

ALNUTH

ALgebraic NUmber THeory and an interface to the KANT System

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Kant part

This package incorporates an interface to some functions of the computer algebra system Kant. Kant is developed by Michael Pohst and his group at the Technische Universität Berlin. The Kant system itself is not part of this interface. It can be obtained at

www.math.tu-berlin.de/~kant/kash.html

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1

Introduction

A number field is a finite extension of the field of rational numbers. This package provides various methods to compute with number fields which are given by a defining polynomial or by generators. For background on number fields we refer to [ST79].

Some of the methods provided in this package are written in Gap code. The other part of the methods is imported from the Computer Algebra System KANT [DCK+97]. Hence this package contains some Gap functions and an interface to some functions in the computer algebra system KANT. We note that not all available functions of KANT are linked to Gap and the KANT system provides much more methods for computations in number fields.

The main methods included in this package are: creating a number field, computing its maximal order (using KANT), computing its unit group (using KANT) and a presentation of this unit group, computing the elements of a given norm of the number field (using KANT) and determining a presentation for a finitely generated multiplicative subgroup (using KANT). For background on algorithms for number fields we refer to [Poh93], [PZ89] and [Coh93].

The functions provided by this package are introduced in the following chapter. Then an example application is outlined. In the final chapter of this manual the installation of the package is described. We note that the computer algebra system KANT itself is not included in the package.

2

Methods for number fields

An algebraic number field is a finite-dimensional extension of the rational numbers \mathbb{Q} . Such a number field has a primitive element and it can be defined by the minimal polynomial of this primitive element. Another important way to define an algebraic number field is by a set of rational matrices which generate a number field.

2.1 Creation of number fields

We provide functions to create number fields defined by rational matrices or by rational polynomials.

- 1 ► `FieldByMatricesNC(matrices)`
 - `FieldByMatrices(matrices)`

Creates a field generated by the rational matrices *matrices*. In the faster NC version, the function assumes that the input generates a field and there are no checks on this performed.

- 2 ► `FieldByMatrixBasisNC(matrices)`
 - `FieldByMatrixBasis(matrices)`

Creates a field with basis *matrices*. The list *matrices* must consist of rational matrices which form a basis for a number field. In the faster NC version, the function assumes that the input is a matrix basis for a field and no checks are performed.

- 3 ► `FieldByPolynomialNC(polynomial)`
 - `FieldByPolynomial(polynomial)`

Creates a field defined by *polynomial*. The polynomial *polynomial* must be an irreducible rational polynomial. In the faster NC version, no checks on the input are performed.

2.2 Methods for number fields

We outline a number of functions for number fields.

- 1 ► `PrimitiveElement(F)`
 - `DefiningPolynomial(F)`

Computes a primitive element and a defining polynomial for the given number field. The defining polynomial is the minimal polynomial of the primitive element. Since *F* contains various primitive elements, `PrimitiveElement` tries to find a primitive element which has a minimal polynomial with small coefficients. Via the global variable `PRIM_TEST` the user can decide how many primitive elements will be compared. The default value is 20.

- 2 ► `IsPrimitiveElement(F, a)`

Checks if the given element generates the field.

- 3 ► `DegreeOverPrimeField(F)`

Returns the degree of *F* over the rationals.

- 4 ▶ `EquationOrderBasis(F)`
- ▶ `MaximalOrderBasis(F)`
- ▶ `IsIntegerOfNumberField(F, k)`

These functions return bases for the equation order or the maximal order of the number field F . Also, they allow to check if a given element is an integer in the given number field.

- 5 ▶ `UnitGroup(F)`
- ▶ `IsomorphismPcpGroup(U)`
- ▶ `IsUnitOfNumberField(F, k)`

These functions determine the unit group of F and an isomorphism to a pcp group. (Recall that the unit group of F is a finitely generated abelian group.) The isomorphism can be used for various computations with the unit group. Also, the last function allows to check whether a given element is a unit in F .

- 6 ▶ `ExponentsOfUnits(F, elms)`

This function determines the exponent vectors of the elements in $elms$ with respect to the generators of the unit group of F . If the unit group of F is not known, then the function computes this unit group also.

- 7 ▶ `IsCyclotomicField(F)`

Check whether F is cyclotomic.

- 8 ▶ `NormCosetsOfNumberField(F, norm)`

Returns a description for the set of all elements of norm $norm$ in F . These elements can be written as a finite union of cosets of the unit group of F . The function returns coset representatives for these cosets.

