*rule partition-lt-gen* [*THEN order-less-le-trans*, *THEN order-less-imp-le*], *auto*)
apply (*drule le-imp-less-or-eq*, *auto*)
apply (*drule-tac* [*2*] *x* = *psize D* **in** *spec*, *auto*)
apply (*drule-tac r* = *Suc n* **in** *partition-ub*)
apply (*drule-tac x* = *n* **in** *spec*, *auto*)
done

lemma *lemma-Integral-rsum-le*:
[| ∀ *x*. *a* ≤ *x* & *x* ≤ *b* −−> *f x* ≤ *g x*;
tpart(*a*,*b*) (*D*,*p*)
|] ==> *rsum*(*D*,*p*) *f* ≤ *rsum*(*D*,*p*) *g*
apply (*simp add*: *rsum-def*)
apply (*auto intro*!: *setsum-mono dest*: *tpart-partition* [*THEN partition-lt-gen2*]
dest!: *lemma-Integral-le*)
done

lemma *Integral-le*:
[| *a* ≤ *b*;
∀ *x*. *a* ≤ *x* & *x* ≤ *b* −−> *f*(*x*) ≤ *g*(*x*);
Integral(*a*,*b*) *f k1*; *Integral*(*a*,*b*) *g k2*
|] ==> *k1* ≤ *k2*
apply (*simp add*: *Integral-def*)
apply (*rotate-tac 2*)
apply (*drule-tac x* = |*k1* − *k2*| /*2* **in** *spec*)
apply (*drule-tac x* = |*k1* − *k2*| /*2* **in** *spec*, *auto*)
apply (*drule gauge-min*, *assumption*)
apply (*drule-tac g* = %*x*. *if ga x* < *gaa x then ga x else gaa x*
in *partition-exists*, *assumption*, *auto*)
apply (*drule fine-min*)
apply (*drule-tac x* = *D* **in** *spec*, *drule-tac x* = *D* **in** *spec*)
apply (*drule-tac x* = *p* **in** *spec*, *drule-tac x* = *p* **in** *spec*, *auto*)
apply (*frule lemma-Integral-rsum-le*, *assumption*)
apply (*subgoal-tac* |(*rsum* (*D*,*p*) *f* − *k1*) − (*rsum* (*D*,*p*) *g* − *k2*)| < |*k1* − *k2*|)
apply *arith*
apply (*drule add-strict-mono*, *assumption*)
apply (*auto simp only*: *left-distrib* [*symmetric*] *mult-2-right* [*symmetric*]
real-mult-less-iff1)
done

lemma *Integral-imp-Cauchy*:

```
  (∃ k. Integral(a,b) f k) ==>
   (∀ e > 0. ∃ g. gauge (%x. a ≤ x & x ≤ b) g &
              (∀ D1 D2 p1 p2.
                 tpart(a,b) (D1, p1) & fine g (D1,p1) &
                 tpart(a,b) (D2, p2) & fine g (D2,p2) -->
                 |rsum(D1,p1) f - rsum(D2,p2) f| < e))
apply (simp add: Integral-def, auto)
apply (drule-tac x = e/2 in spec, auto)
apply (rule exI, auto)
apply (frule-tac x = D1 in spec)
apply (frule-tac x = D2 in spec)
apply ((drule spec)+, auto)
apply (erule-tac V = 0 < e in thin-rl)
apply (drule add-strict-mono, assumption)
apply (auto simp only: left-distrib [symmetric] mult-2-right [symmetric]
              real-mult-less-iff1)
done

lemma Cauchy-iff2:
  Cauchy X =
   (∀ j. (∃ M. ∀ m ≥ M. ∀ n ≥ M. |X m - X n| < inverse(real (Suc j))))
apply (simp add: Cauchy-def, auto)
apply (drule reals-Archimedean, safe)
apply (drule-tac x = n in spec, auto)
apply (rule-tac x = M in exI, auto)
apply (drule-tac x = m in spec, simp)
apply (drule-tac x = na in spec, auto)
done

lemma partition-exists2:
  [| a ≤ b; ∀ n. gauge (%x. a ≤ x & x ≤ b) (fa n) |]
   ==> ∀ n. ∃ D p. tpart (a, b) (D, p) & fine (fa n) (D, p)
by (blast dest: partition-exists)

lemma monotonic-anti-derivative:
 fixes f g :: real => real shows
  [| a ≤ b; ∀ c. a ≤ c & c ≤ b --> f' c ≤ g' c;
     ∀ x. DERIV f x :> f' x; ∀ x. DERIV g x :> g' x |]
   ==> f b - f a ≤ g b - g a
apply (rule Integral-le, assumption)
apply (auto intro: FTC1)
done

end
```