2.3 Presentations of multiplicative subgroups

Suppose that a finite number of invertible elements of a number field are given. Then these elements generate a finitely generated abelian group. However, it is a non-trivial task to provide a presentation for this abelian group. The most useful representation for such groups is as pcp group.

- 1 ▶ `PcpPresentationOfMultiplicativeSubgroup(F, elms)`
- ▶ `IsomorphismPcpGroup(F, elms)`

Determine a pcp presentation for the multiplicative group of $F \setminus \{0\}$ generated by $elms$ and an isomorphism on this presentation. Note, that the method `IsomorphismPcpGroup` is defined in the Polycyclic package [EN00]. We refer to the manual of this package for further background.

- 2 ▶ `Kernel(map)`
- ▶ `ImagesSet(map, fieldelms)`
- ▶ `ImageElm(map, fieldelm)`
- ▶ `PreImagesRepresentative(map, pcpelm)`

These functions can be used to compute with an isomorphism to a pcp presented image. If $fieldelm$ is not contained in the source of map , then the function `ImageElm` returns fail.

In the determination of the Pcp-presentation of a multiplicative subgroup generated by $elms$ the relations between the elements in $elms$ play an important role. Let $elms = \{e_1, \dots, e_l\}$ be a finite subset of a field F . The relation lattice for $elms$ is

$$rl(elms) := \left\{ (h_1, \dots, h_l) \in \mathbb{Z}^l \mid e_1^{h_1} \cdots e_l^{h_l} = 1 \right\}.$$

- 3 ▶ `RelationLattice(F, elms)`

Determines a basis for the relation lattice of the field elements $elms$.

2.4 Methods to compute with subgroups of the unit group

1 ► `RelationLatticeOfUnits(F, elms)`

Determines a basis for the relation lattice of the units *elms* in triangularized form. Note that this method is more efficient than the method `RelationLattice`.

2 ► `IntersectionOfUnitSubgroups(F, gen1, gen2)`

The lists *gen1* and *gen2* are supposed to generate two subgroups U_1 and U_2 of the unit group of F . This function determines the intersection of U_1 with U_2 . The result is returned as a list of vectors generating the lattice $\{e \in \mathbb{Z}^n \mid g_1^{e_1} \cdots g_n^{e_n} \in U_2\}$ for $gen1 = [g_1, \dots, g_n]$.

This function does not check the input for efficiency reasons and it may return wrong results if the input generators do not fulfil the requirements.

2.5 Examples

1 ► `ExampleMatField(l)`

This function returns some examples of fields generated by matrices. There are 9 such example fields provided and they can be obtained by assigning the input *l* to an integer between 1 and 9. Some of the properties of the examples are summarized in the following table.

	degree over Q	number of generators	dimension of generators
<code>ExampleMatField(1)</code>	4	4	4
<code>ExampleMatField(2)</code>	4	4	4
<code>ExampleMatField(3)</code>	4	4	4
<code>ExampleMatField(4)</code>	4	13	4
<code>ExampleMatField(5)</code>	4	13	4
<code>ExampleMatField(6)</code>	4	7	4
<code>ExampleMatField(7)</code>	4	18	4
<code>ExampleMatField(8)</code>	4	13	4
<code>ExampleMatField(9)</code>	4	7	4

3 An example application

In this section we outline two example computations with the functions of the previous chapter. The first example uses number fields defined by matrices and the second example considers number fields defined by a polynomial.

3.1 Number fields defined by matrices

```
gap> m1 := [ [ 1, 0, 0, -7 ],
             [ 7, 1, 0, -7 ],
             [ 0, 7, 1, -7 ],
             [ 0, 0, 7, -6 ] ];;

gap> m2 := [ [ 0, 0, -13, 14 ],
             [ -1, 0, -13, 1 ],
             [ 13, -1, -13, 1 ],
             [ 0, 13, -14, 1 ] ];;

gap> F := FieldByMatricesNC( [m1, m2] );
<field in characteristic 0>

gap> DegreeOverPrimeField(F);
4
gap> PrimitiveElement(F);
[ [ 1, 0, 0, -7 ], [ 7, 1, 0, -7 ], [ 0, 7, 1, -7 ], [ 0, 0, 7, -6 ] ]

gap> Basis(F);
Basis( <field in characteristic 0>,
[ [ [ 1, 0, 0, 0 ], [ 0, 1, 0, 0 ], [ 0, 0, 1, 0 ], [ 0, 0, 0, 1 ] ],
  [ [ 0, 1, 0, 0 ], [ -1, 1, 1, 0 ], [ -1, 0, 1, 1 ], [ -1, 0, 0, 1 ] ],
  [ [ 0, 0, 1, 0 ], [ -1, 0, 1, 1 ], [ -1, -1, 1, 1 ], [ 0, -1, 0, 1 ] ],
  [ [ 0, 0, 0, 1 ], [ -1, 0, 0, 1 ], [ 0, -1, 0, 1 ], [ 0, 0, -1, 1 ] ] ] )

gap> MaximalOrderBasis(F);
Basis( <field in characteristic 0>,
[ [ [ 1, 0, 0, 0 ], [ 0, 1, 0, 0 ], [ 0, 0, 1, 0 ], [ 0, 0, 0, 1 ] ],
  [ [ 1, 0, 0, -1 ], [ 1, 1, 0, -1 ], [ 0, 1, 1, -1 ], [ 0, 0, 1, 0 ] ],
  [ [ 1, 0, -1, 0 ], [ 1, 1, -1, -1 ], [ 1, 1, 0, -1 ], [ 0, 1, 0, 0 ] ],
  [ [ 1, -1, 0, 0 ], [ 1, 0, -1, 0 ], [ 1, 0, 0, -1 ], [ 1, 0, 0, 0 ] ] ] )

gap> U := UnitGroup(F);
<matrix group with 2 generators>

gap> u := GeneratorsOfGroup( U );;
```

```

gap> nat := IsomorphismPcpGroup(U);
[ [ [ 0, 1, -1, 0 ], [ 0, 1, 0, -1 ], [ 0, 1, 0, 0 ], [ -1, 1, 0, 0 ] ],
  [ [ 1, 0, -1, 1 ], [ 0, 1, -1, 0 ], [ 1, 0, 0, 0 ], [ 0, 1, -1, 1 ] ] ] ->
[ g1, g2 ]

gap> H := Image(nat);
Pcp-group with orders [ 10, 0 ]
gap> ImageElm( nat, u[1]*u[2] );
g1*g2
gap> PreImagesRepresentative(nat, GeneratorsOfGroup(H)[1] );
[ [ 0, 1, -1, 0 ], [ 0, 1, 0, -1 ], [ 0, 1, 0, 0 ], [ -1, 1, 0, 0 ] ]

```

3.2 Number fields defined by a polynomial

```

gap> x:=Indeterminate(Rationals);
x_1
gap> g:= x^4-4*x^3-28*x^2+64*x+16;
x_1^4-4*x_1^3-28*x_1^2+64*x_1+16

gap> F := FieldByPolynomialNC(g);
<field in characteristic 0>
gap> PrimitiveElement(F);
(a)
gap> MaximalOrderBasis(F);
Basis( <field in characteristic 0>,
[ !1, (1/2*a), (1/4*a^2), (5/7+1/14*a+1/14*a^2+1/56*a^3) ] )

gap> U := UnitGroup(F);
[ !-1, (-3/7+6/7*a+3/28*a^2-1/28*a^3),
  (13/7+25/14*a+1/28*a^2-3/56*a^3), (36/7-9/7*a-2/7*a^2+3/56*a^3) ]
<group with 4 generators>

gap> natU := IsomorphismPcpGroup(U);
[ !-1, (-3/7+6/7*a+3/28*a^2-1/28*a^3),
  (13/7+25/14*a+1/28*a^2-3/56*a^3), (36/7-9/7*a-2/7*a^2+3/56*a^3)
] -> [ g1, g2, g3, g4 ]

gap> elms := List( [1..10], x-> Random(F) );
[ (4-1/2*a-1*a^2+3/2*a^3), (4/5-2/3*a+4/3*a^3), (1+a+2*a^2-1*a^3),
  (3/4+3*a+3*a^2), (-1-1/5*a^3), (-1/4*a-5/3*a^2), (1-1*a+1/2*a^2),
  (4-3/2*a+1/2*a^2), (-2/5+a-3/2*a^2), (-1*a+a^2+2*a^3) ]

gap> PcpPresentationOfMultiplicativeSubgroup( F, elms );
Pcp-group with orders [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ]

gap> isom := IsomorphismPcpGroup( F, elms );
[ (4-1/2*a-1*a^2+3/2*a^3), (4/5-2/3*a+4/3*a^3),
  (1+a+2*a^2-1*a^3), (3/4+3*a+3*a^2), (-1-1/5*a^3),
  (-1/4*a-5/3*a^2), (1-1*a+1/2*a^2), (4-3/2*a+1/2*a^2),
  (-2/5+a-3/2*a^2), (-1*a+a^2+2*a^3) ]
[ (4-1/2*a-1*a^2+3/2*a^3), (4/5-2/3*a+4/3*a^3), (1+a+2*a^2-1*a^3),