44 Log: Logarithms: Standard Version

theory *Log*

imports *Transcendental*
begin

definition
powr :: [*real*,*real*] => *real* (**infixr** *powr 80*) **where**
— exponentation with real exponent
x powr a = *exp*(*a* * *ln x*)

definition
log :: [*real*,*real*] => *real* **where**
— logarithm of *x* to base *a*
log a x = *ln x* / *ln a*

lemma *powr-one-eq-one* [*simp*]: *1 powr a* = *1*
by (*simp add*: *powr-def*)

lemma *powr-zero-eq-one* [*simp*]: *x powr 0* = *1*
by (*simp add*: *powr-def*)

lemma *powr-one-gt-zero-iff* [*simp*]: (*x powr 1* = *x*) = (*0* < *x*)
by (*simp add*: *powr-def*)
declare *powr-one-gt-zero-iff* [*THEN iffD2*, *simp*]

lemma *powr-mult*:
[| *0* < *x*; *0* < *y* |] ==> (*x* * *y*) *powr a* = (*x powr a*) * (*y powr a*)
by (*simp add*: *powr-def exp-add* [*symmetric*] *ln-mult right-distrib*)

lemma *powr-gt-zero* [*simp*]: *0* < *x powr a*
by (*simp add*: *powr-def*)

lemma *powr-ge-pzero* [*simp*]: *0* <= *x powr y*
by (*rule order-less-imp-le*, *rule powr-gt-zero*)

lemma *powr-not-zero* [*simp*]: *x powr a* ≠ *0*
by (*simp add*: *powr-def*)

lemma *powr-divide*:
[| *0* < *x*; *0* < *y* |] ==> (*x* / *y*) *powr a* = (*x powr a*)/(*y powr a*)
apply (*simp add*: *divide-inverse positive-imp-inverse-positive powr-mult*)
apply (*simp add*: *powr-def exp-minus* [*symmetric*] *exp-add* [*symmetric*] *ln-inverse*)
done

lemma *powr-divide2*: *x powr a* / *x powr b* = *x powr* (*a* − *b*)
apply (*simp add*: *powr-def*)
apply (*subst exp-diff* [*THEN sym*])
apply (*simp add*: *left-diff-distrib*)
done

lemma *powr-add*: $x\ powr\ (a + b) = (x\ powr\ a) * (x\ powr\ b)$
by (*simp add*: *powr-def exp-add* [*symmetric*] *left-distrib*)

lemma *powr-powr*: $(x\ powr\ a)\ powr\ b = x\ powr\ (a * b)$
by (*simp add*: *powr-def*)

lemma *powr-powr-swap*: $(x\ powr\ a)\ powr\ b = (x\ powr\ b)\ powr\ a$
by (*simp add*: *powr-powr real-mult-commute*)

lemma *powr-minus*: $x\ powr\ (-a) = inverse\ (x\ powr\ a)$
by (*simp add*: *powr-def exp-minus* [*symmetric*])

lemma *powr-minus-divide*: $x\ powr\ (-a) = 1/(x\ powr\ a)$
by (*simp add*: *divide-inverse powr-minus*)

lemma *powr-less-mono*: $[|\ a < b;\ 1 < x\ |] ==> x\ powr\ a < x\ powr\ b$
by (*simp add*: *powr-def*)

lemma *powr-less-cancel*: $[|\ x\ powr\ a < x\ powr\ b;\ 1 < x\ |] ==> a < b$
by (*simp add*: *powr-def*)

lemma *powr-less-cancel-iff* [*simp*]: $1 < x ==> (x\ powr\ a < x\ powr\ b) = (a < b)$
by (*blast intro*: *powr-less-cancel powr-less-mono*)

lemma *powr-le-cancel-iff* [*simp*]: $1 < x ==> (x\ powr\ a \leq x\ powr\ b) = (a \leq b)$
by (*simp add*: *linorder-not-less* [*symmetric*])