```



```

(3/4+3*a+3*a^2), (-1-1/5*a^3), (-1/4*a-5/3*a^2), (1-1*a+1/2*a^2),
(4-3/2*a+1/2*a^2), (-2/5+a-3/2*a^2), (-1*a+a^2+2*a^3) ] ->
[ g1, g2, g3, g4, g5, g6, g7, g8, g9, g10 ]

```

```

gap> y := RandomGroupElement( elms );
(-475709724976707031371325/71806328788189775767952976
-379584641261299592239825/13055696143307231957809632*a
-462249188570593771377595/287225315152759103071811904*a^2+
2639763613873579813685/2901265809623829323957696*a^3)

```

```

gap> ImageElm( isom, y );
g1^-1*g3^-2*g6^2*g8^-1*g9^-1
gap> z := last;
g1^-1*g3^-2*g6^2*g8^-1*g9^-1

```

```

gap> PreImagesRepresentative( isom, z );
(-475709724976707031371325/71806328788189775767952976
-379584641261299592239825/13055696143307231957809632*a
-462249188570593771377595/287225315152759103071811904*a^2+
2639763613873579813685/2901265809623829323957696*a^3)

```

4

Installation

This package provides an interface between GAP 4 and KANT respectively KASH, the shell of the computational algebraic number theory system KANT. KASH itself is not part of this package. It has to be obtained and installed independently of this package.

4.1 Getting and installing KASH

KASH is available at

www.math.tu-berlin.de/~kant/download.html

Note that you have to download two files for a complete installation of KASH. For the installation of version 2.4 of KASH on a Linux system you would do the following steps:

1. Download the files `kash_2.4.common.tar.gz` and `kash_2.4.1.linux.tar.gz` into the same directory on your system.
2. Unpack the files using `tar`. This will create a directory `KASH.2.4` containing among other files the KASH executable called `kash`. The place where KASH is put is independent of the place where the Alnuth-package is installed.

4.2 Installing this package

This package is available at

<http://cayley.math.nat.tu-bs.de/software/assmann/Alnuth>

in form of a gzipped tar-archive or as an uncompressed tar-archive.

There are two ways of installing the package. If you have permission to add files to the installation of GAP 4 on your system you may install the Alnuth-package into the `pkg` subdirectory of the GAP installation tree. If you do not have the permission to do that you may install the Alnuth-package in your private area.

4.2.1 Installation in the GAP 4 `pkg` subdirectory on a Unix system.

We assume that the archive file `alnuth.tar.gz` or `alnuth.tar` is present in `pkg` and that the current directory is `pkg`. All that needs to be done is to unpack the archive.

```
bash> tar xzf alnuth.tar.gz          # for the gzipped tar-archive
bash> tar xf alnuth.tar              # for the uncompressed tar-archive
bash> gap4
[... startup messages ...]
gap> RequirePackage("alnuth");
true
gap>
```

Now edit the file `alnuth/read.g` and change the line

```
KANTEXEC := "kash"
```

to something like

```
KANTEXEC := "mykash/kash -l mykash/lib"
```

where `mykash` needs to be replaced with the directory where `kash` was installed.

4.2.2. Installation in a private directory

We assume that we are in a directory called `mygap` which also contains the archive file of the `Alnuth`-package.

```
bash> mkdir pkg
bash> mv alnuth.tar.gz pkg
bash> tar xzf alnuth.tar.gz
```

Now edit the file `alnuth/read.g` and change the line

```
KANTEXEC := "kash"
```

to something like

```
KANTEXEC := "mykash/kash -l mykash/lib"
```

where `mykash` needs to be replaced with the directory where `kash` was installed.

When you start `GAP 4` you have to use the option `-l` in the following manner:

```
gap4 -l ";mygap"
```

Note the semicolon! It is important to have it there. The effect is that the directory `mygap` is appended to the list of directories which `GAP` searches for input data. Note that you do not say `mygap/pkg`.

4.2.3. Running the test suite

Once the package is installed, it is possible to check the correct installation by running the test suite of the package.

```
gap> Read( "mygap/pkg/alnuth/tst/testall.g" );
```

where `mygap` needs to be replaced with the directory where `gap` was installed.

Bibliography

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