lemma *log-ln*: $ln\ x = log\ (exp(1))\ x$
by (*simp add*: *log-def*)

lemma *powr-log-cancel* [*simp*]:
  $[|\ 0 < a;\ a \neq 1;\ 0 < x\ |] ==> a\ powr\ (log\ a\ x) = x$
by (*simp add*: *powr-def log-def*)

lemma *log-powr-cancel* [*simp*]: $[|\ 0 < a;\ a \neq 1\ |] ==> log\ a\ (a\ powr\ y) = y$
by (*simp add*: *log-def powr-def*)

lemma *log-mult*:
  $[|\ 0 < a;\ a \neq 1;\ 0 < x;\ 0 < y\ |]$
   $==> log\ a\ (x * y) = log\ a\ x + log\ a\ y$
by (*simp add*: *log-def ln-mult divide-inverse left-distrib*)

lemma *log-eq-div-ln-mult-log*:
  $[|\ 0 < a;\ a \neq 1;\ 0 < b;\ b \neq 1;\ 0 < x\ |]$
   $==> log\ a\ x = (ln\ b/ln\ a) * log\ b\ x$
by (*simp add*: *log-def divide-inverse*)

Base 10 logarithms

lemma *log-base-10-eq1*: *0 < x ==> log 10 x = (ln (exp 1) / ln 10) ∗ ln x*
by (*simp add*: *log-def*)

lemma *log-base-10-eq2*: *0 < x ==> log 10 x = (log 10 (exp 1)) ∗ ln x*
by (*simp add*: *log-def*)

lemma *log-one* [*simp*]: *log a 1 = 0*
by (*simp add*: *log-def*)

lemma *log-eq-one* [*simp*]: [| *0 < a*; *a ≠ 1* |] *==> log a a = 1*
by (*simp add*: *log-def*)

lemma *log-inverse*:
  [| *0 < a*; *a ≠ 1*; *0 < x* |] *==> log a* (*inverse x*) *= − log a x*
apply (*rule-tac a1 = log a x* **in** *add-left-cancel* [*THEN iffD1*])
apply (*simp add*: *log-mult* [*symmetric*])
done

lemma *log-divide*:
  [|*0 < a*; *a ≠ 1*; *0 < x*; *0 < y*|] *==> log a* (*x/y*) *= log a x − log a y*
by (*simp add*: *log-mult divide-inverse log-inverse*)

lemma *log-less-cancel-iff* [*simp*]:
  [| *1 < a*; *0 < x*; *0 < y* |] *==>* (*log a x < log a y*) *=* (*x < y*)
apply *safe*
apply (*rule-tac* [*2*] *powr-less-cancel*)
apply (*drule-tac a = log a x* **in** *powr-less-mono*, *auto*)
done

lemma *log-le-cancel-iff* [*simp*]:
  [| *1 < a*; *0 < x*; *0 < y* |] *==>* (*log a x ≤ log a y*) *=* (*x ≤ y*)
by (*simp add*: *linorder-not-less* [*symmetric*])

lemma *powr-realpow*: *0 < x ==> x powr* (*real n*) *= x^n*
 apply (*induct n*, *simp*)
 apply (*subgoal-tac real*(*Suc n*) *= real n + 1*)
 apply (*erule ssubst*)
 apply (*subst powr-add*, *simp*, *simp*)
done

lemma *powr-realpow2*: *0 <= x ==> 0 < n ==> x^n =* (*if* (*x = 0*) *then 0*
 else x powr (*real n*))
 apply (*case-tac x = 0*, *simp*, *simp*)
 apply (*rule powr-realpow* [*THEN sym*], *simp*)
done

lemma *ln-pwr*: *0 < x ==> 0 < y ==> ln*(*x powr y*) *= y ∗ ln x*
by (*unfold powr-def*, *simp*)

```
lemma ln-bound: 1 <= x ==> ln x <= x
  apply (subgoal-tac ln(1 + (x - 1)) <= x - 1)
  apply simp
  apply (rule ln-add-one-self-le-self, simp)
done

lemma powr-mono: a <= b ==> 1 <= x ==> x powr a <= x powr b
  apply (case-tac x = 1, simp)
  apply (case-tac a = b, simp)
  apply (rule order-less-imp-le)
  apply (rule powr-less-mono, auto)
done

lemma ge-one-powr-ge-zero: 1 <= x ==> 0 <= a ==> 1 <= x powr a
  apply (subst powr-zero-eq-one [THEN sym])
  apply (rule powr-mono, assumption+)
done

lemma powr-less-mono2: 0 < a ==> 0 < x ==> x < y ==> x powr a <
    y powr a
  apply (unfold powr-def)
  apply (rule exp-less-mono)
  apply (rule mult-strict-left-mono)
  apply (subst ln-less-cancel-iff, assumption)
  apply (rule order-less-trans)
  prefer 2
  apply assumption+
done

lemma powr-less-mono2-neg: a < 0 ==> 0 < x ==> x < y ==> y powr a <
    x powr a
  apply (unfold powr-def)
  apply (rule exp-less-mono)
  apply (rule mult-strict-left-mono-neg)
  apply (subst ln-less-cancel-iff)
  apply assumption
  apply (rule order-less-trans)
  prefer 2
  apply assumption+
done

lemma powr-mono2: 0 <= a ==> 0 < x ==> x <= y ==> x powr a <= y
powr a
  apply (case-tac a = 0, simp)
  apply (case-tac x = y, simp)
  apply (rule order-less-imp-le)
  apply (rule powr-less-mono2, auto)
done
```

```
lemma ln-powr-bound: 1 <= x ==> 0 < a ==> ln x <= (x powr a) / a
  apply (rule mult-imp-le-div-pos)
  apply (assumption)
  apply (subst mult-commute)
  apply (subst ln-pwr [THEN sym])
  apply auto
  apply (rule ln-bound)
  apply (erule ge-one-powr-ge-zero)
  apply (erule order-less-imp-le)
done

lemma ln-powr-bound2: 1 < x ==> 0 < a ==> (ln x) powr a <= (a powr a) *
x
proof -
  assume 1 < x and 0 < a
  then have ln x <= (x powr (1 / a)) / (1 / a)
    apply (intro ln-powr-bound)
    apply (erule order-less-imp-le)
    apply (rule divide-pos-pos)
    apply simp-all
    done
  also have ... = a * (x powr (1 / a))
    by simp
  finally have (ln x) powr a <= (a * (x powr (1 / a))) powr a
    apply (intro powr-mono2)
    apply (rule order-less-imp-le, rule prems)
    apply (rule ln-gt-zero)
    apply (rule prems)
    apply assumption
    done
  also have ... = (a powr a) * ((x powr (1 / a)) powr a)
    apply (rule powr-mult)
    apply (rule prems)
    apply (rule powr-gt-zero)
    done
  also have (x powr (1 / a)) powr a = x powr ((1 / a) * a)
    by (rule powr-powr)
  also have ... = x
    apply simp
    apply (subgoal-tac a ~= 0)
    apply (insert prems, auto)
    done
  finally show ?thesis .
qed

lemma LIMSEQ-neg-powr: 0 < s ==> (%x. (real x) powr - s) ----> 0
  apply (unfold LIMSEQ-def)
  apply clarsimp
```

```
  apply (rule-tac x = natfloor(r powr (1 / − s)) + 1 in exI)
  apply clarify
  proof −
    fix r fix n
    assume 0 < s and 0 < r and natfloor (r powr (1 / − s)) + 1 <= n
    have r powr (1 / − s) < real(natfloor(r powr (1 / − s))) + 1
      by (rule real-natfloor-add-one-gt)
    also have ... = real(natfloor(r powr (1 / −s)) + 1)
      by simp
    also have ... <= real n
      apply (subst real-of-nat-le-iff)
      apply (rule prems)
      done
    finally have r powr (1 / − s) < real n.
    then have real n powr (− s) < (r powr (1 / − s)) powr − s
      apply (intro powr-less-mono2-neg)
      apply (auto simp add: prems)
      done
    also have ... = r
      by (simp add: powr-powr prems less-imp-neq [THEN not-sym])
    finally show real n powr − s < r .
  qed

end
```

45 HLog: Logarithms: Non-Standard Version

```
theory HLog
imports Log HTranscendental
begin


lemma epsilon-ge-zero [simp]: 0 ≤ epsilon
by (simp add: epsilon-def star-n-zero-num star-n-le)

lemma hpfinite-witness: epsilon : {x. 0 ≤ x & x : HFinite}
by auto


definition
  powhr :: [hypreal,hypreal] => hypreal     (infixr powhr 80) where
  x powhr a = starfun2 (op powr) x a

definition
  hlog :: [hypreal,hypreal] => hypreal where
  hlog a x = starfun2 log a x
```

declare *powhr-def* [*transfer-unfold*]
declare *hlog-def* [*transfer-unfold*]

lemma *powhr*: (*star-n X*) *powhr* (*star-n Y*) = *star-n* (%*n*. (*X n*) *powr* (*Y n*))
by (*simp add*: *powhr-def starfun2-star-n*)

lemma *powhr-one-eq-one* [*simp*]: !!*a*. *1 powhr a* = *1*
by (*transfer*, *simp*)

lemma *powhr-mult*:
!!*a x y*. [| *0* < *x*; *0* < *y* |] ==> (*x* ∗ *y*) *powhr a* = (*x powhr a*) ∗ (*y powhr a*)
by (*transfer*, *rule powr-mult*)

lemma *powhr-gt-zero* [*simp*]: !!*a x*. *0* < *x powhr a*
by (*transfer*, *simp*)

lemma *powhr-not-zero* [*simp*]: *x powhr a* ≠ *0*
by (*rule powhr-gt-zero* [*THEN hypreal-not-refl2*, *THEN not-sym*])

lemma *powhr-divide*:
!!*a x y*. [| *0* < *x*; *0* < *y* |] ==> (*x* / *y*) *powhr a* = (*x powhr a*)/(*y powhr a*)
by (*transfer*, *rule powr-divide*)

lemma *powhr-add*: !!*a b x*. *x powhr* (*a* + *b*) = (*x powhr a*) ∗ (*x powhr b*)
by (*transfer*, *rule powr-add*)

lemma *powhr-powhr*: !!*a b x*. (*x powhr a*) *powhr b* = *x powhr* (*a* ∗ *b*)
by (*transfer*, *rule powr-powr*)

lemma *powhr-powhr-swap*: !!*a b x*. (*x powhr a*) *powhr b* = (*x powhr b*) *powhr a*
by (*transfer*, *rule powr-powr-swap*)

lemma *powhr-minus*: !!*a x*. *x powhr* (−*a*) = *inverse* (*x powhr a*)
by (*transfer*, *rule powr-minus*)

lemma *powhr-minus-divide*: *x powhr* (−*a*) = *1*/(*x powhr a*)
by (*simp add*: *divide-inverse powhr-minus*)

lemma *powhr-less-mono*: !!*a b x*. [| *a* < *b*; *1* < *x* |] ==> *x powhr a* < *x powhr b*
by (*transfer*, *simp*)

lemma *powhr-less-cancel*: !!*a b x*. [| *x powhr a* < *x powhr b*; *1* < *x* |] ==> *a* < *b*
by (*transfer*, *simp*)

lemma *powhr-less-cancel-iff* [*simp*]:
1 < *x* ==> (*x powhr a* < *x powhr b*) = (*a* < *b*)
by (*blast intro*: *powhr-less-cancel powhr-less-mono*)

lemma *powhr-le-cancel-iff* [*simp*]:

1 < *x* ==> (*x powhr a* ≤ *x powhr b*) = (*a* ≤ *b*)
by (*simp add*: *linorder-not-less* [*symmetric*])

lemma *hlog*:
hlog (*star-n X*) (*star-n Y*) =
star-n (%*n*. *log* (*X n*) (*Y n*))
by (*simp add*: *hlog-def starfun2-star-n*)

lemma *hlog-starfun-ln*: !!*x*. (**f* ln*) *x* = *hlog* ((**f* exp*) *1*) *x*
by (*transfer*, *rule log-ln*)

lemma *powhr-hlog-cancel* [*simp*]:
!!*a x*. [| *0* < *a*; *a* ≠ *1*; *0* < *x* |] ==> *a powhr* (*hlog a x*) = *x*
by (*transfer*, *simp*)

lemma *hlog-powhr-cancel* [*simp*]:
!!*a y*. [| *0* < *a*; *a* ≠ *1* |] ==> *hlog a* (*a powhr y*) = *y*
by (*transfer*, *simp*)

lemma *hlog-mult*:
!!*a x y*. [| *0* < *a*; *a* ≠ *1*; *0* < *x*; *0* < *y* |]
==> *hlog a* (*x* ∗ *y*) = *hlog a x* + *hlog a y*
by (*transfer*, *rule log-mult*)

lemma *hlog-as-starfun*:
!!*a x*. [| *0* < *a*; *a* ≠ *1* |] ==> *hlog a x* = (**f* ln*) *x* / (**f* ln*) *a*
by (*transfer*, *simp add*: *log-def*)

lemma *hlog-eq-div-starfun-ln-mult-hlog*:
!!*a b x*. [| *0* < *a*; *a* ≠ *1*; *0* < *b*; *b* ≠ *1*; *0* < *x* |]
==> *hlog a x* = ((**f* ln*) *b*/(**f*ln*) *a*) ∗ *hlog b x*
by (*transfer*, *rule log-eq-div-ln-mult-log*)

lemma *powhr-as-starfun*: !!*a x*. *x powhr a* = (**f* exp*) (*a* ∗ (**f* ln*) *x*)
by (*transfer*, *simp add*: *powr-def*)

lemma *HInfinite-powhr*:
[| *x* : *HInfinite*; *0* < *x*; *a* : *HFinite* − *Infinitesimal*;
0 < *a* |] ==> *x powhr a* : *HInfinite*
apply (*auto intro*!: *starfun-ln-ge-zero starfun-ln-HInfinite HInfinite-HFinite-not-Infinitesimal-mult2 starfun-exp-HInfinite*
simp add: *order-less-imp-le HInfinite-gt-zero-gt-one powhr-as-starfun zero-le-mult-iff*)
done

lemma *hlog-hrabs-HInfinite-Infinitesimal*:
[| *x* : *HFinite* − *Infinitesimal*; *a* : *HInfinite*; *0* < *a* |]
==> *hlog a* (*abs x*) : *Infinitesimal*
apply (*frule HInfinite-gt-zero-gt-one*)
apply (*auto intro*!: *starfun-ln-HFinite-not-Infinitesimal*

HInfinite-inverse-Infinitesimal Infinitesimal-HFinite-mult2
simp add: *starfun-ln-HInfinite not-Infinitesimal-not-zero*
hlog-as-starfun hypreal-not-refl2 [*THEN not-sym*] *divide-inverse*)
done

lemma *hlog-HInfinite-as-starfun*:
[| a : *HInfinite*; $0 < a$ |] ==> *hlog* $a\ x = (*f* \ ln)\ x\ /\ (*f* \ ln)\ a$
by (*rule hlog-as-starfun*, *auto*)

lemma *hlog-one* [*simp*]: !!*a*. *hlog* $a\ 1 = 0$
by (*transfer*, *simp*)

lemma *hlog-eq-one* [*simp*]: !!*a*. [| $0 < a$; $a \neq 1$ |] ==> *hlog* $a\ a = 1$
by (*transfer*, *rule log-eq-one*)

lemma *hlog-inverse*:
[| $0 < a$; $a \neq 1$; $0 < x$ |] ==> *hlog* a (*inverse* x) $= -$ *hlog* $a\ x$
apply (*rule add-left-cancel* [*of hlog a x*, *THEN iffD1*])
apply (*simp add*: *hlog-mult* [*symmetric*])
done

lemma *hlog-divide*:
[| $0 < a$; $a \neq 1$; $0 < x$; $0 < y$|] ==> *hlog* $a\ (x/y) =$ *hlog* $a\ x -$ *hlog* $a\ y$
by (*simp add*: *hlog-mult hlog-inverse divide-inverse*)

lemma *hlog-less-cancel-iff* [*simp*]:
!!*a x y*. [| $1 < a$; $0 < x$; $0 < y$ |] ==> (*hlog* $a\ x <$ *hlog* $a\ y$) $= (x < y)$
by (*transfer*, *simp*)

lemma *hlog-le-cancel-iff* [*simp*]:
[| $1 < a$; $0 < x$; $0 < y$ |] ==> (*hlog* $a\ x \leq$ *hlog* $a\ y$) $= (x \leq y)$
by (*simp add*: *linorder-not-less* [*symmetric*])

end

theory *Hyperreal*
imports *Ln Poly Taylor Integration HLog*
begin

end

46 Complex-Main: Comprehensive Complex Theory

theory *Complex-Main*

imports *CLim ../Hyperreal/Hyperreal*
begin

end